TECHNICAL EFFICIENCY EVALUATION: NATURALLY DUAL![*]

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ABSTRACT

We provide a dual perspective on technical efficiency evaluation, in two respects. First, we build on the price assumptions implicitly associated with the notion of technical efficiency in a general equilibrium framework to characterize a set of appropriate references to be used in the technical efficiency evaluation of an input-output vector. Some existing evaluation methods always select an element of this set, but other methods fail to do so. Second, the above framework leads us to assert that a well-grounded measure of technical efficiency is naturally decomposable. One part refers to technical efficiency as captured by the classical Debreu-Farrell measure. The other part refers to technical efficiency resulting from the "implicit allocative efficiency" or "mix efficiency" of the evaluated vector. We present both a quantity-based distance measure and its price-based equivalent to evaluate this complementary dimension of technical efficiency. This generalized perspective encompasses the standard Debreu-Farrell framework for technical efficiency evaluation, and makes it fully consistent with the well-established Koopmans efficiency notion.

Key words: production theory, productivity analysis, technical efficiency, implicit cost minimization, gauge functions.

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1. Introduction

It is a commonplace that technical efficiency analysis, which only uses quantity information, is an attractive second-best option when economic efficiency analysis, which uses both quantity and (reliable) price information, is practically impossible. This statement is true insofar as technical efficiency, the absence of waste in production, is a necessary condition for economic efficiency. But it is also slightly misleading because technical efficiency analysis itself builds on underlying assumptions about *shadow* prices, which may enable to draw minimal inferences about economic efficiency without necessitating price information. Building on this close relationship between technical and economic efficiency, this paper sets out an economically consistent framework for technical efficiency evaluation.

Let us briefly sketch the main points developed on the following pages. Throughout, we take it for granted that “technical efficiency” refers to the well-established Koopmans (1951) definition of technical efficiency. This is an uncontested notion that is rooted in the Pareto optimality criterion and is naturally associated with the fundamental theorems of welfare economics. We will next make the distinction between (a) Koopmans technical inefficiency as captured by the classical measure introduced by Debreu (1951) and Farrell (1957), and (b) Koopmans technical inefficiency that can be considered as “implicit allocative” or “mix” inefficiency. Our use of the Debreu-Farrell gauge (be it here as one component of an overall technical efficiency measure) is inspired by its natural and attractive interpretation in terms of shadow prices. Still, when used exclusively, this gauge does not always indicate Koopmans technical efficiency: it may select allegedly efficient references that are actually inconsistent with the fundamental welfare theorems. The rest of the paper then revolves around the question whether and how the shadow price interpretation of the Debreu-Farell gauge can be preserved while overcoming at the same time its indication problem. The positive answer to that question actually instantiates the need for a complementary measure of mix inefficiency.

Section 2 introduces the class of technologies we consider. Section 3 sets out the framework for our discussion by briefly recapturing established notions of technical efficiency, and by offering a prelude on the concept of mix inefficiency. Sections 4 and 5 form the core of the paper. Basically, they each capture a different dimension of any (economically consistent) technical efficiency evaluation process: (a) identify a suitable Koopmans efficient reference; (b) evaluate the distance between this reference and the vector under evaluation, using a measure that has an economically meaningful interpretation in price terms. Specifically, in Section 4 we characterize the set of Koopmans efficient references that indeed preserve the shadow cost legitimisation associated with the Debreu-Farrell gauge. We also look at some existing reference selection procedures and indicate whether or not they always select references belonging to this set. Section 5 then presents the implicated measure of mix inefficiency and shows that this measure can equivalently be interpreted as a measure of dominance in price space.

Two important practical issues that relate directly to the proposed procedure for an economically justifiable evaluation of technical efficiency are discussed in the following sections. Section 6 discusses commensurability of the mix efficiency measure. Finally, Section 7 presents and discusses the "comprehensive", mix-adjusted Debreu-Farrell measure in its capacity as a generally applicable gauge for Koopmans technical efficiency evaluation. The proofs of our propositions can be found in section 8.

Before we begin our actual discussion, we point out that we deliberately provide a general framework. Our final purpose is not to add yet another specific alternative to an already quite long list of technical efficiency gauges. Rather, we want (a) to provide economic arguments favouring a particular class of evaluation procedures, and particularly (b) to show that the measurement of "implicit allocative" or "mix" efficiency should be considered as a natural and fully-fledged component of technical efficiency evaluation.

2. Production Technologies

We will start from a production technology with graph

$$T = \left\{ (\mathbf{x},\mathbf{u}) \middle| \mathbf{x} \in \Re^n_+ \setminus \{\mathbf{0}_n\} \text{ can produce } \mathbf{u} \in \Re^m_+ \setminus \{\mathbf{0}_m\} \right\},$$

with $\mathbf{x} = (x_1,...,x_n)$ an input vector, $\mathbf{u} = (u_1,...,u_m)$ an output vector and $\mathbf{0}_i$ $(i = n,m)$ the i-dimensional zero vector. Clearly, $T \subseteq \Re^{n+m}_+$. We will consider the following properties of T

(T1) T is non-empty.
(T2) T is closed.
(T3) T is convex.
(T4) $card(V) < +\infty$ for $V = \{(\mathbf{x},\mathbf{u}) | \Lambda(\mathbf{x},\mathbf{u}) = \varnothing\}$ and

$$\Lambda(\mathbf{x},\mathbf{u}) = \left\{ \lambda \in (0,1) \middle| \begin{array}{l} \exists(\mathbf{x}',\mathbf{u}') \in T, (\mathbf{x}'',\mathbf{u}'') \in T : (\mathbf{x}',\mathbf{u}') \neq (\mathbf{x},\mathbf{u}) \\ \wedge (\mathbf{x},\mathbf{u}) = \lambda(\mathbf{x}',\mathbf{u}') + (1-\lambda)(\mathbf{x}'',\mathbf{u}'') \end{array} \right\}.$$

Properties (T1) and (T2) are technical conveniences. Property (T3) implies decreasing marginal rates of technical substitution/transformation between inputs, between outputs and between inputs and outputs. Property (T4) implies that the number of vectors that cannot be reconstructed as convex combinations of other vectors in T is finite. Imposing property (T4) simplifies our following discussion considerably as it allows us to exclude peculiar special cases. We emphasize, however, that our results are easily extended to settings that do not meet (T4), although they would become more ambiguous.[1]

Remark 1: Property (T4) does not seem very restrictive, especially as regards empirical application. For example, it is satisfied by all activity analysis models (see Von Neumann (1945), Koopmans (1951) and Shephard (1970)) and by all models that are summarized under the denomination Data Envelopment Analysis (after Charnes, Cooper and Rhodes (1978); see Färe et al. (1994) for a survey).

[1] For example, (T4) guarantees that marginal rates of substitution/transformation do not smoothly go to zero at a technical efficient point, which conveniently allows for our price characterization of technical Koopmans efficiency in Proposition 1. See for example Mass-Colell et al. (1995, pp. 149-152) for the price characterization of Koopmans technical efficiency when a property like (T4) does not hold.

Finally, it is met by discrete sets. See for example the non-parametric approach (after Afriat (1972) and Hanoch and Rothschild (1972)) that analyses production starting from the discrete set of observed input-output vectors.

Most of our discussion below will focus on input efficiency. For that purpose, we will use the equivalent representation of T in terms of input correspondences (see Shephard (1970))

$$L(\mathbf{u}) = \{\mathbf{x} | (\mathbf{x}, \mathbf{u}) \in T\}.$$

Analogously as before, we will consider $L(\mathbf{u})$ that satisfies

(L1) $L(\mathbf{u})$ is non-empty.

(L2) $L(\mathbf{u})$ is closed.

(L3) $L(\mathbf{u})$ is convex.

(L4) $card(E) < +\infty$ for $E = \{(\mathbf{x}, \mathbf{u}) | \Gamma(\mathbf{x}, \mathbf{u}) = \varnothing\}$ and
$\Gamma(\mathbf{x}, \mathbf{u}) = \{\gamma \in (0,1) | \exists \mathbf{x}' \in L(\mathbf{u}), \mathbf{x}'' \in L(\mathbf{u}) : \mathbf{x}' \neq \mathbf{x} \wedge \mathbf{x} = \gamma\ \mathbf{x}' + (1-\gamma)\mathbf{x}''\}$.

Property (L3) implies decreasing marginal rates of substitution between inputs.
The interpretation of (L4) is similar to that of (T4).

3. Technical Efficiency: A Generalized Perspective

As indicated in the introduction, we employ the Koopmans (1951) notion of technical efficiency. Specifically, a production vector $(\mathbf{x}, \mathbf{u}) \in T$ is called *Koopmans efficient* if and only if[2]

(1) $\{(\mathbf{x}', \mathbf{u}') \in T | (-\mathbf{x}', \mathbf{u}') \geq (-\mathbf{x}, \mathbf{u})\} = \varnothing$.

That is, Koopmans efficiency is achieved if and only if no input can be reduced and no output can be increased for given amounts of the remaining inputs and outputs. This technical efficiency criterion, which is clearly inspired on the Pareto optimality criterion, is widely accepted. It can be given a nice interpretation in terms of "implicit" or "shadow" prices, so revealing the link between technical and economic efficiency. To see this let us first restrict attention to the class of technologies that are convex. We have[3]

Proposition 1: Under (T1), (T2), (T3) and (T4),
$\{(\mathbf{x}', \mathbf{u}') \in T | (-\mathbf{x}', \mathbf{u}') \geq (-\mathbf{x}, \mathbf{u})\} = \varnothing \Leftrightarrow$
$\exists \mathbf{p} \in \Re^n_{++}, \mathbf{w} \in \Re^m_{++} : \mathbf{w} \cdot \mathbf{u} - \mathbf{p} \cdot \mathbf{x} >= \mathbf{w} \cdot \mathbf{u}' - \mathbf{p} \cdot \mathbf{x}' \quad \forall (\mathbf{x}', \mathbf{u}') \in T$.

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[2] We use > (<) to denote "strictly bigger (smaller) than", >= (=<) to denote "bigger (smaller) than or equal to" and $\mathbf{y} \geq \mathbf{z}\ (\mathbf{y} \leq \mathbf{z})$ when $\mathbf{y} >= \mathbf{z}\ (\mathbf{y} =< \mathbf{z})$ and $\mathbf{y} \neq \mathbf{z}$.

[3] We use $\Re^z_{++} = \{\mathbf{y} \in \Re^z_+ | \mathbf{y} > \mathbf{0}_z\}$.

This is a well-known result. See for example Koopmans (1951) for his activity analysis setting. In words, it says that a production vector is Koopmans efficient if and only if it is supportable by profit maximization for a strictly positive price vector. In fact, it restates the fundamental theorems of welfare economics (the first fundamental theorem of welfare economics gives the "if"-part, and the second fundamental theorem of welfare economics the "only if"-part).

A similar result can be derived if T is not necessarily convex. Specifically,

Proposition 2: Under (T1), (T2) and (T4),
$\{(\mathbf{x}',\mathbf{u}') \in T | (-\mathbf{x}',\mathbf{u}') \geq (-\mathbf{x},\mathbf{u})\} = \varnothing \Leftrightarrow$
$\forall(\mathbf{x}',\mathbf{y}') \in T : \exists \mathbf{p} \in \Re^n_{++}, \mathbf{w} \in \Re^m_{++} : \mathbf{w} \cdot \mathbf{y} - \mathbf{p} \cdot \mathbf{x} >= \mathbf{w} \cdot \mathbf{y}' - \mathbf{p} \cdot \mathbf{x}'$.

This result clearly reveals the weaker Koopmans efficiency criterion in price terms when convexity does not hold. Of course, when convexity does hold the price conditions in Propositions 1 and 2 are equivalent. Quinzii (1992) provides an in-depth discussion of the economic (price) meaning of the Koopmans efficiency criterion for non-convex technologies within a general equilibrium context. To focus our further discussion, we will abstract from non-convexities in the following.

Remark 2: Our results are easily extended towards non-convex technologies. The reason is that monotonicity properties do not interfere with the technical efficiency concepts that we consider, and monotone sets can always be redefined as a union of convex sets. See for example Proposition 2 that readily follows from Proposition 1.

In practice, technical efficiency analysis is mostly input- or output-oriented. For example, because outputs are exogenously fixed, the focus of the analysis is on whether inputs are used in a technically efficient way. In the following we will concentrate on input technical efficiency, as this has most often been analyzed in the literature.

Remark 3: Extensions of our results towards output technical efficiency are straightforward. In addition, technical efficiency analysis that is not input or output oriented could be treated along the same lines, starting e.g. from the directional distance function framework for technical efficiency measurement as introduced by Chambers et al. (1998).[4] For example, the McFadden gauge function (that can be fitted in the directional distance function framework) has a dual interpretation in terms of profit levels (see McFadden, 1978).

An input vector $\mathbf{x} \in L(\mathbf{u})$ is called *Koopmans input efficient* if and only if

(2) $\{\mathbf{x}' \in L(\mathbf{u}) | \mathbf{x}' \leq \mathbf{x}\} = \varnothing$.

Koopmans input efficiency is a necessary condition for Koopmans efficiency as defined in (1). (In the remainder we will sometimes use Koopmans efficiency instead

[4] See Cherchye et al. (2000) for a discussion of directional distance functions for non-convex technologies.

of Koopmans input efficiency when the meaning is clear from the context.) The *efficient subset* of $L(\mathbf{u})$ (see Shephard (1970)) contains the Koopmans input efficient points and is defined as

$$Eff\ L(\mathbf{u}) = \{\mathbf{x} \in L(\mathbf{u}) | \mathbf{x}' \leq \mathbf{x} \Rightarrow \mathbf{x}' \notin L(\mathbf{u})\}.$$

The analogue of Proposition 1 is

Proposition 3: Under (L1), (L2), (L3) and (L4),
$\mathbf{x} \in Eff\ L(\mathbf{u}) \Leftrightarrow \exists \mathbf{p} \in \Re^n_{++} : \mathbf{p} \cdot \mathbf{x} =< \mathbf{p} \cdot \mathbf{x}' \quad \forall \mathbf{x}' \in L(\mathbf{u})$.

Thus, Koopmans input efficiency of $(\mathbf{x}, \mathbf{u})$ is equivalent to $\mathbf{x}$ being cost minimizing over $L(\mathbf{u})$ for a strictly positive input price vector.

Koopmans technical efficiency is directly tested for a given specification of *T*. In practice, deviations from Koopmans efficiency are frequently observed. Such observations are based on technical efficiency measures. A well-known and frequently employed technical input efficiency measure is the Debreu (1951)-Farrell (1957) (DF) measure. We will consider an extended version of the original DF measure,[5] i.e. for $\mathbf{x} \in L(\mathbf{u})$

$$E_{DF}(\mathbf{x}, \mathbf{u}) = \min\{\rho \in \Re | \rho\ \mathbf{x} \in M(L(\mathbf{u}))\} \text{ with } M(L(\mathbf{u})) = L(\mathbf{u}) + \Re^n_-.$$

Obviously, $E_{DF}(\mathbf{x}, \mathbf{u}) \in (0,1]$ for $\mathbf{x} \in L(\mathbf{u})$. The (extended) DF measure compares $\mathbf{x}$ to $M(L(\mathbf{u}))$, the monotone hull of $L(\mathbf{u})$. Note that using $M(L(\mathbf{u}))$ instead of $L(\mathbf{u})$ in (2) would not affect the Koopmans efficiency condition, i.e. $Eff\ L(u) = Eff\ M(L(\mathbf{u}))$ (see also Färe (1975)).

Now, in view of the aforementioned solid underpinnings of Koopmans technical efficiency, it is desirable to give an equal economic interpretation in price terms for inefficient observations. The DF measure has a convenient interpretation in terms of shadow prices, which undoubtedly helps to explain its benchmark status in the literature:

Proposition 4: Under (L1), (L2), (L3) and (L4),
$E_{DF}(\mathbf{x}, \mathbf{u}) = \max_{\mathbf{p} \in \Re^n_+} \min_{\mathbf{x}' \in L(\mathbf{u})} \left\{ \frac{\mathbf{p} \cdot \mathbf{x}'}{\mathbf{p} \cdot \mathbf{x}} \middle| \mathbf{p} \cdot \mathbf{x} > 0 \right\}$.

In words, the DF measure applies "benefit-of-the-doubt pricing" for measuring the degree of technical inefficiency. Debreu (1951) originally stated a similar result. Several authors (e.g. Russell (1985)) have emphasized this "price" feature to justify using the DF measure (in a setting, to recall, that typically is quantity-based). Indeed,

[5] Färe and Grosskopf (1983) and Färe et al. (1983), who called it the "weak (input) measure of technical efficiency", first introduced the technical efficiency measure $E_{DF}(\mathbf{x}, \mathbf{u})$. The denomination "extended Farrell measure" is due to Russell (1988, 1990).

the above property implies that the DF input measure provides a natural upper bound to the degree of economic/cost efficiency (i.e. the ratio of minimal to actual cost).

There exists a one-to-one relationship between the DF measure and the *isoquant* of $M(L(\mathbf{u}))$ (see Shephard, 1970)

$$Isoq\, M(L(\mathbf{u})) = \{\mathbf{x} \in M(L(\mathbf{u})) | \rho \in (0,1] \Rightarrow \rho\, \mathbf{x} \notin M(L(\mathbf{u}))\}.$$

Obviously, $E_{DF}(\mathbf{x},\mathbf{u}) = 1 \Leftrightarrow \mathbf{x} \in Isoq\, M(L(\mathbf{u}))$. However, while $Eff\ L(\mathbf{u}) \subseteq Isoq\, M(L(\mathbf{u}))$ both sets do generally not coincide for the class of technologies that we consider. As such, the DF efficiency measure does not *indicate* Koopmans efficiency. This *indication problem* has received much attention in the axiomatic literature on technical efficiency gauges (after Färe and Lovell, 1978). In this paper we deviate from this axiomatic literature precisely by explicitly upholding Koopmans' (and Debreu's) own "economic" shadow price perspective when solving the indication problem. From this perspective the indication problem can be given a specific price interpretation, as from Proposition 4 we have

Corollary 1: Under (L1), (L2), (L3) and (L4),
$\mathbf{x} \in Isoq\, M(L(\mathbf{u})) \Leftrightarrow \exists \mathbf{p} \in \Re^n_+ : 0 < \mathbf{p} \cdot \mathbf{x} =< \mathbf{p} \cdot \mathbf{x}' \quad \forall \mathbf{x}' \in L(\mathbf{u})$.

In fact, for $\mathbf{x} \in Isoq\, M(L(\mathbf{u}))$ it suffices to be cost minimizing over $L(\mathbf{u})$ for a price vector that may contain zero prices, whereas $\mathbf{x} \in Eff\ L(\mathbf{u})$ requires cost minimization over $L(\mathbf{u})$ for a strictly positive price vector.

Example 1: The potential conflict between Koopmans and DF input efficiency is illustrated for a two-input situation in Figure 1, where the input efficiency of a vector $(\mathbf{x},\mathbf{u})$ is to be evaluated. In the diagram $Eff\ L(\mathbf{u})$ corresponds to the facet $\overline{AB}$ whereas $Isoq\, M(L(\mathbf{u}))$ also contains the vertical and horizontal facets $\overline{AC}$ and $\overline{BD}$. The measure $E_{DF}(\mathbf{x},\mathbf{u}) = \|\mathbf{x}_e\| / \|\mathbf{x}\|$ clearly has a benefit of the doubt interpretation in price terms. For example, it exceeds $\|\mathbf{x}_e^{'}\| / \|\mathbf{x}\|$ and –to an even greater extent- $\|\mathbf{x}_e^{''}\| / \|\mathbf{x}\|$, where $\mathbf{x}_e^{'}$ and $\mathbf{x}_e^{''}$ respectively lie on the supporting ("isocost") hyperplanes of the isoquant facets $\overline{AC}$ and $\overline{BD}$. However, the DF reference vector $\mathbf{x}_e$ does not belong to $Eff\ L(\mathbf{u})$, although it would be labeled DF efficient. In price terms, it is only cost minimizing over $L(\mathbf{u})$ when a zero price is accorded to the second input.

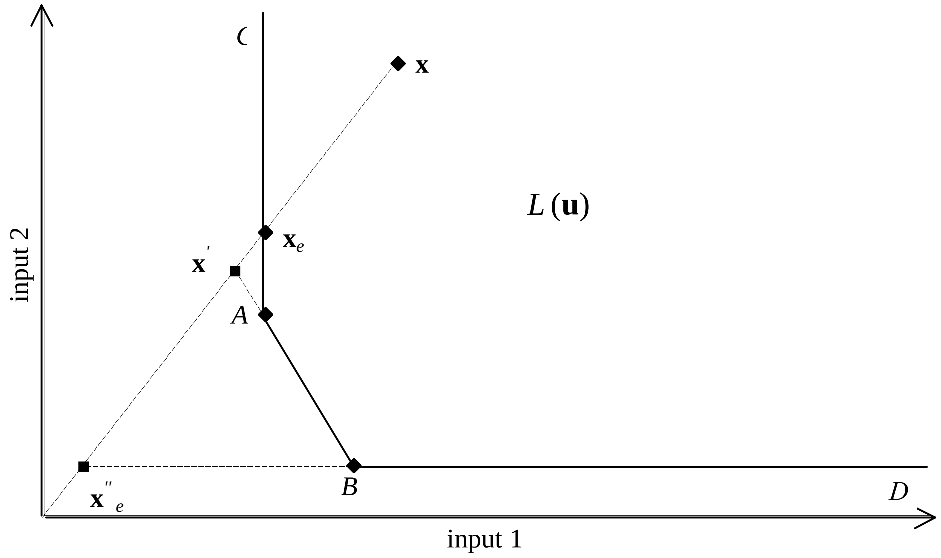

Figure 1

As Figure 1 illustrates, the DF efficiency evaluation procedure may fail to indicate Koopmans efficiency, and, furthermore, technical efficiency may be estimated with respect to points that are themselves not Koopmans efficient. Stated otherwise, applying DF efficiency evaluation does not completely eliminate wasteful production (or 'slack') in the reference vector, a feature which evidently is mirrored by zero shadow prices.

The example also hints at a way to solve the indication problem: it clearly reveals that the part of Koopmans inefficiency which is not captured by the *DF* efficiency gauge pertains to the input *mixes* (or input proportions) of the input vector under evaluation. Hence, we can call it "mix" inefficiency. This denomination refers to the primal variables of analysis, i.e. the physical input quantities. Alternatively, referring to its shadow price interpretation, we can call it "implicit cost" inefficiency, or "implicit allocative" inefficiency when using Farrell's concept of allocative efficiency (i.e. the ratio of economic/cost efficiency over technical efficiency). The denomination "implicit allocative (in)efficiency" hence refers to the dual interpretation of technical efficiency in price terms.

4. Benefit-of-the-Doubt Pricing and Koopmans Efficiency

In the following we will use $P(\mathbf{x},\mathbf{u})$ to denote the correspondence capturing all shadow price vectors under which $\mathbf{x} \in Isoq\,M(L(u))$ is cost minimizing over $L(\mathbf{u})$, i.e.

$$P(\mathbf{x},\mathbf{u}) = \{\mathbf{p} \in \Re^n_+ | 0 < \mathbf{p}\cdot\mathbf{x} =< \mathbf{p}\cdot\mathbf{x}' \quad \forall \mathbf{x}' \in L(\mathbf{u})\} \quad \forall \mathbf{x} \in Isoq\,M(L(\mathbf{u})).$$

$P(\mathbf{x},\mathbf{u})$ defines the (set of) supporting hyperplane(s) tangent to $L(\mathbf{u})$ in $\mathbf{x}$.[6] In addition, we will use the following definitions for notational convenience:

$$\mathbf{x}_{DF}(\mathbf{x},\mathbf{u}) = E_{DF}(\mathbf{x},\mathbf{u})\mathbf{x},$$
$$P_{DF}(\mathbf{x},\mathbf{u}) = P(\mathbf{x}_{DF}(\mathbf{x},\mathbf{u}),\mathbf{u}).$$

Proposition 4 provides a shadow cost efficiency characterization for $\mathbf{x}_{DF}(\mathbf{x},\mathbf{u})$. It is this feature which makes $\mathbf{x}_{DF}(\mathbf{x},\mathbf{u})$ an appealing reference vector from an economic point of view. On the other hand, it may still be that $\mathbf{x}_{DF}(\mathbf{x},\mathbf{u}) \notin \mathit{Eff}\ L(\mathbf{u})$. We now show that there always exist references that preserve both the DF benefit-of-the-doubt interpretation *and* pass the test for Koopmans efficiency. Specifically, these "pivotal" references belong to

$$X_D(\mathbf{x},\mathbf{u}) = \{\mathbf{x}' | \mathbf{x}' \in \mathit{Eff}\ L(\mathbf{u}) \wedge \mathbf{x}' =< \mathbf{x}_{DF}(\mathbf{x},\mathbf{u})\}.$$

Trivially, $P_{DF}(\mathbf{x},\mathbf{u}) \subseteq P(\mathbf{x}_D,\mathbf{u}) \forall \mathbf{x}_D \in X_\mathrm{D}(\mathbf{x},\mathbf{u})$. Hence, all $\mathbf{x}_D \in X_D(\mathbf{x},\mathbf{u})$ preserve the attractive price interpretation of $\mathbf{x}_{DF}(\mathbf{x},\mathbf{u})$ (see Proposition 4). But this point can be strengthened by

Proposition 5: Under (L1), (L2), (L3) and (L4),
$\forall \mathbf{x}_D \in X_\mathrm{D}(\mathbf{x},\mathbf{u}), \mathbf{x}' \in \mathit{Eff}\ L(\mathbf{u}) \setminus X_\mathrm{D}(\mathbf{x},\mathbf{u}), \mathbf{p}_D \in P(\mathbf{x}_D,\mathbf{u}), \mathbf{p}' \in P(\mathbf{x}',\mathbf{u})$:
$$\frac{\mathbf{p}_D \cdot \mathbf{x}_D}{\mathbf{p}_D \cdot \mathbf{x}} >= \frac{\mathbf{p}' \cdot \mathbf{x}'}{\mathbf{p}' \cdot \mathbf{x}}.$$

Thus, for each $\mathbf{x}_D \in X_D(\mathbf{x},\mathbf{u})$ the associated implicit cost ratio will never be lower than for any $\mathbf{x}' \in \mathit{Eff}\ L(\mathbf{u}) \setminus X_\mathrm{D}(\mathbf{x},\mathbf{u})$. The interpretation is analogous to the one of Proposition 4. The important difference is that Proposition 5 is defined with reference to $\mathit{Eff}\ L(\mathbf{u})$, whereas Proposition 4 concerns $\mathit{Isoq}\ L(\mathbf{u})$. Hence, in view of Proposition 3, the result is appealing since it allows us to use strictly positive price vectors in comparisons. Stated otherwise, Proposition 5 reveals that input technical efficiency evaluation of a vector $(\mathbf{x},\mathbf{u})$ by comparing it to a vector $\mathbf{x}_D \in X_D(\mathbf{x},\mathbf{u})$ is consistent with both the fundamental theorems of welfare economics and the convenient price interpretation of the DF input measure.

Example 2: We take again our example of Figure 1. In this case the set $X_D(\mathbf{x},\mathbf{u}) = \{A\}$. Cost ratios for $\mathbf{x}$ as obtained from using any $\mathbf{p} \in P(A,\mathbf{u})$ are not below $\|\mathbf{x}_e'\| / \|\mathbf{x}\|$. In geometric terms, one can always find a supporting hyperplane through A that crosses the radial somewhere between $\mathbf{x}_e'$ and $\mathbf{x}_e$. Obviously, no other element belonging to the facet $\overline{AB}$ would yield a strictly higher ratio value.

Remark 4: In the efficiency measurement literature there have been several proposals that always yield references in $\mathit{Eff}\ L(\mathbf{u})$ (i.e. that "solve" the indication problem of

[6] Our definition of the correspondence $P(\mathbf{x},\mathbf{u})$ is directly related to the single-valued cost function concept (see Shephard (1970)).

DF measures). But some of these proposals may select references that do not necessarily belong to $X_D(\mathbf{x},\mathbf{u})$, e.g. those introduced by Färe and Lovell (1978) and Charnes et al. (1985). Other proposals always select a reference in $X_D(\mathbf{x},\mathbf{u})$, e.g. those introduced by Zieschang (1984), Banker et al. (1984), Coelli (1998) and Cherchye and Van Puyenbroeck (1999a). Loosely stated, this concerns a class of multi-stage procedures which first apply the Debreu-Farrell projection method "to get on the production frontier" and subsequently adjust for any remaining zero shadow prices "by moving along the frontier" (i.e. by selecting an appropriate supporting price hyperplane characterized by prices $\mathbf{p} \in \Re^n_{++}$).

While proposition 3 indicates that an *appropriate* reference (from a price perspective) can be found in $X_D(\mathbf{x},\mathbf{u})$ it does not identify a *unique* reference. Of course, if $\mathbf{x}_{DF}(\mathbf{x},\mathbf{u}) \in \mathit{Eff}\ L(\mathbf{u})$ then $X_D(\mathbf{x},\mathbf{u}) = \{\mathbf{x}_{DF}(\mathbf{x},\mathbf{u})\}$. Also, even if $\mathbf{x}_{DF}(\mathbf{x},\mathbf{u}) \notin \mathit{Eff}\ L(\mathbf{u})$ the set $X_D(\mathbf{x},\mathbf{u})$ will always be a singleton if $n = 2$ (see the example). In general it could be that $X_D(\mathbf{x},\mathbf{u})$ contains more elements. The following proposition demonstrates that, in such cases, each input vector $\mathbf{x}_D \in X_D(\mathbf{x},\mathbf{u})$ is justifiable from a general price perspective:

Proposition 6: Under (L1), (L2), (L3) and (L4),
$\forall \mathbf{x}_D, \mathbf{x}'_D \in X_D(\mathbf{x},\mathbf{u}), \mathbf{x}_D \neq \mathbf{x}'_D : \forall \mathbf{p}'_D \in P(\mathbf{x}'_D,\mathbf{u}), \mathbf{p}'_D > \mathbf{0}_n : \exists \mathbf{p}_D \in P(\mathbf{x}_D,\mathbf{u}), \mathbf{p}_D > \mathbf{0}_n :$
$\dfrac{\mathbf{p}_D \cdot \mathbf{x}_D}{\mathbf{p}_D \cdot \mathbf{x}} >= \dfrac{\mathbf{p}'_D \cdot \mathbf{x}'_D}{\mathbf{p}'_D \cdot \mathbf{x}}$.

As Proposition 6 applies to all $\mathbf{x}_D \in X_D(\mathbf{x},\mathbf{u})$ there is no general economic reason to further discriminate within the set $X_D(\mathbf{x},\mathbf{u})$ in terms of implicit cost efficiency, at least when one is only concerned with a consistent characterization of technical efficiency *per se*.

Remark 5: Given this general orientation, we deliberately leave open the question which reference belonging to $X_D(\mathbf{x},\mathbf{u})$ is the most preferred one if $\mathbf{x}_{DF}(\mathbf{x},\mathbf{u}) \notin X_D(\mathbf{x},\mathbf{u})$ and $n > 2$. Different answers could be provided. As the indication problem pertains to the input mix properties, one could for example take into account feasibility restrictions on input mix adjustments (see e.g. Coelli (1998) and Cherchye and Van Puyenbroeck (1999)). Or, in view of our discussion below on mix efficiency measurement, one can choose a reference selection procedure dependent on the axiomatic properties associated with the concomitant mix efficiency measure (after Färe and Lovell (1978)). But if the sole restriction on price vectors stems from the technical efficiency requirement that there is no wasteful production (i.e. $\mathbf{p} \in \Re^n_{++}$ in shadow price terms), then Proposition 6 states that there may well be different "suitable" Koopmans efficient references for which this restriction holds.

So far we have shown that vectors $\mathbf{x}_D \in X_D(\mathbf{x},\mathbf{u})$ have a sound economic grounding as input references for $(\mathbf{x},\mathbf{u})$. However, we have not yet related this to an efficiency measure. The attractiveness of the DF gauge (see Proposition 4) does not straightforwardly extend when using $X_D(\mathbf{x},\mathbf{u})$. In particular, for $\mathbf{x}_D \neq \mathbf{x}_{DF}(\mathbf{x},\mathbf{u})$ the correspondence $P(\mathbf{x}_D,\mathbf{u})$ will typically be multi-valued. In turn, this leads to a multitude of possible implicit cost ratios.

Actually, one should not be too concerned with this "deadlock". The preceding discussion suggests (a) that there is still room for using the DF measure to rationalize a part of observed Koopmans technical inefficiency, and (b) that the remaining part of inefficiency pertains to the implicit allocative or mix properties of the evaluated vector (which is present as soon as $\mathbf{x}_{DF}(\mathbf{x},\mathbf{u}) \notin \mathit{Eff}\ L(\mathbf{u})$). The question is then whether one can give an economically meaningful interpretation to this second component. How should we assess the difference between the vector $\mathbf{x}_{DF}(\mathbf{x},\mathbf{u})$ and the Koopmans efficient reference $\mathbf{x}_D \in X_D(\mathbf{x},\mathbf{u})$? In the next section we propose a way to evaluate this complementary dimension of suboptimal behavior that falls in line with the dual price interpretation of technical efficiency.

5. The Measurement of Mix Efficiency

To measure mix efficiency, we first construct the vector of directional cosines, which is defined as follows for a vector $\mathbf{x} \in L(\mathbf{u})$

$$\mathbf{cos}_{\mathbf{x}} = \frac{\mathbf{x}}{\|\mathbf{x}\|}.$$

In $\mathbf{cos}_{\mathbf{x}} = \left(\cos_{x_1},\ldots,\cos_{x_n}\right) \in \Re^n_+$ each $\cos_{x_l} \in [0,1] \, (l = 1,\ldots,n)$ gives the cosine of the angle between the vector **x** and the *l*-th input axis (see Charnes and Cooper, 1961, pp. 162-164).

Example 3: To clarify the concept we return to our previous illustration as recaptured in Figure 2. Using $A = (a_1, a_2)$ and $\mathbf{x}_e = (x_{e1}, x_{e2})$ the directional cosines corresponding to A and $\mathbf{x}_e$ are respectively $\cos_{a_l} = \cos\alpha_l = a_l / \|A\|$ and $\cos_{x_{el}} = \cos\beta_l = x_{el} / \|x_e\| \quad (l = 1,2)$.

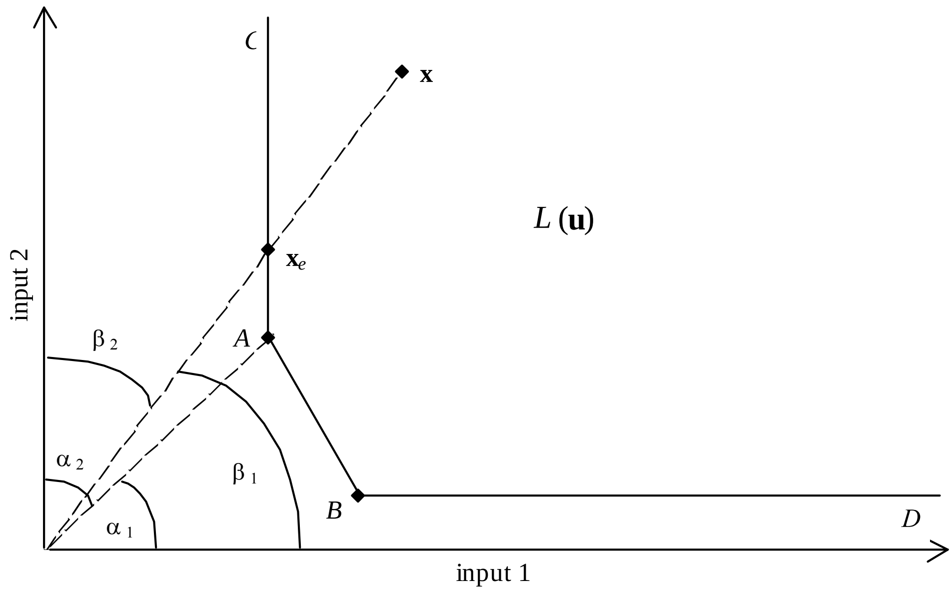

Figure 2

Hence, the vector $\mathbf{cos}_{\mathbf{x}}$ contains dimensional characterizations of the mix of **x**: the proportions between each pair of row entries of **x** can equivalently be represented in terms of directional cosines as $x_i / x_j = \cos_{x_i} / \cos_{x_j} \quad \forall i,j \in \{1,...,n\}$. Two input vectors are collinear if and only if they share the same vector of directional cosines. Formally,

$$(3) \qquad \mathbf{x}, \mathbf{x}' \in L(\mathbf{u}) : \mathbf{x} = \rho\ \mathbf{x}' \ \left(\rho \in (0,+\infty)\right) \Leftrightarrow \mathbf{cos}_{\mathbf{x}} = \mathbf{cos}_{\mathbf{x}'} .$$

Evidently, given (3), when evaluating the input mix efficiency of the vector $(\mathbf{x}, \mathbf{u})$ we can confine attention to its collinear input projection $\mathbf{x}_{DF}(\mathbf{x}, \mathbf{u})$.

In this and the following sections we will assume that an input reference $\mathbf{x}^*_D(\mathbf{x}, \mathbf{u}) \in X_D(\mathbf{x}, \mathbf{u})$ has been selected for $(\mathbf{x}, \mathbf{u})$. Slightly abusing notation, we denote the directional cosine vectors corresponding to $\mathbf{x}_{DF}(\mathbf{x}, \mathbf{u})$ and $\mathbf{x}^*_D(\mathbf{x}, \mathbf{u})$ by respectively $\mathbf{cos}_{\mathbf{x}_{DF}}$ and $\mathbf{cos}_{\mathbf{x}^*_D}$. Each ratio $\cos_{x_{DF,l}} / \cos_{x^*_{D,l}} \ (l = 1,...,n)$ gives an angular representation of mix deviation (e.g. in Figure 2 these are $\left(\cos_{x_{e1}} / \cos_{x_{a1}}\right) < 1$ and $\left(\cos_{x_{e2}} / \cos_{x_{a2}}\right) > 1$). To obtain an overall mix efficiency measure these dimension-specific values should be combined. In analogy to the DF technical efficiency measure we take the ratio of price-weighted sums of the different row elements of $\mathbf{cos}_{\mathbf{x}_{DF}}$ and $\mathbf{cos}_{\mathbf{x}^*_D}$, using $\mathbf{p} \in P_{DF}(\mathbf{x}, \mathbf{u})$ as the weighting vector. The overall mix efficiency measure is hence defined as[7]

$$(4) \qquad E_{ME}(\mathbf{x}, \mathbf{u}) = \frac{\mathbf{p} \cdot \mathbf{cos}_{\mathbf{x}_{DF}}}{\mathbf{p} \cdot \mathbf{cos}_{\mathbf{x}^*_D}} \quad \forall \mathbf{p} \in P_{DF}(\mathbf{x}, \mathbf{u}).$$

---

[7] It will become clear in what follows why the equality in (4) holds for all $\mathbf{p} \in P_{DF}(\mathbf{x}, \mathbf{u})$.

Of course, in view of (3) we can rewrite (4) as

(5) $$E_{ME}(\mathbf{x},\mathbf{u}) = \frac{\mathbf{p}\cdot\mathbf{cos}_{\mathbf{x}}}{\mathbf{p}\cdot\mathbf{cos}_{\mathbf{x}^*_D}} \quad \forall \mathbf{p} \in P_{DF}(\mathbf{x},\mathbf{u}).$$

which links $E_{ME}(\mathbf{x},\mathbf{u})$ directly to the input mix properties of $(\mathbf{x},\mathbf{u})$.

The fact that we use $P_{DF}(\mathbf{x},\mathbf{u})$ rather than e.g. $P(\mathbf{x}^*_D(\mathbf{x},\mathbf{u}),\mathbf{u})$ as the set of admissible shadow price weighting vectors in (4) allows for a convenient third definition of $E_{ME}(\mathbf{x},\mathbf{u})$. As $\mathbf{p}\cdot\mathbf{x}_{DF}(\mathbf{x},\mathbf{u}) = \mathbf{p}\cdot\mathbf{x}^*_D(\mathbf{x},\mathbf{u})\ \forall \mathbf{p} \in P_{DF}(\mathbf{x},\mathbf{u})$ and recalling the definition of $\mathbf{cos}_{\mathbf{x}}$ we get

(6) $$E_{ME}(\mathbf{x},\mathbf{u}) = \frac{\left\|\mathbf{x}^*_D(\mathbf{x},\mathbf{u})\right\|}{\left\|\mathbf{x}_{DF}(\mathbf{x},\mathbf{u})\right\|},$$

from which it is immediate that $E_{ME}(\mathbf{x},\mathbf{u}) \in (0,1]$ and that $E_{ME}(\mathbf{x},\mathbf{u}) = 1$ if and only if $\mathbf{x}_{DF}(\mathbf{x},\mathbf{u}) \in \mathit{Eff}\ L(\mathbf{u})$. The attractiveness of this norm ratio formulation is that it clearly reveals the analogy with the way the DF technical efficiency measure is commonly presented.

Observe that definitions (4), (5) and (6) apply to any $\mathbf{x}^*_D(\mathbf{x},\mathbf{u}) \in X_D(\mathbf{x},\mathbf{u})$ (compare with the "indifference" result in Proposition 6). Conversely, equivalence of the three characterizations crucially depends on the fact that $\mathbf{x}^*_D(\mathbf{x},\mathbf{u}) \in X_D(\mathbf{x},\mathbf{u})$.

The use of $P_{DF}(\mathbf{x},\mathbf{u})$ implies that zero weights are assigned to those input dimensions in which $\mathbf{x}_{DF}(\mathbf{x},\mathbf{u})$ exhibits slack. In general, when $\mathbf{x}_{DF}(\mathbf{x},\mathbf{u})$ exhibits slack in a particular dimension the corresponding directional cosine value will be lower for $\mathbf{x}_{DF}(\mathbf{x},\mathbf{u})$ than for a vector belonging to $X_{\mathrm{D}}(\mathbf{x},\mathbf{u})$. As this cannot be viewed as mix *dominance* it seems natural to accord a zero weight to these cosines.

Example 4: The intuition can be sharpened by means of Figure 2. Only the directional cosines with respect to the first axis are weighted positively as for $\mathbf{x}_e$ the implicit price for the second input equals zero (and $\cos_{x_{e2}} > \cos_{x_{a2}}$). The mix efficiency measure equals $\left(\cos_{x_{e1}} / \cos_{x_{a1}}\right) = \left(\|A\| / \|\mathbf{x}_e\|\right)$.

Evidently, there also exists a dual characterization of $E_{ME}(\mathbf{x},\mathbf{u})$. As we will now show, the proposed mix efficiency measure can as well be interpreted, rather conveniently, as a measure of dominance in price space.

By definition, the cost level associated with $\mathbf{x}_{DF}(\mathbf{x},\mathbf{u})$ and $\mathbf{x}^*_D(\mathbf{x},\mathbf{u})$ is the same for all $\mathbf{p} \in P_{DF}(\mathbf{x},\mathbf{u})$. Let us then reverse the picture and consider all price vectors that imply the same cost level (arbitrarily fixed at unity) for both input vectors. Obviously, this dual perspective allows to consider strictly positive price vectors to compare

$\mathbf{x}_{DF}(\mathbf{x},\mathbf{u})$ and $\mathbf{x}^*_D(\mathbf{x},\mathbf{u})$. Specifically, we consider the following two hyperplanes in price space

$$H(\mathbf{x}_{DF}(\mathbf{x},\mathbf{u}),1) = \{\pi_{DF} \in \Re^n_+ | \pi_{DF} \cdot \mathbf{x}_{DF}(\mathbf{x},\mathbf{u}) = 1\},$$
$$H(\mathbf{x}^*_D(\mathbf{x},\mathbf{u}),1) = \{\pi^*_D \in \Re^n_+ | \pi^*_D \cdot \mathbf{x}^*_D(\mathbf{x},\mathbf{u}) = 1\}.$$

Throughout the domain $\Re^n_+$ isocost hyperplanes associated with $\mathbf{x}^*_D(\mathbf{x},\mathbf{u}) \in X_D(\mathbf{x},\mathbf{u})$ are located at least as far away from the origin as those corresponding to $\mathbf{x}_{DF}(\mathbf{x},\mathbf{u})$. Mix efficiency decreases the more these distances differ.

Example 5: We present these hyperplanes for input vectors $\mathbf{x}_e$ and A of our illustration in Figure 3. At the shadow price vector corresponding to $\mathbf{x}_e$ (with zero weight for the second input) both hyperplanes intersect. The implicit allocative inefficiency of $\mathbf{x}_e$ is revealed by the "dominance" of the hyperplane associated with A over the one associated with $\mathbf{x}_e$. The reason is that, because it uses strictly less of the second input, observation A allows higher (positive) second input prices to imply the same cost level for each possible price assigned to the first input.

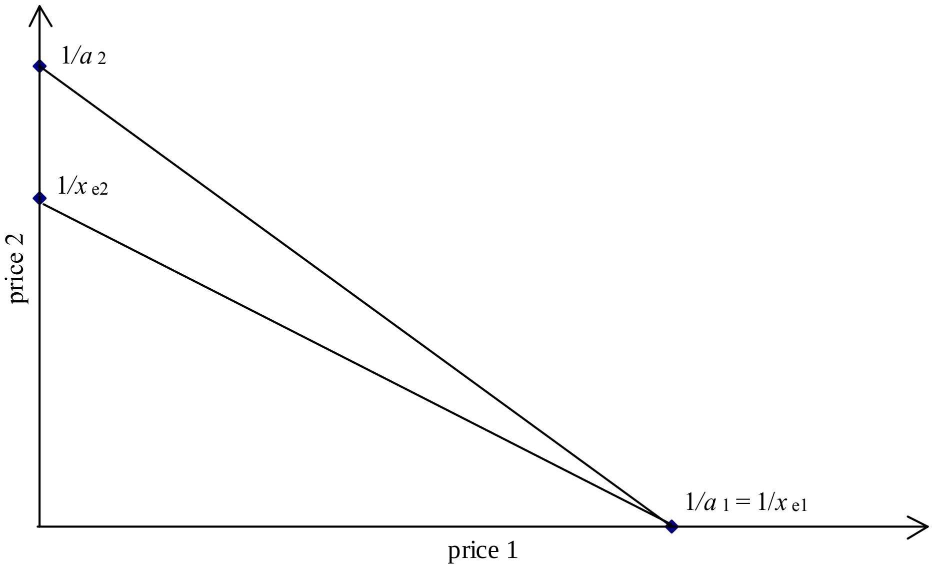

Figure 3

A natural way to evaluate this dominance in price space is to compare the Euclidean distances from these hyperplanes to the origin. These distances are given by

$$d_{\mathbf{x}_{DF}}(\mathbf{x},\mathbf{u}) = \min_{\pi_{DF} \in \Re^n_+} \{\|\pi_{DF}\| \, | \pi_{DF} \cdot \mathbf{x}_{DF}(\mathbf{x},\mathbf{u}) = 1\},$$
$$d_{\mathbf{x}^*_D}(\mathbf{x},\mathbf{u}) = \min_{\pi_{DF} \in \Re^n_+} \{\|\pi^*_D\| \, | \pi^*_D \cdot \mathbf{x}^*_D(\mathbf{x},\mathbf{u}) = 1\}.$$

and we have

(7) $$E_{ME}(\mathbf{x},\mathbf{u}) = \frac{d_{\mathbf{x}_{DF}}(\mathbf{x},\mathbf{u})}{d_{\mathbf{x}^*_D}(\mathbf{x},\mathbf{u})},$$

The equivalence of this price-based dominance characterization with the quantity-based characterization in (6) follows directly from the fact that the distance from a hyperplane to the origin can be retrieved using its normal. In this case, substituting

$$d_{\mathbf{x}_{DF}}(\mathbf{x},\mathbf{u}) = \frac{1}{\left\|\mathbf{x}_{DF}(\mathbf{x},\mathbf{u})\right\|} \text{ and } d_{\mathbf{x}^*_D}(\mathbf{x},\mathbf{u}) = \frac{1}{\left\|\mathbf{x}^*_D(\mathbf{x},\mathbf{u})\right\|}$$

in (7) immediately yields (6).

We conclude this section by showing that characterization (7) allows for coming full circle with our particular choice of $\mathbf{p} \in P_{DF}(\mathbf{x},\mathbf{u})$ as the weighting vector in (4). For this purpose we switch to the Hesse normal form representations of $H(\mathbf{x}_{DF}(\mathbf{x},\mathbf{u}),1)$ and $H(\mathbf{x}^*_D(\mathbf{x},\mathbf{u}),1)$

$$H^n(\mathbf{x}_{DF}(\mathbf{x},\mathbf{u}),1) = \left\{\pi_{DF} \in \Re^n_+ \middle| \pi_{DF} \cdot \mathbf{cos}_{\mathbf{x}_{DF}} = d_{\mathbf{x}_{DF}}(\mathbf{x},\mathbf{u})\right\},$$
$$H^n(\mathbf{x}^*_D(\mathbf{x},\mathbf{u}),1) = \left\{\pi^*_D \in \Re^n_+ \middle| \pi^*_D \cdot \mathbf{cos}_{\mathbf{x}^*_D} = d_{\mathbf{x}^*_D}(\mathbf{x},\mathbf{u})\right\}.$$

Using this representation it can be seen that $E_{ME}(\mathbf{x},\mathbf{u}) = \left[\pi_{DF} \cdot \mathbf{cos}_{\mathbf{x}_{DF}}\right] / \left[\pi^*_D \cdot \mathbf{cos}_{\mathbf{x}^*_D}\right]$ for $\pi_{DF} \in H^n(\mathbf{x}_{DF}(\mathbf{x},\mathbf{u}),1)$ and $\pi^*_D \in H^n(\mathbf{x}^*_D(\mathbf{x},\mathbf{u}),1)$. The only legitimate way to employ a common price-weighting vector in the numerator and the denominator as is done in (4), is to take vectors that lie in the intersection of $H^n(\mathbf{x}_{DF}(\mathbf{x},\mathbf{u}),1)$ and $H^n(\mathbf{x}^*_D(\mathbf{x},\mathbf{u}),1)$. These vectors are indeed all $\mathbf{p} \in P_{DF}(\mathbf{x},\mathbf{u})$ for which $\mathbf{p} \cdot \mathbf{x}_{DF}(\mathbf{x},\mathbf{u}) = \mathbf{p} \cdot \mathbf{x}^*_D(\mathbf{x},\mathbf{u}) = 1$.[8]

6. A Digression on Commensurability

The mix efficiency measure as it has been introduced above is not invariant to the units in which the different input quantities are measured. Russell (1988) has especially advocated the commensurability property.

Remark 6: Before addressing the issue more thoroughly, remember that so far we have assumed the projection $\mathbf{x}^*_D(\mathbf{x},\mathbf{u}) \in X_D(\mathbf{x},\mathbf{u})$ to be given. A first point of interest then concerns the reference selection procedure itself. Zieschang (1984) (see Russell (1988)), Coelli (1998) and Cherchye and Van Puyenbroeck (1999a) suggest procedures for selecting $\mathbf{x}^*_D(\mathbf{x},\mathbf{u}) \in X_D(\mathbf{x},\mathbf{u})$ that satisfy the commensurability property. The procedure proposed by Banker et al. (1984), on the other hand, is incommensurable.

[8] It is immediate that the validity of (7) does not depend on the fact that we considered a cost level of unity. This also implies that this legitimization of (4) holds for all $\mathbf{p} \in P_{DF}(\mathbf{x},\mathbf{u})$.

To obtain units invariance of $E_{ME}(\mathbf{x},\mathbf{u})$ one needs to define the efficiency measure with respect to rescaled input data: dividing each original input quantity by a value expressed in the same measurement unit indeed implies commensurability of this newly defined mix efficiency measure. Briefly elaborating on this point, we now present three examples that yield particularly intuitive reformulations of the mix efficiency measure (computed with respect to the rescaled data) in terms of the original input values.

The norm representation as given in definition (6) provides a convenient point of departure. Using $\mathbf{x}_{DF}(\mathbf{x},\mathbf{u}) = (x_{DF1},...,x_{DFn})$ and $\mathbf{x}_D^*(\mathbf{x},\mathbf{u}) = (x_{D1}^*,...,x_{Dn}^*)$, we re-express it as:

(8) $$E_{ME}(\mathbf{x},\mathbf{u}) = \left[\frac{\sum_{l=1}^{n}(x_{Dl}^*)^2}{\sum_{l=1}^{n}(x_{DFl}^*)^2}\right]^{1/2}.$$

We further use $\hat{x}_l$ to denote the rescaled counterparts of the input entries x_l $(l=1,...,n)$. We similarly use $\hat{E}_{ME}(\mathbf{x},\mathbf{u})$ to indicate the mix efficiency estimate as defined with respect to these rescaled vectors.

Example 6: For $\hat{x}_l = \left(x_l / \sqrt{x_l x_{DFl}}\right)$ (i.e. we divide each input value by the geometric mean of itself and the corresponding value in $\mathbf{x}_{DF}(\mathbf{x},\mathbf{u})$) we get from (8)

(9) $$\hat{E}_{ME}(\mathbf{x},\mathbf{u}) = \left[\frac{\sum_{l=1}^{n}(\hat{x}_{Dl}^*)^2}{\sum_{l=1}^{n}(\hat{x}_{DFl}^*)^2}\right]^{1/2} = \left[\sum_{l=1}^{n}\left(\frac{x_{Dl}^*}{x_{DFl}^*}\right)\Big/ n\right]^{1/2}.$$

This is a monotonic transformation of the arithmetic mean of the original input proportions.[9] It is precisely this arithmetic mean which is minimized in the second stage of the Zieschang (1984) reference selection procedure. It therefore follows that, for this particular rescaling, the Zieschang reference selection procedure is one that minimizes mix efficiency.

Example 7: For $\hat{x}_l = \left(x_l / \sqrt{x_l x_{Dl}^*}\right)$ (i.e. we divide each input value by the geometric mean of itself and the corresponding value in $\mathbf{x}_D^*(\mathbf{x},\mathbf{u})$) we get from (8)

(10) $$\hat{E}_{ME}(\mathbf{x},\mathbf{u}) = \left[n \Big/ \sum_{l=1}^{n}\left(\frac{x_{DFl}}{x_{Dl}^*}\right)\right]^{1/2}.$$

This is a monotonic transformation of the harmonic mean of the original input proportions. The fact that the harmonic mean is obtained instead of the arithmetic mean directly builds on the orientation change implicit in choosing $\mathbf{x}_D^*(\mathbf{x},\mathbf{u})$ as a

9 Strictly speaking the possibility of zero input values is excluded. However, Färe et al. (1983) proposed, a way to circumvent this problem (see also Zieschang (1984)).

basis of comparison. Whereas in (9) one averages over input reductions when going from $\mathbf{x}_{DF}(\mathbf{x},\mathbf{u})$ to $\mathbf{x}_D^*(\mathbf{x},\mathbf{u})$, we now look at input expansions to get from $\mathbf{x}_D^*(\mathbf{x},\mathbf{u})$ to $\mathbf{x}_{DF}(\mathbf{x},\mathbf{u})$ and a mix efficiency estimate is consequently obtained as the inverse of an arithmetic mean. The orientation is thus reversed, resulting in an 'inverse Zieschang procedure' and a correspondingly different mix efficiency estimate (see also Cherchye and Van Puyenbroeck (1999b)).

Example 8: For $\hat{x}_l = \left(x_l / \sqrt{x_{DFl} x_{Dl}^*}\right)$ (i.e. we divide each input value by the geometric mean of the corresponding values in $\mathbf{x}_{DF}(\mathbf{x},\mathbf{u})$ and $\mathbf{x}_D^*(\mathbf{x},\mathbf{u})$) we get from (8)

$$(11)\qquad \hat{E}_{ME}(\mathbf{x},\mathbf{u}) = \left[\left(\sum_{l=1}^{n}\left(\frac{x_{Dl}^*}{x_{DFl}}\right)\Big/ n\right)\cdot\left(n\Big/\sum_{l=1}^{n}\left(\frac{x_{DFl}}{x_{Dl}^*}\right)\right)\right]^{1/2},$$

which is the geometric mean of the estimates in (9) and (10) (ignoring the square roots). Conceptually similar to a Fisher ideal index, this alternative may have particular appeal to some as it avoids having to choose between the two previous orientations.

In each of these examples, the rescaled mix efficiency measure is expressed in terms of the *ratios of the (original) evaluated and reference input vector elements*. Now these are precisely the ratios of the intercepts of the respective associated price hyperplanes, which –as noted in the previous section (see e.g. Figure 3)- reveal information on the dominance of the reference vector in price space. Therefore, the three examples above seem particularly appealing candidates for solving the commensurability problem.

7. A MIX-ADJUSTED DEBREU-FARRELL MEASURE

As we have indicated in the introduction, our aim was to preserve the commonly employed DF measure of technical efficiency in view of its economic intuition while at the same time making it consistent with the well-established notion of Koopmans technical efficiency.. Our discussion makes clear that both the DF and the mix efficiency measures capture separate information about two connected dimensions of technical efficiency performance. However, a measure that merges all this information in a single statistic is sometimes useful. Such a measure can readily be obtained by combining the measures $E_{DF}(\mathbf{x},\mathbf{u})$ and $E_{ME}(\mathbf{x},\mathbf{u})$ into the following *mix-adjusted Debreu-Farrell index*[10]

$$(12)\qquad E_{MA}(\mathbf{x},\mathbf{u}) = E_{DF}(\mathbf{x},\mathbf{u})\cdot E_{ME}(\mathbf{x},\mathbf{u}).$$

In view of definition (6) and given that $E_{DF}(\mathbf{x},\mathbf{u}) = \|\mathbf{x}_{DF}(\mathbf{x},\mathbf{u})\| / \|\mathbf{x}\|$ we can rewrite (12) as

[10] For ease of exposition we assume that the commensurability problem of the mix efficiency component has adequately been dealt with. It should further be noted that the DF component $E_{DF}(\mathbf{x},\mathbf{u})$ satisfies the commensurability property (see Russell (1988)).

$$E_{MA}(\mathbf{x},\mathbf{u}) = \frac{\left\|\mathbf{x}_D^*(\mathbf{x},\mathbf{u})\right\|}{\left\|\mathbf{x}\right\|}$$

which indicates that the mix-adjusted Debreu-Farrell index $E_{MA}(\mathbf{x},\mathbf{u})$ can also be computed directly. The composite index $E_{MA}(\mathbf{x},\mathbf{u})$ then acts as a measure for the relative distance from $\mathbf{x}$ to $Eff\ L(\mathbf{u})$. Moreover, it is well justified by economic/price considerations. Its DF component $E_{DF}(\mathbf{x},\mathbf{u})$ gives the traditional "benefit-of-the-doubt" shadow cost efficiency estimate for $(\mathbf{x},\mathbf{u})$ and measures the distance from $\mathbf{x}$ to $Isoq\ L(\mathbf{u})$. It does not only yield the (maximum) cost efficiency ratio of $(\mathbf{x},\mathbf{u})$ with respect to $\mathbf{x}_{DF}(\mathbf{x},\mathbf{u})$ but also with respect to $\mathbf{x}_D^*(\mathbf{x},\mathbf{u})$ (as $P_{DF}(\mathbf{x},\mathbf{u}) = P(\mathbf{x}_{DF}(\mathbf{x},\mathbf{u}),\mathbf{u}) \subseteq P(\mathbf{x}_D^*(\mathbf{x},\mathbf{u}),\mathbf{u})$ by definition). On the other hand, $E_{ME}(\mathbf{x},\mathbf{u})$ deals with the presence of any slack in (or, alternatively, zero implicit prices associated with) $\mathbf{x}_{DF}(\mathbf{x},\mathbf{u})$ and pertains to the mix properties of $\mathbf{x}$. To conclude, it is worth to emphasize that $E_{MA}(\mathbf{x},\mathbf{u})$ as defined in (12) coincides with $E_{DF}(\mathbf{x},\mathbf{u})$ when this last measure correctly indicates Koopmans efficiency (so that $E_{ME}(\mathbf{x},\mathbf{u}) = 1$), i.e. when zero prices are not needed to reconstruct $E_{DF}(\mathbf{x},\mathbf{u})$. In effect, the mix-adjusted DF efficiency evaluation approach completely encompasses the standard DF approach.

Remark 7: Two final qualifications are in order:

1. Expression (12) provides a comprehensive measure for overall technical efficiency. However, we plead for considering its components separately in order to reveal the fundamental two-stage nature of the reference selection process (first collinearly projecting on the isoquant and next moving along the isoquant towards the efficient subset) more clearly. As our above discussion makes clear, both components provide information about basically two distinct dimensions of technical efficiency performance.
2. Expression (12) provides only one possibility to merge the DF and mix efficiency components. It is attractive because $E_{MA}(\mathbf{x},\mathbf{u})$ coincides with $E_{DF}(\mathbf{x},\mathbf{u})$ when $\mathbf{x}_{DF}(\mathbf{x},\mathbf{u}) \in Eff\ L(\mathbf{u})$, and because $E_{MA}(\mathbf{x},\mathbf{u})$ preserves the norm representation of the $E_{DF}(\mathbf{x},\mathbf{u})$.However, alternative routes could equally well be followed. A thorough discussion falls beyond the scope of the current paper, but we mention one interesting point. One alternative possibility is to select a merging procedure dependent on the axiomatic properties of the resulting mix-adjusted DF measure. In this respect, it is worth to recall the particular rescaling considered in the first example in Section 6 (see (9)). If this would be used to construct a mix-adjusted DF measure (following formula (12)), the result would be quasi-identical to the Zieschang (1984) measure, of which the (attractive) properties are well documented in the axiomatic literature on technical efficiency measures.

8. PROOFS

PROOF OF PROPOSITION 1: First, $\{(\mathbf{x}',\mathbf{u}')\in T \mid (-\mathbf{x}',\mathbf{u}')\geq(-\mathbf{x},\mathbf{u})\}=\varnothing \Leftrightarrow$

$$(13)\qquad \max_{\substack{(\mathbf{x}',\mathbf{u}')\in T\\ \mathbf{s}^I\in\Re_+^n,\mathbf{s}^O\in\Re_+^m}} \left\{\varepsilon\left(\mathbf{e}_n\cdot\mathbf{s}^I+\mathbf{e}_m\cdot\mathbf{s}^O\right)\middle|\mathbf{s}^I =< \mathbf{x}-\mathbf{x}';\quad \mathbf{s}^O =< \mathbf{u}'-\mathbf{u}\right\}=0,$$

where $\mathbf{e}_i$ $(i=n,m)$ is the *i*-dimensional unit vector and $0<\varepsilon<+\infty$ can be chosen arbitrarily small. The right hand side is well defined because of (T1) and (T2).

Next, because of (T3) and (T4) we have

$$T=\left\{(\mathbf{x},\mathbf{u})\middle|\begin{array}{l}\mathbf{x}=\sum_{(\mathbf{x},\mathbf{u})\in V}\lambda_{(\mathbf{x},\mathbf{u})}\mathbf{x}';\quad \mathbf{u}=\sum_{(\mathbf{x},\mathbf{u})\in V}\lambda_{(\mathbf{x},\mathbf{u})}\mathbf{u}';\\ \sum_{(\mathbf{x},\mathbf{u})\in V}\lambda_{(\mathbf{x},\mathbf{u})}=1;\quad \lambda_{(\mathbf{x},\mathbf{u})}\in\Re_+\quad \forall(\mathbf{x}',\mathbf{u}')\in V\end{array}\right\},$$

so that (13) can be re-expressed as

$$\max_{\substack{\lambda_{(\mathbf{x}',\mathbf{u}')}\in\Re_+\ \forall(\mathbf{x}',\mathbf{u}')\in V\\ \mathbf{s}^I\in\Re_+^n,\mathbf{s}^O\in\Re_+^m}}\left\{\varepsilon\left(\mathbf{e}_n\cdot\mathbf{s}^I+\mathbf{e}_m\cdot\mathbf{s}^O\right)\middle|\begin{array}{l}\mathbf{s}^I =< \mathbf{x}-\sum_{(\mathbf{x}',\mathbf{u}')\in V}\lambda_{(\mathbf{x}',\mathbf{u}')}\mathbf{x}';\\ \mathbf{s}^O =< \sum_{(\mathbf{x}',\mathbf{u}')\in V}\lambda_{(\mathbf{x}',\mathbf{u}')}\mathbf{u}'-\mathbf{u};\quad ;\ \sum_{(\mathbf{x}',\mathbf{u}')\in V}\lambda_{(\mathbf{x}',\mathbf{u}')}=1\end{array}\right\}=0,$$

a linear programming problem. Its dual is

$$\min_{\mathbf{p}\notin\Re_+^n,\mathbf{w}\notin\Re_+^m,q\in\Re}\left\{-(\mathbf{p}\cdot\mathbf{x}-\mathbf{w}\cdot\mathbf{y}+q)\middle|\begin{array}{l}-(\mathbf{p}\cdot\mathbf{x}'-\mathbf{w}\cdot\mathbf{y}'+q)>=0\quad(\mathbf{x}',\mathbf{u}')\in V;\\ \mathbf{p}>=\varepsilon\ \mathbf{e}_n;\quad \mathbf{w}>=\varepsilon\ \mathbf{e}_n\end{array}\right\}=0.$$

Because of (T3), (T4) (given the standard result that the maximum value of a linear function over a convex set is always achieved in a vertex point of that set) and $0<\varepsilon<+\infty$ this gives the result.
Q.E.D.

PROOF OF PROPOSITION 2: First construct the monotone hull of *T*

$$M(T)=T+\Re_-^n\times\Re_+^m$$

Immediately, $\{(\mathbf{x}',\mathbf{u}')\in T \mid (-\mathbf{x}',\mathbf{u}')\geq(-\mathbf{x},\mathbf{u})\}=\varnothing \Leftrightarrow$

$$\{(\mathbf{x}',\mathbf{u}')\in M(T) \mid (-\mathbf{x}',\mathbf{u})\geq(-\mathbf{x},\mathbf{u})\}=\varnothing.$$

Since $M(T)=\bigcup_{(\mathbf{x},\mathbf{u})\in T}M(\{\mathbf{x},\mathbf{u}\})$ we have $\{(\mathbf{x}',\mathbf{u}')\in T \mid (-\mathbf{x}',\mathbf{u}')\geq(-\mathbf{x},\mathbf{u})\}=\varnothing \Leftrightarrow$

$$\{(\mathbf{x}'',\mathbf{u}'')\in M(\{(\mathbf{x}',\mathbf{u}')\}) \mid (-\mathbf{x}'',\mathbf{u}'')\geq(-\mathbf{x},\mathbf{u})\}=\varnothing\quad \forall(\mathbf{x}',\mathbf{u}')\in T,$$

where each $M(\{\mathbf{x}',\mathbf{u}'\})((\mathbf{x}',\mathbf{u}')\in T)$ satisfies T1, T2, T3 and T4. Hence, the result follows from Proposition 1 and the obvious fact that $\mathbf{w}\cdot\mathbf{x}'-\mathbf{p}\cdot\mathbf{x}'>=\mathbf{w}\cdot\mathbf{x}''-\mathbf{p}\cdot\mathbf{x}''$ $\forall(\mathbf{x}'',\mathbf{u}'')\in M(\{\mathbf{x}',\mathbf{u}'\})((\mathbf{x}',\mathbf{u}')\in T),\mathbf{p}\in\Re_{++}^n,\mathbf{w}\in\Re_{++}^m$.
Q.E.D.

PROOF OF PROPOSITION 3: Develops along similar lines as the proof of Proposition 1. **Q.E.D.**

PROOF OF PROPOSITION 4: Because of (L1) and (L2) $E_{DF}(\mathbf{x},\mathbf{u})$ is defined for $\mathbf{x} \in L(\mathbf{u})$. Further, under (L3) and (L4) we can redefine $E_{DF}(\mathbf{x},\mathbf{u})$ as

$$E_{DF}(\mathbf{x},\mathbf{u}) = \min_{\gamma_{\mathbf{x}} \in \Re_+ \quad \forall \mathbf{x}' \in E} \left\{ \rho \in \Re \middle| \rho\, \mathbf{x} >= \sum_{\mathbf{x}' \in E} \gamma_{\mathbf{x}'} \mathbf{x}'; \sum_{\mathbf{x}' \in E} \gamma_{\mathbf{x}'} = 1 \right\},$$

a linear programming problem. Its dual is

$$E_{DF}(\mathbf{x},\mathbf{u}) = \max_{\mathbf{q} \in \Re_+^n} \left\{ r \in \Re \middle| \mathbf{q} \cdot \mathbf{x} = 1; \quad \mathbf{q} \cdot \mathbf{x}' >= r \quad \forall \mathbf{x}' \in E \right\},$$

from which the result follows immediately (given (L3), (L4) and the standard result that the minimum value of a linear function over a convex set is always achieved in a vertex point of that set).

Q.E.D.

PROOF OF PROPOSITION 5: For $\mathbf{x}_{DF}(\mathbf{x},\mathbf{u}) \in Eff\ L(\mathbf{u})$ the result follows directly from Proposition 4. Let us then consider $\mathbf{x}_{DF}(\mathbf{x},\mathbf{u}) \notin Eff\ L(\mathbf{u})$. We have to compare input vectors $\mathbf{x}_D \in X_{\mathrm{D}}(\mathbf{x},\mathbf{u})$ and $\mathbf{x}' \in Eff\ L(\mathbf{u}) \setminus X_{\mathrm{D}}(\mathbf{x},\mathbf{u})$. To facilitate exposition we consider scaled shadow price vectors $\mathbf{p}_D \in P(\mathbf{x}_D,\mathbf{u})$ and $\mathbf{p}' \in P(\mathbf{x}_D,\mathbf{u})$ such that $\mathbf{p}_D \cdot \mathbf{x}_D = \mathbf{p}' \cdot \mathbf{x}' = 1$ in the following. We are led to proof that for all pairs $(\mathbf{p}_D, \mathbf{p}')$

(14) $\quad \dfrac{1}{\mathbf{p}_D \cdot \mathbf{x}} >= \dfrac{1}{\mathbf{p}' \cdot \mathbf{x}}$ or $\mathbf{p}' \cdot \mathbf{x} >= \mathbf{p}_D \cdot \mathbf{x}$.

Multiplying both sides of the second inequality in (14) by $E_{DF}(\mathbf{x},\mathbf{u})$ we obtain

(15) $\quad \mathbf{p}' \cdot \mathbf{x}_{DF} >= \mathbf{p}_D \cdot \mathbf{x}_{DF}$,

where $\mathbf{x}_{DF} = \mathbf{x}_{DF}(\mathbf{x},\mathbf{u})$ to save on notation. Of course, from Proposition 4, (15) holds for $\mathbf{p}_D \in P_{DF}(\mathbf{x},\mathbf{u})$. Let us therefore focus on $\mathbf{p}_D \notin P_{DF}(\mathbf{x},\mathbf{u})$. As a preliminary step recall that $P_{DF}(\mathbf{x},\mathbf{u}) \subseteq P(\mathbf{x}_D,\mathbf{u})$ while $P_{DF}(\mathbf{x},\mathbf{u}) \cap P(\mathbf{x}',\mathbf{u}) = \varnothing$ by construction (as $\mathbf{x}_{DF}$ and $\mathbf{x}'$ lie on different hyperplanes supporting the set $L(\mathbf{u})$). Thus, for $\mathbf{p}_{DF} \in P_{DF}(\mathbf{x},\mathbf{u})$ (with again $\mathbf{p}_{DF} \cdot \mathbf{x}_{DF} = 1$ for simplicity) we get

(16) $\quad \mathbf{p}' \cdot \mathbf{x}_{DF} > 1 = \mathbf{p}_{DF} \cdot \mathbf{x}_{DF}$.

Define $\mathbf{p}_\lambda = \lambda \mathbf{p}_{DF} + (1-\lambda)\mathbf{p}_D$. We have $\mathbf{p}_\lambda \in P(\mathbf{x}_D,\mathbf{u})\ \forall \lambda \in [0,1]$. Indeed, $\forall \mathbf{x} \in L(\mathbf{u}): \mathbf{p}_{DF} \cdot \mathbf{x}_D =< \mathbf{p}_{DF} \cdot \mathbf{x} \wedge \mathbf{p}_D \cdot \mathbf{x}_D =< \mathbf{p}_D \cdot \mathbf{x} \Rightarrow \mathbf{p}_\lambda \cdot \mathbf{x}_D =< \mathbf{p}_\lambda \cdot \mathbf{x} \quad \forall \lambda \in [0,1]$.

Now suppose (15) is not met so that

(17) $\quad \mathbf{p}' \cdot \mathbf{x}_{DF} < \mathbf{p}_D \cdot \mathbf{x}_{DF}$.

From (16) and (17), we have $\exists \bar{\lambda} \in (0,1): \mathbf{p}' \cdot \mathbf{x}_{DF} = \mathbf{p}_{\bar{\lambda}} \cdot \mathbf{x}_{DF}$ or

(18) $\quad \bar{\lambda} = \dfrac{\mathbf{p}' \cdot \mathbf{x}_{DF} - \mathbf{p}_D \cdot \mathbf{x}_{DF}}{1 - \mathbf{p}_D \cdot \mathbf{x}_{DF}}$

Note that $\mathbf{p}_{\bar{\lambda}} \in P(\mathbf{x}_D, \mathbf{u}) \Rightarrow 1 = \mathbf{p}_{\bar{\lambda}} \cdot \mathbf{x}_D =< \mathbf{p}_{\bar{\lambda}} \cdot \mathbf{x}_{DF}$. However, under (v) $\mathbf{p}_{\bar{\lambda}} \cdot \mathbf{x}_D =< \mathbf{p}_{\bar{\lambda}} \cdot \mathbf{x}_{DF} \Leftrightarrow 1 >= \mathbf{p}' \cdot \mathbf{x}_{DF}$, which contradicts (16). We therefore conclude that (17) does not hold and (15) is indeed satisfied.

Q.E.D.

PROOF OF PROPOSITION 6: We consider two input vector $\mathbf{x}_D \in X_{\mathrm{D}}(\mathbf{x}, \mathbf{u})$ and $\mathbf{x}_D^{'} \in X_{\mathrm{D}}(\mathbf{x}, \mathbf{u}) \setminus \{\mathbf{x}_D\}$. We assume $\mathbf{p}_D^{'} \in P(\mathbf{x}_D^{'}, \mathbf{u})$ with $\mathbf{p}_D^{'} \in \Re_{++}^n$ given. We have to proof that $\mathbf{p}_D \in P(\mathbf{x}_D, \mathbf{u})$ with $\mathbf{p}_D \in \Re_{++}^n$ can be constructed such that

(19) $$\frac{\mathbf{p}_D \cdot \mathbf{x}_D}{\mathbf{p}_D \cdot \mathbf{x}} >= \frac{\mathbf{p}_D^{'} \cdot \mathbf{x}_D^{'}}{\mathbf{p}_D^{'} \cdot \mathbf{x}}.$$

Dividing both sides by $E_{DF}(\mathbf{x}, \mathbf{u})$ gives the equivalent condition

(20) $$\frac{\mathbf{p}_D \cdot \mathbf{x}_D}{\mathbf{p}_D \cdot \mathbf{x}_{DF}} >= \frac{\mathbf{p}_D^{'} \cdot \mathbf{x}_D^{'}}{\mathbf{p}_D^{'} \cdot \mathbf{x}_{DF}},$$

where $\mathbf{x}_{DF} = \mathbf{x}_{DF}(\mathbf{x}, \mathbf{u})$ to save on notation. We can decompose $\mathbf{x}_{DF}$, $\mathbf{x}_D$ and $\mathbf{x}_D^{'}$ in a slack and non-slack subvector. Let $\mathbf{x}_{DF} = (\overline{\mathbf{x}}_{DF}, \widetilde{\mathbf{x}}_{DF})$, $\mathbf{x}_D = (\overline{\mathbf{x}}_D, \widetilde{\mathbf{x}}_D)$ and $\mathbf{x}_D^{'} = (\overline{\mathbf{x}}_D^{'}, \widetilde{\mathbf{x}}_D^{'})$ such that (for the non-slack subvectors) $\overline{\mathbf{x}}_{DF} = \overline{\mathbf{x}}_D = \overline{\mathbf{x}}_D^{'}$ while (for the slack subvectors) $\widetilde{\mathbf{x}}_D < \widetilde{\mathbf{x}}_{DF}$ and $\widetilde{\mathbf{x}}_D^{'} < \widetilde{\mathbf{x}}_{DF}$. We restate (20):

(21) $$\frac{\overline{\mathbf{p}}_D \cdot \overline{\mathbf{x}}_{DF} + \widetilde{\mathbf{p}}_D \cdot \widetilde{\mathbf{x}}_D}{\overline{\mathbf{p}}_D \cdot \overline{\mathbf{x}}_{DF} + \widetilde{\mathbf{p}}_D \cdot \widetilde{\mathbf{x}}_{DF}} >= \frac{\overline{\mathbf{p}}_D^{'} \cdot \overline{\mathbf{x}}_{DF} + \widetilde{\mathbf{p}}_D^{'} \cdot \widetilde{\mathbf{x}}_D^{'}}{\overline{\mathbf{p}}_D^{'} \cdot \overline{\mathbf{x}}_{DF} + \widetilde{\mathbf{p}}_D^{'} \cdot \widetilde{\mathbf{x}}_{DF}}.$$

with corresponding decompositions $\mathbf{p}_D = (\overline{\mathbf{p}}_D, \widetilde{\mathbf{p}}_D)$ and $\mathbf{p}_D^{'} = (\overline{\mathbf{p}}_D^{'}, \widetilde{\mathbf{p}}_D^{'})$. Obviously, $\forall \mathbf{p}_D, \mathbf{p}_D^{'} \in \Re_{++}^n$

(22) $$\frac{\overline{\mathbf{p}}_D \cdot \overline{\mathbf{x}}_{DF}}{\overline{\mathbf{p}}_D \cdot \overline{\mathbf{x}}_{DF}} = \frac{\overline{\mathbf{p}}_D^{'} \cdot \overline{\mathbf{x}}_{DF}}{\overline{\mathbf{p}}_D^{'} \cdot \overline{\mathbf{x}}_{DF}} = 1 \text{ and } \frac{\widetilde{\mathbf{p}}_D \cdot \widetilde{\mathbf{x}}_D}{\widetilde{\mathbf{p}}_D \cdot \widetilde{\mathbf{x}}_{DF}} < 1, \frac{\widetilde{\mathbf{p}}_D^{'} \cdot \widetilde{\mathbf{x}}_D^{'}}{\widetilde{\mathbf{p}}_D^{'} \cdot \widetilde{\mathbf{x}}_{DF}} < 1.$$

The left and right hand sides of expression (21) are both strictly smaller than one. Let

(23) $$\frac{\overline{\mathbf{p}}_D^{'} \cdot \overline{\mathbf{x}}_{DF} + \widetilde{\mathbf{p}}_D^{'} \cdot \widetilde{\mathbf{x}}_D^{'}}{\overline{\mathbf{p}}_D^{'} \cdot \overline{\mathbf{x}}_{DF} + \widetilde{\mathbf{p}}_D^{'} \cdot \widetilde{\mathbf{x}}_{DF}} = 1 - \varepsilon, \text{ with } \varepsilon > 0.$$

We now proceed by constructing $\mathbf{p}_D \in \Re_{++}^n$ such that (21) is satisfied. First note that each $\mathbf{p}_{DF} \in P_{DF}(\mathbf{x}, \mathbf{u})$ can be decomposed similarly as $\mathbf{p}_D$ and $\mathbf{p}_D^{'}$ above so that $\mathbf{p}_{DF} = (\overline{\mathbf{p}}_{DF}, 0)$. As $\mathbf{x}_D \in X_{\mathrm{D}}(\mathbf{x}, \mathbf{u})$ we can define $\mathbf{p}_D^{*} = (\overline{\mathbf{p}}_{DF}, \widetilde{\mathbf{p}}_D^{*})$ with $\widetilde{\mathbf{p}}_D^{*}$ appropriately chosen such that $\mathbf{p}_D^{*} \in P(\mathbf{x}_D, \mathbf{u})$ and $\mathbf{p}_D^{*} \in \Re_{++}^n$. We have $\forall \mathbf{x} \in L(\mathbf{u})$

(24) $\quad \overline{\mathbf{p}}_{DF} \cdot \overline{\mathbf{x}}_{DF} =< \overline{\mathbf{p}}_{DF} \cdot \overline{\mathbf{x}}$,

(25) $\quad \overline{\mathbf{p}}_{DF} \cdot \overline{\mathbf{x}}_{DF} + \widetilde{\mathbf{p}}^*_D \cdot \widetilde{\mathbf{x}}_D =< \overline{\mathbf{p}}_{DF} \cdot \overline{\mathbf{x}} + \widetilde{\mathbf{p}}^*_D \cdot \widetilde{\mathbf{x}}$,

where again we use the decomposition $\mathbf{x} = (\overline{\mathbf{x}}, \widetilde{\mathbf{x}})$. Convex combinations of the inequalities in (24) and (25) yield $\forall \kappa \in [0,1]$

(26) $\quad \overline{\mathbf{p}}_{DF} \cdot \overline{\mathbf{x}}_{DF} + \kappa(\widetilde{\mathbf{p}}^*_D \cdot \widetilde{\mathbf{x}}_D) =< \overline{\mathbf{p}}_{DF} \cdot \overline{\mathbf{x}} + \kappa(\widetilde{\mathbf{p}}^*_D \cdot \widetilde{\mathbf{x}})$.

Let $\mathbf{p}^\kappa_D = (\overline{\mathbf{p}}_{DF}, \kappa\widetilde{\mathbf{p}}^*_D)$. From (26) we know that $\mathbf{p}^\kappa_D \in P(\mathbf{x}_D, \mathbf{u}) \quad \forall \kappa \in [0,1]$. Moreover, $\mathbf{p}^\kappa_D \in \Re^n_{++} \quad \forall \kappa \in (0,1]$. So, we can shrink the shadow price vector associated with $\widetilde{\mathbf{x}}_D$ independently of that associated with $\overline{\mathbf{x}}_{DF}$. In fact, it can be made infinitesimally small while still remaining positive. Using (22) and for a given value of $\varepsilon > 0$, we can always set $\mathbf{p}_D = \mathbf{p}^{\overline{\kappa}}_D \in \Re^n_{++}$ with $\kappa \in (0,1]$ sufficiently small such that

(27) $$\frac{\overline{\mathbf{p}}_D \cdot \overline{\mathbf{x}}_{DF} + \widetilde{\mathbf{p}}_D \cdot \widetilde{\mathbf{x}}_D}{\overline{\mathbf{p}}_D \cdot \overline{\mathbf{x}}_{DF} + \widetilde{\mathbf{p}}_D \cdot \widetilde{\mathbf{x}}_{DF}} >= 1 - \varepsilon .$$

Combining (24) and (27) yields (19) and thus also (18).
Q.E.D.

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