UNDERSTANDING THE WORLD WOOL MARKET: TRADE, PRODUCTIVITY AND GROWER INCOMES

PART II: THE TOOLBOX*

by

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Chapter 2

The Toolbox

2.1 Preamble

In this chapter we present the tools used to construct the theoretical structure of the model presented in Chapter 3. The theory of the model is highly nonlinear but is specified in linearised form. In deriving the linearised form of the nonlinear functions, we make explicit the optimising behaviour that underlies the tools and their properties. We use the notational convention of expressing the levels form of a variable in capital letters and the percentage-change equivalent in lower case letters. We also discuss how the tools can be combined by assuming separability between functions.

2.2 Differentiation rules

In deriving linearised or percentage-change functional forms we apply some of the rules of differentiation; the rules are derived by totally differentiating the levels expression. The rules are presented below.

Imagine the simple function $Y = K$ where the independent variable Y is the function of the constant variable K. The percentage-change form of this function is $y = 0$, where y is the percentage in Y. This is the constant-function rule.

Imagine the power function $Y = KX^{v}$ where Y is a function of a constant K multiplied by the variable X raised to the power of the parameter v. Here, the percentage-change form is $y = vx$, where x is the percentage change in X. This is the power-function rule.

Imagine the product function $Y = KXN$ where Y is the product of the constant K, and the variables X and N. The percentage-change form of the function is $y = x + n$ where x and n are the percentage changes in X and N. This is the product rule

Imagine the quotient function $Y = K\frac{X}{N}$ where Y is the product of the constant K, and the ratio of the variables X and N. The percentage-change form of the function is $y = x - n$. This is the quotient rule.

Imagine the function $Y = K\left(X + N\right)$ where Y is the product of the constant K, and the summation of the variables X and N. The percentage-change form of the function is $y = \frac{X}{Y}x + \frac{N}{Y}n$ or $Yy = Xx + Nn$. This is the sum rule

2.3 The tools

The following subsections derive the linearised form of the nonlinear functions that underlie the theory of the model presented in Chapter 3. We also discuss the application of separable functions. Equation Section 2

2.3.1 The Leontief production function

Let Z be the firm's activity level, X_i (i = 1,...,n) the inputs used by the firm, and A_i the input-output coefficients, which show the minimum effective input of i required to support a unit of activity. The Leontief production function (Leontief 1937) can then be represented as

$$Z = \min\left(X_1, ..., X_n\right); \tag{2.1}$$

or

$$Z = \min\left(\frac{X_1}{A_1},\ldots,\frac{X_n}{A_n}\right). \tag{2.2}$$

With Z representing the firm's activity level, equation (2.1) implies that inputs X_i are nonspecific to outputs and only provide a general capacity to produce.[1] In Leontief production technology the minimum of the actual units of the n inputs, $X_1,\ldots,X_n$ in equation (2.1), or the *effective* units of the n inputs, $\frac{X_1}{A_1},\ldots,\frac{X_n}{A_n}$ in equation (2.2), is chosen in finding the cost minimum.

The percentage-change forms of (2.1)–(2.2) are

$$z = x_i, \tag{2.3}$$

and

$$z = x_i - a_i \text{ or } z + a_i = x_i. \tag{2.4}$$

Equations (2.3) and (2.4) state that demand for composite input i will move exactly with industry activity levels, or with industry activity levels plus any change in technology in the use of input i. With a_i = 0, the share of each of the n inputs in total inputs will remain fixed. For this reason, Leontief production technology is also known as fixed-coefficients or fixed-proportions technology.

2.3.2 The CES production function

The CES (constant elasticity of substitution) production function (Arrow et al. 1961) can be represented as[2]

[1] This point should be noted for all production functions discussed here where activity level, rather than output, appears as the independent variable.

[2] The above derivation of the percentage-change form of the input demand functions from a CES production function generally follows Dixon et al. (1982), Chapter 3, Section 12.1; Dixon et al. (1992), Chapter 3, Section C; and Horridge et al. (1993), Appendix A.

$$Z = \left[\sum_{i=1}^{n} \delta_i \left(\frac{X_i}{A_i} \right)^{-\rho} \right]^{-1/\rho}, \; 0 < \delta_i < 1, \sum_i \delta_i = 1; \rho \geq -1, \rho \neq 0 \tag{2.5}$$

where Z, X_i and A_i (i = 1,...,n) are as previously defined, and δ_i and ρ are parameters. For use below, we note the percentage-change form of (2.5) as

$$z = \sum_i S_i \left(x_i - a_i \right), \; i = 1,...,n \tag{2.6}$$

where

$$S_i = \frac{\delta_i X_i^{-\rho}}{\sum_{k=1}^{n} \delta_k X_k^{-\rho}}, \; i,k = 1,...,n\,. \tag{2.7}$$

Assume each firm operates in a perfectly competitive environment and is efficient. Perfect competition means firms face given input prices, $P_1,...,P_n$; efficiency means that for any given activity level firms choose each i input so as to minimise total costs, $\sum_{i=1}^{n} P_i X_i$. Also assume that $\rho > -1$ so that corner solutions are avoided.

Setting $\overline{X}_i = \frac{X_i}{A_i}$ and $\overline{P}_i = P_i A_i$,[3] alternatively, in percentage-change form

$$\overline{x}_i = x_i - a_i \text{ and } \overline{p}_i = p_i + a_i, \tag{2.8}$$

the first-order conditions for cost minimisation are

$$\overline{P}_i - \Lambda \frac{\partial Z}{\partial \overline{X}_i} = \overline{P}_i - \Lambda \left[\sum_{k=1}^{n} \delta_k \overline{X}_k^{-\rho} \right]^{-(1+\rho)/\rho} \delta_i \overline{X}_i^{-(1+\rho)} = 0, \tag{2.9}$$

and

[3] Here, $\overline{X}_i$ and $\overline{P}_i$ are the *effective* demand and price of input i.

$$Z - \left[\sum_{i=1}^{n} \delta_i \left(\frac{X_i}{A_i} \right)^{-\rho} \right]^{-1/\rho} = 0, \tag{2.10}$$

where Λ is the Lagrange multiplier. We can use (2.5) to replace (2.9) with

$$\overline{P}_i - \Lambda Z^{(1+\rho)} \delta_i \overline{X}_i^{-(1+\rho)} = 0. \tag{2.11}$$

The percentage-change form of (2.11) is

$$\overline{p}_i - \lambda - (1+\rho) z + (1+\rho) \overline{x}_i = 0 \text{ or } \overline{p}_i = \lambda + (1+\rho) z - (1+\rho) \overline{x}_i, \tag{2.12}$$

where $\overline{p}_i$, λ, z and $\overline{x}_i$ are percentage changes in $\overline{P}_i$, Λ, Z and $\overline{X}_i$.

Dixon et al. (1982), pp. 80-1, show that equation (2.11) implies

$$\frac{\overline{P}_i \overline{X}_i}{\sum_{k=1}^{n} \overline{P}_k \overline{X}_k} = \frac{\delta_i \overline{X}_i^{-\rho}}{\sum_{k=1}^{n} \delta_k \overline{X}_k^{-\rho}}. \tag{2.13}$$

Therefore S_i is the share of input *i* in total costs.

If we define a positive parameter $\sigma = \dfrac{1}{(1+\rho)}$, then we can rewrite (2.12) as

$$\overline{x}_i = -\sigma \overline{p}_i + \sigma \lambda + z. \tag{2.14}$$

Using (2.8), (2.6) can be rewritten as

$$z = \sum_i S_i \overline{x}_i, \; i = 1, \ldots, n. \tag{2.15}$$

Then, substituting (2.14) into (2.15) we get

$$z = \sum_i S_i \left(-\sigma \overline{p}_i + \sigma \lambda + z \right) \text{ or } \lambda = \sum_i S_i \overline{p}_i, \tag{2.16}$$

and substituting (2.16) into (2.14) and rearranging, we get

$$\overline{x}_i = z - \sigma \left[\overline{p}_i - \sum_{k=1}^{n} S_k \overline{p}_k \right], \tag{2.17}$$

which is the percentage-change form of the input demand functions. Note that the summation term on the right-hand side (RHS) of (2.17) is the Divisia price index for inputs (or the price of composite inputs). Equations (2.17) state that the demand for any input *i* is a function of an expansion effect and a substitution effect. If we set the change in relative prices, $\left[\bar{p}_i - \sum_{k=1}^{n} S_k \bar{p}_k\right]$, to zero, then demand for input *i* will move exactly with the firm's activity level, z; i.e., the expansion effect. This reflects the constant nature of returns to scale in the production function (2.5). Alternatively, if we set $z = 0$ then demand for input *i* will be a function of the change in price of input *i* relative to the change in the price of composite inputs, and the size of the (constant) elasticity of substitution between any pair of inputs, σ. So that if the price of input *i* rises relative to the price of composite inputs, demand for input *i* will fall relative to the firm's activity level, i.e., the substitution effect. The size of the substitution effect is determined by the size of σ.

As the production function (2.5) includes technical change terms, we can replace $\bar{x}_k$ and $\bar{p}_k$ in (2.17) with (2.8), yielding

$$x_i - a_i = z - \sigma\left[p_i + a_i - \sum_{k=1}^{n} S_k(p_k + a_k)\right]. \tag{2.18}$$

Equations (2.18) contain technical change terms, and thus they are written in terms of *effective* input quantities and prices. Therefore, the term on the left-hand side (LHS) of (2.18) is the *effective* demand for input *i*. Equivalently, the summation term on the RHS of (2.18) is the Divisia price index of *effective* inputs (or the price of *effective* composite inputs). Note that a *reduction* in the input-output coefficient, $a_k < 1$, represents an improvement in the technology used to apply input *i*, thus reducing the demand for input *i*.

In comparing equation (2.5) with its linearised counterpart (2.18), it is obvious that the latter are much simpler than former. This provides a significant computational saving in applying the linearised version. Furthermore, even though both (2.5) and (2.18) imply the same behaviour with respect to demand for inputs by firms, equations (2.18) are more intuitive than (2.5) as they are in elasticity form.[4] Also, by using (2.18) instead of (2.5) there is no need to calculate initial values of quantities, prices and substitution parameters. This is another significant saving in analytical and computational processes. By using (2.18), the only data we require in levels are cost shares, i.e., values, which is extremely convenient as the benchmark equilibrium data are most naturally available in value terms (Hertel et al. 1992).

Note that in deriving equations (2.18) from (2.5), we have worked with linearised versions of the first-order conditions of (2.5). We could, instead, have derived the levels form of the input demand functions from (2.5) and then linearised to get (2.18), but this would have been a less simple procedure (Dixon et al. 1992).

2.3.3 The CRESH production function

The CRESH (constant ratios of elasticities of substitution, homothetic) production function (Hanoch 1971) can be represented as[5]

$$\sum_{i=1}^{n}\left(\frac{X_i}{Z}\right)^{h_i}\frac{Q_i}{h_i}=\alpha\,,\; 0<h_i<1; Q_i>0, \sum_i Q_i=1\,, \tag{2.19}$$

where Z and X_i are as previously defined, and Q_i, h_i and α are parameters. α can take either sign but if each of the $\frac{Q_i}{h_i}$ s has the same sign, then α must have their common sign.

Assume each firm is competitive, i.e., firms face given input prices $P_1,...,P_n$, and efficient, i.e., for any given activity level, firms choose input levels so as to minimise total

[4] In fact, the CES production function was invented via the elasticity form (see Arrow et al. 1961).

[5] The above derivation of the percentage-change form of the input demand functions from a CRESH production function follows Dixon et al. (1992), Chapter 3, Section C.

costs, $\sum_{i=1}^{n} P_i X_i$. The first-order conditions for cost minimisation are that there exists Λ (the Lagrange multiplier) such that Λ and the X_is satisfy (2.19) and

$$P_i = \Lambda \left[\frac{X_i^{h_i - 1}}{Z^{h_i}} \right] Q_i, \; i = 1, \ldots, n. \tag{2.20}$$

The percentage-change equivalents of (2.19) and (2.20) are

$$\sum_{i=1}^{n} h_i \left(x_i - z \right) W_i = 0 \tag{2.21}$$

and

$$p_i = \lambda + \left(h_i - 1 \right) x_i - h_i z, \; i = 1, \ldots, n, \tag{2.22}$$

where p_i, λ, z and x_i are percentage changes in P_i, Λ, Z and X_i, and

$$W_i = \left[\frac{X_i}{Z_i} \right]^{h_i} \frac{Q_i}{h_i}, \; i = 1, \ldots, n. \tag{2.23}$$

Multiplying both sides of (2.20) by X_i and rearranging gives

$$\frac{h_i W_i}{\sum_{k=1}^{n} h_k W_k} = S_i, \; i = 1, \ldots, n, \tag{2.24}$$

where S_i is the share of input *i* in total costs. Thus, (2.21) can be rewritten as

$$\sum_{i=1}^{n} S_i x_i = z \tag{2.25}$$

Rearranging (2.22) as

$$x_i = \left[\frac{1}{h_i - 1} \right] \left(p_i - \lambda + h_i z \right), \tag{2.26}$$

and substituting into (2.25) gives

$$z = \sum_{i=1}^{n} \frac{S_i}{\left(h_i - 1 \right)} \left(p_i - \lambda + h_i z \right). \tag{2.27}$$

Therefore,

$$\lambda = z + \sum_{i=1}^{n} \overline{S}_i p_i \, , \tag{2.28}$$

where

$$\overline{S}_i = \frac{S_i / (1 - h_i)}{\sum_{k=1}^{n} S_k / (1 - h_k)} \, . \tag{2.29}$$

In (2.29), $\overline{S}_i$ is known as the 'modified' cost share. The percentage-change form of the input demand functions is given by substituting (2.28) into (2.26) giving

$$x_i = z - \sigma_i \left(p_i - \sum_{k=1}^{n} \overline{S}_k p_k \right), \; i = 1, ..., n \, , \tag{2.30}$$

where σ_i is a positive parameter defined as

$$\sigma_i = \frac{1}{(1 - h_i)} \, . \tag{2.31}$$

CRESH input demand functions [(2.30)] are similar to CES input demand functions [(2.17) or (2.18)] with two differences. First, the weights used in calculating the average movement in input prices are the 'modified' costs shares of (2.29) rather than ordinary cost shares. Second, CRESH input demand functions allow the coefficient σ_i to vary across inputs whereas CES input demand functions apply a common elasticity of substitution (σ); thus, the CES input demand functions are a special case of the CRESH input demand functions, the case of $\sigma_i = \sigma$ for all *i*. This follows from the CRESH production function being a generalisation of the CES production function.

Note that x_i and p_i in (2.30) could be redefined as *effective* demands and prices by rewriting them as $\overline{x}_i$ and $\overline{p}_i$ and then substituting in (2.8). (2.30) would then be rewritten as

$$x_i - a_i = z - \sigma_i \left(p_i + a_i - \sum_{k=1}^{n} \overline{S}_k p_k + a_k \right), \; i = 1, ..., n \, . \tag{2.32}$$

2.3.4 Applying separable production functions

The model presented in Chapter 3 applies the Leontief, CES and CRESH production functions presented above. In all instances, the applications assume the production functions are separable. The advantages of this assumption are that it reduces the number of parameters requiring explicit evaluation that, in turn, simplifies the representation of systems of demand equations (Dixon et al. 1992, p. 142).[6]

A function $g\left(X_{\alpha}, X_{\beta},...,\right)$ is separable with respect to the partition $N_1,...,N_n$ if it can be written as[7]

$$g\left(X_{\alpha}, X_{\beta},...,\right) = V\left[f_1\left(X^1\right), f_2\left(X^2\right),..., f_n\left(X^n\right)\right], \tag{2.33}$$

where $N_1,...,N_n$ are a nonoverlapping coverage of the set $\{\alpha,\beta,...\}$, and X^k is the subvector of $\left(X_{\alpha}, X_{\beta},...\right)$ formed by the components of X_{τ} for which $\tau \in N_k$. An application of separable production functions follows.

Equation (2.1) represents the Leontief production function as

$$Z = \min\left(X_1,..., X_n\right); \tag{2.34}$$

where Z is the firm's activity level and X_i (i = 1,...,n) are the inputs used by the firm. If the n inputs are determined by n CES production functions, then equation (2.34) can be rewritten as

$$Z = \min\left[CES\left(X_1\right),..., CES\left(X_n\right)\right]. \tag{2.35}$$

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[6] The discussion of separability in this section closely follows Dixon et al. (1992), Exercise 3.13.

[7] The definition of separability used here is usually referred to as 'weak separability', e.g., see Katzner (1970), p. 28; Chung (1994), pp. 188-9.

If we define the n inputs as nonoverlapping, then (2.35) is combining n separable CES production functions in determining Z. Thus, we are assuming separability between inputs and the activity level, which greatly reduces the number of parameters requiring explicit evaluation. In (2.35), the n CES production functions are nested under the Leontief function, which greatly simplifies the representation of the system of demand equations implied by (2.35).

Equation (2.35) is only one example of how to apply separable production functions: it could be rewritten as

$$Z = \min\left[CRESH\left(X_1\right),...,CRESH\left(X_n\right)\right]. \qquad (2.36)$$

In (2.36), CRESH production functions are nested under a Leontief production function. Both (2.35) and (2.36) are two-level nested production structures with a Leontief function at level 1 and CES or CRESH functions at level 2.

A further example is

$$Z = \min\left\{CES\left[X_1\left\langle CRESH\left(X_{11}, X_{12}\right)\right\rangle,...,X_n\left\langle CRESH\left(X_{n1}, X_{n2}\right)\right\rangle\right]\right\}, \qquad (2.37)$$

where X_{ns} (i = 1,...,n; s = 1,2) are the n inputs used by the firm from the s sources, where source 1 represents domestically-produced inputs and source 2 represents imported inputs. Thus, $CRESH\left(X_{i1}, X_{i2}\right)$ says that X_{i1} and X_{i2} are to be combined according to the CRESH production function (2.19). (2.37) represents a three-level nested production structure with a Leontief function at level 1, CES functions at level 2 and CRESH functions at level 3.

2.3.5 The CET production possibilities frontier

The CET (constant elasticity of transformation) production possibilities frontier (PPF) (Powell and Gruen 1968) can be represented as[8]

$$Z = B\left[\sum_{i=1}^{m}\gamma_i Y_i^{-\rho}\right]^{-1/\rho}, \; B > 0; 0 < \gamma_i < 1, \sum_i \gamma_i = 1; \rho \le -1, \qquad (2.38)$$

where Z is as previously defined, B is a technology parameter, γ_i is a share parameter, Y_i (i = 1,...,m) are the m outputs that the firm produces, and ρ is a substitution parameter. The CET functional form is identical to the CES functional form except for the restrictions placed on ρ; with the CES form $\rho \ge -1$, with the CET form $\rho \le -1$.

Assume Z is exogenous to the choice of the Y_is. Thus, the composition of the firm's outputs is assumed to be determined independently of the firm's inputs. This assumption is appropriate where inputs are of a general-purpose nature and only provide the firm with a capacity to produce.

As before, we assume the firm is competitive so that the output prices, $P_1,...,P_m$, it faces are given, and that it is a profit maximiser, so that it attempts to maximise revenue, $\sum_{i=1}^{m} P_i Y_i$. We also assume $\rho < -1$ to avoid corner solutions. The first-order conditions for revenue maximisation are then

$$P_i - \Lambda \partial Z / \partial Y_i = P_i - \Lambda B\left[\sum_{k=1}^{m}\gamma_k Y_k^{-\rho}\right]^{-(1+\rho)/\rho} \gamma_i Y_i^{-(1+\rho)} = 0, \; i,k = 1,...,m, \qquad (2.39)$$

and

$$Z - B\left[\sum_{i=1}^{m}\gamma_i Y_i^{-\rho}\right]^{-1/\rho} = 0, \; i = 1,...,m. \qquad (2.40)$$

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[8] The above derivation of the percentage-change form of the output supply functions from a CET PPF follows Dixon et al. (1992), Chapter 3, Section C.

By following a procedure similar to that used above to go from (2.9) and (2.10) to (2.17), we can use (2.39) and (2.40) to arrive at

$$y_i = z - \theta\left[p_i - \sum_{k=1}^{m} S_k p_k\right],\ i = 1,\ldots,m\,, \tag{2.41}$$

where

$$\theta = \frac{1}{(1+\rho)} < 0\,, \tag{2.42}$$

and

$$S_i = \frac{P_i Y_i}{\sum_{k=1}^{m} P_k Y_k},\ i = 1,\ldots,m\,. \tag{2.43}$$

Equations (2.41) are CET output supply functions in percentage-change form, and y_i and p_i are the percentage-change forms of Y_i and P_i. Note that the S_is are revenue shares, and that the summation term on the RHS of (2.41) is the Divisia index of output prices (or the price of composite outputs).

Equations (2.41) state that the supply for any output *i* is a function of an expansion effect and a transformation effect. If we set the change in relative prices, $\left[p_i - \sum_{k=1}^{m} S_k p_k\right]$, to zero, then supply for output *i* will move exactly with the firm's activity level, z; i.e., the expansion effect. This reflects the constant nature of returns to scale in the PPF (2.38). Alternatively, if the change in the firm's activity level is set to zero, then supply of input *i* will be a function of the price of output *i* relative to the price of composite outputs and the size of the (constant) elasticity of transformation between any pair of outputs, θ. So that if the price of output *i* rises relative to the price of composite outputs, supply of output *i* will rise relative to *z*, i.e., the transformation effect. The size of this effect is determined by the size of the coefficient θ.

2.3.6 The CRETH production possibilities frontier

The CRETH (constant ratios of elasticities of transformation, homothetic) PPF (Vincent et al. 1980) can be represented as[9]

$$\sum_{i=1}^{m}\left[\frac{Y_i}{Z}\right]^{h_i}\frac{V_i}{H_i}=\beta\ ,\ h_i>1;\beta,V_i>0;\sum_i V_i=1, \tag{2.44}$$

where V_i, h_i and β are parameters. The CRETH PPF is identical to the CRESH production function except for the restrictions on the h_is; the CRESH form requires $0<h_i<1$ whereas the CRETH form requires $h_i>1$.[10]

Applying the method used in going from (2.19) to (2.30), we can derive the following output supply functions from (2.44),

$$y_i=z-\theta_i\left(p_i-\sum_{k=1}^{m}\bar{R}_k p_k\right),\ i=1,...,m, \tag{2.45}$$

where θ_k is a negative parameter defined as

$$\theta_i=\frac{1}{\left(1-h_i\right)}, \tag{2.46}$$

and the $\bar{R}_i$s are 'modified' revenue shares defined as

$$\bar{R}_i=\frac{R_i\Big/\left(1-h_i\right)}{\sum_{k=1}^{m}R_k\Big/\left(1-h_k\right)}. \tag{2.47}$$

The R_is in (2.47) are revenue shares.

CRETH output supply functions (2.45) are similar to CET output supply functions (2.41) with two differences. First, the weights used in calculating the average movement in output prices are the 'modified' revenue shares in (2.45) rather than ordinary revenue

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[9] The above derivation of the percentage-change form of the output supply functions from a CRETH PPF follows Dixon et al. (1992), Chapter 3, Section C.

[10] As before, with Z representing the firm's activity level equation (2.44) implies that the composition of outputs are nonspecific to inputs; therefore, inputs only provide a general capacity to produce.

shares. Second, CRETH output supply functions allow the coefficient θ_i to vary across outputs whereas CET output supply functions apply a common elasticity of transformation (θ); thus, the CET output supply functions are a special case of the CRETH output supply functions, the case of $\theta_i = \theta$ for all *i*. This follows from the CRETH PPF being a generalisation of the CET PPF.

2.3.7 The implicit utility function

Imagine the consumer's problem is to choose the inputs $X_1, ..., X_n$ so as to maximise[11]

$$U\left(X_1, ..., X_n\right) \tag{2.48}$$

subject to

$$\sum_{i=1}^{n} P_i X_i = M \ , \tag{2.49}$$

where *U* is utility, $X_1, ..., X_n$ are the consumer's inputs to utility maximisation, $P_1, ..., P_n$ are the (given) prices of X_i faced by the consumer, and *M* is the consumer's total budget. We assume that the function (2.48) is differentiable and that there is no satiation, so that each marginal utility is positive but diminishes as X_i continues to increase. (2.48) is an implicit utility function because it does not explicitly state how the *n* inputs are transformed into *U*, i.e., it does not have an explicit functional form.

In order to maximise (2.48) subject to (2.49) we form the Lagrangian expression

$$U\left(X_1, ..., X_n\right) - \Lambda\left(\sum_{i=1}^{n} P_i X_i - M\right), \tag{2.50}$$

[11] This section draws on Clements et al. (1995), Theil (1980), Theil and Clements (1987), and Dixon et al. (1992).

where Λ is, as previously, the Lagrange multiplier. The first-order conditions for maximising (2.48) subject to (2.49) are

$$\partial U/\partial X_i = \Lambda P_i \text{ or } \partial U/\partial(P_i X_i) = \Lambda,\ i = 1,...,n. \tag{2.51}$$

The rearranged form of (2.51) shows that Λ is the marginal utility of income.

The first-order conditions (2.49) and (2.51) are of size n+1, and with n+1 unknowns, $X_1,...,X_n$ and Λ, we are able, in principle, to solve these conditions. If we assume that the resulting quantities are unique and positive for relevant values of prices and income, then we can write demand equations of the form

$$X_i = X_i\left(M, P_i,...,P_n\right),\ i = 1,...,n. \tag{2.52}$$

Totally differentiating (2.52) gives

$$dX_i = \frac{\partial X_i}{\partial M}dM + \sum_{j=1}^{n}\frac{\partial X_i}{\partial P_j}dP_j,\ i = 1,...,n. \tag{2.53}$$

or, in percentage-change form,

$$x_i = \frac{\partial X_i}{\partial M}\frac{M}{X_i(.)}m + \sum_{j=1}^{n}\frac{\partial X_i}{\partial P_j}\frac{P_j}{X_i(.)}p_j,\ i = 1,...,n, \tag{2.54}$$

where x_i, p_i and m are the percentage-change equivalents of X_i, P_i and M. The expressions $\frac{\partial X_i}{\partial M}\frac{M}{X_i(.)}$ and $\frac{\partial X_i}{\partial P_j}\frac{P_j}{X_i(.)}$ represent the i-th income elasticity and the (i,j)-th uncompensated price elasticity. Thus, we can rewrite (2.54) as

$$x_i = \eta_i m + \sum_{j=1}^{n}\varepsilon_{ij}p_j,\ i = 1,...,n, \tag{2.55}$$

where η_i and ε_{ij} are the i income elasticities and (i,j) uncompensated price elasticities.

The elasticities η_i and ε_{ij} satisfy Engel's aggregation, i.e.,

$$\sum_{i=1}^{n} S_i\eta_i = 1, \tag{2.56}$$

homogeneity, i.e.,

$$\sum_{j=1}^{n} \varepsilon_{ij} = -\eta_i \,, \tag{2.57}$$

and symmetry, i.e.,

$$S_i \left(\varepsilon_{ij} + \eta_i S_j \right) = S_j \left(\varepsilon_{ji} + \eta_j S_i \right), \; i \neq j \,, \tag{2.58}$$

where S_i is the share of good *i* in the consumer's budget.

Equations (2.55)–(2.58) are written in terms of uncompensated price elasticities. We can rewrite the equations in terms of compensated price elasticities using the Slutsky decomposition, which says

$$\varepsilon_{ij} = \bar{\varepsilon}_{ij} - S_j \eta_i \,, \; i \neq j \,. \tag{2.59}$$

where $\bar{\varepsilon}_{ij}$ are the (*i*,*j*) compensated price elasticities. Thus, (2.59) decomposes ε_{ij} into substitution $\left(\bar{\varepsilon}_{ij} \right)$ and income $\left(S_j \eta_i \right)$ effects. Using (2.59) to replace ε_{ij} in (2.55) gives

$$x_i = \eta_i m + \sum_{j=1}^{n} \left(\bar{\varepsilon}_{ij} - S_j \eta_i \right) p_j \,, \; i = 1,\ldots,n \,. \tag{2.60}$$

Expanding (2.60) and rearranging gives

$$x_i = \eta_i \left(m - \sum_{j=1}^{n} S_j p_j \right) + \sum_{j=1}^{n} \bar{\varepsilon}_{ij} p_j \,, \; i = 1,\ldots,n \,. \tag{2.61}$$

The term in parentheses on the RHS of (2.61) is the percentage change in real income (or expenditure) (*i*), and so can be rewritten as

$$x_i = \eta_i i + \sum_{j=1}^{n} \bar{\varepsilon}_{ij} p_j \,, \; i = 1,\ldots,n \,. \tag{2.62}$$

By substituting the Slutsky decomposition (2.59) into (2.57) and (2.58), the homogeneity and symmetry restrictions for the compensated price elasticities are modified to

$$\sum_{j=1}^{n} \bar{\varepsilon}_{ij} = 0 \,, \tag{2.63}$$

and

$$S_i\bar{\varepsilon}_{ij} = S_j\bar{\varepsilon}_{ji}, \; i \neq j. \tag{2.64}$$

2.3.8 The differential demand system

Unlike other consumer demand theories, the differential approach to consumption theory (Theil 1980) is not developed from an explicit utility function. Instead, it is derived from implicit utility function such as (2.48), reproduced below:[12]

$$U = U\left(X_1, \ldots, X_n\right), \tag{2.65}$$

where X_i is the quantity consumed of good *i*. As above, we assume that the function is differentiable and that there is no satiation, so that each marginal utility is positive but diminishes as X_i continues to increase.

The consumer's budget constraint is

$$I = \sum_{i=1}^{n} P_i X_i, \tag{2.66}$$

where *I* is the income of the consumer, and P_i are the prices paid for the *n* goods in the consumer's budget.[13] We know from Section 2.3.7 that if we maximise the implicit utility function (2.65) subject to the budget constraint (2.66), we derive the first-order conditions

$$\partial U / \partial X_i = \Lambda P_i \text{ or } \partial U / \partial (P_i X_i) = \Lambda, \; i = 1, \ldots, n, \tag{2.67}$$

where Λ is, as previously, the Lagrange multiplier. Solving these conditions and, again, assuming that the resulting quantities are unique and positive for relevant values of prices and income, gives the demand equations of (2.52), reproduced below:

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[12] The following derivation of the differential demand system draws on Theil and Clements (1987), Chapters 1 and 4.

[13] (2.66) assumes that consumer spends all of their income on consumption. Where this is not the case, we replace income with total consumption expenditure.

$$X_i = X_i\left(I, P_i, ..., P_n\right),\ i = 1,...,n\,. \tag{2.68}$$

Taking the differential of the budget constraint (2.66), we get

$$dI = \sum_{i=1}^{n} P_i dX_i + \sum_{i=1}^{n} X_i dP_i\,. \tag{2.69}$$

Define budget shares as

$$W_i = P_i X_i / I \text{ or } I = P_i X_i / W_i\,. \tag{2.70}$$

Dividing (2.69) through by I and multiplying through by 100, gives

$$i = \sum_{i=1}^{n} W_i x_i + \sum_{i=1}^{n} W_i p_i\,,\ i = 1,...,n\,, \tag{2.71}$$

where i, x_i and p_i are percentage changes in I, X_i and P_i. (2.71) can also be written as

$$i = x + p\,, \tag{2.72}$$

where x and p are percentage-change Divisia indices in consumption (X) and prices (P), both defined using the budget shares, W_i, as weights.

Totally differentiating (2.68) gives

$$dX_i = \frac{\partial X_i}{\partial I} dI + \sum_{j=1}^{n} \frac{\partial X_i}{\partial P_j} dP_j\,,\ i = 1,...,n\,. \tag{2.73}$$

Multiplying both sides of (2.73) by P_i / I and 100, and using (2.70), gives

$$W_i x_i = \frac{\partial\left(P_i X_i\right)}{\partial I} i + \sum_{j=1}^{n} \frac{P_i P_j}{I} \frac{\partial X_i}{\partial P_j} p_j\,,\ i = 1,...,n\,, \tag{2.74}$$

which is the percentage-change form of (2.73). Thiel and Clements (1987), p.19, show that

$$\frac{\partial X_i}{\partial P_j} = \Lambda U^{ij} - \frac{\Lambda}{\partial \Lambda / \partial I} \frac{\partial X_i}{\partial I} \frac{\partial X_j}{\partial I} - \frac{\partial X_i}{\partial I} X_i\,,\ i, j = 1,...,n\,, \tag{2.75}$$

where U^{ij} is the (i, j)-th element of U^{-1}, which is the inverse of the Hessian matrix of the utility function (2.65).[14]

We can use (2.75) to replace $\frac{\partial X_i}{\partial P_j}$ in (2.74), which after rearranging becomes

$$W_i x_i = \frac{\partial(P_i X_i)}{\partial I}\left[i - \sum_{j=1}^{n} W_i p_j\right] + \sum_{j=1}^{n}\left[\frac{\Lambda P_i P_j U^{ij}}{I} - \frac{\Lambda/I}{\partial\Lambda/\partial I}\frac{\partial(P_i X_i)}{\partial I}\frac{\partial(P_j X_j)}{\partial I}\right]p_j,$$

$$i = 1,\ldots,n. \tag{2.76}$$

The term $\partial(P_i X_i)/\partial I$ in (2.76) is the marginal budget share for the i-th good, which we set equal to θ_i, where $\sum_i \theta_i = 1$. The first term in the first set of square brackets on the RHS of (2.76) is the Divisia volume index, x, defined in (2.71) and (2.72). We can use θ_i and x to rewrite (2.76) as

$$W_i x_i = \theta_i x + \sum_{j=1}^{n}\left[\frac{\Lambda P_i P_j U^{ij}}{I} - \frac{\Lambda/I}{\partial\Lambda/\partial I}\frac{\partial(P_i X_i)}{\partial I}\frac{\partial(P_j X_j)}{\partial I}\right]p_j,\ i, j = 1,\ldots,n. \tag{2.77}$$

We define the reciprocal of the income elasticity of the marginal utility of income, ϕ, as

$$\phi = \frac{\Lambda/I}{\partial\Lambda/\partial I} = \left(\frac{\partial\Lambda/\Lambda}{\partial I/I}\right)^{-1} < 0. \tag{2.78}$$

ϕ is usually referred to as the *income flexibility*. We also define

$$\theta_{ij} = \frac{\Lambda}{\phi I}P_i P_j U^{ij},\ \sum_{j=1}^{n}\theta_{ij} = \theta_i; i, j = 1,\ldots,n. \tag{2.79}$$

We can use θ_i, ϕ and θ_{ij} to rewrite (2.77) as

$$W_i x_i = \theta_i x + \phi\sum_{j=1}^{n}\theta_{ij}\left[p_j - p'\right],\ i = 1,\ldots,n, \tag{2.80}$$

[14] The Hessian matrix tells us the values of the second-order partial derivatives of the utility function (2.65).

where $p' = \sum_{i=1}^{n} \theta_i p_i$. Thus p' is the *Frisch price index*, which differs from the Divisia price index, *p*, in that the former uses marginal shares as weights and the latter uses budget shares as weights. The term on the LHS of (2.80) can be interpreted in two ways: (i) as the quantity component of the *i*-th budget share; or (ii) as the contribution of good *i* to the Divisia volume index *x*.

Whichever interpretation of $W_i x_i$ is applied, it is made up of two effects; an income (or expenditure) effect and a substitution effect. The first term on the RHS of (2.80) says that $W_i x_i$ will increase as real income (or consumption) rises, adjusted by the marginal share for the *i*-th good – the income (expenditure) effect. The second term on the RHS of (2.80) says that if the price of the *j*-th good rises relative to the *Frisch price index* of the basket of all goods consumed, then $W_i x_i$ will increase adjusted by the term $\phi\theta_{ij}$ – the substitution effect. The term $\phi\theta_{ij}$ is the (*i*, *j*)-th price coefficient, consisting of the *income flexibility* (ϕ) and the *normalised price coefficients* (θ_{ij}).[15] Note that ϕ is always negative. Thus, if θ_{ij} is negative then $\phi\theta_{ij}$ will be positive. In this case, as $\theta_{ij} < 0$ goods *i* and *j* are specific substitutes. But if θ_{ij} positive then $\phi\theta_{ij}$ will be negative. In this case, as $\theta_{ij} > 0$ goods *i* and *j* are specific complements (Houthakker 1960).

The differential approach allows us to derive demand equations of the form (2.80) from a utility function with no explicit functional form; equation (2.65). Thus the coefficients of the demand equations can vary, e.g., they can be functions of income and prices.

[15] θ_{ij} are normalised as $\sum_i \sum_j \theta_{ij} = 1$. Also note that θ_{ij} is symmetric in *i* and *j*.

2.3.9 The differential demand system for additive preferences

Equations (2.80) have been derived from a utility function with no restriction on the nature of preferences. We can adapt equations (2.80) to model demands for goods the preferences for which are additive.

Imagine the representative household's utility function is

$$U = \sum_{i=1}^{n} U_i(X_i),\ i = 1,\ldots,n. \tag{2.81}$$

In (2.81) the utility function is additive; thus marginal utility of good *i* is independent of the consumption of good *j* for $i \neq j$. This characteristic is known as *preference independence*. With (2.81) the Hessian matrix and its inverse are both diagonal, consequently, given (2.79) $\theta_{ij} = 0$ for $i \neq j$, and $\theta_{ii} = \theta_i$; all cross-price coefficients are zero. Equations (2.80) can then be rewritten as

$$W_i x_i = \theta_i x + \phi\theta_i \left[p_i - p' \right],\ i = 1,\ldots,n. \tag{2.82}$$

With all cross-price coefficients zero, equations (2.82) say that no pair of goods is a specific substitute or complement – an intuitive result given the assumption of preference independence.[16]

The above result seems unnecessarily strong. A weaker version of preference independence is *block independence*. Here the additive nature of (2.81) is applied to groups of goods rather than individual goods. If we divide the *n* goods into *G*<*n* groups, $S_1,\ldots,S_G$, and the members of each group are non-overlapping, we can then write the utility function as

$$U = \sum_{g=1}^{G} U_g\left(X_g^*\right),\ g = 1,\ldots,G, \tag{2.83}$$

[16] Although no pair of goods is a specific substitute or complement in (2.82), all pairs are still general substitutes.

where U_g are the G utility functions, each of which is a function of X_g^* – the vector of X_i s within S_G. Under (2.83), the marginal utility of a good only depends on the consumption of goods belonging to the same group. If the goods are numbered appropriately, the Hessian matrix of the utility function and its inverse become block diagonal. In this case, equation (2.83) is known as *block-independent preferences*.

Given (2.79), block independence implies that θ_{ij} is block-diagonal, so that if good i is part of group S_g, equations (2.80) and (2.79) can be written as

$$W_i x_i = \theta_i x + \phi \sum_{j \in S_g} \theta_{ij} \left[p_j - p' \right], \; i \in S_g, \tag{2.84}$$

$$\sum_{j \in S_g} \theta_{ij} = \theta_i \, , \; i \in S_g \, . \tag{2.85}$$

The substitution term in (2.84) implies that demand for good i is dependent upon the price of good i relative to the (Frisch) price index. Also, (2.85) implies that no good is a specific substitute or complement of any good that is not in the same group, i.e., $\theta_{ij} = 0$ for i and j in different groups.

If we sum over $i \in S_g$ for equations (2.84), we get the demand equations for each of the S_G groups,

$$W_g x_g = \theta_g x + \phi \sum_{i \in S_g} \sum_{j \in S_g} \theta_{ij} \left[p_j - p' \right], \; g = 1, \ldots, G, \tag{2.86}$$

where W_g and θ_g are the budget shares and marginal shares of group g, defined as

$$W_g = \sum_{i \in S_g} W_i \text{ and } \theta_g = \sum_{i \in S_g} \theta_i \, , \tag{2.87}$$

and x_g is the demand for group g as a whole, defined as

$$x_g = \sum_{i \in S_g} \frac{W_i}{W_g} x_i \, , \; g = 1, \ldots, G \, . \tag{2.88}$$

Given that θ_{ij} is symmetric in *i* and *j*, (2.85) can be written as

$$\sum_{i \in S_g} \theta_{ij} = \theta_j \text{ , } j \in S_g \text{ .} \qquad (2.89)$$

Using (2.89) and (2.87), (2.86) can be written as

$$W_g x_g = \theta_g x + \phi\theta_g \left[p'_g - p' \right], \ g = 1, \ldots, G \text{ ,} \qquad (2.90)$$

where

$$p'_g = \sum_{i \in S_g} \frac{\theta_i}{\theta_g} p_i \text{ , } g = 1, \ldots, G \text{ ,} \qquad (2.91)$$

so p'_g is the percentage change in the Frisch price index for group *g*. Equations (2.90) are the demand equations for the *g* groups. Thus the (budget-share adjusted) demand for group *g*, $W_g x_g$, is a function of real income (consumption), *x*, and the Frisch price index for the group relative to the Frisch price index for total consumption $\left[p'_g - p' \right]$, adjusted by θ_g and $\phi\theta_g$. Moving W_g to the RHS of (2.90), we note that $\frac{\theta_g}{W_g}$ is the income (expenditure) elasticity of the demand for group *g*, and $\frac{\phi\theta_g}{W_g}$ is the own-price elasticity of demand for group *g*.

The differential demand system with block independence can be extended to commodities *within* groups; the resulting demand equations are known as the *conditional demand equations*. To derive these demand equations we rearrange (2.90) as

$$x = \frac{W_g}{\theta_g} x_g - \phi \left[p'_g - p' \right], \ g = 1, \ldots, G \text{ ,} \qquad (2.92)$$

The RHS of (2.92) is then substituted into (2.84) giving

$$W_i x_i = \theta_i \left[\frac{W_g}{\theta_g} x_g - \phi\left(p'_g - p'\right)\right] + \phi \sum_{j \in S_g} \theta_{ij} \left(p_j - p'\right), \ g = 1, \ldots, G; i \in S_g, \qquad (2.93)$$

which expands to

$$W_i x_i = \theta_i \frac{W_g}{\theta_g} x_g - \theta_i \phi\left(p'_g - p'\right) + \phi \sum_{j \in S_g} \theta_{ij} \left(p_j - p'\right), \ g = 1, \ldots, G; i \in S_g. \qquad (2.94)$$

Using (2.89), the second appearance of θ_i on the RHS of (2.94) can be replaced with $\sum_{j \in S_g} \theta_{ij}$ due to symmetry in *i* and *j*, giving

$$W_i x_i = \theta_i \frac{W_g}{\theta_g} x_g - \phi \sum_{j \in S_g} \theta_{ij} \left(p'_g - p'\right) + \phi \sum_{j \in S_g} \theta_{ij} \left(p_j - p'\right), \ g = 1, \ldots, G; i \in S_g. \qquad (2.95)$$

The second and third terms on the RHS of (2.95) use p' as a deflator, and are multiplied by the same group of variables but of opposite sign, thus the two appearances of p' cancel, giving

$$W_i x_i = \theta_i \frac{W_g}{\theta_g} x_g + \phi \sum_{j \in S_g} \theta_{ij} \left(p_j - p'_g\right), \ g = 1, \ldots, G; i \in S_g. \qquad (2.96)$$

Equations (2.96) apply for all $i \in S_g$, and say that demand for good *i* depends on demand for the group S_g, x_g, and the price of good *i* relative to the Frisch price index for the group S_g, $\left(p_j - p'_g\right)$. Notice that the demands and prices for $i \notin S_g$ do not appear in (2.96). As θ_{ij} is a symmetric matrix in *i* and *j*, then $\theta_{ij} = \theta_{ji}$ where $i, j \in S_g$. This form of the differential demand equation is known as the *conditional demand equation*, whereas (2.84) is known as the *unconditional demand equation*.

2.3.10 The CES utility function

The CES utility function (Burk 1936) can be represented as[17]

$$U_i = \left[\sum_{s=1}^{2} \delta_{is} X_{is}^{-\rho}\right]^{-1/\rho}, \; 0 < \delta_{is} < 1, \sum_s \delta_{is} = 1; \rho \geq -1, \rho \neq 0 \qquad (2.97)$$

where U_i is utility from good i (i = 1,...,n) of the representative household, X_{is} (s = *domestic*, *imported*) are the goods consumed by the household differentiated by place of production, and δ_{is} and ρ are parameters. Here we are assuming that the U_is are determined elsewhere.[18]

For given prices of X_{is}, i.e., P_{is}, and assuming the absence of corner solutions, i.e., $\rho > -1$, the household chooses the X_{is}s so as to maximise utility subject to the budget constraint

$$\sum_{i=1}^{n} \sum_{s=1}^{2} P_{is} X_{is} = M \; , \qquad (2.98)$$

where M is the household's budget.

We know from Section 2.3.2 that in solving the cost minimising problem subject to a CES production function yields the percentage-change input demand equations (2.17), reproduced below

$$\overline{x}_i = z - \sigma\left[\overline{p}_i - \sum_{k=1}^{n} S_k \overline{p}_k\right], \; i = 1,...,n \, . \qquad (2.99)$$

Equations (2.99) state that the (effective) demand for any input i, $\overline{x}_i$, is a function of the firm's activity level, z, (the expansion effect) and the change in price of (the effective) input i relative to the change in the price of (effective) composite inputs, and the size of the elasticity of substitution between any pair of inputs, σ, (the substitution effect).

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[17] There exists a symmetry between the CES production and utility functions (Chung 1994).

[18] The summation operator in (2.97) suggests that the function is additive, but it is not, as $\partial^2 U_i / \partial X_{is} \partial X_{ir} \neq 0$ (Chung 1994, p. 58).

The CES utility maximisation problem (2.97) subject to the budget constraint (2.98) is similar to the cost minimisation problem subject to a CES production function. Thus, we can adapt equations (2.99) to the current problem, giving percentage-change household demand functions for inputs by source of the form

$$x_{is} = x_i - \sigma_i \left[p_{is} - \sum\nolimits_{s=1}^{2} S_{is} p_{is} \right], \ i = 1, \ldots, n; s = imported, domestic, \qquad (2.100)$$

where x_{is} and p_{is} are the percentage-change equivalents of X_{is} and P_{is}, x_i is the percentage-change in the demand for good *i*, σ_i is the elasticity of substitution between alternative types of good *i* defined as $\sigma = \frac{1}{(1+\rho)} > 0$, and S_{is} is the share of the household's expenditure on good *i* which is devoted to good *i* from source *s* and $\sum\nolimits_{s=1}^{2} S_{is} = 1$. Like equations (2.99), (2.100) are subject to expansion and substitution effects.

2.3.11 Applying separable utility functions

The model presented in Chapter 3 applies the utility theories presented in Sections 2.3.7–2.3.10. In all instances, the applications assume the utility functions are separable. Here we illustrate the application of separable utility using the utility functions presented earlier.[19]

Imagine the consumer's problem is to choose the inputs $X_1, \ldots, X_n$ so as to maximise

$$U\left(X_1, \ldots, X_n\right), \qquad (2.101)$$

[19] For a definition of separability and its advantages, see Section 2.3.4.

where U is utility. (2.101) is the implicit utility function. Section 2.3.7 shows how consumer demand equations can be derived from the implicit utility function (2.65). Thus, we able to rewrite (2.101) as

$$U = U\left[Imp\left(X_1\right),...,Imp\left(X_n\right)\right], \tag{2.102}$$

where $Imp\left(X_i\right)$ says that the i inputs are determined by the demand equations derived from the implicit utility function (2.101).

Equation (2.102) is one example of how to apply separable utility functions: it could be rewritten as

$$U = U\left\{ Imp\left[X_1\left(\sum_{k\in S_1} Diff\left\langle X_k\right\rangle\right)\right],...,Imp\left[X_n\left(\sum_{k\in S_n} Diff\left\langle X_k\right\rangle\right)\right]\right\}, \tag{2.103}$$

where $X_1,...,X_n$ now represent $X_{S_1},...,X_{S_n}$ groups of goods whose elements are nonoverlapping. Here, $X_n\left(\sum_{k\in S_n} Diff\left\langle X_k\right\rangle\right)$ says that the elements in each of the n groups are determined by the differential demand system. This represents a two-level utility structure with the implicit utility function demand system at level 1 and the differential demand system at level 2.

We could imagine that the elements in each of the $X_{S_1},...,X_{S_n}$ groups are sourced from domestic and imported sources. In this case, each of the elements in the n groups may be a CES combination of the domestic and imported versions of the good. This would add a third level to the nested utility structure of (2.103).

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