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A Gibbard-Satterthwaite Theorem for Public Good Economies*

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Abstract

We study the properties of mechanisms for deciding upon the provision of public goods when the feasible set is exogenously given (by financial and/or technological constraints), and individuals' preferences are represented by continuous, increasing and concave utility functions, and we establish a result analog to the Gibbard-Satterthwaite Theorem: strategy-proof mechanisms are dictatorial. Further, efficient and strategy-proof mechanisms are strongly dictatorial (i.e., maximize the dictator's welfare on the entire feasible set.)

Keywords: Strategy-proofness, Public goods provision, Economic incentives.

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1 Introduction

In this paper we study the properties of *mechanisms* for deciding upon the provision of public goods. In our setting, the feasible set is exogenously given (by financial and/or technological constraints), and individuals' preferences have the common properties assumed in economic environments (*viz.*, they can be represented by utility functions that are continuous, increasing and concave). We show that all mechanisms have very unappealing properties: they are either incompatible with individual incentives or *dictatorial* – i.e., select outcomes in the range of the mechanism attending to maximize the welfare of a single individual. The exact formulation of these results, which may be regarded as not surprising, as well as the geometry of these problems and the counterexamples we find, reveal new and interesting features of the problem of public good provision, and show the extent to which individuals' interests are irreconcilable.

In our study we adopt the social choice framework, although adapted to introduce the additional structure common to economic problems. In this framework, a mechanism is a mapping which associates a feasible outcome with each profile of utility functions. (Restricting attention to *direct mechanism*, for which each individual's strategy space is the set of his possible utility functions, is justified by the Revelation Principle.) Since individuals' preferences are private information, we require that mechanisms be *strategy-proof*; i.e., that each individual be best off reporting a utility function representing his true preferences, independently of the utility functions the other individuals report. Strategy-proof mechanisms are therefore compatible with individual incentives regardless of the information and prior belief each individual has about the preferences of the other individuals.

The properties of strategy-proof mechanisms depend largely on the domain of preferences on which decisions are to be made, as well as on the set of feasible outcomes. Gibbard[3] and Satterthwaite[7] have shown that when the set of feasible outcomes is finite and individuals' preferences are unrestricted, strategy-proof mechanisms whose range contains three or more outcomes are dictatorial. Barberà and Peleg[2] have established that the Gibbard-Satterthwaite Theorem remains valid even if the set of

feasible outcomes is infinite (any metric space) and preferences are restricted to be continuous. When preferences monotonic as well as continuous, Moreno[5] has shown that a weaker version of this result still holds: strategy-proof mechanisms whose *undominated range*¹ contains three or more outcomes are dictatorial on the subdomain of profiles of strictly increasing utility functions.

When preferences are convex as well as continuous there are mechanisms with a one dimensional range that are both strategy-proof and non-dictatorial. (Moulin[6] characterizes this class as median voter mechanisms; see also Barberà and Jackson[1].) Zhou[8] has shown, however, that also on this domain the conclusion of the Gibbard-Satterthwaite Theorem arises whenever the range of the mechanism contains a two dimensional set. (The dimensionality condition effectively requires that for any two outcomes in the range there be a third outcome in this set which is not a convex combination of the first two, so that any preference ordering over the three outcomes be possible.)

The results of the present paper establish a version of the Gibbard-Satterthwaite Theorem for the domain of preferences commonly associated with public goods in economic environments; namely, when individuals' preferences are described by continuous, concave and *increasing* utility functions: strategy-proof mechanisms whose undominated range satisfies a dimensionality condition are dictatorial on the subdomain of profiles of strictly increasing utility functions. The dimensionality condition on the undominated range is the counterpart (in the geometry imposed by the monotonicity of preferences) of Zhou's dimensionality condition; specifically, it is required that for any two outcomes in the undominated range there be a third outcome in this set which is not dominated by a convex combination of the first two, so that any preference ordering over the three outcomes be possible. We provide examples showing that this condition is essential to obtain the result. Interestingly, there are examples of strategy-proof mechanisms that are not dictatorial on the subdomain of profiles of strictly increasing utility functions, and do not seem to be median

¹The undominated range of a mechanism is the subset of its range containing the outcomes for which there are no other outcomes in the range containing no less of any good and more of at least one good.

voter type mechanisms; e.g., the mechanism discussed in Example 2 below produces a range which cannot be ordered in such a way that individuals' preferences be "singled peaked," which suggests that this mechanism cannot be written as a median voter mechanism.

The conclusions we obtain, albeit weaker than those obtained when satiated preferences are admissible, are undoubtedly negative: strategy-proof mechanisms are dictatorial on the subdomain of profiles of strictly increasing utility functions. Of course, it is on this subdomain, which is dense subset of the domain in the topologies commonly used in these contexts, that individual's conflicting interests manifest more clearly (and crudely). Further, we show that if a mechanism is also *non-wasteful* (that is, it is such that no outcome on its range is dominated by another feasible outcome) then it must be (fully) dictatorial. In addition, we establish that when the (undominated) feasible set satisfies the dimensionality condition, strategy-proof and *efficient* mechanisms (that is, it is such that the outcome it associates to each profile of preferences is Pareto optimal with respect to this profile) are *strongly dictatorial*; i.e., they maximize the dictator's welfare on the entire feasible set, without any other constraint. Finally, we also show that when the set of admissible utility functions is the set of all continuous, concave and strictly increasing utility functions, then only (fully) dictatorial mechanisms are strategy-proof. The present results therefore leave little doubt that the conclusion of the Gibbard-Satterthwaite Theorem is pervasive in public good economies.

Even though we are unable to use the existing results directly, our approach relies heavily on the literature. Barberà & Peleg[2] provide a direct proof (i.e., a proof that does not appeal to Arrow's Theorem) of the Gibbard-Satterthwaite Theorem, as well as of their own theorem. The argument of their proof is very general, and can be adapted to other domains. Using this approach Zhou[8] establishes his theorem, overcoming the need of "bimodal" utility functions used in the proof of the Barberà & Peleg Theorem by providing new and interesting arguments that reveal the role of the dimensionality condition. Moreno[5] also uses the argument introduced by Barberà & Peleg showing that it can be adapted to the geometry of increasing utility

functions, even though the argument yields conclusions that are weaker than those obtained by Barberà & Peleg. In the present paper we also follow the argument of Barberà & Peleg, and in order to resolve the difficulties that arise in dealing with a smaller domain we borrow both from Zhou[8] and from Moreno[5]. Adapting the argument turned out to be a difficult task, which required that we refine some of the intermediate results and establish new ones. In the end, however, we show once again this approach to be fruitful.

We present our results, discuss their implications, and present some examples in Section 2. The proofs are given in Section 3.

2 Results and Examples

We consider allocations problems where a group of individuals $N = \{1, \dots, n\}$ must decide the provision of l public goods. The set of feasible outcomes (i.e., bundles of public goods) is denoted by X , a compact subset of \mathbb{R}_+^l . Individuals' preferences are described by utility functions, $u : \mathbb{R}_+^l \rightarrow \mathfrak{R}$. (For simplicity, individuals' consumption sets are taken to be \mathbb{R}_+^l). For $a, b \in \mathbb{R}_+^l$, we write $a \geq b$ ($a \gg b$) if $a_k \geq b_k$ ($a_k > b_k$) for $k = 1, \dots, l$; and we write $a > b$ if $a_k \geq b_k$ and $a \neq b$. A utility function is *increasing* if for each $a, b \in \mathbb{R}_+^l$, $a > b$ implies $u(a) \geq u(b)$, and $a \gg b$ implies $u(a) > u(b)$; it is *strictly increasing* if $a > b$ implies $u(a) > u(b)$. We denote by U the set of all continuous, increasing and concave utility functions. For $\mathbf{u} \in U^n$ and $S \subset N$, \mathbf{u}_{-S} is the profile obtained from \mathbf{u} by deleting the utility functions of the members of S .

An *allocation mechanism* (or simply a *mechanism*) is a mapping $f : U^n \rightarrow X$. A mechanism f is *manipulable by Individual i* at $\mathbf{u} = (\mathbf{u}_{-i}, u_i) \in U^n$ if there is $u' \in U$ such that $u_i(f(\mathbf{u}_{-i}, u')) > u_i(f(\mathbf{u}))$. A mechanism is *strategy-proof* if it is not manipulable by any $i \in N$ at any $\mathbf{u} \in U^n$. A mechanism f is *dictatorial on $\Omega \subset U^n$* if there is an individual $i \in N$ such that for each $\mathbf{u} \in \Omega$, $f(\mathbf{u})$ maximizes u_i on $f(\Omega)$; when this is the case, we refer to Individual i as a *dictator for f on Ω* . A mechanism f is *dictatorial* if it is dictatorial on U^n ; we refer to a dictator for f on U^n as a *dictator*.

Strategy-proofness requires that “truthful revelation” be always an “equilibrium.” When preferences are private information and individuals’ prior beliefs about the preferences of the other individuals are unknown, this condition is necessary for a mechanism to be incentive compatible. Dictatorial mechanisms always select the outcome on the mechanism’s range attending to maximize the dictator’s welfare. Thus, when there are conflicting interests between the dictator’s and the other individuals, the dictator’s interest prevails.

The literature has generally found that strategy-proof mechanisms have very unappealing properties: *viz.*, strategy-proof mechanisms are dictatorial (see, e.g., Gibbard[3], Satterthwaite[7], Barberà and Peleg[2], Zhou[8], Moreno[5]). The public good economies considered here, however, are outside the scope of the existing results.

Zhou[8], in particular, establishes that if all continuous and concave utility functions are admissible, then strategy-proof mechanisms whose range contains a two dimensional set are dictatorial.² The following example shows that for the domain considered here there are strategy-proof and non-dictatorial mechanisms with a two dimensional range, and therefore that the analog of Zhou’s Theorem does not hold when admissible utility functions are further restricted to be increasing.³

Example 1. There are two public goods, and the set of feasible outcomes is

$$X = \{(x_1, x_2) \in \mathfrak{R}_+^2 \mid x_1^2 + x_2^2 \leq 1\}.$$

For each $u \in U$, let $m(u) = (m_1(u), m_2(u))$ denote the maximizer of u on X . Consider the mechanism $f : U^n \rightarrow X$ which for each $\mathbf{u} \in U^n$, selects

$$f(\mathbf{u}) = \left(x_1, \sqrt{1 - x_1^2} \right),$$

where x_1 is the “median” of $m_1(u_1), \dots, m_1(u_n)$. (If n is even, pick an arbitrary point in $[0, 1]$ and calculate the median adding this point.) This is a non-dictatorial

²Actually, Zhou’s Theorem is a slightly more general result (see Theorem 2 in Zhou[8] for a precise statement). We provide this formulation to facilitate comparison to the present results.

³Zhou[8] also provides an application of his theorem to provision problems where the feasible is linear and individuals’ preferences are described by continuous, strictly concave and strictly increasing utility functions. We discuss this application in connection with Example 4 below.

mechanism whose range is the set

$$f(U^n) = \{(x_1, x_2) \in \mathfrak{R}_+^2 \mid x_1^2 + x_2^2 = 1\},$$

a two dimensional set. It is easy to prove that this mechanism is also strategy-proof: let $i \in N$ and $\mathbf{u} \in U^n$ be such that $f(\mathbf{u}) \neq m(u_i)$; then either (1) $f_1(u'_i, \mathbf{u}_{-i}) \geq f_1(\mathbf{u}) > m_1(u_i)$ for each $u'_i \in U$, or (2) $f_1(u'_i, \mathbf{u}_{-i}) \leq f_1(\mathbf{u}) < m_1(u_i)$ for each $u'_i \in U$; thus, as u_i is increasing and concave one has

$$u_i(f(\mathbf{u})) \geq u_i(f(u'_i, \mathbf{u}_{-i})),$$

for each $u'_i \in U$. Hence f is strategy-proof.

A key feature of this example is that the underlying set of preferences that can be represented by increasing and concave utility functions are *single peaked* in the range of the mechanism; i.e., there is a *natural order* \succeq on $f(U^n)$ (namely, $x \succeq x'$ if $x_1 \geq x'_1$) such that for each $u \in U$ there is $x^* \in f(U^n)$ with the property that for each $x, x' \in f(U^n)$ with $x^* \succeq x \succ x'$ (or $x' \succ x \succeq x^*$), one has $u(x) > u(x')$. As the following example shows, however, there are strategy-proof and non-dictatorial mechanisms whose range is a two dimensional set on which preferences represented by increasing and concave utility functions need not be single peaked. A *selection on X* is a mapping $\sigma : 2^X \rightarrow X$ that assigns to each non-empty set $A \subset X$, a point $\sigma(A) \in A$.

Example 2. Consider the decision problem in Example 1. Let

$$A = \left\{ (1, 0), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \right\},$$

let

$$B = A \cup \left\{ \left(\frac{1}{4}, \frac{3}{4} \right), (0, 1) \right\},$$

and let σ be an arbitrary selection on X . For $u \in U$ and $Z \subset X$, write $M(u, Z)$ for the set of maximizers of u on Z , and let the mechanism $f : U^n \rightarrow X$ be given for each $\mathbf{u} \in U^n$ by

$$f(\mathbf{u}) = \begin{cases} \sigma(M(u_1, B)) & \text{if } (1, 0) \notin M(u_1, B), \\ \sigma(M(u_2, A)) & \text{otherwise.} \end{cases}$$

This mechanism is not dictatorial. Moreover, it is strategy-proof; clearly, individuals $2, \dots, n$ cannot manipulate; further, Individual 1 cannot manipulate either as $f(\mathbf{u}) \notin M(u_1, B)$ implies $\{(1, 0)\} \in M(u_1, B)$ and $f(\mathbf{u}) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, and therefore as u_1 is increasing and concave, one has $u_1(f(\mathbf{u})) \geq u_1(x)$ for each $x \in B \setminus \{(1, 0)\}$. Also it is easy to see that there is no “order” on $B = f(U^n)$ according to which all continuous increasing and concave utility functions represent single-peaked preferences.

These examples suggest that when preferences are described by increasing as well as concave utility functions, assuming that the range of the mechanism is at least a two dimensional set does not imply that a strategy-proof mechanism must be dictatorial as in Zhou’s Theorem. It turns out, however, that under an analogous condition a slight weaker conclusion arises.

For $A \subset \mathbb{R}^l$, write \widehat{A} for the set $\{a \in A \mid \nexists a' \in A : a' > a\}$. (Thus, if $A \subset \mathbb{R}^2$, then \widehat{A} is the “northeast” boundary of A .) Given a mechanism f , we refer to the set $\widehat{f(U^n)}$ as the *undominated range* of the mechanism. For each $a, b \in \mathbb{R}^l$, by $[a, b]$ the (closed) segment connecting a and b , i.e.,

$$[a, b] = \{x \in \mathbb{R}^l \mid x = \lambda a + (1 - \lambda) b, \lambda \in [0, 1]\}.$$

Also write $\Lambda(a, b)$ for the set containing the vectors on and above $[a, b]$; i.e.,

$$\Lambda(a, b) = \{x \in \mathbb{R}^l \mid \exists x' \in [a, b] : x \geq x'\}$$

Three vectors $a, b, c \in \mathbb{R}^l$ are said to be *monotonically affinely independent (m.a.i.)* if $a \notin \Lambda(b, c)$, $b \notin \Lambda(a, c)$, and $c \notin \Lambda(a, b)$ (i.e., if no point is greater than or equal to a convex combination of the other two). Condition D below plays a role in Theorem 1 below analog to the condition in Zhou’s Theorem that “the range of a mechanism be at least a two dimensional set.”

Condition D. A set $A \subset \mathbb{R}_+^l$ satisfies *Condition D* if it is not a singleton, and if for each $a, b \in A$, there is $c \in A$ such that a, b, c are monotonically affinely independent.

Note that the dimension of a set $A \subset \mathbb{R}^l$ is at least two if (and only if) for all $a, b \in A$ there is $c \in A$ such that a, b , and c are affinely independent (or equivalently,

such that no point is a convex combination of the other two). Hence every set satisfying Condition D contains a two-dimensional subset, although the converse does not hold; e.g., the range of the mechanisms defined in examples 1 and 2 are both two-dimensional sets, but they do not satisfy Condition D. The set $\{(1, 0), (0, 1), (\frac{1}{3}, \frac{1}{3})\}$, or the simplex in \mathbb{R}^3 are examples of sets satisfying Condition D. Note that Condition D effectively requires that at least two public goods be provided.

Theorem 1 below establishes a result analog to Zhou’s Theorem for the domain of utility profiles whose coordinates are continuous increasing and concave utility functions. Denote by \dot{U} the set of admissible utility functions that are strictly increasing.

Theorem 1. *Let f be a strategy-proof mechanism such that $\widehat{f(U^n)}$ satisfies Condition D. Then f is dictatorial on \dot{U}^n . Moreover, if Individual i is the dictator for f on \dot{U}^n , then for each $\mathbf{u} \in \dot{U}^n$, $f(\mathbf{u})$ maximizes u_i on $f(U^n)$.*

Theorem 1 identifies an important subdomain where every strategy-proof mechanism satisfying the assumptions of Theorem 1 must be dictatorial; namely, the set of profiles whose coordinates are strictly increasing utility functions.⁴ Moreover, on this subdomain the mechanism selects the dictator’s most preferred outcome *on the entire range*.

The conclusion of Theorem 1 is weaker than that of Zhou’s Theorem, but as Example 3 shows a dictator on \dot{U}^n need not be a (full) dictator.

Example 3. There are three public goods, and the set of feasible outcomes is $X = \{x \in \mathbb{R}_+^3 \mid x_1 + x_2 + x_3 \leq 1\}$. Let a mechanism $f : U^n \rightarrow X$ be given for each $\mathbf{u} \in U^n$ by $f(\mathbf{u}) = (0, 0, 0)$ if $u_2(x_1, x_2, x_3) = x_1x_2x_3$; otherwise let $f(\mathbf{u})$ be some arbitrary maximizer of u_1 on $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. Note that f is strategy-proof and $\widehat{f(U^n)} = \hat{X}$ satisfies Condition D. Thus, f is within the scope of Theorem 1, and is therefore dictatorial on \dot{U}^n . Nonetheless, it is not (fully) dictatorial. (See Moreno[5] for further discussion of this example).

Two features of the mechanism in Example 3 are worth pointing out: the mechanism is both *wasteful* (i.e., it does not always exhaust the existing resources), and

⁴In addition, the set \dot{U} is a dense subset of U in the most commonly used topologies; e.g., the compact-open topology, or the topology of closed convergence – see Hildenbrand[4].

inefficient (when the dictator for f on \dot{U}^n does not get one of his most preferred outcome, it is possible to make him better off without making anybody else worse off). Propositions 1 and 2 below establish that these features are common to all mechanisms that satisfy the assumptions of Theorem 1 but are not (fully) dictatorial. In particular, Proposition 1 establishes that mechanisms that satisfy the assumptions of Theorem 1 and are *non-wasteful* (i.e., satisfy $f(U^n) \subset \hat{X}$) are (fully) dictatorial. Note that since admissible utility functions need not be strictly increasing, wastefulness does not imply inefficiency.

Proposition 1. *Every strategy-proof and non-wasteful mechanism f such that $f(U^n)$ satisfies Condition D is dictatorial.*

Proposition 2 below establishes that only strongly dictatorial mechanisms are strategy-proof and efficient. (A mechanism f is *efficient* if for each $\mathbf{u} \in U^n$, $f(\mathbf{u})$ is Pareto optimal with respect to \mathbf{u} .) Note that in Proposition 2 Condition D is imposed on the *undominated feasible set*, \hat{X} , rather than on the range of the mechanism. In the proof of Proposition 2 it is shown that the range of a strategy-proof and efficient mechanism coincides with \hat{X} , which therefore satisfies Condition D. This also provides a stronger condition to the implication of Proposition 2, since it is shown as mechanisms satisfying its assumptions maximize the dictator's welfare *on the entire feasible set*. In the literature, mechanisms with this property are referred to as *strongly dictatorial*. (Formally, a mechanism f is *strongly dictatorial* if there is an individual $i \in N$ such that for each $\mathbf{u} \in U^n$, $f(\mathbf{u})$ maximizes u_i on X .)

Proposition 2. *If \hat{X} satisfies Condition D, then every strategy-proof and efficient mechanism $f : U^n \rightarrow X$ is strongly dictatorial.*

Finally, Proposition 3 establishes a conclusion stronger than that of Theorem 1 for mechanisms whose domain is the set of profiles of utility functions whose coordinates are continuous, concave and *strictly* increasing: For this domain strategy-proof mechanisms whose range satisfies Condition D are dictatorial.

Proposition 3. *Every strategy-proof mechanism $f : \dot{U}^n \rightarrow X$ such that $f(\dot{U}^n)$ satisfies Condition D is dictatorial.*

Note that, as in Theorem 1, Condition D is essential in propositions 1 to 3: even

though the range of the mechanisms in the scope of these results contain only undominated outcomes, imposing that this be at least a two dimensional set does not imply that the mechanism be dictatorial, as Example 1 above shows.

We finish this section with a discussion of the applications of our results to an interesting class of economies, which is described in Example 4 below.

Example 4. There are $l \geq 3$ public goods, and the set of feasible outcomes is $X = \{x \in \mathbb{R}_+^l \mid px \leq G\}$, where the coordinates of $p \in \mathbb{R}_{++}^l$ can be interpreted as the prices of the public goods that can be provided, and $G > 0$ as the government's budget for public expenditure.

Every strategy-proof mechanism $f : U^n \rightarrow X$ such that $\widehat{f(U^n)}$ satisfies Condition D is dictatorial on \dot{U}^n (Theorem 1); if in addition f is non-wasteful (i.e., it exhausts the budget), then it is (fully) dictatorial (Proposition 1). And since \hat{X} satisfies Condition D, every strategy-proof and efficient mechanism $f : U^n \rightarrow X$ is strongly dictatorial.

For the economies described in Example 4, Zhou[8] shows (Theorem 3) that if a mechanism $f : \dot{U}^n \rightarrow X$ is strategy-proof and *unanimous* (i.e., such that for each $u \in \dot{U}$, $f(u, \dots, u)$ maximizes u on X), then it is dictatorial. This conclusion can be obtained as a corollary of Proposition 3, since unanimity implies $f(\dot{U}^n) \supseteq \hat{X}$, and because $f(\dot{U}^n) = \widehat{f(\dot{U}^n)}$ (see the proof of Proposition 3 in the Section 3), we have $\widehat{f(\dot{U}^n)} \supseteq \hat{X}$, and thus $f(\dot{U}^n) = \widehat{f(\dot{U}^n)} = \hat{X}$; hence f satisfies the assumptions of Proposition 3. (Note that, as in Proposition 2 above, f is dictatorial in the stronger sense that it maximizes the dictator's welfare on the entire feasible set.)

3 Proofs

Throughout this section, let U be the set of all the continuous, concave and increasing utility functions, and let $f : U^n \rightarrow X$ be a strategy-proof mechanism. For each $u \in U$ let the *mechanism* $f^u : U^{n-1} \rightarrow X$ be given, for each $\mathbf{u}_{-n} \in U^{n-1}$, by $f^u(\mathbf{u}_{-n}) = f(\mathbf{u}_{-n}, u)$. Note that each f^u is also a strategy-proof mechanism. For each $u \in U$, write $O(u) = f^u(U^{n-1}) \cap \widehat{f(U^n)}$. The set $O(u)$ provides the "options" (i.e.,

outcomes in the undominated range) that are attainable by individuals $1, \dots, n - 1$, when Individual n reports the utility function u . Also write U^* for the set containing the utility functions in U with a unique maximizer on $f(U^n)$, and \dot{U}^* for the set $U^* \cap \dot{U}$. Note that the maximizer of each $u \in U^*$ on $f(U^n)$, denoted by $m(u)$, is a member of $\widehat{f(U^n)}$.

The proof proceeds by showing that a mechanism f satisfying the assumptions of Theorem 1 must be dictatorial on $(\dot{U}^*)^n$. With this result it is straightforward to show that f is also dictatorial on \dot{U}^n . In proving this the correspondence O plays a fundamental role. The key insight is to note that if f is dictatorial on $(\dot{U}^*)^n$, then O has a peculiar structure: if Individual n is the dictator, then for each $u \in \dot{U}^*$ the set of options for individuals $1, \dots, n - 1$ is a singleton, $m(u)$, whereas if the dictator is somebody else, then for each $u \in \dot{U}^*$ the set of options for individuals $1, \dots, n - 1$ must be constant and equal the entire set $\widehat{f(U^n)}$. Showing that O has this structure (Lemma 10) turns out to be a difficult task in our setting, and requires that we establish first a number of preliminary results.

We begin by stating some known properties of strategy-proof mechanisms. The proof of $P1$ can be found in Zhou[8] (Lemma 1), whereas the proofs of $P2 - P6$ are given in Moreno[5] (lemmas 3 to 7). Given an arbitrary set $A \subset \mathbb{R}^l$, denote by \bar{A} its closure.

P1. (*Unanimity*) For each $u \in U$, $f(u, \dots, u)$ maximizes u on $f(U^n)$.

P2. For each $u \in U^*$, $m(u) \in O(u)$.

P3. $f(\dot{U}^n) \subset \widehat{f(U^n)}$.

P4. For each $x \in \overline{f(U^n)}$, there is $x' \in f(U^n)$ such that $x' \geq x$.

P5. For each $u \in U$, $O(u)$ is closed relative to $\widehat{f(U^n)}$.

P6. For each $x \in f(U^n) \setminus \widehat{f(U^n)}$, there is $x' \in \widehat{f(U^n)}$ such that $x' > x$.

We now establish a number of intermediate results. For each $a, b \in \mathbb{R}^l$, write by $(a, b) = [a, b] \setminus \{a, b\}$ for the open segment connecting a and b .

Lemma 1. Let x_1, x_2, x_3, x_4 be four distinct points in $\widehat{f(U^n)}$.

L1.1. If $x_4 \in \Lambda(x_1, x_2)$ and $x_2 \in \Lambda(x_3, x_4)$, then $x_2 \in \Lambda(x_1, x_3)$ and $x_4 \in \Lambda(x_1, x_3)$.

L1.2. If $x_4 \in \Lambda(x_1, x_2)$, then either $x_4 \notin \Lambda(x_1, x_3)$ or $x_4 \notin \Lambda(x_2, x_3)$.

L1.3. If $x_4 \in \Lambda(x_1, x_2)$ and $x_1 \in \Lambda(x_2, x_3)$, then $x_4 \notin \Lambda(x_1, x_3)$.

Proof: Let x_1, x_2, x_3, x_4 be four distinct points in $\widehat{f(U^n)}$.

We proof L1.1. Assume $x_4 \in \Lambda(x_1, x_2)$ and $x_2 \in \Lambda(x_3, x_4)$; then

$$x_4 \geq \lambda x_1 + (1 - \lambda)x_2,$$

and

$$x_2 \geq tx_3 + (1 - t)x_4$$

for some $\lambda, t \in (0, 1)$. Thus

$$x_4 \geq \lambda x_1 + (1 - \lambda)(tx_3 + (1 - t)x_4),$$

and therefore

$$x_4 \geq \beta x_1 + (1 - \beta)x_3,$$

where $\beta = \frac{\lambda}{\lambda + t(1 - \lambda)} \in (0, 1)$. Hence $x_4 \in \Lambda(x_1, x_3)$. The proof that $x_2 \in \Lambda(x_1, x_3)$ is analogous.

We prove L1.2. Denote by r the line passing through x_1 and x_2 , and for $x \in \mathbb{R}^l$ write x_r for the orthogonal projection of x on r . Note that because $x_1, x_2, x_3 \in \widehat{f(U^n)}$ we have $x_1 \neq (x_3)_r$ and $x_2 \neq (x_3)_r$. Now, assume by way of contradiction that $x_4 \in \Lambda(x_1, x_2) \cap \Lambda(x_1, x_3) \cap \Lambda(x_2, x_3)$. Since $x_4 \in \Lambda(x_1, x_2)$ and $x_1, x_2, x_4 \in \widehat{f(U^n)}$ then $(x_4)_r \in (x_1, x_2)$. Likewise since $x_4 \in \Lambda(x_1, x_3)$, then $(x_4)_r \in (x_1, (x_3)_r)$, and since $x_4 \in \Lambda(x_2, x_3)$, then $(x_4)_r \in (x_2, (x_3)_r)$. Therefore

$$(x_4)_r \in (x_1, x_2) \cap (x_1, (x_3)_r) \cap (x_2, (x_3)_r) = \emptyset,$$

which is a contradiction.

Finally, we prove L1.3. Assume by way of contradiction that $x_1 \in \Lambda(x_2, x_3)$ and $x_4 \in \Lambda(x_1, x_2) \cap \Lambda(x_1, x_3)$. Since $x_1 \in \Lambda(x_2, x_3)$ we have $\Lambda(x_1, x_2) \subset \Lambda(x_2, x_3)$, and therefore $x_4 \in \Lambda(x_2, x_3)$. This contradicts L1.2. \square

Lemma 2. Let $u \in \dot{U}^*$. For each $x \in \overline{O(u)}$ there is $x' \in O(u)$ such that $x' \geq x$.

Proof: Let $u \in \dot{U}^*$ and let $x \in \overline{O(u)}$. Assume by way of contradiction that there is no $x' \in O(u)$ such that $x' \geq x$. Let $\bar{x} \in \arg \max \sum x_k$ on $(x + \mathbb{R}_+^l) \cap \overline{O(u)}$. Then

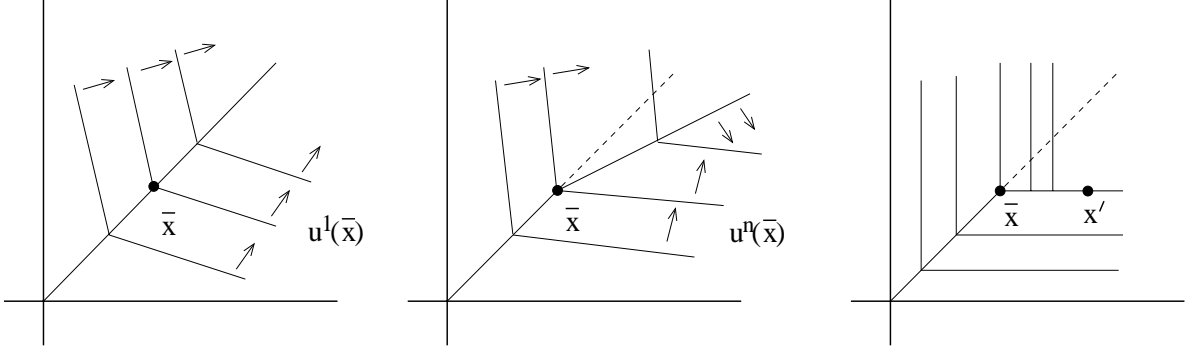


Figure 1: The sequence $\{u^q\}$ in Lemma 2.

$\bar{x} \in \overline{O(u)} \setminus O(u)$ and $(\bar{x} + \mathbb{R}_+^l) \setminus \{\bar{x}\} \cap \overline{O(u)} = \emptyset$. Since $\bar{x} \in \overline{O(u)}$, then $\bar{x} \in \overline{f(U^n)}$, and therefore by *P4* and *P6* there is $x' \geq \bar{x}$ such that $x' \in \widehat{f(U^n)}$; moreover, since $x' \notin O(u)$ by assumption and $O(u)$ is closed relative to $\widehat{f(U^n)}$ (by *P5*), then $x' \notin \overline{O(u)}$. Hence $x' \neq \bar{x}$, and $x' > \bar{x}$. Let \hat{u} be such that $m(\hat{u}) = x'$, and let $\{u^q\} \subset \dot{U}$ be a sequence of utility functions whose upper contour sets on \bar{x} approach $\bar{x} + \mathbb{R}_+^l$, and whose upper contour sets on x' approach $x' + \mathbb{R}_+^l$ (see Figure 1).

Since the sequence $\{f(u^q, \dots, u^q, u)\} \subset \overline{O(u)}$ is bounded, it has a convergent subsequence $\{f(u^{q_s}, \dots, u^{q_s}, u)\}$. We show that the limit of $\{f(u^{q_s}, \dots, u^{q_s}, u)\}$ is \bar{x} . Assume, by way of contradiction, that $\lim f^u(u^{q_s}, u^{q_s}, \dots, u^{q_s}) = \hat{x} \neq \bar{x}$; since $((\bar{x} + \mathbb{R}_+^l) \setminus \{\bar{x}\}) \cap \overline{O(u)} = \emptyset$ and $\lim f^u(u^{q_s}, u^{q_s}, \dots, u^{q_s}) \in \overline{O(u)}$, then $\hat{x} \notin (\bar{x} + \mathbb{R}_+^l) \setminus \{\bar{x}\}$. Therefore there is \hat{s} be sufficiently large that $u^{q_s}(\bar{x}) > u^{q_s}(\hat{x})$ for $q_s \geq q_{\hat{s}}$, and there are $\bar{s} > \hat{s}$ sufficiently large that $f^u(u^{q_s}, u^{q_s}, \dots, u^{q_s})$ is so close to \hat{x} , and $\bar{x}' \in O(u)$ so close to \bar{x} that $u^{q_s}(\bar{x}') > u^{q_s}(f^u(u^{q_s}, u^{q_s}, \dots, u^{q_s}))$. This contradicts *P1*. Hence $\{f(u^{q_s}, \dots, u^{q_s}, u)\}$ converges to \bar{x} . Analogously, the sequence $\{f(u^{q_s}, \dots, u^{q_s}, \hat{u})\} \subset \overline{O(\hat{u})}$ has a convergent subsequence, $\{f(u^{q_{s_k}}, \dots, u^{q_{s_k}}, \hat{u})\}$, whose limit is x' . Now, since f is strategy-proof, for each q_{s_k} we have

$$u(f(u^{q_{s_k}}, \dots, u^{q_{s_k}}, u)) \geq u(f(u^{q_{s_k}}, \dots, u^{q_{s_k}}, \hat{u})),$$

and therefore in the limit

$$u(\bar{x}) \geq u(x'),$$

which is a contradiction. \square

Lemma 3. Let $u \in \dot{U}^*$. For each $\bar{x} \in \overline{O(u)}$ and $y \in \widehat{f(U^n)} \setminus O(u)$ there is $\hat{x} \in \Lambda(\bar{x}, y) \cap O(u)$ such that $(\Lambda(\hat{x}, y) \setminus \{\hat{x}\}) \cap \overline{O(u)} = \emptyset$.

Proof: Let $u \in \dot{U}^*$, $\bar{x} \in \overline{O(u)}$ and $y \in \widehat{f(U^n)} \setminus O(u)$. For $x \in \Lambda(\bar{x}, y) \cap \overline{O(u)}$ define $g(x) = \lambda$, where λ is the smallest number in $[0, 1]$ such that $x \geq \lambda\bar{x} + (1 - \lambda)y$. Since g is a continuous function it has a minimum on $\Lambda(\bar{x}, y) \cap \overline{O(u)}$ (a compact set). Let $\hat{x} \in \arg \min_{x \in \Lambda(\bar{x}, y) \cap \overline{O(u)}} g(x)$ be such that there is no $\hat{x}' \in \arg \min_{x \in \Lambda(\bar{x}, y) \cap \overline{O(u)}} g(x)$ such that $\hat{x}' > \hat{x}$. Since $y \in \widehat{f(U^n)} \setminus O(u)$ and $O(u)$ is closed relative to $\widehat{f(U^n)}$ by P5, $y \notin \overline{O(u)}$. (If $y \in \overline{O(u)}$, then $y \in \widehat{f(U^n)} \cap \overline{O(u)}$ and therefore $y \in O(u)$, which is a contradiction.) Thus $\hat{x} \neq y$, and since $y \in \widehat{f(U^n)}$, we have $g(\hat{x}) > 0$. We show that $(\Lambda(\hat{x}, y) \setminus \{\hat{x}\}) \cap \overline{O(u)} = \emptyset$. Suppose that there is $x \in (\Lambda(\hat{x}, y) \setminus \{\hat{x}\}) \cap \overline{O(u)}$. Then there is $\lambda \in (0, 1]$ such that

$$x \geq \lambda\hat{x} + (1 - \lambda)y \geq \lambda(g(\hat{x})\bar{x} + (1 - g(\hat{x}))y) + (1 - \lambda)y;$$

i.e.,

$$x \geq \lambda g(\hat{x})\bar{x} + (1 - \lambda g(\hat{x}))y.$$

Hence $g(x) \leq \lambda g(\hat{x})$. If $\lambda = 1$, we have $g(x) = g(\hat{x})$, $x \geq \hat{x}$, and (because $x \neq \hat{x}$) $x > \hat{x}$, which contradicts our choice of \hat{x} . If $\lambda \neq 1$, then $g(x) < g(\hat{x})$, and since $x \in \Lambda(\bar{x}, y) \cap \overline{O(u)}$ (because $\Lambda(\hat{x}, y) \subset \Lambda(\bar{x}, y)$) this contradicts that $\hat{x} \in \arg \min_{x \in \Lambda(\bar{x}, y) \cap \overline{O(u)}} g(x)$. Now $\hat{x} \in O(u)$, for if $\hat{x} \in \overline{O(u)} \setminus O(u)$ then by L2 there is $x \in O(u)$ such that $x \geq \hat{x}$; but then $x \in (\Lambda(\hat{x}, y) \setminus \{\hat{x}\}) \cap \overline{O(u)}$ which is a contradiction.

Lemma 4. Let $u, u' \in \dot{U}^*$ be such that $m(u) = m(u')$. Then $O(u) = O(u')$.

Proof: Let $u, u' \in \dot{U}^*$ such that $m(u) = m(u') = \bar{x}$, and assume that $\exists y \in O(u) \setminus O(u')$. Since $\bar{x} \in \overline{O(u')}$ (by P2) and $y \in \widehat{f(U^n)} \setminus O(u')$, then by Lemma 3 there is $\hat{x} \in \Lambda(\bar{x}, y) \cap O(u')$ such that $(\Lambda(\hat{x}, y) \setminus \{\hat{x}\}) \cap \overline{O(u')} = \emptyset$. Let $\{u^q\} \subset \dot{U}$ be a sequence of utility functions such that their upper contour sets on \hat{x} approach $\Lambda(\hat{x}, y)$ and their upper contour sets on y approach $y + \mathbb{R}_+^l$ (see Figure 2). Since $\{f(u^q, \dots, u^q, u')\} \subset O(u')$ is a bounded sequence, it has a convergent subsequence $\{f(u^{q_s}, \dots, u^{q_s}, u')\}$ whose limit is \hat{x} – this is established by an argument analogous to that given in the proof of Lemma 2. Also the sequence $\{f(u^{q_s}, \dots, u^{q_s}, u)\} \subset \overline{O(u)}$ has a convergent subsequence $\{f(u^{q_{s_k}}, \dots, u^{q_{s_k}}, u)\}$ whose limit is y . Now, since f is

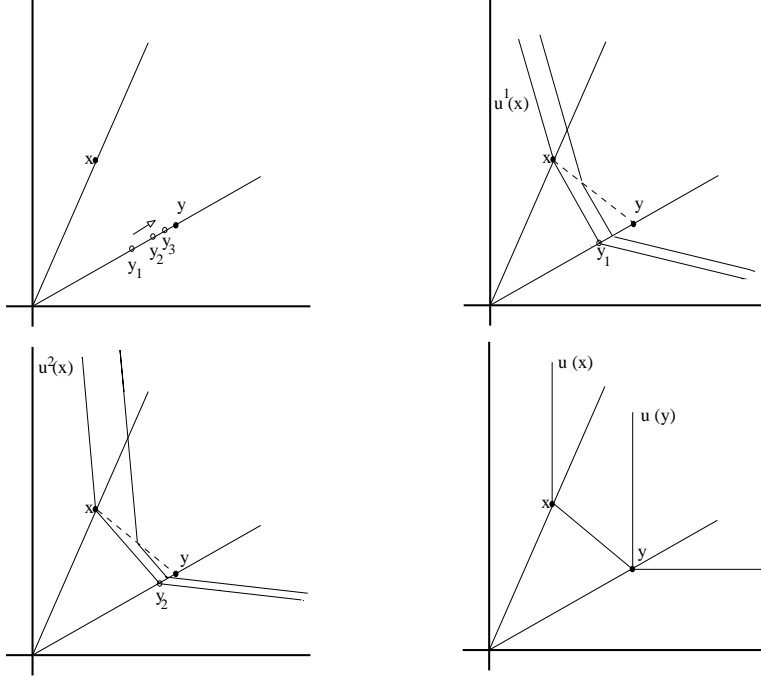


Figure 2: The sequence $\{u^q\}$ in Lemma 5.

strategy-proof, for each q_{s_k} we have

$$u(f(u^{q_{s_k}}, \dots, u^{q_{s_k}}, u)) \geq u(f(u^{q_{s_k}}, \dots, u^{q_{s_k}}, u')),$$

and therefore

$$u(y) \geq u(\hat{x}),$$

which is a contradiction since $m(u) = \bar{x}$, $\hat{x} \in \Lambda(\bar{x}, y) \setminus \{y\}$ and u is strictly increasing and concave. \square

Lemma 5. *Let $u \in \dot{U}^*$, $\bar{x} \in \overline{O(u)} \setminus \{m(u)\}$, and $y \in \widehat{f(U^n)} \setminus O(u)$. Then either $\bar{x} \in \Lambda(m(u), y)$ or $(\Lambda(\bar{x}, y) \setminus \{\bar{x}\}) \cap \overline{O(u)} \neq \emptyset$.*

Proof: Let $u \in \dot{U}^*$, $\bar{x} \in \overline{O(u)} \setminus \{m(u)\}$, and $y \in \widehat{f(U^n)} \setminus O(u)$. Assume, by way of contradiction, that $\bar{x} \notin \Lambda(m(u), y)$ and $(\Lambda(\bar{x}, y) \setminus \{\bar{x}\}) \cap \overline{O(u)} = \emptyset$. Since $\bar{x} \notin \Lambda(m(u), y)$, by Lemma 4 we can assume, without loss of generality, that $u(y) > u(\bar{x})$. Let $\{u^q\} \subset \dot{U}$ be a sequence of utility functions whose upper contour sets at \bar{x} approach $\Lambda(\bar{x}, y)$, and at y approach $y + \mathbb{R}_+^l$ (see Figure 2). The sequence $\{f(u^q, \dots, u^q, u)\} \subset \overline{O(u)}$ has a convergent subsequence $\{f(u^{q_s}, \dots, u^{q_s}, u)\}$ whose limit

can only be \bar{x} . (The proof of this fact is analogous to that given in Lemma 2.) Likewise the sequence $\{f(u^{q_s}, \dots, u^{q_s}, \hat{u})\}$, where $\hat{u} \in \dot{U}^*$ is such that $m(\hat{u}) = y$, has a convergent subsequence $\{f(u^{q_{s_k}}, \dots, u^{q_{s_k}}, \hat{u})\}$ whose limit is y . However, since f is strategy-proof, for each q_{s_k} we have

$$u(f(u^{q_{s_k}}, \dots, u^{q_{s_k}}, u)) \geq u(f(u^{q_{s_k}}, \dots, u^{q_{s_k}}, \hat{u})),$$

and therefore in the limit $u(\bar{x}) \geq u(y)$, which is a contradiction. \square

Lemma 6. *Let $u \in \dot{U}^*$, $\bar{x} \in \overline{O(u)}$, $y \in \widehat{f(U^n)} \setminus O(u)$. If $m(u), \bar{x}, y$ are m.a.i., then there is $\hat{x} \in (\Lambda(m(u), y) \setminus \{m(u)\}) \cap O(u)$ such that $(\Lambda(\hat{x}, y) \setminus \{\hat{x}\}) \cap \overline{O(u)} = \emptyset$.*

Proof: Let $u \in \dot{U}^*$, $\bar{x} \in \overline{O(u)}$, and $y \in \widehat{f(U^n)} \setminus O(u)$ be such that $m(u), \bar{x}, y$ are m.a.i. By Lemma 3 there is $\hat{x} \in \Lambda(\bar{x}, y) \cap O(u)$ such that $(\Lambda(\hat{x}, y) \setminus \{\hat{x}\}) \cap \overline{O(u)} = \emptyset$. Moreover, $\hat{x} \neq m(u)$ for if $\hat{x} \in \Lambda(\bar{x}, y)$ then $m(u), \bar{x}, y$ would not be m.a.i.. Lemma 5 therefore implies that $\hat{x} \in \Lambda(m(u), y)$. \square

The next lemma is the analog of Zhou's step three (adapted to the geometry of increasing utility functions). A set $A \subset \mathbb{R}_+^l$ is *monotonically-star-shaped relative to* $B \subset \mathbb{R}_+^l$ with respect to a base point $x \in A$ if for each $y \in A$, $\Lambda(x, y) \cap B \subset A$.

Lemma 7. *For each $u \in \dot{U}^*$, $O(u)$ is monotonically-star-shaped relative to $\widehat{f(U^n)}$ with respect to $m(u)$.*

Proof: Let $u \in \dot{U}^*$ and $\bar{x} \in O(u)$, and assume that there is $y \in \Lambda(\bar{x}, m(u)) \cap \widehat{f(U^n)}$ such that $y \notin O(u)$. By Lemma 3 there is $\hat{x} \in \Lambda(\bar{x}, y) \cap O(u)$ such that $(\Lambda(\hat{x}, y) \setminus \{\hat{x}\}) \cap \overline{O(u)} = \emptyset$. If $\hat{x} = \bar{x}$, then we have $(\Lambda(\bar{x}, y) \setminus \{\bar{x}\}) \cap \overline{O(u)} = \emptyset$ and $\bar{x} \notin \Lambda(m(u), y)$, which contradicts Lemma 5. If $\hat{x} \neq \bar{x}$, then $\hat{x} \notin \Lambda(y, m(u))$ by L1.3. Then we have $\hat{x} \in \overline{O(u)} \setminus \{m(u)\}$ and $y \in \widehat{f(U^n)} \setminus O(u)$ such that $\hat{x} \notin \Lambda(m(u), y)$ and $(\Lambda(\hat{x}, y) \setminus \{\hat{x}\}) \cap \overline{O(u)} = \emptyset$, which again contradicts Lemma 5. \square

Lemma 8. *Let $u, u' \in \dot{U}^*$ and $\bar{x} \in O(u) \setminus \Lambda(m(u), m(u'))$. Then $\bar{x} \in O(u')$.*

Proof: Let $u, u' \in \dot{U}^*$ and $\bar{x} \in O(u) \setminus \Lambda(m(u), m(u'))$. If $m(u) = m(u')$, $\bar{x} \in O(u')$ by Lemma 4. Assume that $m(u) \neq m(u')$ and $\bar{x} \notin O(u')$. Since $\bar{x} \notin \Lambda(m(u), m(u'))$ by Lemma 4 we can assume, without loss of generality, that $u(\bar{x}) < u(m(u'))$. By Lemma 3 there is $\hat{x} \in \Lambda(m(u'), \bar{x}) \cap O(u')$ such that $(\Lambda(\hat{x}, \bar{x}) \setminus \{\hat{x}\}) \cap \overline{O(u')} = \emptyset$. Let $\{u^q\} \subset \dot{U}$ be a sequence of utility functions whose upper counter sets at \hat{x} approach

$\Lambda(\hat{x}, \bar{x})$, and at \bar{x} approach $\bar{x} + \mathbb{R}_+^l$ (see Figure 2). The sequence $\{f(u^q, \dots, u^q, u')\} \subset \overline{O(u')}$ has a subsequence $\{f(u^{q_s}, \dots, u^{q_s}, u')\}$ whose limit is \hat{x} . Likewise, the sequence $\{f(u^{q_s}, \dots, u^{q_s}, u)\}$ has a convergent subsequence $\{f(u^{q_{s_k}}, \dots, u^{q_{s_k}}, u)\}$ whose limit is \bar{x} . However, since f is strategy-proof, for each q_{s_k} we have

$$u(f(u^{q_{s_k}}, \dots, u^{q_{s_k}}, u)) \geq u(f(u^{q_{s_k}}, \dots, u^{q_{s_k}}, u')),$$

and therefore in the limit $u(\bar{x}) \geq u(\hat{x})$, which is a contradiction, since $u(\bar{x}) < u(m(u'))$ and $\hat{x} \in \Lambda(\bar{x}, m(u'))$. \square

Lemma 9. *Assume that $\widehat{f(U^n)}$ satisfies Condition D. Then for each $u \in \dot{U}^*$, either $O(u) = \{m(u)\}$ or $O(u) = \widehat{f(U^n)}$.*

Proof: Assume that $\widehat{f(U^n)}$ satisfies Condition D. Suppose, by way of contradiction, that there are $u \in \dot{U}^*$, $y \in O(u) \setminus \{m(u)\}$, and $z \in \widehat{f(U^n)} \setminus O(u)$. Then by Lemma 3 there is $\bar{x} \in \Lambda(y, z) \cap O(u)$ such that $(\Lambda(\bar{x}, z) \setminus \{\bar{x}\}) \cap \overline{O(u)} = \emptyset$. We distinguish two cases: $\bar{x} = m(u)$ (Case I), and $\bar{x} \neq m(u)$ (Case II). We show that in both cases we reach a contradiction.

Case 1: Assume that $\bar{x} = m(u)$. Let $u' \in \dot{U}^*$ be such that $m(u') = z$. Since $\widehat{f(U^n)}$ satisfies Condition D there is $c \in \widehat{f(U^n)}$ such that $m(u), m(u'), c$ are m.a.i. Moreover, we have

$$(1.a) \quad \Lambda(m(u'), c) \cap O(u) = \emptyset,$$

$$(1.b) \quad c \notin O(u') \text{ and}$$

$$(1.c) \quad m(u) \in O(u').$$

(These facts are established below.) Thus, since $m(u) \in O(u')$ and $c \in \widehat{f(U^n)} \setminus O(u')$, by Lemma 3 there is $\tilde{x} \in \Lambda(m(u), c) \cap O(u')$ such that $(\Lambda(\tilde{x}, c) \setminus \{\tilde{x}\}) \cap \overline{O(u')} = \emptyset$. Note that $\tilde{x} \neq m(u')$ since otherwise $m(u'), m(u), c$ would not be m.a.i. (because $\tilde{x} = m(u') \in \Lambda(m(u), c)$). Hence $\tilde{x} \in \Lambda(m(u'), c)$ by Lemma 5. Moreover, since $\tilde{x} \in \Lambda(m(u), c)$ and $\tilde{x} \in \Lambda(m(u'), c)$, L1.2 implies $\tilde{x} \notin \Lambda(m(u'), m(u))$. Thus $\tilde{x} \in O(u') \setminus \Lambda(m(u'), m(u))$, and therefore $\tilde{x} \in O(u)$ by Lemma 8, which contradicts (1.a) since $\tilde{x} \in \Lambda(m(u'), c)$.

Case 2: Assume that $\bar{x} \neq m(u)$. By Lemma 5 we have $\bar{x} \in \Lambda(m(u), z)$. Let $\hat{u} \in \dot{U}^*$ be such that $m(\hat{u}) = \bar{x}$. We prove below that

$$(2.a) \ (\Lambda(m(\hat{u}), z) \setminus \{m(\hat{u})\}) \cap \overline{O(\hat{u})} = \emptyset.$$

If $y \notin \Lambda(m(u), m(\hat{u}))$, we have $y \in O(\hat{u})$ (this is implied by $y \in O(u)$ and Lemma 8). But then we are in the situation described in Case I (with \hat{u} playing the role of u) which has been shown cannot occur.

If $y \in \Lambda(m(u), m(\hat{u}))$, since $\widehat{f(U^n)}$ satisfies Condition D there is $d \in \widehat{f(U^n)}$ such that $m(u), m(\hat{u}), d$ are m.a.i. We prove below that

$$(2.b) \ (\Lambda(m(\hat{u}), m(u)) \setminus \{m(\hat{u})\}) \cap O(\hat{u}) = \emptyset, \text{ and}$$

$$(2.c) \ \text{there is } x \in \Lambda(m(\hat{u}), d) \cap O(\hat{u}) \text{ such that } m(\hat{u}), m(u), x \text{ are m.a.i..}$$

Since $x \in \overline{O(\hat{u})}$, $m(u) \notin O(\hat{u})$ (by (2.b)) and $m(\hat{u}), m(u), x$ are m.a.i., then by Lemma 6 there is $\tilde{x} \in \Lambda(m(\hat{u}), m(u)) \setminus \{m(\hat{u})\} \cap O(\hat{u})$, which contradicts (2.b).

We now prove claims (1.a) – (1.c) and (2.a) – (2.c)

Proof of (1.a): Assume by way of contradiction that there is $x \in \Lambda(m(u'), c) \cap O(u)$. We show that $m(u), m(u'), x$ are m.a.i.. We have (i) $m(u) \notin \Lambda(x, m(u'))$ for if $m(u) \in \Lambda(x, m(u'))$, then $\Lambda(m(u), m(u')) \subset \Lambda(x, m(u')) \subset \Lambda(m(u'), c)$ which contradicts that $m(u), m(u'), c$ are m.a.i.; also (ii) $x \notin \Lambda(m(u), m(u'))$, since $x \in O(u)$ and $(\Lambda(m(u), m(u')) \setminus \{m(u)\}) \cap \overline{O(u)} = \emptyset$; finally, (iii) $m(u') \notin \Lambda(m(u), x)$ for if $m(u') \in \Lambda(m(u), x)$, then since $x \in \Lambda(m(u'), c)$ L1.1 implies $m(u') \in \Lambda(m(u), c)$, which contradicts that $m(u), m(u'), c$ are m.a.i. Now, since $m(u), m(u'), x$ are m.a.i. as $x \in \overline{O(u)}$, and $m(u') \notin O(u)$, there is $\hat{x} \in (\Lambda(m(u), m(u')) \setminus \{m(u)\}) \cap O(u)$ by Lemma 6, which is a contradiction.

Proof of (1.b): Since $c \notin \Lambda(m(u), m(u'))$ (because $m(u), m(u'), c$ are m.a.i.), and $c \notin O(u)$ (by (1.a)), Lemma 8 implies $c \notin O(u')$.

Proof of (1.c): Since $m(u) \in \Lambda(y, m(u'))$, then $y \notin \Lambda(m(u), m(u'))$, and since $y \in O(u)$, then $y \in O(u')$ (by Lemma 8). Thus, since $m(u) \in \Lambda(y, m(u'))$, $y \in O(u')$ implies (by Lemma 7) that $m(u) \in O(u')$.

Proof of (2.a): Assume by way of contradiction that there is $x \in (\Lambda(m(\hat{u}), z) \setminus \{m(\hat{u})\}) \cap \overline{O(\hat{u})}$. By Lemma 2 there is $\tilde{x} \in O(\hat{u})$ such that $\tilde{x} \geq x$. Thus, $\tilde{x} \in (\Lambda(m(\hat{u}), z) \setminus \{m(\hat{u})\}) \cap O(\hat{u})$, and since $m(\hat{u}) \in \Lambda(m(u), z) \setminus \{m(u)\}$, and $\tilde{x} \in (\Lambda(m(\hat{u}), z))$, we have $\tilde{x} \notin \Lambda(m(u), m(\hat{u}))$ by L1.3. Also $\tilde{x} \in O(\hat{u})$ and $\tilde{x} \notin \Lambda(m(u), m(\hat{u}))$ implies (by Lemma 8) that $\tilde{x} \in O(u)$, which contradicts that $(\Lambda(m(\hat{u}), z) \setminus \{m(\hat{u})\}) \cap \overline{O(u)} = \emptyset$.

Proof of (2.b): Assume by way of contradiction that there is $x \in (\Lambda(m(\hat{u}), m(u)) \setminus \{m(\hat{u})\}) \cap O(\hat{u})$. Then $z \notin \Lambda(x, m(\hat{u}))$ (otherwise $z \in O(\hat{u})$ by Lemma 7, which contradicts (2.a)). Also since $x \in O(\hat{u})$ and $(\Lambda(m(\hat{u}), z) \setminus \{m(\hat{u})\}) \cap \overline{O(\hat{u})} = \emptyset$ by (2.a), we have $x \notin \Lambda(m(\hat{u}), z)$. If $m(\hat{u}) \notin \Lambda(x, z)$, then $m(\hat{u}), x, z$ are m.a.i., and therefore by Lemma 6 there is $x' \in (\Lambda(m(\hat{u}), z) \setminus \{m(\hat{u})\}) \cap \overline{O(\hat{u})}$, which contradicts (2.a). If $m(\hat{u}) \in \Lambda(x, z)$, since $x \in O(\hat{u}) \setminus \{m(\hat{u})\}$, $z \in \widehat{f(U^n)} \setminus O(\hat{u})$, $m(\hat{u}) \in \Lambda(x, z)$ and $(\Lambda(m(\hat{u}), z) \setminus \{m(\hat{u})\}) \cap \overline{O(\hat{u})} = \emptyset$, we are in the situation described in Case I (with \hat{u} and x playing the role of u and y , respectively), which cannot occur.

Proof of (2.c): Since $(\Lambda(m(\hat{u}), m(u)) \setminus \{m(\hat{u})\}) \cap O(\hat{u}) = \emptyset$ by (2.b), then $m(u) \notin O(\hat{u})$, and since $m(\hat{u}), m(u), d$ are m.a.i., then $d \notin O(\hat{u})$ by Lemma 6. Also since $d \notin \Lambda(m(\hat{u}), m(u))$, then $d \notin O(u)$ by Lemma 8. Thus, $m(\hat{u}) \in \overline{O(u)}$ and $d \notin O(u)$, and therefore by Lemma 3 there is $x \in \Lambda(m(\hat{u}), d) \cap O(u)$ such that $(\Lambda(x, d) \setminus \{x\}) \cap \overline{O(u)} = \emptyset$. Moreover, $x \neq m(u)$ (because $x \in \Lambda(m(\hat{u}), d)$ and $m(\hat{u}), m(u), d$ are m.a.i.), and $x \neq m(\hat{u})$ (because $x \in \Lambda(m(u), d)$ by Lemma 5, and $m(\hat{u}), m(u), d$ are m.a.i.). And since $x \in \Lambda(m(\hat{u}), d)$ and $x \in \Lambda(m(u), d)$, L1.2 implies $x \notin \Lambda(m(u), m(\hat{u}))$, and therefore since $x \in O(u)$, we have $x \in O(\hat{u})$ by Lemma 8. Hence $x \in \Lambda(m(\hat{u}), d) \cap O(\hat{u})$.

We show that $m(\hat{u}), m(u), x$ are m.a.i.. As seen above, we have $x \notin \Lambda(m(u), m(\hat{u}))$. Suppose that $m(\hat{u}) \in \Lambda(m(u), x)$; since $x \in \Lambda(m(\hat{u}), d)$, we have $m(\hat{u}) \in \Lambda(m(u), d)$ by L1.1, which contradicts that $m(\hat{u}), m(u), d$ are m.a.i.. Hence $m(\hat{u}) \notin \Lambda(m(u), x)$. Finally, suppose that $m(u) \in \Lambda(m(\hat{u}), x)$; since $x \in \Lambda(m(\hat{u}), d)$, then $\Lambda(m(\hat{u}), x) \subset \Lambda(m(\hat{u}), d)$, and therefore $m(u) \in \Lambda(m(\hat{u}), d)$, which again contradicts that $m(\hat{u}), m(u), d$ are m.a.i. Hence $m(u) \notin \Lambda(m(\hat{u}), x)$. \square

Lemma 10. *Assume that $\widehat{f(U^n)}$ satisfies Condition D. If there is $u \in \dot{U}^*$ such that $O(u) = \{m(u)\}$, then for each $u \in \dot{U}^*$, one has $O(u) = \{m(u)\}$.*

Proof: Assume that $\widehat{f(U^n)}$ satisfies Condition D. Assume by way of contradiction that there are $u, u' \in \dot{U}^*$ such that $O(u) = \{m(u)\}$ and $O(u') \neq \{m(u')\}$. Lemma 9 implies $O(u') = \widehat{f(U^n)}$. Since $\widehat{f(U^n)}$ satisfies Condition D there is $\bar{x} \in \widehat{f(U^n)} \setminus \{m(u), m(u')\}$ such that $\bar{x} \notin \Lambda(m(u), m(u'))$. Since $\bar{x} \in O(u')$ and $\bar{x} \notin \Lambda(m(u), m(u'))$, we have $\bar{x} \in O(u)$ by Lemma 8, which contradicts that $O(u) = \{m(u)\}$. \square

With these results in hand we prove now Theorem 1.

PROOF OF THEOREM 1: Let f be a strategy-proof mechanism satisfying the assumptions of Theorem 1. First, it is shown by induction on the number of individuals that f is dictatorial on $(\dot{U}^*)^n$. When $n = 1$, this is an implication of $P1$. Assume that this claim is true for every mechanism for which $n \leq k - 1$. It is shown that the claim holds for $n = k$.

We show that f is dictatorial on $(\dot{U}^*)^k$. By Lemma 10, either $O(u) = \{m(u)\}$ for each $u \in \dot{U}^*$, or $O(u) = \widehat{f(U^k)}$ for each $u \in \dot{U}^*$. Assume that $O(u) = \{m(u)\}$ for each $u \in \dot{U}^*$, then $f(\mathbf{u}) = m(u_k)$ for each $\mathbf{u} \in (\dot{U}^*)^k$, and therefore Individual k is a dictator for f on $(\dot{U}^*)^k$. Assume $O(u) = \widehat{f(U^k)}$ for each $u \in \dot{U}^*$, and consider the mechanism $f^u : U^{k-1} \rightarrow X$ given for each $\mathbf{u}_{-k} \in U^{k-1}$ by $f^u(\mathbf{u}_{-k}) = f(\mathbf{u}_{-k}, u)$. Each f^u satisfies the assumptions of Theorem 1 – obviously each f^u is strategy-proof, and it is easy to see that $f^u(\widehat{U^{k-1}}) = \widehat{f(U^k)}$; see the proof of Theorem 1 in Moreno[5]. Thus the induction hypothesis implies that each f^u is dictatorial on $(\dot{U}^*)^{k-1}$. We show that a single individual i (always the same) is the dictator for each f^u on $(\dot{U}^*)^{k-1}$; i.e., that for each $\mathbf{u} \in (\dot{U}^*)^k$, $f^u(\mathbf{u}_{-k}) = f(\mathbf{u}) = m(u_i)$, which establishes that f is dictatorial on $(\dot{U}^*)^n$. Suppose not; w.l.o.g., assume that Individual 1 is the dictator for f^u on $(\dot{U}^*)^{k-1}$, and Individual 2 is the dictator for $f^{u'}$ on $(\dot{U}^*)^{k-1}$. Let $\mathbf{u}_{-\{1,2,k\}} \in (\dot{U}^*)^{k-3}$ arbitrary, and let $u_1, u_2 \in \dot{U}^*$ be such that $m(u_1) \neq m(u)$, and $m(u_2) = m(u)$. Hence $f^u(u_1, u_2, \mathbf{u}_{-\{1,2,k\}}) = f(u_1, u_2, \mathbf{u}_{-\{1,2,k\}}, u) \neq m(u)$, and $f^{u'}(u_1, u_2, \mathbf{u}_{-\{1,2,k\}}) = f(u_1, u_2, \mathbf{u}_{-\{1,2,k\}}, u') = m(u)$, and therefore f is manipulable by Individual k at $(u_1, u_2, \mathbf{u}_{-\{1,2,k\}}, u)$, which is a contradiction.

Assume, without loss of generality, that Individual 1 is the dictator for f on $(\dot{U}^*)^n$. We show that Individual 1 is a dictator for f on $\dot{U}^* \times \dot{U}^{n-1}$, and then that he is also a dictator for f of \dot{U}^n , which establishes Theorem 1.

Suppose, by way of contradiction, that there is $\mathbf{u} \in \dot{U}^* \times \dot{U}^{n-1}$ such that $f(\mathbf{u}) \neq m(u_1)$. Since $\widehat{f(U^n)}$ satisfies Condition D and $f(\mathbf{u}), m(u_1) \in \widehat{f(U^n)}$, there is $z \in \widehat{f(U^n)}$ such that $f(\mathbf{u}), m(u_1)$ and z are m.a.i.. Let $\bar{u} \in \dot{U}^*$ be such that $m(\bar{u}) = z$ and $\bar{u}(f(\mathbf{u})) > \bar{u}(m(u_1))$. Then $P1$ (applied to f^{u_1}) implies

$$\bar{u}(f^{u_1}(\bar{u}, \dots, \bar{u})) = \bar{u}(f(u_1, \bar{u}, \dots, \bar{u})) \geq \bar{u}(f(\mathbf{u})) > \bar{u}(m(u_1)),$$

and therefore $f(u_1, \bar{u}, \dots, \bar{u}) \neq m(u_1)$, which contradicts that Individual 1 is the dictator for f on $(\dot{U}^*)^n$.

Finally, suppose, again by way of contradiction, that there are $\mathbf{u} \in \dot{U}^n$ and $\tilde{x} \in f(U^n)$ such that $u_1(\tilde{x}) > u_1(f(\mathbf{u}))$. Let $\tilde{u} \in \dot{U}^*$ be such that $u_1(m(\tilde{u})) > u_1(f(\mathbf{u}))$. Since Individual 1 is the dictator for f on $\dot{U}^* \times \dot{U}^{n-1}$, one has

$$u_1(f(\tilde{u}, \mathbf{u}_{-1})) = u_1(m(\tilde{u})) > u_1(f(\mathbf{u})),$$

and therefore f is manipulable by Individual 1 at \mathbf{u} , contradicting that f is strategy-proof. \square

Lemma 11 below provides an auxiliary result that will be useful in the proofs of propositions 1 to 3. The proof is this lemma given in Moreno[5] applies to the present context without any change.

Lemma 11. *If f is a mechanism satisfying the assumptions of Theorem 1, then there is $i \in N$ such that for each $(u_i, \mathbf{u}_{-i}) \in U^* \times U^{n-1}$, $f(u_i, \mathbf{u}_{-i}) \leq m(u_i)$.*

Now Proposition 1 can be easily proved.

PROOF OF PROPOSITION 1: Assume, without loss of generality, that Lemma 11 is satisfied for $i = 1$. It is shown that Individual 1 is a dictator for f . Suppose by way of contradiction that there are $\mathbf{u} = (u_1, \dots, u_n) \in U^n$ and $x \in f(U^n)$ such that $u_1(x) > u_1(f(\mathbf{u}))$. Because f is non-wasteful, we have $f(U^n) \subset \hat{X}$, and therefore $f(U^n) = \widehat{f(U^n)}$. Hence $x \in \widehat{f(U^n)}$. Let $u \in U^*$ be such that $m(u) = x$. Lemma 11 yields $f(u, \mathbf{u}_{-1}) \leq x$, and since $f(u, \mathbf{u}_{-1}) \in \widehat{f(U^n)}$, we have $f(u, \mathbf{u}_{-1}) = x$. Then

$$u_1(f(u, \mathbf{u}_{-1})) = u_1(x) > u_1(f(\mathbf{u})),$$

and therefore f is manipulable by Individual 1 at \mathbf{u} , contradicting that f is strategy-proof. \square

PROOF OF PROPOSITION 2: The proof of Proposition 2 is straightforward: Let f be strategy-proof and efficient mechanism $f : U^n \rightarrow X$, where \hat{X} satisfies condition D. It is easy to see that efficiency implies that $\hat{X} \subset f(U^n)$. (For $x \in \hat{X}$, let $u \in U^*$ be such that it is maximized on X at x ; then efficiency implies $f(u, \dots, u) = x$.) Thus, $\widehat{f(U^n)} = \hat{X}$. Hence f satisfies the assumptions of Theorem 1, and therefore Lemma

11 applies. An argument analogous to that of Proposition 1 establishes that f is dictatorial. Moreover, since $f(U^n) \supset \hat{X}$, it is strongly dictatorial. \square

PROOF OF PROPOSITION 3: Let $f : \dot{U}^n \rightarrow X$ be a strategy-proof mechanism such that $f(\dot{U}^n)$ satisfies Condition D. We prove Proposition 3 by showing that f can be extended to a mechanism $F : U^n \rightarrow X$ satisfying the assumptions of Theorem 1. The mechanism F will therefore be dictatorial on \dot{U}^n by Theorem 1, which establishes that f is (fully) dictatorial, since F coincides with f on \dot{U}^n .

The mechanism F is defined as follows: Let σ is an arbitrary selection on X , and let $F \equiv F^n$, where F^n is defined recursively as $F^0 \equiv f$, and $F^i : U^i \times \dot{U}^{n-i} \rightarrow X$ is given by

$$F^i(\mathbf{u}) = \begin{cases} F^{i-1}(\mathbf{u}) & \text{if } \mathbf{u} \in U^{i-1} \times \dot{U}^{n-i+1} \\ \sigma(M_i(\mathbf{u})) & \text{otherwise,} \end{cases}$$

where $M_i(\mathbf{u}) \equiv M_i^n(\mathbf{u})$ is defined by letting $M_i^0(\mathbf{u})$ be the set of maximizers of u_i on $F^{i-1}(\dot{U}, \mathbf{u}_{-i})$, and $M_i^j(\mathbf{u})$ be the set of maximizers of u_j on $M_i^{j-1}(\mathbf{u})$, for $j = 1, \dots, n$. (Here U^i and U^{n-i} are the power sets; e.g., $U^i = U \times \dots \times U$, i times.) We show by induction that F is well defined, it is strategy-proof, and satisfies $\widehat{F(U^n)} = f(\dot{U}^n)$ (and hence satisfies Condition D).

We show by induction that each F^i , $i \in \{0, \dots, n\}$, is well defined, it is strategy-proof, and satisfies $F^i(U^i \times \dot{U}^{n-i}) = \widehat{f(\dot{U}^n)} = \widehat{f(\dot{U}^n)} = F^i(\widehat{U^i \times \dot{U}^{n-i}})$. First we note that f satisfies P3 – the proof given in Moreno[5] (Lemma 4) applies to f without any change. Hence $f(\dot{U}^n) \subset \widehat{f(\dot{U}^n)}$, and therefore $f(\dot{U}^n) = \widehat{f(\dot{U}^n)}$. Hence, since $F^0 = f$, F^0 is well defined, it is strategy-proof, and satisfies $F^0(U^n) = f(\dot{U}^n) = \widehat{f(\dot{U}^n)} = \widehat{F^0(U^n)}$. Assume that the claim holds for F^0, \dots, F^{i-1} . We show that F^i is well defined, it is strategy-proof, and satisfies $F^i(U^i \times \dot{U}^{n-i}) = \widehat{f(\dot{U}^n)} = \widehat{f(\dot{U}^n)} = F^i(\widehat{U^i \times \dot{U}^{n-i}})$.

Let $\mathbf{u} \in (U^i \times \dot{U}^{n-i}) \setminus (U^{i-1} \times \dot{U}^{n-i+1})$. We prove by induction that the sets $M_i^0(\mathbf{u}), \dots, M_i^n(\mathbf{u}) = M_i(\mathbf{u})$ are non-empty subsets of $F^{i-1}(\dot{U}, \mathbf{u}_{-i})$, and therefore that F^i is well defined on $(U^i \times \dot{U}^{n-i}) \setminus (U^{i-1} \times \dot{U}^{n-i+1})$. Since F^i coincides with F^{i-1} on $U^{i-1} \times \dot{U}^{n-i+1}$, this establishes that F^i is well defined on its entire domain. Moreover, since $F^i(\mathbf{u}) = \sigma(M_i(\mathbf{u})) \in M_i(\mathbf{u}) = M_i^n(\mathbf{u}) \subset F^{i-1}(\dot{U}, \mathbf{u}_{-i}) \subset f(\dot{U}^n)$ (the last

inclusion is implied by the induction hypothesis) for each $\mathbf{u} \in \left(U^i \times \dot{U}^{n-i} \right) \setminus (U^{i-1} \times \dot{U}^{n-i+1})$, we have $F^i(U^i \times \dot{U}^{n-i}) \subset f(\dot{U}^n)$. And since $f(\dot{U}^n) = F^{i-1}(U^{i-1} \times \dot{U}^{n-i+1}) = F^i(U^{i-1} \times \dot{U}^{n-i+1}) \subset F^i(U^i \times \dot{U}^{n-i})$ (because F^i coincides with F^{i-1} on $U^{i-1} \times \dot{U}^{n-i+1}$), we have $F^i(U^i \times \dot{U}^{n-i}) = f(\dot{U}^n) = \widehat{f(\dot{U}^n)} = F^i(\widehat{U^i \times \dot{U}^{n-i}})$.

Recall that $M_i^0(\mathbf{u})$ is the set of maximizers of u_i on $F^{i-1}(\dot{U}, \mathbf{u}_{-i})$. Hence $M_i^0(\mathbf{u}) \subset F^{i-1}(\dot{U}, \mathbf{u}_{-i})$. We show that $M_i^0(\mathbf{u})$ is non-empty. Note that F^{i-1} is strategy-proof by the induction hypothesis. It is easy to see that F^{i-1} satisfies *P4* and *P6* (the proofs in Moreno[5], lemmas 5 and 7, respectively, apply to F^{i-1} without any change). Therefore for each $x \in \overline{F^{i-1}(\dot{U}, \mathbf{u}_{-i})}$ there is $x' \in F^{i-1}(\dot{U}, \mathbf{u}_{-i})$ such that $x' \geq x$. Moreover, because $F^{i-1}(U^{i-1} \times \dot{U}^{n-i+1}) = f(\dot{U}^n) = \widehat{f(\dot{U}^n)} = F^{i-1}(\widehat{U^{i-1} \times \dot{U}^{n-i+1}})$ by the induction hypothesis, we have $F^{i-1}(\dot{U}, \mathbf{u}_{-i}) = F^{i-1}(\dot{U}, \mathbf{u}_{-i})$. Thus, u_i has a maximizer on $F^{i-1}(\dot{U}, \mathbf{u}_{-i})$, and therefore $M_i^0(\mathbf{u})$ is non-empty subset of $F^{i-1}(\dot{U}, \mathbf{u}_{-i})$. Assume that $M_i^0(\mathbf{u}), \dots, M_i^{k-1}(\mathbf{u})$ are non-empty subsets of $F^{i-1}(\dot{U}, \mathbf{u}_{-i})$, where $k \leq n$. We show that $M_i^k(\mathbf{u})$ is a non-empty subset of $F^{i-1}(\dot{U}, \mathbf{u}_{-i})$. Recall that $M_i^k(\mathbf{u})$ is the set of maximizers of u_k on $M_i^{k-1}(\mathbf{u})$. Let $\bar{x} \in \overline{M_i^{k-1}(\mathbf{u})}$; then there is $\bar{x}' \in F^{i-1}(\dot{U}, \mathbf{u}_{-i}) = \overline{F^{i-1}(\dot{U}, \mathbf{u}_{-i})}$ such that $\bar{x}' \geq \bar{x}$. Moreover, for each $l < k$, we have $u_l(\bar{x}') \geq u_l(\bar{x}) \geq u_l(x)$ for each $x \in M_i^{l-1}(\mathbf{u})$, and therefore $\bar{x}' \in M_i^{k-1}(\mathbf{u})$. Thus u_k has a maximizer on $M_i^{k-1}(\mathbf{u})$. Hence $M_i^k(\mathbf{u})$ is non-empty subset of $F^{i-1}(\dot{U}, \mathbf{u}_{-i})$.

Finally, we prove that F^i is strategy-proof. It is easy to see that F^i is not manipulable by Individual i . Simply note that for each $\mathbf{u} \in U^i \times \dot{U}^{n-i}$, $F^i(\mathbf{u})$ maximizes u_i on $F^{i-1}(\dot{U}, \mathbf{u}_{-i}) \supseteq F^i(U, \mathbf{u}_{-i})$, and hence $u_i(F^i(\mathbf{u})) \geq u_i(x)$, for $x \in F^i(U, \mathbf{u}_{-i})$; i.e., $u_i(F^i(\mathbf{u})) \geq u_i(F^i(u'_i, \mathbf{u}_{-i}))$ for $u'_i \in U$. We show F^i is neither manipulable by any individual $j \in N \setminus \{i\}$. Assume, by way of contradiction, that there are $j \in N \setminus \{i\}$, $\mathbf{u} \in U^i \times \dot{U}^{n-i}$ and $u'_j \in U$ such that $u_j(F^i(\mathbf{u})) < u_j(F^i(u'_j, \mathbf{u}_{-j}))$. Write $F^i(u'_j, \mathbf{u}_{-j}) = x$, $F^i(\mathbf{u}) = y$, $\Theta(u_j) = F^{i-1}(\dot{U}, \mathbf{u}_{-\{i,j\}}, u_j)$, and $\Theta(u'_j) = F^{i-1}(\dot{U}, \mathbf{u}_{-\{i,j\}}, u'_j)$. (Note that the sets $\Theta(\cdot)$ are analog to the option sets defined in the proof of Theorem 1.) By the definition of F^i , we have $x \notin M_i^j(\mathbf{u})$. Suppose that $x \in \Theta(u_j)$; then there is $\bar{j} \in \{1, \dots, j-1\} \cup \{i\}$ such that $u_{\bar{j}}(y) > u_{\bar{j}}(x)$, and $u_{j'}(y) = u_{j'}(x)$ for $j' \in \{1, \dots, \bar{j}-1\} \cup \{i\}$. As $F^i(\mathbf{u}_{-j}, u'_j) = x$, then $y \notin M_i^{\bar{j}}(\mathbf{u}_{-j}, u'_j)$, and as $u_{j'}(y) = u_{j'}(x)$ for $j' \in \{1, \dots, \bar{j}-1\} \cup \{i\}$, one must

have $y \notin \Theta(u'_j)$. Therefore there are two possible cases: either

(P3.1) $x \in \Theta(u'_j)$ and $y \notin \Theta(u'_j)$, or

(P3.2) $y \in \Theta(u_j)$ and $x \notin \Theta(u_j)$.

We show that (P3.1) leads to contradiction. Assume that (P3.1) holds. An argument analogous to that given in Lemma 3 establishes that there is $\hat{x} \in \Lambda(x, y) \cap \overline{\Theta(u'_j)}$ such that $(\Lambda(\hat{x}, y) \setminus \{\hat{x}\}) \cap \overline{\Theta(u'_j)} = \emptyset$. Note that because u_j is strictly increasing and concave, $u_j(x) > u_j(y)$ and $\hat{x} \in \Lambda(y, x)$, we have $u_j(\hat{x}) > u_j(y)$. Let $\{u^q\} \subset \dot{U}$ be a sequence of utility functions such that their upper contour sets at \hat{x} approach $\Lambda(\hat{x}, y)$, and their upper contour sets at y approach $y + \mathbb{R}_+^l$ (see Figure 2). The sequence $\{F^{i-1}(u^q, \mathbf{u}_{-\{i,j\}}, u'_j)\}$ has a convergent subsequence, $\{F^{i-1}(u^{q_s}, \mathbf{u}_{-\{i,j\}}, u'_j)\}$, whose limit is \hat{x} . Similarly, the sequence $\{F^{i-1}(u^{q_s}, \mathbf{u}_{-\{i,j\}}, u_j)\}$ has a convergent subsequence, $\{F^{i-1}(u^{q_{s_k}}, \mathbf{u}_{-\{i,j\}}, u_j)\}$, whose limit is y . Since F^{i-1} is strategy-proof, for each q_{s_k} we have

$$u_j(F^{i-1}(u^{q_{s_k}}, \mathbf{u}_{-\{i,j\}}, u_j)) \geq u_j(F^{i-1}(u^{q_{s_k}}, \mathbf{u}_{-\{i,j\}}, u'_j)),$$

and therefore in the limit we have $u_j(y) \geq u_j(\hat{x})$, which is a contradiction.

Finally, assume that (P3.2) holds. Then there is $\hat{x} \in \Lambda(y, x) \cap \Theta(u_j)$ such that $(\Lambda(\hat{x}, x) \setminus \{\hat{x}\}) \cap \overline{\Theta(u_j)} = \emptyset$. Since $u_j(\hat{x}) > u_j(y)$, by the definition of F^i , $F^i(\mathbf{u}_{-j}, u_j) = y$ implies the existence of $\bar{j} \leq j$ such that $u_{\bar{j}}(y) > u_{\bar{j}}(\hat{x}) > u_{\bar{j}}(x)$. And since $F^i(\mathbf{u}_{-j}, u'_j) = x$ and $u_k(y) = u_k(x)$ for $k < \bar{j} - 1$, then we must have $y \notin \Theta(u'_j)$. Hence (P3.1) holds, which leads to a contradiction. \square

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