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FOUNDATIONS FOR CONTEST SUCCESS FUNCTIONS*

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Abstract

We examine two approaches to contest success functions. In the first we analyze the implications of contestants´ incomplete information concerning the `type´ of the contest administrator. While in the case of two contestants this approach can rationalize prominent contest success functions, we show that it runs into difficulties when there are more agents.
Our second approach interprets contest success functions as sharing rules and establishes a connection to bargaining and claims problems which is independent of the number of contestants. Both approaches provide foundations for popular contest success functions and guidelines for the definition of new ones.

Keywords: Endogenous Contests, Contest Success Function.

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"The strategic approach also seeks to combine axiomatic cooperative solutions and non-cooperative solutions. Roger Myerson recently named this task the 'Nash program'." (Rubinstein (1985), p. 1151)

1 Introduction

A contest is a game in which players exert effort in order to win a certain prize. Contests have been used to analyze a variety of situations including lobbying, rent-seeking and rent-defending contests, advertising, litigation, political campaigns, conflict, patent races, arms races, sports events or R&D competition. A crucial determinant for the equilibrium predictions of contests is the specification of the so-called contest success function (CSF) which relates the players' efforts and win probabilities. Justifications for a particular CSF can be twofold. A justification can be on normative grounds, because it is the unique CSF fulfilling certain axioms, or essential properties. A justification can also be positive when it can be shown that the CSF arises from the strategic interaction of players, thereby yielding a description of situations when it can be expected to be realistic. The purpose of the present paper is to contribute to our understanding of CSFs in both dimensions.

Formally, a contest success function associates, to each vector of efforts $\boldsymbol{G}$, a lottery specifying for each agent a probability p_i of getting the object. That is, $p_i = p_i(\boldsymbol{G})$ is such that, for each contestant $i \in N := \{1, ..., n\}$, $p_i(\boldsymbol{G}) \geq 0$, and $\sum_{i=1}^{n} p_i(\boldsymbol{G}) = 1$. For notational ease in the present paper we consider vectors of efforts $\boldsymbol{G}$ with at least two strictly positive entries.

The canonical example of a contest situation is rent-seeking. In a pioneering paper, Tullock (1980) proposed a special form of the contest success function, namely, given a positive scalar R,

$$p_i = \frac{G_i^R}{\sum_{j=1}^{n} G_j^R}, \text{ for } i = 1, ..., n. \tag{1}$$

Gradstein (1995, 1998) postulated the following variation of this form where, given $q_i > 0$ for all $i \in N$,

$$p_i = \frac{G_i q_i}{\sum_{j=1}^{n} G_j q_j}, \text{ for } i = 1, ..., n. \tag{2}$$

A generalization that comprises both previous functional forms is, given $a_i > 0$ for all $i \in N$,

$$p_i = \frac{G_i^R q_i + a_i}{\sum_{j=1}^{n} (G_j^R q_j + a_j)}, \text{ for } i = 1, ..., n. \tag{3}$$

A different functional form, the logit model, was proposed by Hirshleifer (1989) where, given a positive scalar k,

$$p_i = \frac{e^{kG_i}}{\sum_{j=1}^{n} e^{kG_j}}, \text{ for } i = 1, ..., n. \tag{4}$$

Note that the four expressions (1) – (4) are specific instances of the following functional form

$$p_i = \frac{f_i(G_i)}{\sum_{j=1}^{n} f_j(G_j)}, \text{ for } i = 1, ..., n. \tag{5}$$

The so-called effectivity functions f_i are usually interpreted as determining how 'effective' agent i's effort is in affecting the win probability of agent i. Most papers dealing with contest models in the literature analyze a CSF which is a special case of the form in (5) (Nitzan (1994), Konrad (2004)). Consequently, the present paper will be mainly concerned with deriving foundations for CSFs of this form.

However, there are also some CSFs in the literature which are not special cases of the form in (5). The first two consider the case of two contestants and build on the idea that only differences in effort should matter – an idea introduced by Hirshleifer in (4). Baik (1998) proposed the following form, given a positive scalar σ,

$$p_1 = p_1(\sigma G_1 - G_2) \text{ and } p_2 = 1 - p_1. \tag{6}$$

Che and Gale (2000) postulate the following piece-wise linear difference-form

$$p_1 = max\left\{min\left\{\frac{1}{2} + \sigma(G_1 - G_2), 1\right\}, 0\right\} \text{ and } p_2 = 1 - p_1. \tag{7}$$

Recently, Alcalde and Dahm (2006) proposed a CSF that is neither a difference-form contest nor a special case of (5). Given a positive scalar R, the serial contest is defined as

$$p_i = \sum_{j=i}^{n} \frac{G_j^R - G_{j+1}^R}{j \cdot G_1^R}, \text{ for } i = 1, ..., n \text{ with } G_{n+1} = 0. \tag{8}$$

As for justifications of CSFs the most systematic approach has been normative and the seminal paper is Skaperdas (1996). He proposed the following five axioms:

(A1) Imperfect Discrimination: For all $i \in N$, if $G_i > 0$, then $p_i > 0$.

(A2) Monotonicity: For all $i \in N$, p_i is increasing in G_i and decreasing in $G_j, j \neq i$.

(A3) Anonymity: For any permutation function π on the set of bidders we have

$$p(\pi \boldsymbol{G}) = \pi p(\boldsymbol{G}) \text{ for all } \boldsymbol{G}.$$

While these axioms are standard, the next two properties are more specific and relate win probabilities in contests with different sets of active contestants.

(A4) Consistency: Let $p_i^M(\boldsymbol{G})$ be contestant i's probability of winning a contest played by a subset $M \subset N$ of contestants. For all $i \in M$, and for all $M \subset N$ with at least two elements,

$$p_i^M(\boldsymbol{G}) = \frac{p_i(\boldsymbol{G})}{\sum_{j \in M} p_j(\boldsymbol{G})}.$$

(A5) Independence: For all $i \in M$, $p_i^M(\boldsymbol{G})$ is independent of G_j for all $j \notin M$.

Together both axioms imply that the CSF satisfies Luce's Choice Axiom (Clark and Riis (1998)). This is an independence of irrelevant alternatives property: the probability that contestant i wins if player k does not participate is equal to the probability that i wins when k participates given that k does not win. This axiom holds recursively for any subset of non-participating bidders. A different interpretation is to require that if a subset $M \subset N$ of contestants wishes to reallocate the total win probability assigned to them by the same contest success function, each contestant should obtain the same win probability as before.

Skaperdas showed that (A1)–(A5) are equivalent to assume a CSF of the form given in (5) with $f_i(\cdot) = f(\cdot)$ for all $i \in N$, where $f(\cdot)$ is a positive increasing function of its argument. Moreover, if the CSF is assumed to be homogeneous of degree zero[1], i.e.
(A6) Homogeneity: For all $\boldsymbol{G}$ and each positive scalar λ

$$p_i(\boldsymbol{G}) = p_i(\lambda \boldsymbol{G}) \text{ for all } \lambda > 0 \text{ and for all } i \in N$$

then, Properties (A1)–(A6) are equivalent to assume a CSF of the form given in (1).[2]

An extension of Skaperdas results is given by Clark and Riis (1998) by dropping the anonymity assumption. Skaperdas also axiomatized the logit model (4) by showing that it is equivalent to properties (A1)–(A5) above plus an additional property that asserts that the winning probability of each player depends only on the difference in the efforts made by all players.

In this paper we want to approach the issue of justifying CSFs mainly from a positive point of view. We first notice that the plausibility of the above properties depends on the context. In particular it is easy to find examples where Property (A4) does not hold, as we show next.

Example 1 *Let $n = 3$. Players have to play against each other twice. If a player wins the match, she gets 3 points. If she gets a draw she gets 1 point and if she loses she gets 0 points. Suppose that the level of effort is arbitrarily given. There are two equally likely states of the world. In the first state of the world, the scores obtained by the players are the following;*

Player 1 against Player 2: 1 gets 4 points and 2 gets 1 point.

Player 1 against Player 3: 1 gets 0 points and 3 gets 6 points.

Player 2 against Player 3: 2 gets 6 points and 3 gets 0 points.

Thus, in this case, player 2 wins the contest because she gets 7 points (players 3 and 1 get 6 and 4 points respectively). In the second state of the world, results are identical to those above except that

Player 1 against Player 3: 1 gets 6 points and 3 gets 0 points.

In this case, player 1 wins the contest because she gets 10 points (players 2 and 3 get 7 and 0 points respectively).
Thus, for a given level of effort, the probability that player 1 wins the contest is 1/2. However, if player 3 is removed from the contest and we assume that the score obtained in any particular match is independent of the scores obtained in other matches, player 1 wins with probability 1. Thus the ratio between probabilities of success of players 1 and 2 is altered when the third player is not considered in the contest.

[1] Corchón (2000) has shown that homogeneity of degree zero implies that if $p_i(\cdot)$ is continuous at 0 effort for all players, $p_i(\cdot)$ must be constant on G_i/G_j any i, j, which is clearly absurd. Another possibility is that $p_i(\cdot)$ is undefined at the point of 0 effort by all players, as in the case considered by Tullock (see above).

[2] In this case, as noticed by Esteban and Ray (1999), the rent-seeking model becomes a particular case of the Cournot model.

The example shows that complementing the normative approach through a positive perspective can be insightful. By taking into account specific institutional details of specific conflict situations, it allows to obtain a description of situations when a particular CSF can be expected to be realistic.

In order to develop a positive perspective the present paper proposes two approaches to CSFs. In the first we investigate the implications of the – in our view – most natural approach to probabilistic (or imperfectly discriminating) CSFs. We postulate the existence of a contest administrator who allocates the prize to one of the contestants. However, contestants have incomplete information about the type of the contest administrator. We show that this approach can generate non-deterministic CSFs for any number of contestants. However, we are only able to obtain prominent contest success functions for case of two contestants. When there are further contestants, CSFs are likely to be less tractable.

Our second approach interprets contest success functions as *sharing rules* and establishes a connection to bargaining and claims problems which is independent of the number of contestants. The analysis exploits the observation that these problems are mathematically related – but not equivalent – to the problem of assigning win probabilities in contests. A main result here is that the generalization of the class of contest success functions axiomatized by Skaperdas given in (5) can be understood as the weighted Nash bargaining solution where efforts represent the weights of the agents. We turn then to the framework of bargaining with claims (Chun and Thomson (1992)) to incorporate explicitly the contestants' efforts in the description of the problem. This allows to associate prominent solution concepts in this framework to the previously mentioned class of contest success functions and to a generalized version of Che and Gale's difference form contest.

Foundations for contest success functions have been reviewed by Konrad (2004) and we have already described the most important results of the normative approach. Our paper contributes to this literature indirectly by making connections to related problems which are well understood from a normative point of view. For instance, we establish a relationship between Che and Gale's difference-form CSF (7) and the principle of equal sacrifice. However, the main aim of the present paper is to provide a systematic positive approach to CSFs tailored to capture key elements of the situation to be modelled. Given that this approach makes the resulting foundation necessarily specific to the situation at hand, our paper complements existing positive justifications tailored for different situations. We are not aware of any work understanding CSFs as sharing rules as our second approach does.[3] However, our first approach is related to other works. Fullerton and McAfee (1999) and Baye and Hoppe (2003) offer micro-foundations for a subset of CSFs of the form in (1) for the context of innovation tournaments and patent races. A major conceptional

[3] Anbarci, Skaperdas and Syropoulos (2002) present a model in which a two party conflict over a resource can either be settled through bargaining over the resource or through a contest. The contest defines the disagreement point of the bargaining problem to which three different bargaining solutions are applied. In contrast, in our framework we interpret bargaining to be over win probabilities and derive contest success functions as bargaining rules.

difference between our approach and theirs is the nature of the underlying uncertainty. While in our case the uncertainty has a centralized character (it comes from a single decision-maker), in these models it is decentralized (it comes from the actions of the contestants). This is also true in Hillman and Riley (1988) where a model of the political process is offered in which the political impact of effort is uncertain. Similarly to our second approach Hillman and Riley succeed to derive a CSF of the form in (1) only for the case of two contestants.[4]

2 External Decider

Assume that one person has to decide to award a prize to one of two contestants. In the situation we have in mind contestants are uncertain about a characteristic of the decider that is relevant for his decision. So rent-seekers exert effort without knowing the realization of the characteristic and then the decision-maker decides whom to give the prize based both on the contestants' efforts and his type.

Let Θ be the set of states of the world. Let θ be an arbitrary element of Θ. We assume that $\Theta = [0, 1]$ and that θ is uniformly distributed. Let V_i be the decider's payoff if the prize is awarded to contestant $i = 1, 2$. V_i is assumed to depend on the state of the world, i.e. $V_i = V_i(\theta)$. This may reflect the uncertainty in the contestants' minds about the preferences of the decider. We will assume the following single-crossing property.

(SC) $V_1(\theta)$ is decreasing in θ and $V_2(\theta)$ is strictly increasing in θ.

Taking into account efforts, let $U_i(V_i(\theta), G_i)$ be the decider's payoff if the prize is awarded to contestant $i = 1, 2$. This function is assumed to be increasing in both arguments. For the sake of interpretation let G_i be interpreted as the level of advertisement (resp. quality) made (resp. provided) by contestant $i = 1, 2$. Let

$$\theta' = \begin{cases} 1 \text{ if } U_1(\theta, G_1) > U_2(\theta, G_2), \forall \theta \in \Theta \\ 0 \text{ if } U_1(\theta, G_1) < U_2(\theta, G_2), \forall \theta \in \Theta \\ \{\theta | U_1(\theta, G_1) = U_2(\theta, G_2)\} \text{ otherwise.} \end{cases} \tag{9}$$

Under our assumptions θ' is well-defined and unique. Moreover, θ' equals p_1, the probability that contestant 1 gets the prize. Thus, we can solve p_1 as a function of G_1 and G_2 and obtain

$$p_1 = p_1(G_1, G_2).$$

Summing up, the contest success function can be derived as arising from the maximization of the payoff function of the decider. We now provide several examples:

[4] There is also a relationship to probabilistic voting models (Coughlin (1992)) and to Dahm and Porteiro (2006). The latter paper derives specific instances of CSFs of the form in (5) for the case of two interest groups lobbying a political decision-maker whose decision depends both on the lobbies' efforts and on the information the decision-maker has.

Example 2 *Let* $U_1(\theta, G_1) = V_1(\theta) + a_1 G_1$ *and* $U_2(\theta, G_2) = V_2(\theta) + a_2 G_2$*, where* $a_1, a_2 > 0$*. Thus,* $a_1 G_1 - a_2 G_2 = V_2(\theta) - V_1(\theta) \equiv z(\theta)$*, say. Since* $z(\cdot)$ *is invertible we get,* $p_1 = z^{-1}(a_1 G_1 - a_2 G_2)$ *which is the form in (6) considered by Baik (1998).*[5] *Notice that this procedure is identical to the one used in models of spatial differentiation in order to obtain the demand function (see Hotelling (1929)).*

Example 3 *Let* $U_1(\theta, G_1) = \theta + 2\sigma G_1 - 1/2$ *and* $U_2(\theta, G_2) = -\theta + 2\sigma G_2 + 1/2$*, where* σ *is a positive scalar. In this case, it is easily calculated that* $p_1 = max\{min\{1/2 + \sigma(G_1 - G_2), 1\}, 0\}$*. We obtain (7) the family of difference-form contest success functions analyzed by Che and Gale (2000).*

Example 4 *Let* $U_1(\theta, G_1) = (1-\theta) f_1(G_1)$ *and* $U_2(\theta, G_2) = \theta f_2(G_2)$*. Here we obtain* $p_1 = f_1(G_1)/(f_1(G_1) + f_2(G_2))$*. This is an instance of (5) a generalization of the family of contest success functions axiomatized by Skaperdas (1996).*

Example 5 *Let* $U_1(\theta, G_1) = f_1(G_1)$ *and* $U_2(\theta, G_2) = 2\theta f_2(G_2)$ *if* $\theta \leq 1/2$ *and* $U_2(\theta, G_2) = f_2(G_2)/(2(1-\theta))$ *if* $1/2 \leq \theta < 1$*. Analogous reasoning as before yields* $p_1 = f_1(G_1)/(2 f_2(G_2))$ *if* $f_1(G_1) \leq f_2(G_2)$ *and* $p_1 = 1 - f_2(G_2)/(2 f_1(G_1))$ *otherwise. This expression is a generalization of the family of serial contests in (8) analyzed in Alcalde and Dahm (2006).*

The question is if any contest success function can be derived from payoff maximization of the decider in the way we did in the examples above. Let us consider the following definition, in which we deal with the general case of two or more rent-seekers:

Definition 2.1 *The contest success function* $p_i = p_i(G_1, G_2, ..., G_n)$ *is rationalizable if there is a list of payoff functions* $U_i = U_i(\theta, G_i)$ *with* $U_i(\cdot)$ *strictly increasing on* $G_i, i = 1, 2, ..., n$ *such that*

$$p_i(G_1, G_2, ..., G_n) = probability\{U_i(\theta, G_i) > U_j(\theta, G_j), \forall j \neq i\}.$$

Proposition 2.1 *Let* $p_1(G_1, G_2)$ *be strictly increasing on* G_1 *and strictly decreasing on* G_2*. Then* $p_1(\cdot, \cdot)$ *is rationalizable by a payoff function fulfilling the single crossing condition (SC).*[6]

Proof. Let G_2 be given, say $G_2 = G_2'$. Thus, $p_1 = p_1(G_1, G_2')$. Since this function is strictly increasing on G_1 it is invertible. This inverse depends on G_2', so let us write it as $G_1 = P(p_1, G_2')$. Repeating this argument for all possible values of G_2', we obtain $G_1 = H(p_1, G_2)$, say. Now let

[5] Alternatively, we may assume that the payoff function of the decider is $U_i = V_i(\theta) - a_j G_j$, $i \neq j$, reflecting the disutility received from the effort (bribe) made by contestant 2, if the prize is awarded to rent-seeker 1. The same applies to Example 3 and to Example 4 by taking $U_1 = (1-\theta)/f_2(G_2)$ and $U_1 = \theta/f_1(G_1)$.

[6] To be fully precise the statement of this proposition refers to contest success functions that are exhaustive in the sense that there is competition for the whole probability mass. Formally, for a given G_2', as G_1 goes to infinity, p_1 goes to 1 (and similarly, for a given G_1', as G_2 goes to infinity, p_1 goes to 0). Notice that in Examples 2 – 5 this property is fulfilled. If the contest success function specifies that there is a rent-seeker who obtains a certain probability of winning the prize no matter what effort levels are, then the argument of the proof can be adapted to rationalize the part of the probability mass which is subject to competition.

$U_1 = G_1$ and $U_2 = H(\theta, G_2)$. By the properties of $p_1(\cdot,\cdot)$, $H(\cdot,\cdot)$ is strictly increasing on θ and G_2. Also U_1 is strictly increasing on G_1 and constant on θ, so the SC assumption holds. By construction, θ' (as defined in equation (9)) equals p_1 so the result is proved. ■

In the case of three rent-seekers the previous argument does not yield microfoundations for the class of contest success functions axiomatized by Skaperdas. There are two reasons for that which are explained in Examples 6 and 7 below. In both examples we assume that there are three rent-seekers and that the single crossing property (SC) above holds for contestants 1 and 3. The first example shows that it might be impossible to partition Θ in three non-empty intervals in which case Skaperdas' axiom (A1) will be violated. Moreover, the win probability of a given contestant might not be responsive to changes in the efforts of all other rent-seekers, as in (5).

Example 6 *Let us assume that $U_i(1/2, G_i) \to \infty$ when $G_i \to \infty, i = 1, 3$. This assumption is fulfilled in the payoff functions used in Examples 2 and 3 above. In the case of Example 4 and 5 this assumption is fulfilled if $f_i(G_i) \to \infty$ when $G_i \to \infty$. U_2 is assumed to be continuous in θ and G_2. Let $U_2'(G_2) = max\, U_2(\theta, G_2)$, $\theta \in \Theta$. The maximum exists and varies continuously with G_2 (by Berge's maximum theorem). By taking G_1 and G_3 large enough, say G_1' and G_3', the property (SC) and the assumption made at the beginning of the example imply,*

$$\begin{aligned} U_1(\theta, G_1') &> U_2'(G_2), \forall \theta \in [0, 1/2) \\ U_3(\theta, G_3') &> U_2'(G_2), \forall \theta \in (1/2, 1]. \end{aligned}$$

Thus, rent-seeker 2 never obtains the prize. Moreover, because $U_2'(\cdot)$ is continuous in G_2, small variations in G_2 do not affect neither p_1 nor p_3. Thus, the class of functions axiomatized by Skaperdas can not be obtained from payoff maximization.

The next example shows that, even if Θ can be partitioned in three non-empty intervals, the form axiomatized by Skaperdas can not be obtained from payoff maximization.

Example 7 *Suppose that $\Theta = [0, \theta''] \cup [\theta'', \theta'''] \cup [\theta''', 1]$ and that*

$$\begin{aligned} U_1(\theta, G_1) &> U_j(\theta, G_j), j = 2, 3, \forall \theta \in [0, \theta'') \\ U_2(\theta, G_2) &> U_j(\theta, G_j), j = 1, 3, \forall \theta \in (\theta'', \theta''') \\ U_3(\theta, G_3) &> U_j(\theta, G_j), j = 1, 2, \forall \theta \in (\theta''', 1]. \end{aligned}$$

If $U_j(\cdot,\cdot)$ are continuous in θ, the previous equations imply that $U_1(\theta'', G_1) = U_2(\theta'', G_2)$, and $U_2(\theta''', G_2) = U_3(\theta''', G_3)$. Thus, $p_1 = \theta''$, $p_2 = \theta''' - \theta''$ and $p_3 = 1 - \theta'''$. It is clear that p_1 (resp. p_3) does not depend on G_3 (resp. G_1) for small variations of this variable. Thus, the required functional form can not be obtained in this case.

Albeit this difficulty in deriving the class of functions axiomatized by Skaperdas for more than three contestants, contestants' uncertainty about the type of the contest administrator seems to be a reasonable approach to CSFs. Therefore, it is an important research program to find contest success functions that are rationalizable according to Definition 2.1 above and to work out the consequences of these new functional forms on equilibrium, comparative statics, etc. We show now that although this route appears to be promising, it is not free from difficulties. We will work out two examples and we will show that in both cases:

- Contest success functions are neither differentiable nor concave.
- Despite the symmetric nature of basic data, no symmetric Nash equilibrium exists.

However, we also point out that these are problems that have been successfully dealt with in the analysis of the serial contest (Alcalde and Dahm (2006)).

Example 8 *Let $U_1(\theta, G_1) = (1-\theta)G_1$, $U_2(\theta, G_2) = G_2 2/3$ and $U_3(\theta, G_3) = \theta G_3$. Notice that if $G_1 = G_2 = G_3$, $p_1 = p_2 = p_3 = 1/3$. We will compute the best reply of contestant 1.*
If $G_2 2/3 < G_3$ we have two cases: First, if $G_1 < G_2 2/3$, then $p_1 = 0$. Second, if $G_1 \geq G_2 2/3$, then

$$p_1 = \begin{cases} (G_1 - G_2 2/3)/G_1 & \text{if } G_1 < (G_3 G_2 2/3)/(G_3 - G_2 2/3) \\ G_1/(G_1 + G_3) & \text{otherwise.} \end{cases}$$

If $G_2 2/3 \geq G_3$ we have again two cases

$$p_1 = \begin{cases} 0 & \text{if } G_1 < G_2 2/3 \\ (G_1 - G_2 2/3)/G_1 & \text{otherwise.} \end{cases}$$

In a symmetric equilibrium $\hat{\mathbf{G}}$ we have $G_1 \geq G_2 2/3$ and $G_1 < (G_3 G_2 2/3)/(G_1 - G_2 2/3)$. Thus, rent-seeker 1 maximizes $V(G_1 - G_2 2/3)/G_1 - G_1$, where V is the value of the prize. If the equilibrium is symmetric it must be at positive level of effort. Thus, the maximum is interior and the first order condition yields the best reply, namely $G_1 = (V G_2 2/3)^{1/2}$.
For $\hat{G}_1 = \hat{G}_2$ this yields $\hat{G}_1 = V\,2/3$. We now have to make sure that this payoff is larger than the payoff associated to $G_1 = 0$ (yielding a p_1 and a payoff equal to 0). This is equivalent to $\hat{G}_2 \leq V\,27/100$, which contradicts $\hat{G}_1 = \hat{G}_2 = V\,2/3$.

Example 8 can be criticized because the existence of endpoints (0 and 1) which we know are troublesome when dealing with this kind of models (see d'Aspremont, Gabszewicz and Thisse (1979)). Thus, we now adapt the model of Salop (1979) to our framework.

Example 9 *Suppose that rent-seekers are symmetrically distributed in the unit circle, which is now our set of states of the world. Assume that 1 competes only with 2 (resp. 3) for $\theta \in [0, 1/3]$ (resp. $\theta \in [2/3, 1]$). If the prize is awarded to 1 (resp. 2), the decider's payoffs for $\theta \in [0, 1/3]$ are $U_1(\theta, G_1) = u - k + G_1^\alpha$ (resp. $U_2(\theta, G_2) = u - k(1/3 - \theta) + G_2^\alpha$), where u, k and α are positive scalars and $\alpha < 1$. Thus, the state of the world for which, given efforts, the decider is indifferent between both candidates is*

$$\theta' = \min\{\max\{1/6 + (G_1^\alpha - G_2^\alpha)/(2k), 0\}, 1/3\}.$$

A similar reasoning in the case of 1 and 3 yields

$$\theta'' = \min\{\max\{5/6 + (G_3^\alpha - G_1^\alpha)/(2k), 2/3\}1\}.$$

Define

$$\theta''' = \min\{\max\{1/3 + (2G_1^\alpha - G_2^\alpha - G_3^\alpha)/(2k), 0\}2/3\}.$$

Thus, the probability that 1 gets the prize is as follows

$$p_1 = \begin{cases} 0 & \text{if } \theta' = 0 \text{ and } \theta'' = 1 \\ \theta' & \text{if } 0 < \theta' \leq 1/3 \text{ and } \theta'' = 1 \\ 1 - \theta'' & \text{if } \theta' = 0 \text{ and } 2/3 \leq \theta'' < 1 \\ \theta''' = \theta' + 1 - \theta'' & \text{if } 0 < \theta' \leq 1/3 \text{ and } 2/3 \leq \theta'' < 1. \end{cases}$$

A symmetric equilibrium $\hat{\mathbf{G}}$ *requires that* $\hat{G}_1$ *maximizes 1's payoffs, given* $\hat{G}_2$ *and* $\hat{G}_3$ *and that* $\hat{G}_1 = \hat{G}_2 = \hat{G}_3$*. Thus,* $\hat{G}_1$ *maximizes* $\theta''' V - G_1$*, where* V *is the value of the prize. If the maximum is interior,* $\hat{G}_1 = (\alpha V/k)^{1/(1-\alpha)}$*. Thus if payoffs for 1 for this value of efforts are negative, 0 effort is the best reply and no symmetric equilibrium exists.*

Note that it is straightforward to extend the last example to more than three contestants. The so derived CSF can be seen as an extension of Che and Gale's linear difference form (given in (7)) to more than two contestants. Note that competition among contestants in this model is in the spirit of the 'chain market' model of monopolistic competition (see Chamberlin (1956) and Friedman (1979)).

3 Contest Success Functions as Sharing Rules

Notice that, although the approach pursued in the previous section explicitly models a politician or contest administrator, the role of the latter is just to choose mechanically the policy alternative that maximizes his utility given his type and the effort levels exerted. Is it possible go even further and provide a rationale for contest success functions without even modelling the contest administrator? In order to provide such a rationale we establish now a connection to bargaining and claims problems that only involves contestants. In order to highlight this relationship the exposition and the proof of Proposition 3.1 follow as closely as possible Dagan and Volij (1993).

3.1 'Classical' Bargaining

A contest problem is a vector $\boldsymbol{f}(\boldsymbol{G}) = (f_1(G_1), ..., f_n(G_n))$ with at least two entries each of which strictly positive.[7] Since we consider a fixed vector of efforts $\boldsymbol{G}$, we will simply use the notation f_i instead of $f_i(G_i)$ and $\boldsymbol{f}$ instead of $\boldsymbol{f}(\boldsymbol{G})$. An allocation in a contest problem is a n-tuple $\boldsymbol{p} = (p_1, ..., p_n) \in \mathbb{R}^n$ with $0 \leq p_i \leq 1$ and $\sum_{i=1}^n p_i = 1$. A contest success function is a function that assigns a unique allocation to each contest problem.

[7] If $f_i(G_i) = 0$ for some contestant i, assign zero win probability to this agent and consider the reduced vector in which the entry corresponding to agent i is missing.

We define now a bargaining problem associated with each contest problem. A bargaining problem is a pair $(S, \boldsymbol{d})$ where $S \subset \mathbb{R}^n$ is a compact convex set, $\boldsymbol{d} \in S$ and there exists $\boldsymbol{s} \in S$ such that $s_i > d_i, i = 1, ..., n$. The set S, the feasible set, consists of all utility vectors attainable by the n contestants through unanimous agreement. The disagreement point $\boldsymbol{d}$ is the utility vector obtained if there is no agreement. In our context it seems natural to define

$$S = \left\{ \boldsymbol{p} \in \mathbb{R}^n \middle| 0 \leq p_i \leq 1 \text{ and } \sum_{i=1}^{n} p_i \leq 1 \right\} \text{ and } \boldsymbol{d} = \boldsymbol{0}.$$

This can be interpreted as contestants bargaining over all possible assignments of win probabilities. If no agreement is reached, all win probabilities are zero.

A bargaining solution is a function ψ assigning to each bargaining problem $(S, \boldsymbol{d})$ a unique element in S. We are interested in the weighted Nash solution with weights $\boldsymbol{\alpha}$.

Definition 3.1 *Let $\alpha_i > 0$ for all $i = 1, ..., n$. The α-asymmetric Nash solution is defined as*

$$\psi^{\alpha} = \arg\max_{p \in S} \Pi_{i=1}^{n} (p_i - d_i)^{\alpha_i}.$$

Next result is an adaptation of a result obtained by Dagan and Volij (1993) in a different framework.

Proposition 3.1 *The α-asymmetric Nash solution for $\boldsymbol{\alpha} = \boldsymbol{f}$ induces a generalization of the class of contest success functions axiomatized by Skaperdas (given by equation (5)).*

Proof. Let $\boldsymbol{f}$ be a contest problem, consider the associated bargaining problem and let $\psi^{\alpha} = \boldsymbol{p}^*$. The first-order conditions of the maximization problem defining the asymmetric Nash solution imply that

$$p_j^* = \frac{\alpha_j}{\alpha_i} p_i^*, \text{ for all } i, j \in N.$$

Given the Pareto optimality of the asymmetric Nash solution we have that $\sum_{j=1}^{n} p_j = 1$. This implies $p_i^* = \alpha_i / \sum_{j=1}^{n} \alpha_j$. ■

Since the preceding result sheets light on the class of contest success functions axiomatized by Skaperdas from a very different angel than the approach of the previous section, it is of interest in its own right. However, it also opens the door to understand CSFs as the outcome of strategic bargaining models based on Rubinstein's alternating offers game. Since it is well known that under certain conditions the asymmetric Nash solution can be supported by such a game, it follows that alternative conditions thought to reflect reasonable properties of underlying institutional details can yield alternative CSFs.

3.2 Bargaining with Claims

It might seem odd that, while the effort vector $\boldsymbol{f}$ defines a contest problem, this information is not used in the description of the associated bargaining problem $(S, \boldsymbol{d})$. If we want to incorporate this information in the description of the problem, the relevant framework is the one of bargaining problems with claims (Chun and Thomson (1992)).[8] A contest bargaining problem is then a triple $(S, \boldsymbol{d}, \boldsymbol{f})$ with the following interpretation: Contestants bargain over all possible assignments of win probabilities. The contestants' effectivity functions translate individual effort into an 'aspiration point' $\boldsymbol{f}$. Thus, $\boldsymbol{f}(\boldsymbol{G})$ measures the social merit that society or the decider awards to the vector of efforts $\boldsymbol{G}$.

If no unanimous agreement is reached, all win probabilities are zero. A contest bargaining solution ϕ assigns to each such triple a unique element in S. A maximal point $\boldsymbol{p}$ of S is a point such that $\sum_{j=1}^{n} p_j = 1$. The proportional solution is defined as follows.

Definition 3.2 *The proportional solution ϕ^P is defined as the maximal point $\boldsymbol{p}$ of S on the segment connecting the disagreement point $\boldsymbol{d}$ and the aspiration point $\boldsymbol{f}$.*

Proposition 3.2 *The proportional solution induces a generalization of the class of contest success functions axiomatized by Skaperdas (given by equation (5)).*

Proof. Let $\boldsymbol{f}$ be a contest problem, consider the associated bargaining problem with claims and let $\phi^P = \boldsymbol{p}^*$. The line which passes through the two points $\boldsymbol{d}$ and $\boldsymbol{f}$ is the set of vectors $\boldsymbol{x}$ of the form $\boldsymbol{x} = (1-t)\boldsymbol{d} + t\,\boldsymbol{f} = t\boldsymbol{f}$, with $t \in \mathbb{R}$. Given that $\boldsymbol{p}^*$ is a maximal point, we have that $t = 1/\sum_{j=1}^{n} f_j$. This implies $p_i^* = f_i/\sum_{j=1}^{n} f_j$. ■

The richer description of bargaining problems with claims has allowed to define an alternative solution that also explicitly builds on the aspiration point $\boldsymbol{f}$. Bossert (1993) analyzes the claim-egalitarian solution. For the purpose of the next proposition it suffices to consider the case of two contestants. The following definition is adapted to our context because in contest problems there is no upper bound on individual effort levels, that is, $\boldsymbol{f}$.

Definition 3.3 *Let $n = 2$ and w.l.o.g. fix $f_h \geq f_l$. The claim-egalitarian solution ϕ^E is defined as the maximal point $\boldsymbol{p}$ of S such that $f_h - p_h = f_l - p_l$ if $f_h - f_l \leq 1$. Otherwise $p_h = 1$ and $p_l = 0$.*

The claim-egalitarian solution selects a point on the Pareto frontier of S such that the loss of each contestant compared with his aspiration level is the same for all agents (if such a point exists). This is an egalitarian solution in the sense that the absolute amount each agent has to give up is equalized across contestants. The next proposition says that this idea is the same as saying that only differences in effort matter.

Proposition 3.3 *For $n = 2$, the claim-egalitarian solution induces a generalization of Che and Gale's difference form contest success function, that is,*

$$\phi_i^E = p_i^{CG'}(\boldsymbol{G}) := \max\left\{\min\left\{\frac{1}{2} + \frac{1}{2}(f_i - f_j), 1\right\}, 0\right\} \text{ for } i = 1, 2.$$

[8] Notice that a contest problem is not equivalent to a bargaining problem with claims. One important difference is that in contest problems there is no upper bound on individual effort levels, that is, $\boldsymbol{f}$.

Proof. The fact that if $|f_i - f_j| \geq 1$ then $\phi_i^E = p_i^{CG'}(\boldsymbol{G})$ is obvious. Suppose $|f_i - f_j| \leq 1$. Since $p_j = 1 - p_i$, we have $f_i - p_i = f_j - (1 - p_i)$. Rearranging yields the desired expression. ■

Notice that when $f_i(G_i) = 2\sigma G_i$ for $i = 1, 2$ where σ is a positive scalar, we obtain (7), the class of linear difference-form functions analyzed in Che and Gale (2000). While this contest is only defined for the case of two contestants, analyzing the strategic implications of difference-form contests with more contestants is an interesting task for future research. However, it is not clear how the contest success function should be extended. Notice that the Salop model of Example 9 and the bargaining solution by Bossert give different recommendations for this extension. The appropriate extension depends therefore on the application and institutional details the contest model is intended to capture.

4 Concluding Remarks

This paper has investigated foundations for prominent contest success functions based on two different approaches. The first analyzes the implications of contestants' incomplete information concerning the 'type' of the contest administrator. The second understands CSFs as sharing rules and makes a connection to bargaining and claims problems. Both approaches provide foundations for popular contest success functions and guidelines for the definition of new ones. The results of this paper suggest two lines for future research on contest success functions.

On the normative side, the implications of linking the problem of assigning win probabilities in contests to bargaining, claims and taxation problems are twofold. On one hand, this connection might yield an improved understanding of existing contest success functions, while, on the other hand, it suggests guidelines for the definition of new ones. As for the former, for instance, proportionality principles have been defended at least since the philosophers of ancient Greece. Therefore, it seems possible to obtain different characterizations of the class of contest success functions axiomatized by Skaperdas using ideas of characterizations of proportionality stressed in these related problems.[9] As for the latter, different normative principles might lead to the formulation of different classes of contest success functions. A case in point here is the claim-egalitarian solution that gives a recommendation how to extend the difference-form functions analyzed in Che and Gale to more than two contestants.

On the positive side, the implications for future research parallel the normative ones. On one hand, strategic foundations of solution concepts in bargaining, claims and taxation problems that can be related to popular contest success functions might yield rationales for the latter. An example is to link contests with the Bilateral Principle that has proved a fruitful way to incorporate Luce's Choice Axiom into game theory. Dagan *et al.* (1997) have provided a game form capturing the non-cooperative dimension of the consistency property of bankruptcy rules.[10]

[9]Note that the class of problems in which win probabilities are assigned has a particularly simple structure. This implies that a characterization of a solution for a larger class of problems does not need to characterize a solution for contests.

[10]Notice that a contest problem is not equivalent to a bankruptcy problem in which the estate is equal to one, since in contest problems there is no lower bound on the sum of individual effort levels, that is, $\sum_{j=1}^{n} f_j$.

An adaptation of their result in our framework shows that a generalization of the class of contest success functions axiomatized by Skaperdas (given by equation (5)), can be supported by a pure strategy subgame perfect equilibrium of a certain non-cooperative game.

On the other hand, both approaches pursued in the present paper highlight that strategic foundations for contest success functions are sensitive to details. By incorporating realistic details of contest situations novel contest success functions can be derived. Examples are the recommendation of the Salop model how to extend Che and Gale's difference-form function to more than two contestants or the effects of modifying Rubinstein's alternating offers bargaining game.

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