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GENERALIZED EXTERNALITY GAMES

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Abstract

Externality games are studied in Grafe *et al.* (1998). We define a generalization of this class of games and show, using the methodology in Izquierdo and Rafels (1996 and 2001), some properties of the new class of generalized externality games. They include, among others, the algebraic structure of the game, convexity, and their implication for the study of cooperative solutions. Also the proportional rule is characterized for this class of games.

Keywords: Cooperative games; Externality games; Proportional Rule.

JEL Classification: *C72*

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1 Introduction

In the literature of cooperative games there has been an interest on characteristic functions that can be obtained after a more primitive model. Moulin (1989) suggested that, given a non-cooperative game, different characteristic-form games could be defined after different specifications of a characteristic function. In this fashion he defined the α-core and the β-core of a normal form game as the core of the associated cooperative games. More recently, several authors have defined subclasses of cooperative games for which the characteristic function has an economic interpretation. Examples of this approach that are relevant to our work include externality games in Grafe et al. (1998) and financial games in Izquierdo and Rafels (1996). Having more structure than general cooperative games, it is only natural to ask whether, for each subclass of games, one can find more interesting properties or define more appealing solution concepts.

In the present work we define the class of generalized externality games, GEG, which include externality games as defined by Grafe et al. (1998). The characteristic function of GEG can be separated in two functions, one that depends on the totality of the resources belonging to the coalition, and another that depends on the number of members of the coalition. Then, using the methodology in Izquierdo and Rafels (1996 and 2001), we study some properties of this new class of games. In particular, we find that each of the families that form the class of GEG has a vectorial space structure, and furthermore, that minimum participation games form an interesting base. Next GEG are shown to be semi-convex, but not convex, and show sufficient conditions for convexity. It is also shown that GEG belong to the family of average monotonic games.

The importance of these properties becomes clear when we study different solution concepts. In the spirit of many other works we define a proportional solution for GEG and present an axiomatic characterization. The vectorial space structure of GEG and the fact that minimum participation games constitute a base are used in showing this result. Interestingly enough the axiomatization of the proportional solution for GEG is the same as for financial games, but is not a generalization of the axiomatization for externality games.

From the above mentioned properties it follows that the core of GEG is non empty, as the proportional solution is always in it, and that the core and the bargaining set coincide. The conditions for convexity are useful if one is interested in GEG for which the Shapley value is in the core. Finally, the property of semi-convexity allows us to use a simple formula for another solution concept, the τ-value.

In section 2 we define generalized externality games and provide some economic examples. In section 3 we prove some properties. Section 4 characterizes the proportional solution. Section 5 discussed other solution concepts. Section 6 concludes.

2 Definition and examples

Externality games were introduced by Grafe et al. (1998) as a class of cooperative games. In this section we present a generalization of these games and show some interesting economic situations that can be interpreted as generalized externality games.

Using conventional notation, Γ_N will denote the set of characteristic form games of N players. In these games, each subset $S \subset N$ (called a coalition) is associated with a value $v(S)$.

Definition 1 A game $v \in \Gamma_N$ is a generalized externality game, GEG, if there exists a vector $\beta = (\beta_i)_{i \in N}$ in $\Re^N_+$, a parameter $\alpha \geq 1$, and a non-decreasing function $N \to \Re_+$, such that $v(S) = (\sum_{i \in S} \beta_i)^\alpha r(s)$, where s denotes the cardinal of coalition S. The set of generalized externality games of N players will be denoted by GEG_N.

When $\alpha = 1$, this is the definition of externality games. Generalized externality games can be interpreted as a situation in which players contribute both with their endowments (β_i) and their presence (through the function r) to the coalition where they belong. One can easily check that these games are monotone and superadditive. Monotonicity requires that $v(S) \leq v(T)$ whenever $S \subset T$, whereas superadditivity means that $v(S) + v(T) \leq v(S \cup T)$ for all coalitions such that $S \cap T = \emptyset$.

Example 1 (Joint venture): Suppose that a group of n firms decides to collaborate in a joint venture, and that each firm participates with two factors. One of them, L_i (e.g., labor) is idiosyncratic to each firm and the other, K (e.g., capital) must be equal for all firms. If the technology of the joint venture can be represented by a Cobb-Douglas function we can write $f(K; \sum L_i) = (nK)^\alpha (\sum L_i)^\beta$. When we consider the possibility of coalitions of firms having their own joint venture, a generalized externality game is defined if $\alpha \geq 0$. Firms may use a solution of this game to decide upon a division of revenues generated by this activity.

Example 2 (Provision of public goods): Consider the following model in Moulin (1992). Let A be a set of public decisions and denote by $c(a)$ the cost of financing decision a. A set of agents, $N = \{1, 2, ..., n\}$, must share the cost of decision a. A feasible outcome is a vector $(a, y_1, ..., y_n)$ where

$a \in A$; $\sum_{i \in N} y_i = c(a)$, and y_i is agent i's cost share. Preferences are represented by $u_i(a; y_i)$. Suppose now that we have a quadratic cost function and linear utilities; i.e., $c(a) = a^2/2$ and $u_i(a; y_i) = \beta_i a - y_i$, where the parameter β_i is agent i's marginal rate of substitution between private and public goods. If we compute the surplus $v(S)$ generated by coalition S standing alone ($\sum_{i \in S} y_i = c(a)$) as $v(S) = \max \sum_{i \in S}(\beta_i a - y_i) = (\sum_{i \in S} \beta_i)^2/2$ a generalized externality game is defined with $r(s) = \frac{1}{2}$ and $\alpha = 2$.

3 Properties of generalized externality games.

It is well known that characteristic form games are a vectorial space of dimension 2^{N-1}, and that unanimity games constitute a base of this space. It is useful to know, for a given class of characteristic form games, whether it preserves the structure of vectorial space and whether one can find an interesting base. In the next proposition we show that this is indeed the case for each one of the subclasses that constitute the class of generalized externality games. To this end we need the following definition. Coalitions will be denoted by upper case letters and their cardinality by the corresponding lower case letter.

Definition 2 A game of minimum participation associated to a vector β and a coalition T is denoted by $v_{T;\beta}$ and defined as:

$$
\begin{array}{lll}
v_{T;\beta}(S) = 0 & \text{if } s < t \text{ or} & \\
 & \text{if } s = t \text{ and } \sum_{i \in S} \beta_i < \sum_{i \in T} \beta_i, & \\
v_{T;\beta}(S) = \sum_{i \in S} \beta_i & \text{if } s \geq t \text{ or} & \\
 & \text{if } s = t \text{ and } \sum_{i \in S} \beta_i \geq \sum_{i \in T} \beta_i. &
\end{array}
$$

The set of games of minimum participation associated to a generalized externality game v, denoted by v_β, is defined by $v_\beta = (v_{T;\beta})_{T \in 2^N}$.

For a given game consider a maximal set of coalitions satisfying that, for every two coalitions S and T, either their cardinal is different, $s \neq t$, or $\sum_{i \in S} \beta_i \neq \sum_{i \in T} \beta_i$. Then define the set $L = \{S_0; S_1; \ldots; S_m\}$, where (i) $s_{k-1} < s_k$ or (ii) if $s_{k-1} = s_k$, then $\sum_{i \in S_{k-1}} \beta_i < \sum_{i \in S_k} \beta_i$. Coalitions out of this set will be identified with a coalition in the set with the same cardinal and same amount of resources. Now we can state a proposition about the algebraic structure of the GEG_N that will be useful when studying solutions for these games. Denote by $GEG_N(\alpha; \beta)$ the subset of GEG_N with parameter α and with vector β of endowments of coalitions.

Proposition 1 Given a game in $GEG_N(\alpha; \beta)$, define a set L of coalitions as before. Then, the set of minimum participation games associated to coalitions in L, $v_\beta = (v_{T;\beta})_{T \in L}$, form a base of $GEG_N(\alpha; \beta)$.

Proof. First show that games in $v^{\beta} = (v_{T,\beta})_{T \in L}$ are linearly independent. This means that

$$\sum_{T \in L} \lambda_T v_{T,\beta} = 0_N; \qquad (1)$$

where 0_N is the vector in $\Re^N$ with a zero in each component, has $\lambda_T = 0$ for all T. Suppose that this is not the case and that there exists a $\lambda_T \neq 0$. Select a coalition $S \in L$ such that $\lambda_S \neq 0$, $s \leq t$ for all $T \in L$, and, whenever, $s = t$, $\sum_{i \in S} \beta_i < \sum_{i \in T} \beta_i$. We can rewrite (1) as

$$\begin{aligned}
v_{S,\beta}(S) &= \sum_{T \neq S \in L} -\frac{\lambda_T}{\lambda_S} v_{T,\beta}(S) \\
&= \sum_{T:t<s} -\frac{\lambda_T}{\lambda_S} v_{T,\beta}(S) + \sum_{T:t>s} -\frac{\lambda_T}{\lambda_S} v_{T,\beta}(S) \\
&\quad + \sum_{T:t=s,\ \sum_{i \in T} \beta_i > \sum_{i \in S} \beta_i} -\frac{\lambda_T}{\lambda_S} v_{T,\beta}(S) + \sum_{T:t=s,\ \sum_{i \in T} \beta_i < \sum_{i \in S} \beta_i} -\frac{\lambda_T}{\lambda_S} v_{T,\beta}(S) \\
&= \sum_{T:t<s} -\frac{\lambda_T}{\lambda_S} v_{T,\beta}(S) = \sum_{T:t<s} -\frac{\lambda_T}{\lambda_S} \sum_{i \in S} \beta_i
\end{aligned}$$

Notice that in the expression in the middle, all terms are zero except for the first. In the second and third, $v_{T,\beta}(S) = 0$ because of the definition of minimum participation games, and in the fourth $\lambda_T = 0$ for all T because of the way S was chosen. From $v_{S,\beta}(S) = \sum_{T:t<s} -\frac{\lambda_T}{\lambda_S} \sum_{i \in S} \beta_i$ we have $\sum_{T:t<s} -\frac{\lambda_T}{\lambda_S} = 1$ as $v_{S,\beta}(S) = \sum_{i \in S} \beta_i$. But this means that $\lambda_T \neq 0$ for some T, in contradiction with the way S was chosen.

Now we show that every $v \in GEG_N(\alpha,\beta)$ can be written as a linear combination of games of minimum participation. To this end notice that, given any $v \in GEG_N(\alpha,\beta)$, for any $S \subset N$; there exists a coalition $S_h \in L$ such that $\sum_{i \in S_h} \beta_i = \sum_{i \in S} \beta_i$ and $s_h = s$. Now consider the linear combination $\sum_{S_k \in L} \lambda_{S_k} v_{S_k,\beta}$, with λ_{S_k} defined as

$$\lambda_{S_k} = \frac{v(S_k)}{\sum_{i \in S_k} \beta_i} - \frac{v(S_{k-1})}{\sum_{i \in S_{k-1}} \beta_i}$$

then we have

$$\begin{aligned}
\sum_{S_k \in L} \lambda_{S_k} v_{S_k,\beta}(S) &= \sum_{S_k \in L} \left[\frac{v(S_k)}{\sum_{i \in S_k} \beta_i} - \frac{v(S_{k-1})}{\sum_{i \in S_{k-1}} \beta_i} \right] v_{S_k,\beta}(S) \\
&= \sum_{S_k \in L: s_k < s} \left[\frac{v(S_k)}{\sum_{i \in S_k} \beta_i} - \frac{v(S_{k-1})}{\sum_{i \in S_{k-1}} \beta_i} \right] \sum_{i \in S} \beta_i
\end{aligned}$$

$$+\sum_{S_k\in L: s_k=s,\ \sum_{i\in S_k}\beta_i \le \sum_{i\in S}\beta_i}\left[\frac{y(S_k)}{\sum_{i\in S_k}\beta_i}-\frac{y(S_{k-1})}{\sum_{i\in S_{k-1}}\beta_i}\right]\sum_{i\in S}\beta_i$$

$$+\sum_{S_k\in L: s_k=s,\ \sum_{i\in S_k}\beta_i > \sum_{i\in S}\beta_i}\left[\frac{y(S_k)}{\sum_{i\in S_k}\beta_i}-\frac{y(S_{k-1})}{\sum_{i\in S_{k-1}}\beta_i}\right]v_{S_k;\beta}(S)$$

$$+\sum_{S_k\in L: s_k>s}\left[\frac{y(S_k)}{\sum_{i\in S_k}\beta_i}-\frac{y(S_{k-1})}{\sum_{i\in S_{k-1}}\beta_i}\right]v_{S_k;\beta}(S)$$

$$= v(S)$$

This completes the proof. ■

This property of generalized externality games will be used in the next section, when the proportional rule is axiomatized for these games.

Another interesting property for cooperative games is convexity as it allows to relate different solution concepts. However, convexity is a too strong concept for many purposes. Weaker versions of this concept have been developed, among them, semiconvexity. Next we show that GEG_N are semiconvex, but not convex.

Definition 3 (Driessen and Tijs, 1983) A cooperative game $(N;v)$ is semiconvex if (i) $v(N)-v(N\setminus\{i\}) \ge v(\{i\})$; and (ii) $v(S)-\sum_{j\in S\setminus\{i\}}(v(N)-v(N\setminus\{j\})) \le v(\{i\})$ for all individuals and coalitions.

Proposition 2 Generalized externality games are semiconvex.

Proof. To show that (i) in the definition is satisfied recall that $\beta_i^\alpha r(1) \le \beta_i^\alpha r(n)$. Then

$$\begin{aligned}
\beta_i^\alpha r(1) &\le \Big(\sum_{j\in N\setminus i}\beta_j\Big)^\alpha (r(n)-r(n-1))\\
&\quad+\binom{\alpha}{1}\Big(\sum_{j\in N\setminus i}\beta_j\Big)^{\alpha-1}\beta_i r(n)+\dots+\binom{\alpha}{\alpha}\beta_i^\alpha r(n)\\
&= \Big(\sum_{i\in N}\beta_i\Big)^\alpha r(n)-\Big(\sum_{j\in N\setminus i}\beta_j\Big)^\alpha r(n-1).
\end{aligned}$$

To show (ii):

$$\beta_i^\alpha r(1) \ \ge\ \Big(\sum_{j\in S}\beta_j\Big)^\alpha r(s)$$

$$
\begin{aligned}
&- (s-1)\Big(\sum_{i\in S}\beta_i + \sum_{i\in N\setminus S}\beta_i\Big)^{\alpha} r(n) - \sum_{j\in S\setminus i}\Big(\sum_{i\in N\setminus j}\beta_i\Big)^{\alpha} r(n-1)\\
=\;& \Big(\sum_{j\in S}\beta_j\Big)^{\alpha} r(s) - (s-1)\Big(\sum_{i\in N}\beta_i\Big)^{\alpha} r(n) - \sum_{j\in S\setminus i}\Big(\sum_{i\in N\setminus j}\beta_i\Big)^{\alpha} r(n-1)\\
=\;& \Big(\sum_{j\in S}\beta_j\Big)^{\alpha} r(s) - \sum_{j\in S\setminus i}\Big(\sum_{i\in N}\beta_i\Big)^{\alpha} r(n) - \Big(\sum_{i\in N\setminus j}\beta_i\Big)^{\alpha} r(n-1).
\end{aligned}
$$

■

This property of GEG_N will be helpful to provide a simple formula to compute the τ-value, a cooperative solution.

Generalized externality games are not convex in general, as the following example shows. Next we show a sufficient condition for a class of generalized externality games to be convex.

Definition 4 A cooperative game $(N;v)$ is convex if $v(S\cup\{i;j\}) - v(S\cup\{j\}) \geq v(S\cup\{i\}) - v(S)$ for all $S \subset N$, $i;j \notin S$.

Counterexample: Consider the generalized externality game defined by $N = \{1;2;3\}$, $\beta = (1;2;20)$, $\alpha = 2$, and $r(1) = 1$, $r(2) = 3$, and $r(3) = 4$. This game is not convex as, for example, $v(\{3\}\cup\{2;1\}) - v(\{3\}\cup\{1\}) = 793$, whereas $v(\{3\}\cup\{2\}) - v(\{3\}) = 1;052$.

Proposition 3 Let $(N;v)$ be a symmetric generalized externality game with $\alpha \in N$, then, if $\frac{r(s+1)}{r(s)} > 2$, the game v is convex.

Proof. Recall that symmetry means that $v(S\cup\{j\}) = v(S\cup\{i\})$ for all $S \subset N$, $i;j \in N$. The condition of convexity for symmetric games can be written as

$$2v(S\cup\{i\}) - v(S) \leq v(S\cup\{i;j\}) \text{ for all } i;j \notin S. \qquad (2)$$

In the case of generalized externality games symmetry implies $\sum_{i\in S}\beta_i = \sum_{i\in T}\beta_i$ whenever $s = t$; and we can write

$$
\begin{aligned}
& 2v(S\cup\{i\}) - v(S)\\
=\;& 2\Big(\sum_{k\in S}\beta_k + \beta_i\Big)^{\alpha} r(s+1) - \Big(\sum_{k\in S}\beta_k\Big)^{\alpha} r(s)\\
=\;& 2\Big(\sum_{k\in S}\beta_k + \beta_i\Big)^{\alpha}\frac{r(s+1)}{r(s+2)} - \Big(\sum_{k\in S}\beta_k\Big)^{\alpha}\frac{r(s)}{r(s+2)}\\
=\;& \Big(\sum_{k\in S}\beta_k\Big)^{\alpha}\frac{2r(s+1) - r(s)}{r(s+2)} + \left[\binom{\alpha}{1}\Big(\sum_{k\in S}\beta_k\Big)^{\alpha-1}(\beta_i) + \ldots\right]\frac{2r(s+1)}{r(s+2)}
\end{aligned}
$$

$$\leq \left(\sum_{k\in S}\beta_k\right)^{\alpha}\frac{2r(s+1)-r(s)}{r(s+2)} + 2\binom{\alpha}{1}\left(\sum_{k\in S}\beta_k\right)^{\alpha-1}(\beta_i)\frac{r(s+1)}{2r(s+2)} + \ldots$$

$$\leq \left[\left(\sum_{k\in S}\beta_k\right)^{\alpha} + \binom{\alpha}{1}\left(\sum_{k\in S}\beta_k\right)^{\alpha-1}(\beta_i+\beta_j) + \ldots + (\beta_i+\beta_j)^{\alpha}\right]$$

$$= v(S\cup\{i,j\})$$

as required by convexity. ■

Another property of interest relates GEG with the class of average monotonic games (Izquierdo and Rafels, 2001). This is formalized in the next proposition.

Definition 5 (Izquierdo and Rafels, 2001) A cooperative game $(N;v)$ is average monotonic if

(i) $v(S)\geq 0$ for all coalitions $S\subset N$, and

(ii) there exists a vector $\alpha\in\mathbb{R}^N_+\setminus\{0\}$ such that $(\sum_{i\in T}\alpha_i)v(S)\leq(\sum_{i\in S}\alpha_i)v(T)$ for $S\subset T\subset N$.

Proposition 4 GEG are average monotonic.

Proof. To show (i) in definition 5 see that $\beta_i\geq 0$ and $r(s)\geq 0$ imply $v(S)\geq 0$. To show (ii) let $\alpha=\beta$. Then

$$\frac{v(S)}{\sum_{i\in S}\alpha_i} = \frac{(\sum_{i\in S}\beta_i)^{\alpha}r(s)}{\sum_{i\in S}\beta_i} = \left(\sum_{i\in S}\beta_i\right)^{\alpha-1}r(s)$$

$$\leq \left(\sum_{i\in T}\beta_i\right)^{\alpha-1}r(t) = \frac{v(T)}{\sum_{i\in T}\alpha_i}.$$

The inequality holds because both r and $x^{\alpha-1}$ with $\alpha\geq 1$ are increasing functions. ■

4 The proportional solution

Proportional solutions have been suggested in many context, like bankruptcy problems (see Chun, 1988; O'Neill, 1982, and Thomson, 1995). For GEG we provide the following definition.

Definition 6 (Adapted from Izquierdo and Rafels, 2001 to GEG). Let $(N;v)$ be a generalized externality game with $\beta=(\beta_1,\ldots,\beta_n)$ as players' endowments, then the proportional solution, $p(v;\beta)\in\mathbb{R}^n$ is defined as

$$p(v;\beta)=(p_i(v;\beta))_{i\in N}=\left(\frac{\beta_i}{\sum_{i\in N}\beta_i}v(N)\right)_{i\in N}.$$

Grafe et al. (1998) define the proportional rule for externality games as $\Pi(v;\beta) = (\Pi_i(v;\beta))_{i\in N} = (\beta_i v(N))_{i\in N}$. For these games, the definition above gives $p_i(v;\beta) = \frac{\beta_i}{\sum_{i\in N}\beta_i} v(N) = \frac{\beta_i}{\sum_{i\in N}\beta_i}\sum_{i\in N}\beta_i v(n) = \beta_i v(n) = \Pi_i(v;\beta)$. Thus definition 6 generalizes the proportional solution for externality games.

Izquierdo and Rafels (1996) and Grafe et al. (1998) present an axiomatic characterization of the proportional solution for financial games (a subset of average monotonic games) and externality games, respectively. We show that, for GEG, the characterization in Izquierdo and Rafels (1996) applies, but that the one in Grafe et al. (1998) does not. It is immediate to show that generalized externality games satisfy the following properties (listed in Izquierdo and Rafels (1996)).

-Individual pseudo-rationality (IPR): if $v(N) \geq \sum_{i\in N}\beta_i$, then $p_i(v;\beta) \geq \beta_i$. This means that, if the grand coalition can get more than the total of endowments provided by the individuals, each player gets, at least, her endowment.

-Efficiency (EF): $\sum_{i\in N} p_i = v(N)$.

-Restricted linearity (RL): Let v_1 and v_2 be two games in GEG_N with the same vector β, then (i) $p_i(v_1;\beta) + p_i(v_2;\beta) = p_i(v_1+v_2;\beta)$, and (ii) $p_i(\lambda v;\beta) = \lambda p_i(v;\beta)$ for all $\lambda \in \Re^+$:

The next proposition shows the sufficiency of these properties to characterize the proportional solution for GEG_N.

Proposition 5 The proportional solution is the only solution that satisfies IPR, EF and RL within the set of GEG_N.

Proof. Let $v \in GEG_N$, and consider a solution Φ^n that satisfies IPR, EF and RL, the we show that it coincides with the proportional solution. By proposition 1 v can be expressed as $v = \sum_{k=1}^{m}\lambda_{S_k} v_{S_k;\beta}$, where $\lambda_{S_k} = \left(\frac{v(S_k)}{\sum_{i\in S_k}\beta_i} - \frac{v(S_{k-1})}{\sum_{i\in S_{k-1}}\beta_i}\right)$, and $v_{S_k;\beta}$ are minimum participation games.

Using properties IPR and RL we can write

$$\begin{aligned}\Phi_i(v;\beta) &= \Phi_i\Big(\sum_{k=1}^{m}\lambda_{S_k} v_{S_k;\beta};\beta\Big) = \sum_{k=1}^{m}\lambda_{S_k}\Phi_i(v_{S_k;\beta};\beta) \\ &\geq \sum_{k=1}^{m}\lambda_{S_k}\beta_i = \frac{v(N)}{\sum_{i\in N}\beta_i}\beta_i = p_i(v;\beta):\end{aligned}$$

By EF of the proportional solution it must be $\Phi_i(v;\beta) = p_i(v;\beta)$ for all $i \in N$. ■

Grafe et al. (1998) provide a characterization of the proportional solution for externality games (GEG with $\alpha = 1$). However, this result cannot be generalized to generalized externality games. The axioms in Grafe et al.(1998) for a solution Φ on EG_N are:

-Individual rationality (IR): for all v in EG_N, $\Phi_i(v) \geq \beta_i r(1)$.

-Monotonicity (M): for all $v(\beta; r)$ and $v(\beta; r')$ in EG_N: if $r(t) \leq r'(t)$ for all $t \in \{1, \ldots, n\}$ then $\Phi_i(v(\beta; r)) \leq \Phi_i(v(\beta; r'))$.

-Efficiency (EF): As before.

It is straightforward to show that the proportional solution verifies these axioms for the class of generalized externality games. However, it is not characterized by them, as there are other solutions that satisfy the same set of axioms. For instance, take the solution Ψ defined by $\Psi_i = \frac{\beta_i^\alpha}{\sum_{j \in N} \beta_j^\alpha} v(N)$. It is immediate to show that this solution satisfies EF and M. To show that it also satisfies IR write

$$\begin{aligned}
\Psi_i &= \frac{\beta_i^\alpha}{\sum_{j \in N} \beta_j^\alpha} v(N) = \frac{\beta_i^\alpha}{\sum_{j \in N} \beta_j^\alpha} \Big(\sum_{j \in N} \beta_j\Big)^\alpha r(n) \\
&\geq \frac{\beta_i^\alpha}{\sum_{j \in N} \beta_j^\alpha} \Big(\sum_{j \in N} \beta_j\Big)^\alpha r(1) = \frac{\left(\sum_{i \in N} \beta_j\right)^\alpha}{\sum_{j \in N} \beta_j^\alpha} v(i) \geq v(i).
\end{aligned}$$

The last inequality holds because $\left(\sum_{i \in N} \beta_j\right)^\alpha \geq \sum_{j \in N} \beta_j^\alpha$ for $\alpha \geq 1$.

5 Other solutions of GEG_N:

Izquierdo and Rafels (2001) show that the core of average monotonic games is non empty and that it contains the proportional solution. They also show that the core coincides with the two most important definitions of bargaining set presented in Aumann and Maschler (1964) and in Mas-Colell (1989). Since generalized externality games are average monotonic, the same properties apply.

Grafe et al. (1998) show an example of an externality game (and, a fortiori, a GEG) where the Shapley value is not in the core. When the game is convex, the Shapley value is in the core. Proposition 4 provided a sufficient condition for GEG to be convex.

Tijs (1981) proposes the solution concept called the $\tau-$value. The motivation for this value is that it represents a compromise among players, as it is gives every player a payoff between a superior and an inferior bound. The superior bound is defined as $M^v = (M_i^v)_{i \in N}$, with $M_i^v = v(N) - v(N \backslash \{i\})$; while the inferior bound is $m^v = (m_i^v)_{i \in N}$, with $m_i^v = \max_{S \subset N \backslash \{i\}}[v(S \cup \{i\}) -$

$\sum_{j\in S} M_j^v)]$. See that the superior bound has the marginal contribution of every player to the grand coalition, and that the inferior bound has the minimum payoff that players have after the other players in the coalition are given their superior bound. The τ-value is defined only for quasi-equilibrated games. These are games that satisfy $\sum_{i\in S} m_i^v \le v(N) \le \sum_{i\in S} M_i^v$ and $m_i^v \le M_i^v$ for all $i \in N$. Games with a non-empty core are quasi-equilibrated. Thus the τ-value is well defined for GEG.

Definition 7 The τ-value of a quasi-equilibrated game is defined as the only efficient point that lies on the segment joining the superior and inferior bounds.

Driessen and Tijs (1983) show that, for games with a non-empty core, the τ-value can be computed using the formula $\tau(v) = (1-\delta)m^v + \delta M^v$, where

$$\delta = \frac{v(N) - \sum_{i\in S} m_i^v}{\sum_{i\in S} M_i^v - \sum_{i\in S} m_i^v}.$$

This formula can be used for games in GEG_N.

Driessen and Tijs (1983) also show that, for balanced semiconvex games with at most four players, the τ-value belongs to the core. As generalized externality games are both balanced and semiconvex, the same applies for these games whenever $N \le 4$. For the general case, Driessen and Tijs (1983) provide a necessary and sufficient condition for the τ-value to belong to the core in semiconvex games, which include GEG_N.

6 Conclusion

We have defined GEG as a generalization of externality games. The different families of GEG have the structure of a vectorial space with minimum participation games as a base. This property makes possible an interesting characterization of the proportional solution. The relations between other properties of GEG and solutions are also explored. One interesting feature that may deserve more attention is the relation between GEG and financial games, as they share many properties although the two classes of games are not related by inclusion.

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