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ESTIMATION METHODS FOR STOCHASTIC VOLATILITY MODELS: A SURVEY

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Abstract

The empirical application of Stochastic Volatility (SV) models has been limited due to the difficulties involved in the evaluation of the likelihood function. However, recently there has been fundamental progress in this area due to the proposal of several new estimation methods that try to overcome this problem, being at the same time, empirically feasible. As a consequence, several extensions of the SV models have been proposed and their empirical implementation is increasing. In this paper, we review the main estimators of the parameters and the volatility of univariate SV models proposed in the literature. We describe the main advantages and limitations of each of the methods both from the theoretical and empirical point of view. We complete the survey with an application of the most important procedures to the S&P 500 stock price index.

Keywords: GMM, Indirect Inference, Kalman filter, Long Memory, Maximum Likelihood, MCMC, QML, SV-M.

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1 Introduction

During the last decade there has been an increasing interest in modelling the dynamic evolution of the volatility of high frequency series of financial returns using Stochastic Volatility (SV) models. In the simplest framework, the series of returns, y_t, is modelled as the product of two stochastic processes. If the logarithm of volatility, σ_t, is assumed to follow an AR(1) process, the corresponding model, known as Autoregressive SV (ARSV(1)) model, is given by

$$\begin{aligned} y_t &= \sigma_* \varepsilon_t \sigma_t, \\ \log(\sigma_t^2) &= \phi \log(\sigma_{t-1}^2) + \eta_t, \end{aligned} \tag{1}$$

where σ_* is a scale parameter that removes the necessity of including a constant term in the log-volatility equation, ε_t and η_t are mutually independent Gaussian white noise processes with variances 1 and σ_η^2 respectively. Some detailed reviews of ARSV models are given by Taylor (1994), Shephard (1996), Ghysels *et al.* (1996), Capobianco (1996) and Barndorff-Nielsen and Shephard (2001).

Carnero *et al.* (2001) show that the ARSV(1) model is more adequate than the more popular GARCH (1,1) model to represent the empirical regularities often observed in financial time series. However, one of the most important limitations of SV models is that the distribution of y_t conditional on past observations up to time $t-1$ is unknown. Consequently, the exact likelihood function is difficult to evaluate. In order to derive it, the vector of the unobserved volatilities has to be integrated out of the joint probability distribution. If we denote by $y = \{y_1, ..., y_T\}$ the vector of observations, $h = \{h_1, ..., h_T\}$ the vector of log-volatilities, i.e. $h_t = \log(\sigma_t^2)$, and $\theta = \{\sigma_*, \phi, \sigma_\eta^2\}$, the corresponding parameter vector, the likelihood is given by,

$$f(y \mid \theta) = \int f(y \mid h, \theta) f(h \mid \theta) dh. \tag{2}$$

The dimension of the integral in (2) is equal to the sample size, T, and, in order to evaluate it, numerical methods must be used. Therefore, Maximum Likelihood (ML) estimation of the parameters of SV models is not straightforward.

Furthermore, one of the main purposes when fitting a heteroscedastic model to a series of returns is to obtain estimates of the volatility. As the volatility is unobservable in SV models, it is necessary to use filters to yield such estimates. Several estimation techniques have been proposed in the literature for both the parameters and volatilities of SV models.

On the other hand, it has often been empirically observed that the log-volatilities could be characterized as a long-memory process. Consequently, Breidt *et al.* (1998) and Harvey (1998) propose independently the Long-Memory SV (LMSV) model, where the log-volatility is modelled as an ARFIMA(p, d, q) process. Denote by ARLMSV$(1, d, 0)$ the following process

$$\begin{aligned} y_t &= \sigma_* \varepsilon_t \sigma_t, \\ (1-\phi L)(1-L)^d \log(\sigma_t^2) &= \eta_t. \end{aligned} \tag{3}$$

where d is the long memory parameter. Model (3) encompasses the short memory model in (1) when $d = 0$.

In this paper, we review the main estimators of the parameters proposed for univariate SV models describing what is known about their asymptotic and finite sample properties. We also summarize the main advantages and limitations of each of the estimators considered, focusing on their ability to deal with non-zero conditional means, Student-t errors[1] and asymmetric behavior of volatility.

The paper is organized as follows. Section 2 describes the alternative estimators of the parameters of ARSV models, their asymptotic and finite sample properties as well as their main advantages and limitations. This section is completed with an illustration with simulated data. Section 3 describes the estimation methods proposed for LMSV models. Section 4 illustrates the results with an empirical application that compares the principal estimators by fitting

[1] Although we focus on the generalization of the estimators to Student-t errors, there are other alternative distributions proposed in the literature, as, for example, (i) the exponential power distribution, suggested by Nelson (1988); (ii) the normal inverse Gaussian distribution, proposed by Barndorff-Nielsen (1997) or (iii) the skewed exponential power distribution, suggested by Steel (1998).

the ARSV model to daily S&P 500 returns. Finally, section 5 concludes the paper.

2 Estimation of the parameters of ARSV(1) models

Estimation methods of the parameters of the short-memory ARSV(1) model in (1) can be classified into two general groups. The first class is comprised by estimators based directly on the statistical properties of y_t. There are three main classes of estimators within this group: (i) estimators based on the method of moments (MM); (ii) estimators based on the ML principle; and (iii) estimators based on an auxiliary model.

Alternatively, taking logarithms of squared observations, the following non-Gaussian linear state space model is obtained:

$$\begin{aligned} \log(y_t^2) &= \mu + h_t + \log(\varepsilon_t^2), \\ h_t &= \phi h_{t-1} + \eta_t. \end{aligned} \tag{4}$$

where $\mu = \log(\sigma_*^2)$. The parameters of the ARSV(1) model can also be estimated using the linear model in (4).

In subsection 2.1, we describe the estimators within the first group while subsection 2.2 deals with the estimators based on $\log(y_t^2)$.

2.1 Methods based on y_t

2.1.1 Method of moments

The simplest estimator within this group is the MM used by, for example, Taylor (1986) and Chesney and Scott (1989). Later, Melino and Turnbull (1990) proposed to estimate the parameters of the ARSV(1) model using generalized method of moments (GMM). These estimators are based on the convergence of selected sample moments to their unconditionally expected values. The procedure also implies the estimation of a weighting matrix that takes into account the non-iid property of the moment discrepancies. The problem is that the score function in ARSV models cannot be computed and, consequently, the proper set of moments to be used should be guessed. An alternative approach to estimate SV models using the MM principle, is the simulated method of moments (SMM), proposed by Duffie and Singleton (1993), that replaces the vector of analytical moments is replaced by the vector of moments of a simulated process. Finally, Andersen and Sørensen (1996) propose to improve the MM procedure by using a penalty function to ensure stationarity of the model and employing a modified weighting matrix.

Hansen (1982) demonstrates that the GMM estimator is consistent and asymptotically normal. Another advantage of this estimator is that it is very simple. For this reason, there are numerous empirical applications where the parameters of SV models are estimated using GMM estimators. For example, Andersen (1994) applies this procedure to a series of IBM returns and Ghyshels and Jasiak (1996) study the S&P stock index. Vetzal (1997) fit the SV model to short term interest rates and Flemming *et al.* (1998) to the S&P 500, the T-bond and the T-bill returns to study the volatility linkages in the stock, bond and money markets.

However, these procedures have poor finite sample properties and their efficiency is suboptimal with respect to ML methods. Jacquier *et al.* (1994) find substantial bias and sampling variability, especially for the estimates of σ_η^2, and show that the performance of the technique worsens when there is high persistence (high ϕ) and low coefficient of variation, defined as $C.V. = V(\sigma_t^2)/(E(\sigma_t^2))^2 = \exp\{\frac{\sigma_\eta^2}{1-\phi^2}\} - 1$. As real time series of returns are characterized by highly persistent volatilities, it seems that the methods based on the MM principle are not suitable to estimate the parameters of SV models. Furthermore, the GMM criterion surface for the ARSV(1) model is highly irregular. Therefore, optimization fails to converge, specially for small sample sizes. A large amount of non-converging estimations has also been reported by Andersen and Sørensen (1996). This problem can be caused by imprecise estimates of the long-run covariance matrix, and, consequently, would disappear with an accurate approximation of the true weighting matrix; see Andersen *et al.* (1999).

Another important limitation of the GMM based estimators is that they do not generate estimates of the volatility so other procedures as, for example, the Kalman filter should be used to get those estimates; see, for example, Andersen (1994) and Ghysels and Jasiak (1996).

2.1.2 Maximum likelihood estimators

Recently, ML estimators of the parameters of SV models have experienced a big progress thanks to the development of numerical methods based on importance sampling and Monte Carlo Markov Chain (MCMC) simulation techniques; see Wong (2002) for a description of MCMC algorithms in the framework of ARSV models.

The main attractive of importance sampling over MCMC algorithms is that is less computationally demanding and avoid convergence problems. Furthermore, the accuracy of the estimators can be increased by augmenting the simulation sample size. However, MCMC algorithms are more flexible than importance sampling because they allow large dimensional problems to be split into smaller dimensional tasks.

The first importance sampling algorithm applied to SV models was Geweke (1994), who also introduced the possibility of using MCMC in this setting. Although his procedure has not been directly applied to real time series, it is important because it was the basis of many other procedures proposed afterwards, for example, Danielsson (1994), Shephard and Pitt (1997) and Sandmann and Koopman (1998).

The first MCMC estimator of the parameters of ARSV(1) models was proposed by Jacquier *et al.* (1994), JPR onwards, who consider a slightly different parameterization of the ARSV(1) model, given by

$$\begin{aligned} y_t &= \varepsilon_t \sigma_t, \qquad (5) \\ \log(\sigma_t^2) &= \gamma + \phi \log(\sigma_{t-1}^2) + \eta_t, \end{aligned}$$

where $\gamma = \log(\sigma_*^2)(1-\phi)$. The JPR procedure treat the volatilities, σ_t, $t = 1, ..., T$, as if it were a vector of unknown parameters to be estimated, so that they focus on the joint posterior $\pi(\theta^*, \sigma^2 \mid y)$ where $\theta^* = (\gamma, \phi, \sigma_\eta^2)$. As commonly used in standard Bayesian analysis of regression models, the authors consider the Normal-Gamma prior for θ^*. Draws of σ_t^2 are performed considering the univariate conditional distributions $p(\sigma_t^2 \mid \sigma_{t-1}^2, \sigma_{t+1}^2, \theta^*, y)$ for $t = 1, ..., T$ and combining an accept/reject and a Metropolis algorithm. Inference about θ^* is carried out by means of the marginal distribution $\pi(\theta^* \mid y)$ and the smoothing problem is solved by means of the marginal distribution $\pi(\sigma^2 \mid y)$. Therefore, the JPR procedure permits to obtain simultaneously: (i) sample inference about the parameters, (ii) smooth estimates of the unobserved variances and (iii) predictive distributions of multistep forecasts of volatility taking into account the inherent model variability and the parameter uncertainty. However, Shephard and Kim (1994) point out that the sampler for the smoothing of the volatility is computationally inefficient. This sampler is slow if ϕ is close to one and σ_η^2 is small because of the almost singularity of the matrix $Corr(\sigma \mid y)$. A feasible alternative could be the efficient multi-move algorithm by Shephard and Pitt (1997) that outperforms the single block JPR approach in computational terms; see Sandmann and Koopman (1998) and Wong (2002)[2]. However, Wong (2002) suggest to use single-move algorithms because, although they are not robust, its implementation is simpler. On top of that, Andersen (1994) also notes that the algorithm is not well suited to the filtering problem of updating the posterior of σ_t as more data are accumulated.

If the simulation size is large, the JPR estimator has asymptotically the same distribution as the ML estimator.

By means of an extensive Monte Carlo study Jacquier *et al.* (1994) show that MCMC outperforms both quasi-maximum likelihood (QML)[3] and GMM

[2] A multi-move (or block-move) algorithm is based on sampling a block of states at a time instead of sampling only one state, as in Geweke (1994) or JPR. That is, let $h_{t,k} = (h_t, ..., h_{t+k})$, then the Shephard and Pitt (1997) algorithm samples from $p(h_{t,k} \mid h_{t-1}, h_{t+k+1}, y, \theta)$, where h_{t-1} and h_{t+k+1} are stochastic knots.

[3] The QML estimator will be described in section 2.2.

estimators in efficiency, demonstrating that the convergence of their Markov chain is rapid and reliable. They consider the following parameter design: $\phi = (0.9, 0.95, 0.98)$ and $C.V. = (10.0, 1.0, 0.1)$. That is, for each parameter ϕ considered in the experiment, there are three associated values of σ_η^2.[4]

However, the procedure is rather complicated and requires substantial computations that contrast with the simplicity of alternative methods. Recently, and despite the complexity of extending this approach, the generalization to fat tails and correlated errors has been implemented in Jacquier *et al.* (2002).

On the other hand, Andersen (1994a) points out that the JPR procedure is not easily applicable to analyze SV-in-mean (SV-M) models that relate the conditional mean return and an estimate of volatility based on current information. In general, due to its lack of flexibility, the JPR algorithm needs to be non-trivially modified for extensions such as the introduction of explanatory variables, as well as the multivariate approach; see Jacquier *et al.* (1995). Furthermore, increasing the number of parameters suppose a big computational cost to the MCMC procedure. For example, Jacquier *et al.* (1994) always remove the AR(1) component and the monthly effects in their empirical applications in order to decrease the number of parameters.

Because of its computational cost, the JPR algorithm has not been used very often in empirical applications, even though is an efficient method.

An alternative estimation procedure within this group is the simulated maximum likelihood (SML) estimator of Danielsson and Richard (1993). The primary purpose of this technique is to obtain the likelihood in (2) by factorizing the joint density $f(y, h \mid \theta)$ in an importance sampling function $\psi(h \mid y)$ and a remainder function $\rho(h, y)$, so that

$$f(y, h \mid \theta) = \rho(h, y)\psi(h \mid y). \tag{6}$$

The expected value of the remainder function is $E\left[\rho(h, y)\right] = f(y \mid \theta)$. Then a

[4]In this Monte Carlo setting, when $\phi = 0.9$, $\sigma_\eta^2 = (0.4556, 0.1317, 0.0182)$; in the case $\phi = 0.95$, $\sigma_\eta^2 = (0.2337, 0.0676, 0.0092)$ and finally if $\phi = 0.98$, $\sigma_\eta^2 = (0.0948, 0.0274, 0.0037)$.

natural estimator of $f(y \mid \theta)$ is the MC sample mean, namely

$$\hat{f}_T(y \mid \theta) = \frac{1}{T}\sum_{t=1}^{T} \rho(\Lambda_t, y), \tag{7}$$

where $\{\Lambda_t\}_{t=1}^{T}$ is a random sample generated from $\psi(h \mid y)$. The variance of the remainder function $\rho(h, y)$ is minimized by means of the Accelerated Gaussian Importance Sampling (AGIS) algorithm of Danielsson and Richard (1993). Once the likelihood is evaluated by simulation, estimates are obtained by means of a derivative-free optimizer. The asymptotic distribution of the SML estimator is the same as for the ML estimator.

Danielsson (1994) analyses the finite sample properties of the SML procedure and compares them with MM, QML and JPR. He concludes that SML clearly outperforms the MM and QML estimators, while SML and JPR have similar RMSE. However, his conclusions are based on just one experiment with $\gamma = -0.736$, $\phi = 0.9$ and $\sigma_\eta = 0.363$ and $T = 2000$.

With respect to the solution to the smoothing problem, Liesenfeld and Jung (2000) provide a method to estimate the volatility using the conditional expectation $E(h \mid y, \theta)$ evaluated at the SML estimates of θ.

However, the SML procedure suffers from several drawbacks. Jacquier *et al.* (1994) criticize the fact that the way in which SML evaluates the integral is not direct, so that it is hard to measure the accuracy of the proposed approximation. In fact, the likelihood can only be exactly evaluated for $\phi = 0$, and for the rest of parameter values is not available. To solve this problem, JPR propose to nest the SML procedure in a wider Bayesian framework using Monte Carlo methods of numerical integration. However, this extension of the SML procedure is not jet developed. Furthermore, the method is not easily generalizable to other SV models, as the adaptation of the importance function is not straightforward. Danielsson (1998) considered the multivariate extension of the SML procedure. Finally, Shephard (2000) presents evidence that typical importance samplers for the ARSV model may not posses a variance and, consequently, not obey a standard central limit theorem.

There are several empirical applications of the SML procedure along the literature. For example, Danielsson (1994) analyze the S&P 500 index, Liesenfeld (1998) applies the SML estimator to German stock-market returns and Liesenfeld and Jung (2000) fit the ARSV(1) model with Normal and Student-t errors to six series of financial returns. Finally, Liesenfeld (2001) applies the method to estimate an ARSV model and a bivariate mixture model for closing prices of the stocks of IBM and Kodak.

Alternatively, Fridman and Harris (1998) propose a direct ML estimation method that calculates the likelihood function directly by means of the recursive numerical integration procedure suggested by Kitagawa (1987) for non-Gaussian filtering problems. This method can be considered as an extended Kalman filter. Through a small simulation experiment, Fridman and Harris (1998) conclude that their direct ML procedure performs better than QML and GMM, and similar to SML, the simulated expectation maximization (SEM) algorithm by Kim and Shephard (1994), and the JPR procedure. The smoothed sequence of volatilities can be obtained with an additional forward-recursion step in the extended filter. They apply their technique to the S&P 500 index considering both Gaussian and Student-t errors.

Finally, Watanabe (1999) proposes a non-linear filtering natural extension of QML. He obtains the exact form of the likelihood by means of a non-linear filter that makes use of the conditional probability density functions of the log-volatility and the observed series[5]. Watanabe (1999) carries out a small Monte Carlo comparative study with the same design proposed by JPR and shows that his non-linear filtering maximum likelihood (NFML) procedure outperforms QML and GMM in efficiency, and is close to SML and JPR. He also proposes a smoothing algorithm to estimate the volatilities that is shown to be superior to the standard smoothing solution used in QML.

The NFML procedure can be used in models with a linear structure in the

[5]This is not the first attempt to use of non-linear filters in order to estimate the log-normal SV model. For example, Brigo and Hanzon (1998) use projection filters in order to estimate the model.

mean where the errors are assumed to be normal. Finally, Watanabe (1999) uses this procedure to fit the Tokyo Stock Price Index (TOPIX).

However, one of the main drawbacks of methods based on an extended Kalman filter (as Fridman and Harris (1998) or Watanabe (1999)) is their slow convergence. These procedures also involve choosing a priori a fixed grid over which the process will be integrated, and the optimal grid may not exist; see Sandmann and Koopman (1998).

Recently Yu *et al.* (2002) propose a new class of SV models, the nonlinear SV model, where the volatility is transformed according to the Box-Cox power function. This new specification includes the lognormal SV model. The proposed estimation procedure is another MCMC algorithm where the simulation efficiency of the single-move algorithm is improved by updating the components of h_t sequentially. Properties of this algorithm compared with other MCMC procedures developed in the SV setting are not already established. The authors apply the new procedure to the Dollar/Pound exchange rate.

2.1.3 Estimation procedures by means of an auxiliary model

The methods described in this subsection choose an auxiliary model easy to estimate, as an instrument to estimate a model easy to simulate. Notice that SV models are easy to simulate although they are difficult to estimate. There are two main methods proposed within this group: the Indirect Inference and the Efficient Method of Moments (EMM) methods.

The indirect inference estimator proposed by Gourieroux *et al.* (1993) is given by

$$\tilde{\theta}_T^H = \underset{\theta \in \Theta}{\arg\min} \left(\hat{\beta}_T - \tilde{\beta}_{TH}(\theta)\right)' \hat{\Omega}_T \left(\hat{\beta}_T - \tilde{\beta}_{TH}(\theta)\right), \tag{8}$$

where $\hat{\beta}_T$ is obtained by maximizing an auxiliary criterion from the auxiliary model $Q_T(\beta, \mathbf{y}_t)$, $\tilde{\beta}_{TH}(\theta)$ is an estimate of the binding function obtained by maximizing $Q_{TH}(\beta, \mathbf{y}_{TH}(\theta))$ and $\mathbf{y}_{TH}(\theta) = (\, y_1, ..., y_{tH})$ is a vector of simulated observations. Gourieroux *et al.* (1993) propose the quasi-likelihood function of

Harvey *et al.* (1994) as an auxiliary criterion to estimate a continuous time SV process. Pastorello *et al.* (1994) also estimate continuous time SV models using Indirect Inference. Engle and Lee (1996) and Calzorali *et al.* (2001) use GARCH as the auxiliary model. Monfardini (1998) fits the log-normal SV model through indirect inference with AR(m) and ARMA(1,1) as auxiliary models. Finally, Fiorentini *et al.* (2002) fit the SV model proposed by Heston (1993) to call options on the Spanish IBEX-35 index using a nonlinear asymmetric GARCH (NAGARCH) as auxiliary process.

The EMM approach introduced by Bansal *et al.* (1993, 1995) and Gallant and Tauchen (1996) is similar, but is based on score calibrating the criterion function. EMM improves the Indirect Inference approach in computational terms, as there is no need to refit the score to each simulated realization, while in the Indirect Inference procedure the binding function should be computed at each step. The EMM procedure has been implemented by Engle (1994) and Ghysels and Jasiak (1994) to estimate SV models in continuous time. Then, Gallant *et al.* (1997) and Jiang and van der Sluis (2000) analyze some discrete time SV models by EMM. In van der Sluis (1997a, 1997b), the EMM procedure is applied to analyze exchange rate and S&P index series making use of the EGARCH model as the auxiliary process[6]. During the last years EMM has been mainly applied for SV models in continuous time, as in Andersen and Lund (1997), Chernov and Ghysels (2000) or Dai and Singleton (2001).

Gourieroux *et al.* (1993) demonstrate that the Indirect Inference and EMM estimators are asymptotically equivalent. Both estimators are consistent and asymptotically normal. Furthermore, Tauchen (1997) and Gallant and Long (1997) show that the estimated covariance matrix of the EMM estimator approaches that of maximum likelihood, as the score generator approaches the true conditional density.

There is available only reduced Monte Carlo evidence on the finite sample properties of the Indirect Inference and EMM. For example, Monfardini (1998)

[6]The code in Ox 2.0 to implement the EMM procedure proposed in van der Sluis (1997a, 1997b) is available at: *http://www.geocities.com/WallStreet/Exchange/6851/*

concludes that the Indirect Inference estimator performs well in finite samples, but JPR procedure and the SEM of Kim and Shephard (1994) are more efficient. Andersen *et al.* (1999) analyze the finite sample properties of EMM estimators of SV models and observe an improvement in efficiency compared with GMM, although they are not as efficient as JPR. As they point out, the EMM procedure can be considered to be conceptually between the GMM and the methods that infer the exact shape of the likelihood. They also state that the efficiency of the method for finite samples strongly depends on the choice of the auxiliary model, specially for small samples. Calzolari et al. (2000) also develop a Monte Carlo study based on the Jacquier *et al.* (1994) design. Their procedure is shown to be less efficient than the approach of Jacquier *et al.* (1994), as the auxiliary model does not nest the model of interest and no prior information is used.

One of the main drawbacks of the Indirect Inference and EMM estimators is that none of them provide exact filtering and smoothing solutions for the associated volatility. Therefore, alternative procedures are necessary to obtain volatility estimates. Furthermore, these two procedures are very expensive in computational terms.

Table 1 summarizes the properties of the estimators described in this subsection. The finite sample properties of the MM estimators are not appropriate when the volatility is highly persistent, as is often the case in real series of returns. On the other hand, the methods based on an auxiliary model, seem adequate for continuous time models but are not, in general, efficient. The efficiency of these methods depend on the auxiliary model and there is not consensus about the best auxiliary model for discrete ARSV models. On top of that, these methods do not estimate directly the volatility. The procedures that seem more relevant are JPR and Watanabe (1999). The JPR procedure is the basis of many posterior procedures. However, Wong (2002) shows that it is not the most efficient among the single-move MCMC procedures. On the other hand, the method proposed by Watanabe (1999) is not computationally very intensive and seems very promising for future developments and empirical implementations. The lack of a deep study of the finite sample properties of this procedure encourages future research on the method.

2.2 Methods based on $\log(y_t^2)$

2.2.1 Quasi-maximum likelihood (QML)

The QML estimator, proposed independently by Nelson (1988) and Harvey *et al.* (1994), is based on linearizing the SV model by taking the logarithms of the squares of the observations as in expression (4). Treating $\log(\varepsilon_t^2)$ as if it were Gaussian, the Kalman filter can be applied in order to obtain the quasi-likelihood function of $\log(y_t^2)$ which, ignoring constants, is given by

$$\log L(\log(y^2) \mid \theta) = -\frac{1}{2}\sum_{t=1}^{T} \log F_t - \frac{1}{2}\sum_{t=1}^{T} \frac{\nu_t^2}{F_t}, \tag{9}$$

where ν_t is the one-step-ahead prediction error of $\log(y_t^2)$, and F_t is the corresponding mean squared error. Ruiz (1994) shows that the QML estimator is consistent and asymptotically normal. However, the QML procedure is inefficient as the method does not rely on the exact likelihood of $\log(y_t^2)$. Note that approximating the density of $\log(\varepsilon_t^2)$ by a normal density instead of using the true $\log(\chi_1^2)$ density could be rather inappropriate; see Figure 1 for a comparison of both densities. The effects of this approximation depend on the true parameter values and are worse as the variance of the volatility equation σ_η^2 decreases. This is an important problem as in empirical applications to financial data, σ_η^2 is typically rather small.

On the other hand, an inlier problem often arises when dealing with the log-squared transformation. When returns, y_t, are very close to zero, the log-squared transformation yields large negative numbers. In the extreme case, if the asset return is equal to zero, the log-squared transformation is not defined. In Figure 1 large negative values in the distribution of $\log(\chi_1^2)$ reflect the presence of those inliers. To solve this problem, Fuller (1996) proposes the following modification of the log-squared transformation

$$y_t^* = \log(y_t^2 + \tau s^2) - \frac{\tau s^2}{y_t^2 + \tau s^2} \tag{10}$$

where s^2 is the sample variance of y_t and τ is a small constant. In several studies this constant has been set equal to 0.02; see, Fuller (1996), Breidt and Carriquiry (1996) and Bollerslev and Wright (2001).

The finite sample properties of the QML estimator have been analyzed by Ruiz (1994) who shows that the bias of the QML estimator of σ_η^2 increases when σ_η^2 decreases. Jacquier *et al.* (1994) also find that QML has a worse behavior in situations of high persistence and low σ_η^2. Furthermore, they also remark that, in this case, there is a huge degradation in the filtering performance in terms of the RMSE. However, Sandmann and Koopman (1998) and Breidt and Carriquiry (1996), state that although QML is inefficient, is not as bad as Jacquier *et al.* (1994) show. These authors suppose that the bad results of QML in Jacquier *et al.* (1994) may be due to an inefficient implementation of the procedure (poor starting values, different convergence criteria, etc.). Finally, Jacquier *et al.* (1994) point out that although γ and ϕ are separately identified, the high correlation between their estimates indicates that these parameters are underidentified. On the other hand, Andersen and Sorensen (1996) show that the QML estimator dominates the GMM estimator for models with a high degree of persistence. Deo (2002) provides theoretical intuition for this finding.

Given estimates of the parameters, the log-volatility, h_t, can also be estimated by means of a smoothing algorithm; see Harvey *et al.* (1994) for further details. Notice that, as $\log(y_t^2)$ is not Gaussian, the Kalman filter yields minimum mean square linear estimators (MMSLE) of h_t rather than minimum mean square estimators (MMSE).

Despite the limitations of the method, QML procedure is very flexible and several generalizations of the method have been proposed. For instance, the QML estimator can be directly implemented to estimate models with heavy-tailed, such as the Student-t or GED, errors. The QML procedure can also be easily extended to models with explanatory variables or other ARMA models for the log-volatility. Furthermore, the multivariate generalization is straightforward in this context; see Harvey *et al.* (1994). Missing or irregularly spaced observations can also be easily handled. Another interesting generalization of the QML procedure is to capture the *"leverage effect"* by which the response of the volatility is larger when the return is negative than when it is positive; see Black (1986). To deal with the *"leverage effect"*, Harvey and Shephard (1996) introduce correlation between the disturbances ε_t and η_t in model (1), where $Corr(\varepsilon_t, \eta_t) = \rho$, so that an increase in predicted volatility tends to be associated with falls in the stock price. The estimate of ρ is obtained after imposing certain distributional assumptions about ε_t and η_t. The corresponding QML estimators are consistent and asymptotically normal.

The QML method is very easy to implement[7] and has often been implemented in practice. For instance, Hwang and Satchell (2000) use QML to analyze four FTSE 100 stock index. Lien and Wilson (2001) fit a multivariate ARSV model to weekly data on spot prices and future prices of crude oil markets. McMillan (2001) also estimates by QML a multivariate ARSV model for the deutschemark/dollar and french franc/dollar exchange rates. Finally, Yu (2002) applies the procedure to fit an SV model to New Zealand stock market data.

The last contribution in this area, at the moment, is the recent work by Alizadeh *et al.* (2002) who propose a new QML estimation procedure based on using the range as a proxy of the volatility. The range is defined as the difference between the highest and lowest log security prices over a fixed sampling interval. The new procedure is applied to five exchange rates.

2.2.2 Other methods based on linearization

Kim and Shephard (1994) propose a simulated expectation maximization (SEM) algorithm using a mixture of seven normal distributions to match the first four moments of $\log(\varepsilon_t^2)$. Later, Mahieu and Schotman (1998) propose a more flexible

[7] QML procedure for SV models is already implemented in the software STAMP 6.0 by Koopman, Harvey, Doornik and Shephard, London: Timberlake consultants, (2000).

mixture model in order to accommodate a wider range of shapes of $\log(\varepsilon_t^2)$. The main advantage of approximating the distribution of $\log(\varepsilon_t^2)$ by mixtures of normals is that, conditional on the mixture component, the state space in (4) is Gaussian. Besides, the use of mixtures makes the procedure more robust than QML to the inlier problem. A Gibbs sampling technique that extends the usual Gaussian Kalman filter can then be applied in both cases, offering the possibility of using a multi-move Gibbs sampler, as explained by Shephard (1994) and Carter and Kohn (1994).

The performance of the SEM procedure for finite samples has been studied by Fridman and Harris (1998), who show that this procedure is similar to MCMC and SML in terms of efficiency.

Later, Kim, Shephard and Chib (1998) (KSC) propose a procedure that nests and improves several aspects of the SEM estimator; see Jacquier *et al.* (1994) for several criticisms of the SEM procedure. Specifically, they propose a MCMC algorithm that samples all the unobserved volatilities simultaneously by means of an approximating offset mixture of normals model, together with an importance reweightening procedure to correct the linearization error. The KSC procedure provides efficient inferences, likelihood evaluation, filtered volatility estimates, diagnostics for model failure and computation of statistics for comparing non-nested volatility models. However, its finite sample properties have not been compared with other estimators. Finally, Kim *et al.* (1998) implement the KSC method to analyze daily returns of the exchanges rates of the U.K. Sterling, German Deutschemark, Yen and Swiss Franc against the U.S. dollar.

The generalization of the KSC method to Student-t errors was proposed by Chib *et al.* (2002) who also include a generalization of the method for the SV model with jumps.

Alternatively, Sandmann and Koopman (1998) propose approximating the likelihood function by a Gaussian part constructed via the Kalman filter plus a correction for departures from the Gaussian assumption relative to the true unknown model. Their procedure is based on previous work by Shephard and Pitt (1997) and Durbin and Koopman (1997), who perform methods for constructing the likelihood function for general state space models using importance sampling. The procedure, known as Monte Carlo Likelihood (MCL), generates simultaneously estimates of the parameters and the latent volatility. The Kalman filter smoother applied to the approximating Gaussian SV model in the MCL procedure effectively computes the posterior mode estimates of the volatility without obtaining the marginal of $\pi(h, \theta \mid y)$ as in Jacquier *et al.* (1994). Using the same design of Jacquier *et al.* (1994), Sandmann and Koopman (1998) compare the finite sample properties of QML, JPR and MCL. They conclude that with a small number of draws, the MCL procedure gives similar results to MCMC in terms of efficiency and it is not so expensive in computational terms.

The MCL estimation procedure can be easily generalized to heavy-tailed errors, explanatory variables or to the inlier problem. The multivariate case is not yet available. Although Sandmann and Koopman (1998) also mention that the MCL can be implemented in models with correlated ε_t and η_t and with stochastic seasonal components, these extensions have not yet been developed in the literature.

Brandt and Kang (2002) propose an interesting application of this procedure for monthly returns on the CRSP index and use MCL to fit a VAR model where the first equation describes the dynamics of the conditional mean and the second equation is a SV model.

The MCL procedure can also be generalized to the SV-M model. The incorporation of the unobserved volatility as an explanatory variable in the mean equation is analyzed by Koopman and Uspensky (2002) who fit the SV-M model to three different financial series. In this empirical application, estimates of the volatility are obtained by means of the particle filtering technique of Pitt and Shephard (1999).

Recently, Singleton (2001) and Knight *et al.* (2002) have proposed a new estimation procedure based on the empirical characteristic function (ECF). As $\log(y^2)$ is the convolution of an AR(1) process and an iid logarithmic χ_1^2 sequence, there is a closed form expression for the characteristic function so that the model is fully and uniquely parameterized by it. Knight and Yu (2002) establish the strong consistency and asymptotic normality for the ECF estimators with a general weight function. The procedure is applied in Knight *et al.* (2002) to the Australian/New Zealand dollar exchange rate.

Table 3 summarizes the main estimators of the parameters of the ARSV(1) model described in this subsection, together with their asymptotic properties and their main advantages. It seems that the most competitive procedure based on the linearization $\log(y_t^2)$ is the MCL approach, both in terms of the efficiency of the estimators and its computational requirements.

2.3 Illustration with simulated data

To illustrate the differences between the estimates obtained by the some of the alternative estimators previously described, in this subsection, the ARSV model is fitted to simulated series generated by two different models. The first model considered (M1) has parameters $\phi = 0.95$, $\sigma_\eta^2 = 0.05$ and $\sigma_* = 1.0$ and the second (M2) $\phi = 0.98$, $\sigma_\eta^2 = 0.02$ and $\sigma_* = 1.0$. The parameters have been selected to represent values often found in empirical applications of daily data. One series of size $T = 5,000$ is generated by each of the models. The parameters are estimated using the whole series and using also the first 500 and $1,500$ observations by GMM, JPR, QML, MCL and KSC. With respect to GMM, estimates of the covariance of the differences between the sample and populational moments have been obtained as in Melino and Turnbull (1990)[8]. We have not obtained the original code of the JPR estimator and, consequently, we have implemented our own code following Wong (2002) with the number of iterations suggested by JPR, i.e. 1500 burn-in iterations and 2500 iterations[9]. The convergence of the resulting algorithm is extremely slowly[10]. The MCL procedure is implemented

[8]Probably, best results for GMM method would have been obtained by using the other procedures, as using the Barlett window; see Andersen *et al.* (1996).

[9]We are very grateful to Mike Wiper for his continuos help to develop this code.

[10]Geweke in his comments on Jacquier *et al.* (1994) proposes an alternative algorithm that uses an adaptive rejection sampling algorithm (Wild and Gilks (1993)) given that the log-conditional probability density of $log(\sigma_t^2)$ is globally concave. Although this alternative is simpler and would produce a faster sampler, no detailed analysis of this suggestion is provided

using the library SsfPack 2.3 of Koopman, Shephard and Doornik (1999)[11]. The program employed for estimation is named sv_mcl_est.ox and can be downloaded from the webside of Siem Koopman[12]. The KSC procedure is implemented in BUGS following the approach by Meyer and Yu (2000)[13]. Alternatively, the KSC procedure can also be implemented using the software package SVPack 2.1 by Neil Shephard but we do not use this approach as it only provides Monte Carlo standard errors of the parameters, that cannot be compared with those of the rest of procedures[14].

Estimates of σ_* are obtained in different ways depending on the estimation procedure. In the QML case, $\hat{\sigma}_*$ is the sample standard deviation of the observations standardized by the smoothed estimates of the volatility. The approximation of the asymptotic variance of $\hat{\sigma}_*$ is given by $Var(\hat{\sigma}_*^2) = \frac{(\pi^2/2)(\hat{\sigma}_*^2)^2}{T}$; see Harvey and Shephard (1993). In MCL and JPR, the corresponding algorithms obtain estimates of γ so estimates of σ_* and its corresponding standard deviation are obtained after transformation. Finally, KSC estimates directly σ_*.

Table 4, that reports the estimation results for model M1, shows that, for moderate sample sizes, $T = 500$, the GMM estimates of ϕ and σ_η^2 could underestimate the true values, even can lead to conclude that the volatility is close to be constant over time. Furthermore, the standard errors of these estimates are clearly greater that for any of the other methods considered. Finally, we would like to point out that, for this moderate sample size, there are important difficulties for the convergence of the GMM estimation algorithm. The QML estimator of ϕ also seems to underestimate the true value when the sample size is moderate. The estimates of these parameters obtained by the other methods are rather similar. With respect to their standard errors, the biggest corresponds to the QML estimator and the smallest to the KSC. Looking at the

and the implementation of the algorithm exceeds the scope of this survey.

[11] More information at *http://www.ssfpack.com*

[12] More information at *http://www.econ.vu.nl/koopman/sv/*

[13] BUGS is available free of charge from *http://www.mcr-bsu.cam.ac.uk/bugs/welcome.shtml*

[14] More information at *http://www.nuff.ox.ac.uk/users/shephard/ox/*

estimates of the scale parameter, it can be observed that all of them are very similar. However, in this case, the smallest standard error corresponds to the GMM estimator and the biggest to the KSC estimator. Finally, for the larger sample sizes, $T = 1000, 5000$, the estimates of all the parameters are similar independently of the estimator used. The main differences between the estimators arise in the standard errors, with the GMM standard errors of $\widehat{\phi}$ and $\widehat{\sigma}_\eta^2$ being rather big compared with the other estimators. Notice that the results for QML and KSC are remarkably similar. Once more, the smallest standard errors of the estimates of σ_* are obtained using the GMM and QML estimators.

The results for the M2 model are reported in Table 5 and are very similar to the ones reported in table 4. Although the results in tables 4 and 5 have just scratch the surface of the problem, it seems that the best estimator of the scale parameter is based on using the corresponding sample moment. Therefore, as suggested by Harvey and Shephard (1993), it seems to be a good strategy to standardize the original observations using the estimated marginal standard deviation before the ARSV model is estimated and then to estimate the scale parameter by the sample standard deviation of the observations standardized by the estimated conditional standard errors. Furthermore, for the sample sizes usually encountered in empirical applications, it seems that, with the exception of the GMM estimator, the results obtained by the methods considered in this example are rather similar and, consequently, it is not worth to use the more computer intensive methods to estimate the parameters. However, before a conclusion can be reached it seems necessary to know whether the asymptotic standard errors reported in tables 4 and 5 are adequate approximations of the standard errors of the sample distributions of each of the estimators. It is clear that a more deep study is due.

With respect to the estimation of the latent volatilities, figure 2 plots the corresponding volatilities estimated by the QML, JPR, KSC and MCL filters, together with the underlying simulated volatilities for the first model considered ($\phi = 0.95$, $\sigma_\eta^2 = 0.05$ and $\sigma_* = 1$) and $T = 1500$. Notice that the KSC estimates are too volatile while the QML estimates are too smooth. The corresponding

Root Mean Square Errors (RMSE) are 0.32 and 0.29 respectively. On the other hand, the JPR and MCL estimates are rather similar. In both cases, the RMSE is 0.27. Finally, figure 3 plots the estimated volatilities obtained using the QML, JPR and MCL filters with the parameters fixed at the GMM estimates together with the simulated volatilities. Once more, the shapes of the JPR and MCL estimates are quite similar with RMSE of 0.27. As before, the QML estimates have a smoother shape and the RMSE is slightly bigger with a value of 0.29.

3 Estimation methods for LMSV models

The autocorrelation function of the squared high frequency returns is usually characterized by its slow decay towards zero. This decay is neither exponential, as in short-memory processes, neither implies a unit root, as in integrated processes; see, for example, Ding *et al.* (1993). Consequently, it has been suggested that squared returns may be modelled as a long memory process, whose autocorrelations decay at an hyperbolic rate. Evidence of long-memory in second order moments is also found in Dacorogna *et al.* (1993) and Lobato and Savin (1998), among others.

Model (3) with $\phi = 0$ is the basic Long-Memory SV (LMSV) model introduced by Harvey (1998). Breidt *et al.* (1998) propose a LMSV model where the log-variance h_t follows an ARFIMA(p, d, q) process. In both papers, estimation is carried out by QML, maximizing the Whittle discrete approximation to the Gaussian likelihood function of $\log(y_t^2)$ in the frequency domain, given by

$$\tilde{L}(\theta \mid y) = -\frac{1}{2T}\sum_{j=1}^{T-1}\left[\log f(\lambda_j, \theta) + \frac{I(\lambda_j)}{f(\lambda_j, \theta)}\right], \tag{11}$$

where $f(\lambda_j, \theta)$ is the spectral density, $\lambda_j = \frac{2\pi j}{T}$ the corresponding frequencies and $I(\lambda_j)$ the sample spectrum. Breidt *et al.* (1998) demonstrates that the QML estimator is strongly consistent. Furthermore, Deo (2002) proves the asymptotic normality of the QML estimator obtained maximizing the time domain likelihood. He conjectures that this result may also hold for the Whittle estimator. Breidt *et al.* (1998) also analyze the finite sample properties of the Whittle estimator, concluding that it performs properly. However, their parameter design is not very realistic and Pérez and Ruiz (2001) extend their analysis and conclude that, when parameters are close to non-stationarity ($d \approx 0.5$) and/or to homoscedasticity ($\sigma_\eta^2 \approx 0$), the properties of the QML estimator are very poor, so that huge sample sizes are needed to obtain reliable inferences.[15]

Harvey (1998) suggest an algorithm to obtain the smoothed estimates of the volatility. The empirical implementation of the algorithm is developed in Pérez (2000) and applied to a real series in Pérez and Ruiz (2001).

Estimation of LMSV models has also been carried out by means of Bayesian procedures. Hsu and Breidt (1997) obtain the posterior distribution of the parameters and the smoothed estimates of the volatilities of LMSV processes by means of a Gibbs sampling based algorithm. So (1999) develops a new algorithm based on the state space formulation of Gaussian time series models with additive noise where full Bayesian inference is implemented through MCMC techniques. This algorithm can be applied to model outliers in long memory time series and long memory stochastic volatility models. The performance of the algorithm for finite samples is not checked. So (2002) applies the algorithm to S&P500 data. Finally, Chan and Petris (2000) propose a Bayesian approach to perform inference in the time domain based on the truncated likelihood method in Chan and Palma (1998) where the LMSV model is expressed as a linear state space model. Nevertheless, there is no finite sample Monte Carlo analysis, so that the behavior of this procedure compared with other methods is unknown.

All three Bayesian based approaches are computationally intensive and not easy to generalize to more complicated models, for example, when the log-volatility equation has an ARMA component.

Deo and Hurvich (2001) suggest a semiparametric estimator of the parameter d based on the GPH estimator of Geweke and Porter-Hudak (1983) for ARFIMA models and derive its asymptotic bias and variance. Under certain

[15] Notice that, in the unit root case the parameters of the ARLMSV model are not identified.

conditions, the corresponding asymptotic distribution is normal, although the convergence rate is $\sqrt{m}$, where m is the number of Fourier frequencies used in the corresponding log-periodogram regression. Furthermore, the generalization to the ARLMSV case has not been proposed. A small Monte Carlo experiment is carried out for a unique sample size of $T = 6,144$.

Finally, Wright (1999) proposes a GMM estimator of LMSV models and demonstrates its consistency and asymptotic normality, provided that $-1/2 < d < 1/4$. However, numerous studies have found that usually the estimates of d for high frequency returns are between 0.3 and 0.47; see, for example, Andersen *et al.* (2001). Therefore, the case $d \geq 1/4$ is most interesting from an empirical point of view. In fact, no application with real data is shown. Monte Carlo experiments comparing the GMM and the frequency domain estimators show that the asymptotic distribution in GMM is not a very good approximation for finite samples, even for the biggest sample size considered, $T = 4,000$. Furthermore, it can be observed that, although both estimators have similar standard errors, the biases are bigger for GMM. Deo (2002) shows that, with the moment conditions that have been commonly used, the rate of convergence of the GMM estimator is $T^{1/2-d}$. Alternatively, he proposes a new set of moment conditions based on the linear transformation $\log(y_t^2)$ and shows that, in this case, the GMM estimator is $\sqrt{T}$ consistent and asymptotically normal.

Table 6 provides a summary of the main characteristics of the estimators proposed for LMSV models.

4 Empirical application

In this section, the ARSV(1) model is fitted to daily observations of the S&P 500 stock price index. The series is observed daily from 19 February 1997 to 15 February 2002 and $T = 1,303$[16]. The prices, p_t, are transformed to returns, r_t, in the usual way, i.e. $r_t = 100 \cdot \log(p_t/p_{t-1})$, and centered around the sample mean. Figure 4 plots the series of returns and the corresponding autocorrelations

[16]The series is obtained from: *http://www.spglobal.com/indexmaineuro350.html.*

of the squared returns. Table 7 reports several sample moments of the daily and squared returns. As usual, returns are leptokurtic and uncorrelated although not independent given that squared returns have significant autocorrelations. However, there is not evidence of long memory in squared returns. Therefore, we fit the short memory ARSV model to the series r_t.

Table 8 shows the parameter estimates of the ARSV model obtained by GMM, JPR, QML, MCL and KSC. It can be observed that the estimates of the persistence parameter, ϕ, are rather similar being the smallest 0.93 obtained when the MCL estimator is used and the biggest 0.96 when the parameters are estimated by GMM or JPR. However, the corresponding standard errors are rather different. When the parameters are estimated by JPR, MCL or KSC, the standard errors are approximately 0.02 but when are estimated by GMM or QML, they are much bigger, being 0.05 and 0.07 respectively. With respect to the estimates of the variance of volatility, σ_η^2, there are important differences depending on the estimator used. For example, if the parameters are estimated by JPR or QML, the estimates are approximately, 0.01 and, consequently, the evolution of the volatility is very smooth. However, when the parameters are estimated by GMM, MCL or KSC, the variance is much bigger, 0.05 and, therefore, the volatility is more volatile. Notice, that these differences on the estimates of the variance are also reflected on the corresponding estimates of the *C.V.* that go from 0.1164 to 1.0006 when the parameters are estimated by QML or GMM respectively. In this example, the standard errors of the estimates of σ_η^2 are rather similar. Finally, looking at the estimates of the scale parameter, σ_*, all of them are around 1. Once more, the main differences appear in the standard errors although, in this case, the smallest corresponds to GMM and the biggest to JPR.

The estimates of the volatility obtained by the QML, JPR, MCL and KSC procedures are plotted in Figure 5. All four volatility estimates detect the dynamic evolution of the latent variable, although the QML procedure produces smoother estimates than MCL and KSC which are rather similar. The variance of volatility is too small so that this could be expected. The smoothness of MCL estimates can be controlled by the number of replications in the importance sampler. The sharpy form of KSC volatility estimates is a direct consequence of the Bayesian approach used and was also observed in the simulated data. Finally, as the GMM procedure does not generate estimates of the volatility, we have obtained these estimates using the QML, JPR and MCL filters with the parameters fixed at the GMM estimates. The results appear in figure 6, where it can be observed that, once the parameters are fixed at the GMM estimates, all the filters give similar estimates of the volatility.

5 Summary and conclusions

In this paper, the main estimation procedures of the ARSV(1) and LMSV models have been revised. For each method we describe, when they are available, the asymptotic and finite sample properties of the estimators, the main advantages and limitations and possible generalizations of these procedures.

There are several methods that seem to match the benchmark efficiency established by the MCMC procedure of Jacquier *et al.* (1994), like MCL, SML, ML of Fridman and Harris (1998) or the NFML approach of Watanabe (1999) being, at the same time, simpler and, consequently, easier for empirical implementation. On the other hand, there are very simple methods as, for example, QML that, although are not efficient, can be easily implemented in real time series and, consequently, generalized to more complicated models as, for example, multivariate systems. Whether the lost of efficiency for the sample sizes usually encountered in financial time series, compensates of using the more computationally demanding methods is still an open question. It seems that the MCL method could be a reasonable compromise between efficiency and computational simplicity. Despite that, there is no detailed comparative study that analyze all those methods. In fact, some of the Monte Carlo experiments are so restrictive that they are only available for one set of parameter values. It seems worth to carry out an extensive and detailed comparison between the estimation methods of SV models that have proven to be more attractive both in terms of their asymptotic and finite sample properties and their simplicity for empirical applications.

With respect to results on the properties of the alternative procedures to estimate the latent volatilities, only Jacquier *et al.* (1994) compare volatility estimates calculated by MCMC with Kalman filter estimates obtained using as parameters for the filter the QML and GMM estimates. In this paper, we also compare the estimates of volatility obtained by the alternative algorithms considered, using both, simulated and real data. It seems that the estimates obtained by the simplest QML filter are too smooth while, on the other hand, the estimates of the KSC algorithm are too sharp. In between, the JPR and MCL filters give similar estimates. Finally, we have observed that when the filters are run with the parameters fixed at the GMM estimates, all of them give similar estimates.

The main conclusion is that, at the moment, it is not possible to recommend any of the methods to estimate SV models. Further research should be carried out to compare the efficiency of the estimates of the parameters and the volatility, the flexibility of the procedure to more general situations as asymmetries or multivariate systems, the robustness of the method and the ease of computation.

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Table 1: Summary of estimators of parameters of ARSV(1) model based on series y_t.

Method	Authors	Asymptotic Properties	Finite Sample Experiments	Advantages/Limitations
MM	Taylor (1986)	Consistent and	Efficiency	Not computationally intensive
	Chesney and Scott (1989)	As. Normal	depends on moments	h_t stationary
GMM	Melino and Turnbull(1990)	(Hansen (1982))	Less efficient than MCMC	Bad results for ϕ close to 1
SMM	Duffie and Singleton (DS) (1993)	Consistent and As. Normal (DS (1993))	Not available	Analytical moments not used
Import. sampling	Geweke (1994)	Simulation based	Not available	Bad results if T is big or dimension of h_t bigger than 1 (a); (e)
MCMC	Jacquier et al.(1994)	Simulation based	More efficient than GMM or QML	Computationally intensive Non trivial extensions (b); (c); (d); (e)

Last column codes are the following: (a) Dynamics on the conditional mean; (b) Fat-tailed distributed errors; (c) Multivariate extension; (d) Leverage effect; (e) Estimation of the latent volatilities.

Table 2: Summary of estimators of parameters of ARSV(1) model based on series y_t. (Continued)

Method	Authors	Asymptotic Properties	Finite Sample Experiments	Advantages/Limitations
SML	Danielsson and Richard(1993) Danielsson(1994)	As ML estimators	Not available	(b); (c); (e)
ML	Fridman and Harris(1998)	As ML estimators	Outperforms: GMM, QML Similar: SML,MCMC, SEM	Slow convergence (b); (e)
NFML	Watanabe(1999)	As ML estimators	Outperforms GMM, QML Similar to MCMC and SML	Slow convergence (a); (e)
Indirect Inference	Gourieroux et al.(1993) (GMR)	Consistent and As. Normal (GMR)	Outperformed by MCMC and SEM	(b)
EMM	Gallant and Tauchen(1996)	Consistent and As. Normal (GMR)	Outperforms GMM Outperformed by MCMC	

Last column codes are the following: (a) Dynamics on the conditional mean; (b) Fat-tailed distributed errors; (c) Multivariate extension; (d) Leverage effect; (e) Estimation of the latent volatilities.

Table 3: Summary of estimators of parameters of ARSV(1) model based on series $\log(y_t^2)$.

Method	Authors	Asymptotic Properties	Finite Sample Experiments	Advantages and limitations
QML	Harvey et al.(1994)	Consistent and As. Normal (Ruiz (1994))	Inefficient	Not computationally intensive Inlier problem (b); (c); (d); (e)
SEM	Kim and Shephard(1994)	Simulation based	Similar to ML methods	
KSC	Kim et al.(1998)	Simulation based	Not available comparative results	(b); (e)
MCL	Sandmann and Koopman(1998)	As ML estimators	Similar to MCMC	(a); (b); (e)

ECF

Last column codes are the following: (a) Dynamics on the conditional mean; (b) Fat-tailed distributed errors; (c) Multivariate extension; (d) Leverage effect; (e) Estimation of the latent volatilities.

Table 4: Estimation results for one series generated by ARSV(1).

		JPR	GMM	QML	MCL	KSC
$T = 500$	ϕ=0.95	0.9510	0.7376	0.9267	0.9429	0.9578
		(0.0357)	(1.3524)	(0.0557)	(0.0300)	(0.0216)
	σ_η^2=0.05	0.0461	0.0168	0.0656	0.0571	0.0398
		(0.0465)	(0.3967)	(0.0664)	(0.0315)	(0.0188)
	σ_*=1.0	0.8480	0.9837	0.9275	0.9318	0.9865
		(0.0855)	(0.0304)	(0.0461)	(0.0920)	(0.1301)
$T = 1500$	ϕ=0.95	0.9652	0.9622	0.9511	0.9496	0.9576
		(0.0310)	(0.1491)	(0.0254)	(0.0174)	(0.0134)
	σ_η^2=0.05	0.0456	0.0501	0.0496	0.0570	0.0459
		(0.0296)	(0.0749)	(0.0204)	(0.0194)	(0.0141)
	σ_*=1.0	0.9754	1.0013	1.0973	1.0694	1.0830
		(0.0770)	(0.0035)	(0.0314)	(0.0784)	(0.0770)
$T = 5000$	ϕ=0.95	0.9612	0.9510	0.9527	0.9622	0.9632
		(0.0140)	(0.0276)	(0.0112)	(0.0073)	(0.0069)
	σ_η^2=0.05	0.0406	0.0541	0.0443	0.0371	0.0363
		(0.0121)	(0.0163)	(0.0130)	(0.0072)	(0.0072)
	σ_*=1.0	0.9887	0.9993	1.0136	1.0037	1.0080
		(0.0372)	(0.0074)	(0.0159)	(0.0382)	(0.0394)

Standard errors in parenthesis

Table 5: Estimation results for one series generated by ARSV(1).

		JPR	GMM	QML	MCL	KSC
$T = 500$	ϕ=0.98	0.9700	0.9836	0.9535	0.9718	0.9742
		(0.0318)	(2.5064)	(0.0474)	(0.0171)	(0.0156)
	σ_η^2=0.02	0.0111	0.0084	0.0235	0.0207	0.0186
		(0.0265)	(1.3450)	(0.0322)	(0.0121)	(0.0097)
	σ_*=1.0	0.9068	0.9979	0.8744	0.8784	0.9348
		(0.0759)	(0.3169)	(0.0434)	(0.1011)	(0.1307)
$T = 1500$	ϕ=0.98	0.9871	0.9909	0.9890	0.9864	0.9861
		(0.0197)	(0.1586)	(0.0056)	(0.0058)	(0.0056)
	σ_η^2=0.02	0.0186	0.0066	0.0112	0.0158	0.0173
		(0.0162)	(0.0301)	(0.0060)	(0.0055)	(0.0047)
	σ_*=1.0	0.9596	0.9997	1.1276	1.0908	1.0820
		(0.0444)	(0.0118)	(0.0323)	(0.1511)	(0.1253)
$T = 5000$	ϕ=0.98	0.9891	0.9759	0.9905	0.9895	0.9895
		(0.0085)	(0.1284)	(0.0027)	(0.0026)	(0.0025)
	σ_η^2=0.02	0.0184	0.0182	0.0102	0.0121	0.0124
		(0.0074)	(0.0227)	(0.0029)	(0.0022)	(0.0019)
	σ_*=1.0	0.9661	0.9978	1.0089	1.0000	1.0070
		(0.0217)	(0.0035)	(0.0158)	(0.0369)	(0.0739)

Standard errors in parenthesis

Table 6: Summary of estimators of parameters of LMSV model .

Method	Authors	Asymptotic Properties	Finite Sample Experiments	Advantages and limitations
QML	Harvey(1998) Breidt et al.(1998)	Consistent	Poor properties when $d \approx 0.5$ and/or $\sigma_\eta^2 \approx 0$	Estimation of volatility
MCMC	Hsu and Breidt(1997) So(1999) Chan and Petris(2000)	Simulation based	Not available	Computationally intensive Not easy to generalize Estimation of volatility
Semi- parametric	Deo and Hurvich(1998)	Results only for $\hat{d}$ As. Normal under certain conditions	Not exhaustive Monte Carlo design	
GMM	Wright(1999)	Consistent and As. Normal if $d < 1/4$	Similar to QML	Unknown asymptotics for $d > 1/4$

Table 7: Summary statistics of r_t

	Mean	SD	SK	Kurtosis	Max.	Min.	Q (10)
r_t	0.000	1.249	-0.232	.813*	4.965	-7.136	13.777
r_t^2	1.560	3.425	7.270	82.469	50.926	0.000	82.799*

	$\rho(1)$	$\rho(2)$	$\rho(3)$	$\rho(4)$	$\rho(5)$	$\rho(6)$	$\rho(7)$	$\rho(8)$	$\rho(9)$	$\rho(10)$
r_t	-0.005	-0.049	-0.046	0.006	-0.021	-0.020	-0.048	0.000	0.000	0.052
r_t^2	0.162 *	0.118 *	0.042	0.019	0.076 *	0.074 *	0.051	0.043	0.060 *	0.041

* Significant at the 5% level

Table 8: Empirical estimates of ARSV model for the S&P 500

	ARSV				
	GMM	*JPR*	*QML*	*MCL*	*KSC*
$\widehat{\phi}$	0.9602	0.9596	0.9401	0.9288	0.9392
	(0.0479)	(0.0203)	(0.0699)	(0.0249)	(0.0237)
$\widehat{\sigma}_\eta^2$	0.0541	0.0172	0.0128	0.0499	0.0405
	(0.0219)	(0.0196)	(0.0222)	(0.0190)	(0.0168)
C.V.	1.0006	0.2427	0.1164	0.4381	0.4099
$\hat{\sigma}_*$	1.0005	0.9673	1.2051	1.1248	1.1260
	(0.0157)	(0.0639)	(0.0371)	(0.0542)	(0.0619)

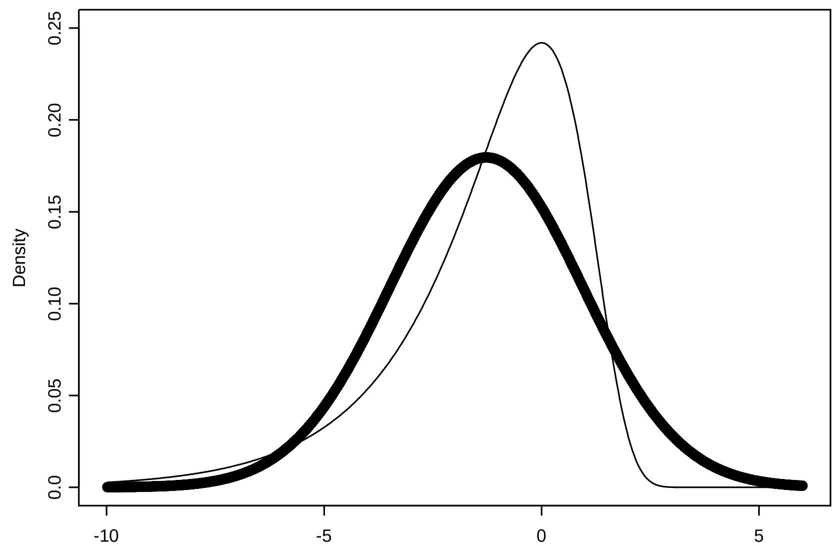

Figure 1: The $\log(\chi_1^2)$ density (solid line) and its corresponding normal approximation (thick solid line)

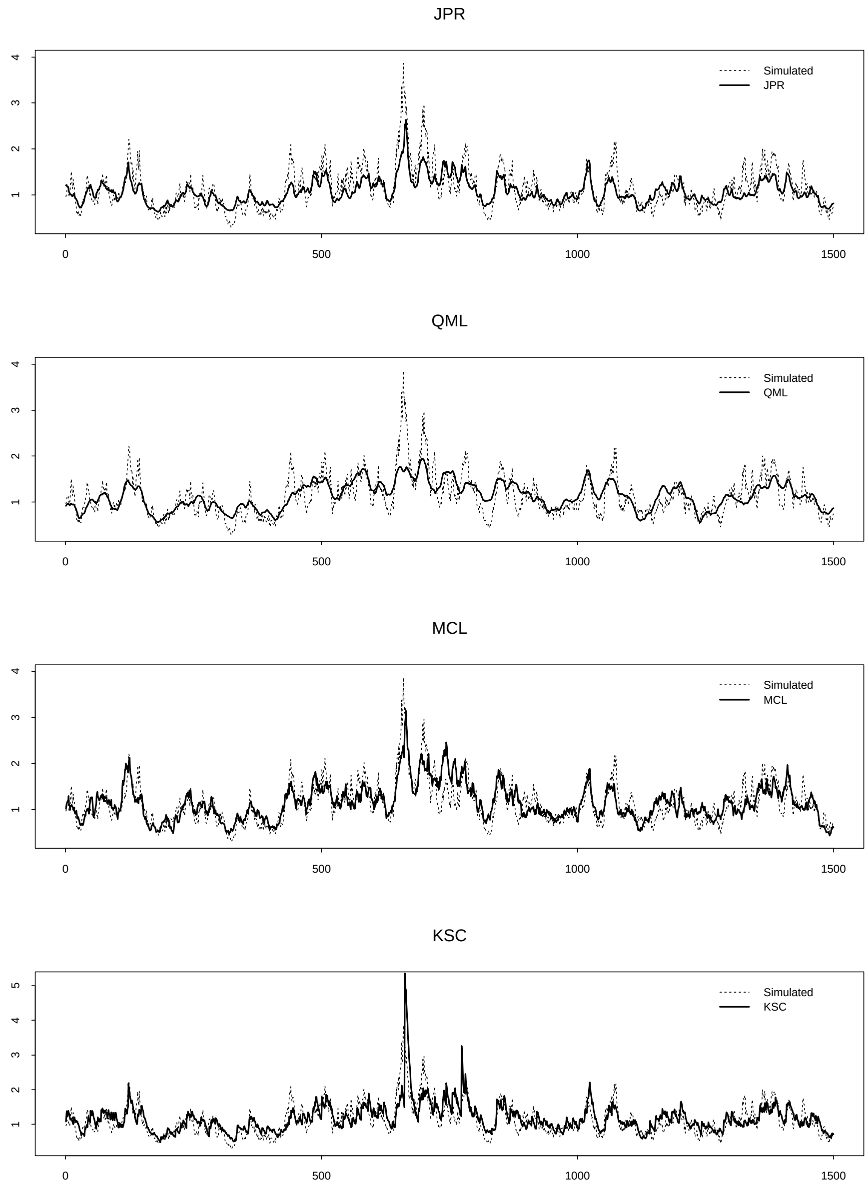

Figure 2: Estimated volatilities of a simulated series with $\phi = 0.95$ and $\sigma_\eta^2 = 0.05$ and $T = 1,500$.

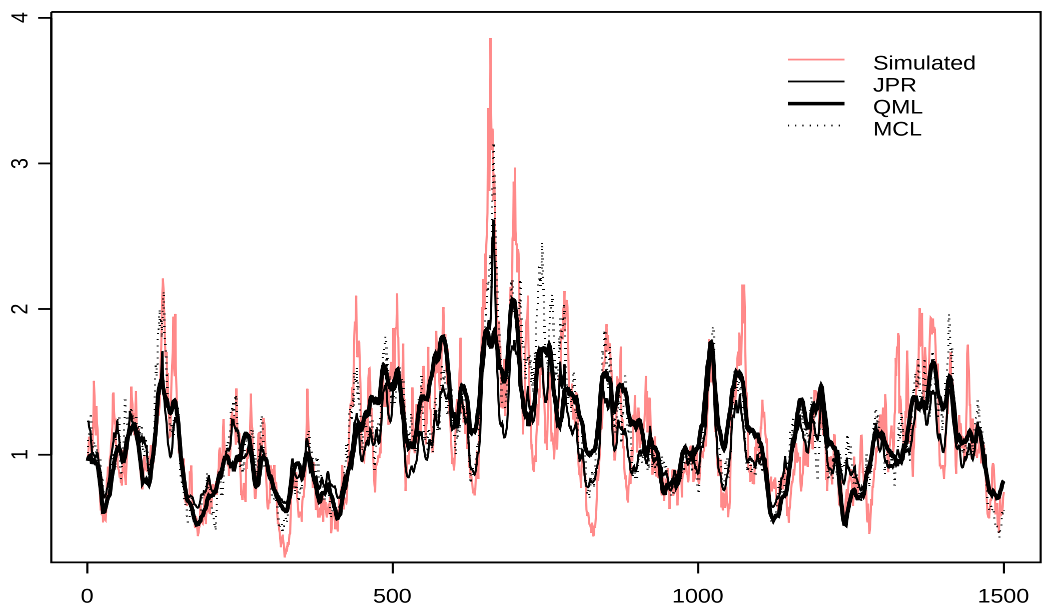

Figure 3: Estimated volatilities of a simulated series with $\phi = 0.95$ and $\sigma_\eta^2 = 0.05$ estimated by JPR, QML and MCL using estimates obtained by GMM. $T = 1,500$.

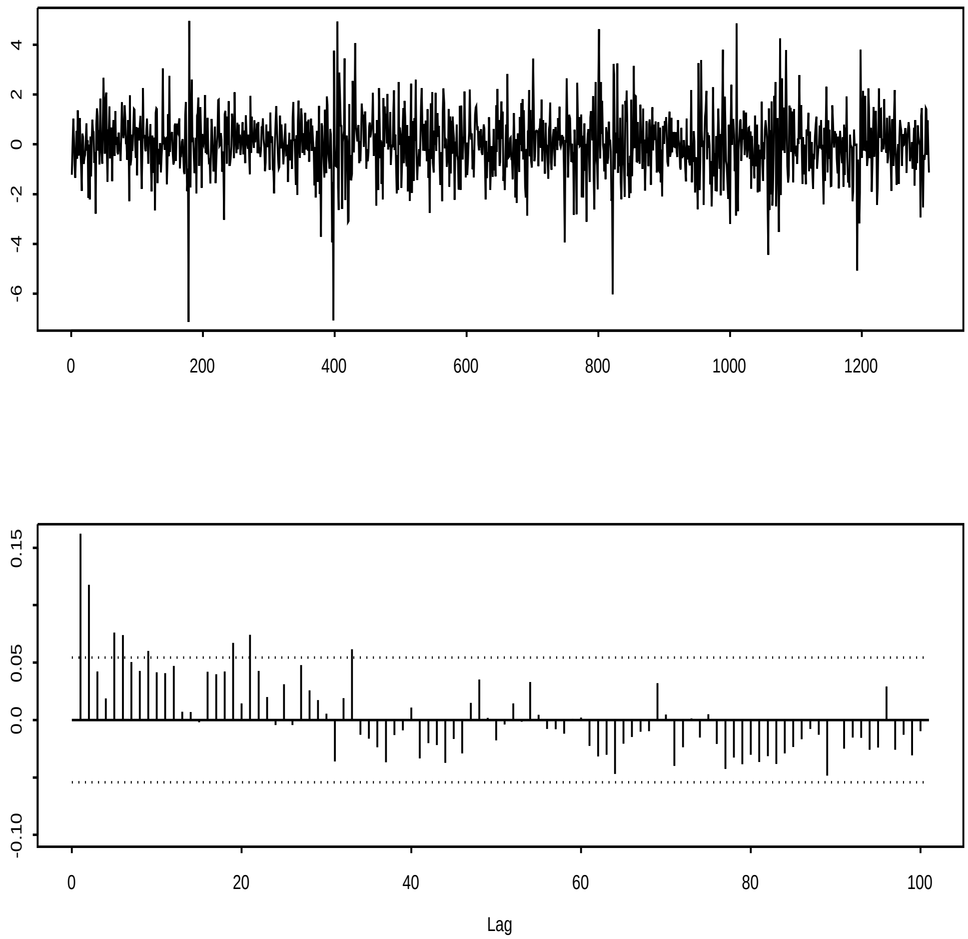

Figure 4: S&P 500 Stock Index returns observed dayly from 19 February 1997 to 15 February 2002 and correlogram of the squared returns.

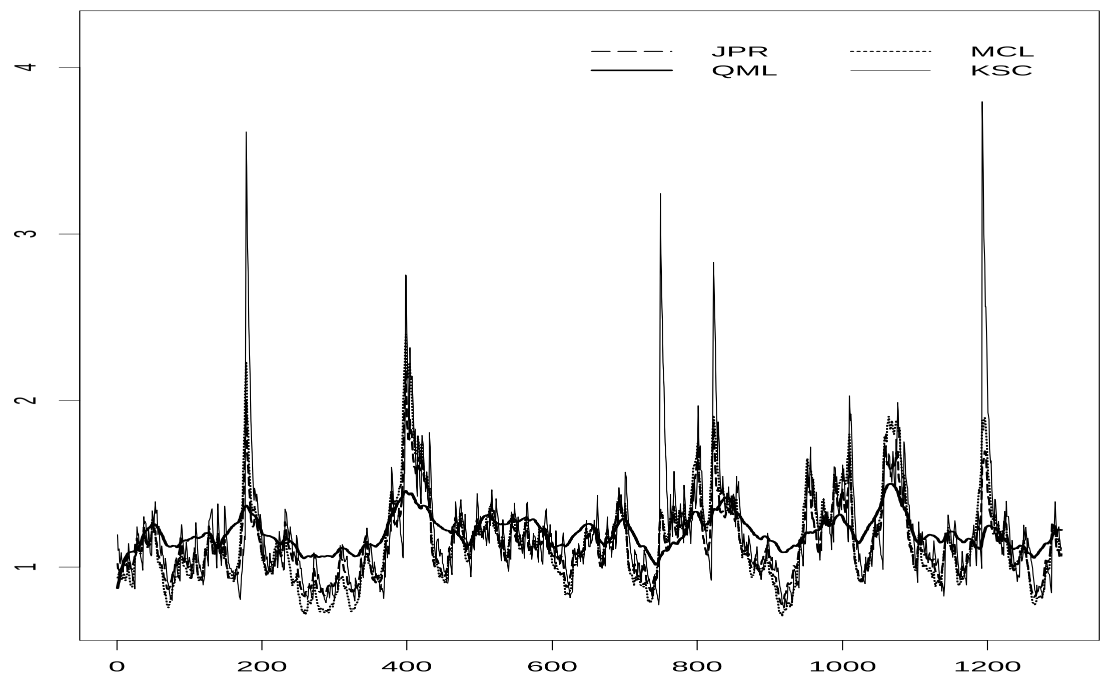

Figure 5: Estimated volatilities of a S& P 500 by JPR, QML, KSC and MCL.

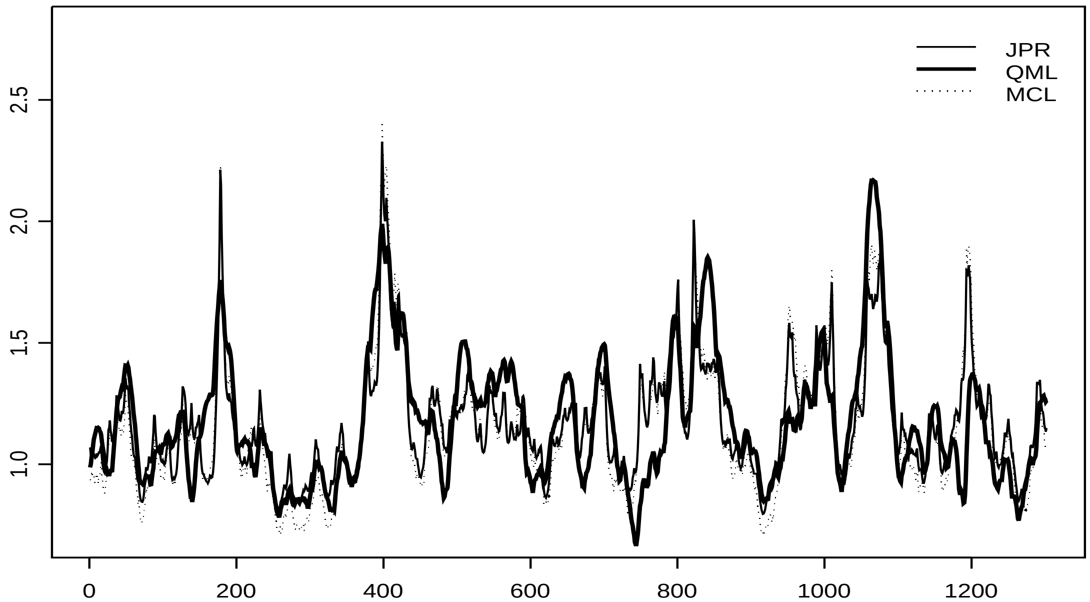

Figure 6: Estimated volatilities of S& P 500 by JPR, QML and MCL using estimated parameter values obtained by GMM.