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ANOTHER LOOK AT THE ESTIMATION OF DYNAMIC PROGRAMMING MODELS WITH CENSORED DECISION VARIABLES

Rocío Sánchez-Mangas*

Abstract

In this paper we propose a new approach to estimate the structural parameters in the context of a censored continuous decision model. Instead of handling with the original model, we consider an approximate model in which the decision variable has been discretized in a finite number of values. In this sense, an ordered choice model becomes a natural approximation to an underlying and more complicated censored continuous one. We extend the kind of Hotz-Miller estimators proposed for the estimation of binary or multinomial choice models to the context of ordered choice models. The estimation approach is based on the existence of a one-to-one mapping from conditional choice value functions to conditional choice probabilities. Exploiting the invertibility of that mapping it is possible to obtain structural parameter estimates without solving the dynamic programming problem.

Keywords:
Structural estimation, dynamic programming, censored decision models, ordered choice models.

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Another look at the estimation of dynamic programming models with censored decision variables*

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1 Introduction

The estimation of decision dynamic models is an area that has experienced a great development over the last decade. The reason for this increasing concern is that many behavioral economic models can be described as sequential choice optimization problems with uncertainty about future events. These models can be found in fields such as industrial organization, labor economics, health economics and public finance among others.

Despite the interest of these models to capture and describe economic behaviour, the extent of applications has been dissimilar, depending on the continuous or discrete nature of the decision variable. In this paper we concentrate on the estimation of dynamic programming models with censored continuous decision variables. It is well known that these models share characteristics of both continuous and discrete decision processes.

In the continuous case, the usual estimation approach is based on the Euler equations, which result from combining marginal optimality conditions at two consecutive periods. Sample counterparts of the orthogonality conditions provided by the Euler equations can be constructed and exploited to obtain estimates of the parameters of interest by the Generalized Method of Moments. Therefore, in continuous decision processes there is no need to solve the dynamic programming problem to estimate the structural parameters. In the discrete case, however, the non-differentiability of the value function with respect to the control variable prevent the use of the Euler equations. In fact, the optimality rule in this case is determined by a set of inequality conditions which involve the evaluation of the value functions conditional on the choice of different alternatives. Until very recently, the structural estimation required the solution of the dynamic programming problem at the expense of great computational burden. In an attempt to reduce it, the model specification had to be very parsimonious in terms of the state variables. This is the reason why the empirical applications of discrete choice dynamic programming models have been scarce. With regard to this, a major contribution has been the conditional independence assumption introduced by Rust (1987), which greatly reduces the dimensionality of the dynamic programming problem.

In the case of both continuous and discrete decision variables, Pakes (1994) proposed to estimate the structural parameters using some modified Euler equations which take into account the number of periods between two consecutive interior solutions. This approach has been extended to the case of dynamic programming models with censored decision variables by Aguirregabiria (1997). However, this method has some limitations in the context of censored decision models. If corner solutions are relatively important, the use of only interior solutions leads to selection bias in the estimation. Besides, there can be some parameters that just can be estimated by exploiting the discrete decision, which in some cases will be more informative than the continuous one. Given the limitations of the Euler equations, an alternative approach to tackle the estimation of censored decision models has been to exploit the discrete decision and to estimate nonparametrically the continuous one. Empirical applications of this technique can be found in Aguirregabiria (1999) or Slade (1999).

In this paper we propose a new approach to estimate the structural parameters in the context of a censored continuous decision model. Instead of handling with the original model, we propose an approximate model in which the decision variable has been discretized in a finite number of values. In this sense, an ordered choice model becomes a natural approximation to an underlying and more complicated censored continuous one. We extend to the context of ordered choice models the estimation technique proposed by Hotz and Miller (1993) for the estimation of binary or multinomial choice models. It is based on the existence of a one-to-one mapping from conditional choice value functions to conditional choice probabilities. Exploiting the invertibility of that mapping it is possible to obtain structural parameter estimates without solving the dynamic programming problem.

The invertibility of that mapping is the keypoint not only in the conditional choice probability estimator by Hotz and Miller (1993), but also in the pseudo-likelihood estimation technique developed by Aguirregabiria and Mira (2002), which builds on the former but means a great improvement in terms of efficiency.

Hence, this paper extends the application field of the estimation methods based on the invertibility of the mapping from conditional choice value functions to conditional choice probabilities to the case of ordered choice models. We apply this extension to the estimation of censored decision models, considering two different sources of censoring: non-negativity constraints and lump sum costs. In both cases, we establish the invertibility of the mapping from conditional choice value functions to conditional choice probabilities. While in the former case the inverse mapping has a closed expression, in the latter it must be obtained numerically.

The rest of the paper is organized as follows. In Section 2 we introduce the notation and review the econometric issues in the estimation of censored continuous decision models. In Section 3 we formulate a model with non-negativity constraints and establish the optimal decision rule and the invertibility of the mentioned mapping in the corresponding ordered choice model. In Section 4 we incorporate lump-sum costs as a source of censoring and establish the optimal decision rule as well as the invertibility of the mapping in the corresponding ordered choice model. Section 5 describes the algorithm for the estimation of the structural parameters. Conclusions and further lines of research are presented in Section 6.

2 Censored continuous decision models: econometric issues

Consider an agent, i.e, individual or firm, whose objective is to maximize the expected discounted value of current and future returns (utility or profits). To do so, she observes some state variables, s, and takes a decision, represented by a control variable d. We assume that the vector of state variables, s, can be decomposed into two types of variables: $s = (x, \xi)'$, where x is the subvector of state variables observable for the econometrician and the decision-maker, whereas ξ is observable for the decision-maker, but not for the econometrician. We also assume that the decision variable has a continuous range of variation and, without loss of generality, is censored at zero. Let D be the set of feasible choices. For example, if we consider a firm which decides its investment in order to maximize the expected discounted value of current and future profits and we think that investment is completely irreversible and there are not physical or financial constraints, the set of feasible choices is $D = [0, \infty)$.

We also assume that the decision-maker has uncertainty about future values of state variables. Her beliefs about these uncertain values can be represented by a Markov transition density function $f(s_{t+1}|\, s_t, d_t)$. The one-period return function can be represented by $\pi(s_t, d_t; \theta)$,where θ is the vector of structural parameters. Time is discrete and indexed by t. The time horizon of the decision problem is infinite. The agent's discount rate is represented by a parameter $\beta \in (0, 1)$.

An agent is represented by the set of primitives $\{\pi, f, \beta\}$. The decision problem of an agent at period t is to find the optimal decision rule $\delta_t(s_t, \theta)$ that satisfies:

$$\delta_t(s_t, \theta) = \arg\max_{d_t \in D} \left\{ \pi(s_t, d_t, \theta) + \beta \int V(s_{t+1}, \theta)\ f(ds_{t+1}|\, s_t, d_t) \right\}$$

where $V(\cdot)$ is defined recursively as the solution to Bellman's equation:

$$V(s_t, \theta) = \max_{d_t \in D} \left\{ \pi(s_t, d_t, \theta) + \beta \int V(s_{t+1}, \theta)\ f(ds_{t+1}|\, s_t, d_t) \right\}$$

In continuous decision processes, that is, when the decision variable is continuous, the usual estimation approach is based on the Euler equations. We can construct sample counterparts of the orthogonality conditions provided by the Euler equations and use them to estimate the parameters of interest by the Generalized Method of Moments. But these equations result from combining marginal conditions of optimality at two consecutive periods. That is, they are based on the assumption that, conditional on the information at period t, an interior solution will hold with probability one at period $t + 1$. However, for censored decision models, interior solutions do not occur at each period with probability one. When an agent makes her decision, she assigns a non-zero probability to the event "corner solution at period $t + 1$". For example, in an irreversible investment model, a firm can choose inaction, that is, zero investment (i.e. corner solution). Therefore the decision variable is censored at zero. Hence the standard Euler equations do not hold. One of the limitations of the Euler approach in the context of censored decision models is that, given that marginal optimality conditions only hold for interior solutions, if there is a significant frequency of corner solutions in the sample and we drop them, we are inducing a selection bias. Another limitation is that there can be situations under which not all the structural parameters can be estimated using only the marginal optimality conditions, since some parameters may be identified only by exploiting the discrete choice "interior solution vs. corner solution". This is the case, for example, when censoring is due to the existence of lump-sum costs. To overcome these limitations, Pakes (1994) proposed some modifications of the Euler equations in a model with both a continuous and a discrete control variable. This method has been extended to the context of censored decision models by Aguirregabiria (1997), for a model of price and inventory decisions. However, if corner solutions are relatively frequent in the sample, the discrete choice between interior solution and corner solution may contain more information about the structural parameters than the modified Euler equations, as it is stressed in Aguirregabiria (1999). There exist some contributions in the literature with censored decision variables that only exploit the discrete choice interior solution vs. corner solution, for example, Slade (1999) or Aguirregabiria (1999). In these cases, the interior solution is estimated nonparametrically.

This paper establishes a new approach to estimate the structural parameters in the context of a censored decision model. Instead of handling with the original censored model, we propose an approximate model in which the decision variable has been discretized in a finite number of values. In this sense, an ordered choice model becomes a natural approximation to an underlying and more complicated censored continuous model. We consider two sources of censoring: non-negativity constraints and lump-sum costs.

3 A censored decision model with non-negativity constraints

Consider a model in which the decision variable is censored, without loss of generality, at zero. For example, in a model of capital investment with total irreversibility, firms can decide whether not to invest (inaction) or to invest a strictly positive amount, but once the new capital has been acquired, it cannot be sold.[1] In such a model, the non-negativity constraint implies a positive probability of corner solutions. The set

[1] This feature can be relaxed: for instance, we can think of an imperfect second-hand market for physical capital in which, due to informational asymmetries, the selling price is lower than the purchasing price. This fact creates a kink in the one-period profit function at $d = 0$.

of feasible choices is $D = [0, \infty)$. The optimal decision rule in this model is as follows:

$$\delta(s, \theta) = \begin{cases} \delta^*(s, \theta) & \text{if} \quad \delta^*(s, \theta) > 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

where $\delta^*(s, \theta)$ is the optimal interior solution characterized by:

$$\pi_d [s, \delta^*(s, \theta), \theta] + \beta EV_d [s, \delta^*(s, \theta), \theta] = 0 \tag{2}$$

where $\pi_d \equiv \partial\pi/\partial d$ and $EV_d = \partial EV/\partial d$.

That is, there is a first order condition of optimality for the interior solution, given by (2), and there is a discrete choice between corner and interior solution, given by (1). This is the main feature of the decision rule in censored continuous decision processes.

Let us assume now that our decision variable d has been discretized in a finite number of values, such that the set of feasible choices is given by:

$$0 = d^0 < d^1 < ... < d^M$$

For example, in a completely irreversible investment model, if we consider that the decision variable is the investment rate, we could group the feasible decisions for a firm in some categories: zero investment, investment of 10 percent of the installed capital, investment of 20 percent, etc.

We establish the following assumptions:

ASSUMPTION A1: The one-period profit associated with each alternative $m = 0, 1, ..., M$ can be decomposed additively as

$$\pi^m(s_t, \theta) = \pi^m(x_t, \theta) - g^m(x_t)\ \varepsilon_t$$

where $g^m(\cdot)$ is a function of the decision variable and the observable state variables and ε_t is an unobservable state variable.

ASSUMPTION A2: The functions $g^m(x_t) \equiv g(x_t, d^m)$ are positive and strictly increasing in d^m, for $m \geq 1$.

ASSUMPTION A3: Conditional Independence assumption (Rust, 1987): The transition probability of state variables can be factorized as:

$$f(ds_{t+1}|\, s_t, d_t^m) \equiv f(x_{t+1}, \varepsilon_{t+1}|\, x_t, \varepsilon_t, d_t^m) = f(\varepsilon_{t+1}|\, x_{t+1})\; f(x_{t+1}|\, x_t, d_t^m) \qquad (3)$$

ASSUMPTION A4: The domain of observable state variables is finite: $x \in X = \left\{x^1, ..., x^H\right\}$.

Regarding the additive decomposition in *A1*, the unobservable state variable ε stands for heterogeneity in some component of the one period return function. For example, in an irreversible investment model where firms decide whether not to invest or to invest a positive amount among a finite set of alternatives, ε can be related to heterogeneity in variable adjustment costs that firms face when they decide to invest.[2]

The implications of *A3* are, on the one hand, that conditional on the discrete choice and the current value of the observable state variables, the future observable state variables do not depend on unobservables. On the other hand, this assumption rules out the existence of autocorrelated unobservables that would make the estimation of the structural parameters very cumbersome.

The conditional value function will be given by:

$$V^m(s_t, \theta) = \pi^m(s_t, \theta) + \beta EV^m(s_t, \theta)$$

where

$$EV^m(s_t, \theta) = \int V(s_{t+1}, \theta)\; f(ds_{t+1}|\, s_t, d_t^m)$$

[2]For example, let us consider an irreversible investment model for which the one-period profit function is:

$$\pi^m(s_t, \theta) = F(K_t) - K_t i_t^m - AC(s_t, i_t^n)$$

where i_t^m is the investment rate (decision variable), K_t is the installed capital (observable state variable) and θ_L and θ_Q are parameters related with linear and quadratic capital adjustment costs respectively. $F(K_t)$ is the revenue function, $K_t i_t^m$ the purchase cost of capital and $AC(s_t, i_t^n) = K_t i_t^m(\theta_L + \varepsilon_t + \theta_Q i_t^m)$ is the capital adjustment cost function.

The decomposition in *A1* will be given by:

$$\pi^m(x_t, \theta) = F(K_t) - K_t i_t^m(1 + \theta_L + \theta_Q i_t^m)$$

$$g^m(x_t) = K_t i_t^m$$

Let us define the expectation of the value function conditional on the state variables, where the expectation is taken with respect to unobservables:

$$V(x_t, \theta) \equiv \int V(x_t, \varepsilon_t, \theta)\ f(d\varepsilon_t |\, x_t)$$

Under *A1, A2* and *A4*, the Bellman's equation for this problem can be writtten as:

$$V(x_t, \theta) = \int \max_{d^m \in D} \left[\pi^m(x_t, \theta) - g^m(x_t)\ \varepsilon_t + \beta \sum_{x_{t+1}} V(x_{t+1}, \theta)\ f(x_{t+1} |\, x_t, d_t^m) \right]\ f(d\varepsilon_t |\, x_t) \tag{4}$$

The functional equation (4) is a contraction mapping, $V(x_t, \theta)$ being its unique fixed point. The conditional choice probability of alternative d^m given the vector of observable state variables is given by:

$$\Pr(d_t^m |\, x_t) = \int I\left\{d^m = \arg\max \left[v(x_t, d^m) - g^m(x_t)\ \varepsilon_t\right]\right\}\ f(d\varepsilon_t |\, x_t), \tag{5}$$

where the conditional choice value function in terms of observable variables is simply

$$v(x_t, d^m) = \pi^m(x_t, \theta) + \beta \sum_{x_{t+1}} V(x_{t+1}, \theta)\ f(x_{t+1} |\, x_t, d_t^m) \tag{6}$$

The underlying model, in which the decision variable has not been discretized, is a censored continuous decision model. The ordered choice model that arises as an approximation of the original one, can be viewed as the result of evaluating the underlying censored model only in a finite number of points of the feasible choice set. Since the intertemporal profit function is concave, the optimal decision rule in the ordered choice model can be defined by a set of inequalities. Such inequalities arise from the comparison of the value attained at each alternative with the value attained at previous and subsequent alternatives:

Optimal decision rule

Under assumptions *A1-A4*, the optimal decision rule in the ordered choice model is:

$$d^*(s_t, \theta) = \begin{cases} d^m > 0 & if \quad e^{m+1}(x_t) < \varepsilon_t \leq e^m(x_t) \\ 0 & if \quad \varepsilon_t > e^1(x_t) \end{cases} \tag{7}$$

where $e^{M+1}(x_t) = -\infty$, and for $m \leq M$,

$$e^m(x_t) = \frac{\Delta v^m(x_t)}{\Delta g^m(x_t)}$$

with $\Delta v^m(x_t) = v(x_t, d^m) - v(x_t, d^{m-1})$ and $\Delta g^m(x_t) \equiv g^m(x_t) - g^{m-1}(x_t)$. The interpretation of this optimal decision rule is simple. Consider first the decision among alternatives $d^m > 0$, that is, the decision among $\{d^m; m = 1, ..., M\}$. Suppose that the optimal decision is $d^*(s_t) = d^m > 0$, for certain $m > 0$. In this case, the concavity of the intertemporal profit function of the underlying model implies that whereas the "slope" of this function to the left of d^m should be positive (i.e., $\Delta v^m(x_t) - \Delta g^m(x_t)\varepsilon_t > 0$), the "slope" to the right should be negative (i.e., $\Delta v^{m+1}(x_t) - \Delta g^{m+1}(x_t)\varepsilon_t < 0$). It is clear that these two inequalities imply that the shock ε_t should lie in the interval $[e^{m+1}(x_t), e^m(x_t)]$. Notice that concavity of the value function in the underlying model implies that the threshold $e^{m+1}(x_t)$ is always smaller than $e^m(x_t)$.

Consider now the decision between $d^0 = 0$ and $d^m > 0$, for $m \geq 1$. In the case of an irreversible investment model, consider the decision of inaction (zero investment) vs. a strictly positive investment. If the "slope" of the intertemporal profit function of the underlying model at $d^0 = 0$ is negative, the maximum intertemporal profit would be reached at a negative value of d. However, since we are considering that there exists a non-negativity constraint, the optimal decision would be $d^0 = 0$. In an irreversible investment model, if the maximum intertemporal profit is reached at a negative investment value, the optimal decision would be inaction, due to irreversibility. That condition for the "slope" of the intertemporal profit function is given by $\Delta v^1(x_t) - \Delta g^1(x_t)\varepsilon_t < 0$, which is equivalent to $\varepsilon_t > e^1(x_t)$.

Let $e(x_t)$ be the M-dimensional vector of conditional choice value functions, $(e^1(x_t), ..., e^M(x_t))'$ and $P(x_t)$ the M-dimensional vector of conditional choice probabilities $\left(P^1(x_t), ..., P^M(x_t)\right)'$.

Notice that this vector does not include the probability of choosing the alternative $d^0 = 0$, which will be trivially computed from the elements in $P(x_t)$. As we can see in the optimal decision rule, there is a mapping from conditional value functions to conditional choice probabilities. For any $m \geq 1$, the conditional choice probability

$P^m(x_{nt})$ is given by

$$P^m(x_{nt}) = \Pr\left\{e^{m+1}(x_{nt}) < \varepsilon_{nt} \leq e^m(x_{nt})\right\} =$$

$$= F_\varepsilon\left(e^m(x_{nt})\right) - F_\varepsilon\left(e^{m+1}(x_{nt})\right) \tag{8}$$

where $e^{M+1}(x_{nt}) = -\infty$, and $F_\varepsilon(\cdot)$ stands for the cumulative distribution function of ε.

Let $Q : R^M \longrightarrow [0,1]^M$ be that mapping:

$$P(x_t) = Q(e(x_t)) \tag{9}$$

More specifically, the choice probability $P^m(x_t)$ depends on the vector $e(x_t)$ only through its m-th and $(m+1)$-th components. That is,

$$P^m(x_{nt}) = Q^m(e^m(x_{nt}), e^{m+1}(x_{nt}))$$

Lemma 1 *Given $g^m(\cdot)$, under assumptions A1-A4, the mapping Q is invertible.*

Proof: See Appendix.

The invertibility of the mapping can be exploited to obtain an alternative representation of the value funcions in terms of choice probabilities. The Bellman's equation (4) can be written as:

$$V(x_t,\theta) = \sum_{m=0}^{M} P^m(x_t)\left\{\pi^m(x_t,\theta) - g^m(x_t)\ E\left[\varepsilon_t|\, x_t, d_t^* = d^m\right] + \beta \sum_{x_{t+1}} V(x_{t+1},\theta)\ f(x_{t+1}|\, x_t, d_t^m)\right. \tag{10}$$

where $E\left[\varepsilon_t|\, x_t, d_t^* = d^m\right]$ is the expectation of the unobservable ε conditional on the optimal choice of the alternative d^m. This expectation is a function of the components m and $m+1$ of the vector of unknown functions $e(x_t)$:

$$E\left[\varepsilon_t|\, x_t, d_t^* = d^m\right] = E\left[\varepsilon_t|\, x_t, e^{m+1}(x_t) < \varepsilon_t \leq e^m(x_t)\right] = \tag{11}$$

$$= \frac{1}{P^m(x_t)} \int_{e^{m+1}(x_t)}^{e^m(x_t)} \varepsilon\ f(\varepsilon)\ d\varepsilon$$

However, by exploiting the invertibility of the mapping Q we can write this expectation in terms of conditional choice probabilities, that we denote as $h^m(P)$. In the context of our ordered choice model, that expectation becomes (omitting the argument x_t for the sake of notational simplicity):

$$h^m(P) = \frac{1}{P^m} \int_{F^{-1}(P^{m+1}+...+P^M)}^{F^{-1}(P^m+P^{m+1}+...+P^M)} \varepsilon \ f(\varepsilon) \ d\varepsilon \tag{12}$$

Including these expressions into (10), and solving the fixed-point equations, we can write the value function in terms of the conditional choice probabilities, in compact matrix notation, as follows:

$$V(x) = \left(I_H - \beta \sum_{m=0}^{M} P^m(x) * F^m(x) \right)^{-1} \left\{ \sum_{m=0}^{M} P^m(x) \ * [\pi^m(x) - g^m(x_t) \ h^m(P)] \right\} \tag{13}$$

where $*$ is the element-by-element product, H is the dimension of the discretized space of observable state variables and $F^m(x)$ is the $H \times H$ matrix of conditional transition probabilities.

4 A censored model with non-negativity constraints and lump-sum costs

Consider now a continuous decision model in which there are two sources of censoring: non-negativity constraints and lump-sum costs. For example, a model of investment such that installed capital cannot be sold (total irreversibility) and there are some fixed adjustment costs that firms face if they decide to invest, irrespective of the amount of investment. The set of feasible choices is $D = [0, \infty)$, as in the earlier section. However, in this case, the optimal decision can be $d = 0$ not only due to non-negativity constraints, but also due to lump-sum costs. The presence of lump-sum costs introduces a discontinuity in the one-period profit function in $d = 0$, what complicates the estimation of the structural parameters.

As in the earlier section, we begin by establishing the optimal decision in the censored continuous decision problem. Let $\pi(s_t, \theta)$ be the one-period profit function and $\tilde{\pi}(s_t, \theta)$ the one-period profit function gross of fixed costs. That is,

$$\pi(s_t, \theta) = \tilde{\pi}(s_t, \theta) - FC(s_t, \theta) \ I(d_t > 0)$$

where $FC(\cdot)$ is the fixed-cost function. The discontinuity of the profit function at $d = 0$ induces nonconcavity in the value function. The characterization of the optimal decision rule in this case requires to exploit properties of K-concave functions.[3] The optimal decision rule for this censored decision problem is:

$$\delta(s,\theta) = \begin{cases} \delta^*(s,\theta) & \text{if} \quad \delta^*(s,\theta) > 0 \text{ and } \gamma(s,\theta) > 0 \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

where $\delta^*(s,\theta)$ is the optimal interior solution characterized by

$$\tilde{\pi}_d(s,\delta^*(s,\theta),\theta) + \beta EV_d(s,\delta^*(s,\theta),\theta) = 0, \tag{15}$$

with $\tilde{\pi}_d \equiv \partial\tilde{\pi}/\partial d$ and $EV_d = \partial EV/\partial d$ and the function $\gamma(s,\theta)$ is given by

$$\tilde{\pi}(s,\delta^*(s,\theta),\theta) - FC(s,\theta) - \tilde{\pi}(s,0,\theta) + \beta\left[EV(s,\delta^*(s,\theta),\theta) - EV(s,0,\theta)\right]. \tag{16}$$

That is, there is a first order condition of optimality for the interior solution, given by (15), and there are two conditions for the discrete choice between corner and interior solution. The first one concerns $\delta^*(s,\theta)$, which is related with the non-negativity constraint, so that the interior solution will be optimal only if it is positive. If condition (15) holds for a negative value, we will choose $\delta(s,\theta) = 0$. The second condition is given in terms of $\gamma(s,\theta)$, which is related with the presence of fixed costs. If $\gamma(s,\theta) > 0$, it means that the fixed costs are not high enough to lead the firm to decide not to invest.

It is possible to formulate an ordered choice model as an approximation to the censored continuous model in order to get a representation of conditional value functions in terms of conditional choice probabilities. Suppose that the decision variable in the original problem has been discretized in a finite number of values, such that the set of feasible choices is given by:

$$0 = d^0 < d^1 < ... < d^M$$

Consider assumptions *A2, A3* and *A4* from the earlier section and these additional assumptions:

[3] See Aguirregabiria (1999) for details on the characterization of the optimal decision rule in a censored decision problem with lump sum costs.

ASSUMPTION B1: The one-period profit associated with each alternative $m = 0, 1, ..., M$ can be decomposed as :

$$\pi^m(s_t, \theta) = \tilde{\pi}^m(x_t, \theta) - g^m(x_t)\ \varepsilon_t - FC(s_t, \theta)\ I(d_t^m > 0)$$

and the fixed cost function is decomposed in observable and unobservable components as follows:

$$FC(s_t, \theta) = c_0(x_t, \theta) + c_1(x_t)\ \omega_t$$

where $g^m(\cdot)$ is a function of the decision variable and the observable state variables, $c_0(\cdot)$ and $c_1(\cdot)$ are functions of the observable state variables and ε and ω are unobservable state variables.

ASSUMPTION B2: The function $c_1(x_t)$ is strictly positive.

ASSUMPTION B3: Independence between unobservable state variables:

$$f(\varepsilon_t, \omega_t | \, x_t) = f(\varepsilon_t | \, x_t)\ f(\omega_t | \, x_t)$$

Regarding assumption *B1*, the unobservable state variables ε and ω stand for heterogeneity in some components of the one-period return function. In the case of an irreversible investment model where firms decide whether not to invest or to undertake an investment project among a finite set of alternatives, ε can represent heterogeneity in variable adjustment costs that firms face when they decide to invest and ω heterogeneity in fixed costs.[4]

[4] See the investment model described in note 2. Let us assume that when the firm decides to invest, it faces fixed adjustment costs, which are, for example, proportional to the installed capital. Let us consider the following one-period profit function:

$$\pi^m(s_t, \theta) = F(K_t) - K_t i_t^m - AC(s_t, i_t^n)$$

where $AC(s_t, i_t^n) = K_t i_t^m(\theta_L + \varepsilon_t + \theta_Q i_t^m) + (\theta_F + \omega_t) K_t 1(i_t^m > 0)$. In this case, the decomposition in *B1* is as follows:

$$\tilde{\pi}^m(x_t, \theta) = F(K_t) - K_t i_t^m(1 + \theta_L + \theta_Q i_t^m)$$

$$\pi^m(x_t, \theta) = \tilde{\pi}^m(x_t, \theta) - c_0(x_t, \theta)\ 1(i_t^m > 0)$$

$$g^m(x_t) = K_t i_t^m, \qquad c_0(x_t, \theta) = \theta_F K_t, \qquad c_1(x_t) = K_t$$

Taking into account assumption *B3*, the conditional independence assumption *A3* can be written as:

$$f(ds_{t+1}|\, s_t, d_t^m) = f(\varepsilon_{t+1}|\, x_{t+1})\ f(\omega_{t+1}|\, x_{t+1})\ f(x_{t+1}|\, x_t, d_t^m)$$

Under assumptions *A3, A4, B1* and *B3* the Bellman's equation for this problem can be written as:

$$V(x_t, \theta) = \int \max_{d^m \in D} [\pi^m(x_t, \theta) - g^m(x_t)\ \varepsilon_t - c_1(x_t)\ \omega_t\ I(d_t^m > 0) +$$

$$\left. + \beta \sum_{x_{t+1}} V(x_{t+1}, \theta)\ f(x_{t+1}|\, x_t, d_t^m) \right]\ f(d\varepsilon_t|\, x_t)\ f(d\omega_t|\, x_t) \tag{17}$$

The functional equation (17) is a contraction mapping, $V(x_t, \theta)$ being its unique fixed point. The conditional choice probability of alternative d^m given the vector of observable state variables is given by:

$$\Pr(d_t^m|\, x_t) = \int I\left\{d^m = \arg\max\left[v(x_t, d^m) - g^m(x_t)\ \varepsilon_t - c_1(x_t)\ \omega_t I(d_t^m > 0)\right]\right\}\ f(d\varepsilon_t|\, x_t) f(d\omega_t|\, x \tag{18}$$

where the conditional choice value function in terms of observable variables is simply

$$v(x_t, d^m) = \pi^m(x_t, \theta) + \beta \sum_{x_{t+1}} V(x_{t+1}, \theta)\ f(x_{t+1}|\, x_t, d_t^m) \tag{19}$$

and $\pi^m(x_t, \theta) = \tilde{\pi}^m(x_t, \theta) - c_0(x_t, \theta)\ I(d_t^m > 0)$.

Similarly to the case without fixed costs, the ordered choice model may be considered as an approximation of the underlying censored continuous decision model. Let us establish the optimal decision rule for the ordered choice model:

Optimal decision rule

Under assumptions *A2-A4* and *B1-B3*, the optimal decision rule for the ordered choice model can be written as:

$$d^*(s_t, \theta) = \begin{cases} d^m > 0 & \text{if} \quad e^{m+1}(x_t) < \varepsilon_t < e^m(x_t) \text{ and } \omega_t < w^m(x_t) - b^m(x_t)\ \varepsilon_t \\ d^0 = 0 & \text{if} \quad \varepsilon_t > e^1(x_t) \text{ or } \omega_t > w^m(x_t) - b^m(x_t)\ \varepsilon_t \text{ for any } m > 0 \end{cases} \tag{20}$$

where $e^{M+1}(x_t) = -\infty$, and for $m \leq M$,

$$e^m(x_t) = \frac{\Delta \tilde{v}^m(x_t)}{\Delta g^m(x_t)}$$

$$b^m(x_t) = \frac{g^m(x_t)}{c_1(x_t)}$$

$$w^m(x_t) = \frac{1}{c_1(x_t)} \left[\tilde{v}^m(x_t) - \tilde{v}^0(x_t) - c_0(x_t, \theta) \right]$$

$$\Delta \tilde{v}^m(x_t) = \tilde{v}^m(x_t) - \tilde{v}^{m-1}(x_t)$$

$$\Delta g^m(x_t) = g^m(x_t) - g^{m-1}(x_t)$$

$$\tilde{v}^m(x_t) = \tilde{\pi}^m(x_t, \theta) + \beta \sum_{x_{t+1}} V(x_{t+1}, \theta)\ f(x_{t+1}| \, x_t, d_t^m)$$

What is the interpretation of this optimal decision rule? Consider first the decision on the intensive margin and suppose that $d^*(s_t) = d^m > 0$. In this case, the concavity of the intertemporal profit function $\tilde{v}$ implies that the "slope" of this function to the left of d^m should be positive (i.e., $\Delta \tilde{v}^m(x_t) - \Delta g^m(x_t)\varepsilon_t > 0$), and the "slope" to the right should be negative (i.e., $\Delta \tilde{v}^{m+1}(x_t) - \Delta g^{m+1}(x_t)\varepsilon_t < 0$). It is clear that these two inequalities imply that the shock ε_t should lie in the interval $(e^{m+1}(x_t), e^m(x_t)]$. Notice that concavity of $\tilde{v}$ implies that the threshold $e^{m+1}(x_t)$ is always smaller than $e^m(x_t)$.

Consider now the decision on the extensive margin. There are two sources of inaction in this model: non-negativity constraints and lump-sum costs. Even without lump-sum costs, the maximum intertemporal profit may take place at $d_t = 0$. This is the case when the "slope" of the profit function at $d_t = 0$ is negative, i.e., $\Delta \tilde{v}^1(x_t) - \Delta g^1(x_t)\varepsilon_t < 0$, which is equivalent to $\varepsilon_t > e^1(x_t)$. Once we account for lump sum costs, optimal decision can be also $d_t = 0$ because the maximum intertemporal profit with $d_t^m > 0$ is smaller than this profit under $d_t = 0$. This is represented by the condition: for any $m > 0$, $\omega_t > w^m(x_t) - b^m(x_t)\varepsilon_t$.

Hence, there are two sources of censoring, non-negativity constraints and lump-sum costs, which are indistinguishable for the econometrician. If we think of an irreversible investment model with fixed adjustment costs, the interpretation of the optimal decision rule is as follows: when the intertemporal profit, gross of fixed adjustment costs, is maximized for a negative value of investment, the optimal decision is inaction due to irreversibility. When it is maximized for a positive level of investment, but the value obtained with this level is lower than the value obtained with zero investment, the optimal decision is inaction due to the presence of fixed adjustment costs.

As in the earlier section, let $e(x_t)$ be the M-dimensional vector of conditional choice value functions, $(e^1(x_t), ..., \ e^M(x_t))'$ and $P(x_t)$ the M-dimensional vector of conditional choice probabilities $\left(P^1(x_t), ..., P^M(x_t)\right)'$.

The mapping from conditional value functions to conditional choice probabilities is given by the following expressions, for $m > 0$:

$$P^m(x_t) = \Pr\left\{e^{m+1}(x_t) < \varepsilon_t \leq e^m(x_t) \ , \omega_t < w^m(x_t) - b^m(x_t)\varepsilon_t\right\} =$$

$$= \int_{e^{m+1}(x_t)}^{e^m(x_t)} F_\omega\left(w^m(x_t) - b^m(x_t)\varepsilon_t\right) \ f_\varepsilon(\varepsilon) \ d\varepsilon \tag{21}$$

where $e^{M+1}(x_{nt}) = -\infty$, and $F_\omega(\cdot)$ and $F_\varepsilon(\cdot)$ stand for the cumulative distribution function of ω and ε respectively.

It is clear that the threshold functions $w^m(x_t)$ can be expressed in terms of the vector $\Delta\tilde{v}(x_t) = \left(\Delta\tilde{v}^1(x_t), ..., \Delta\tilde{v}^M(x_t)\right)'$ and thus in terms of the components of the vector $e(x_t)$, in the following way:

$$w^m(x_t) = \frac{1}{c_1(x_t)}\left[\sum_{k=1}^{m} \Delta\tilde{v}^k(x_t) - c_0(x_t,\theta)\right] = \frac{1}{c_1(x_t)}\left[\sum_{k=1}^{m} \Delta g^k(x_t)e^k(x_t) - c_0(x_t,\theta)\right] \tag{22}$$

Let $Q : R^M \longrightarrow [0,1]^M$ be the mapping from conditional value functions to conditional choice probabilities:

$$P(x_t) = Q(e(x_t))$$

As it can be seen in (22), the choice probability $P^m(x_t)$ depends on the vector $e(x_t)$ through the first $m+1$ components. That is, for $m = 1, ..., M$:

$$P^m(x_t) = Q^m(e^1(x_t), ..., e^m(x_t), e^{m+1}(x_t)) =$$

$$= \int_{e^{m+1}(x_t)}^{e^m(x_t)} F_\omega \left[\frac{1}{c_1(x_t)} \left(\sum_{k=1}^{m} \Delta g^k(x_t) e^k(x_t) - c_0(x_t, \theta) \right) - b^m(x_t)\varepsilon_t \right] f_\varepsilon(\varepsilon)\, d\varepsilon \qquad (23)$$

Lemma 2 *Given $c_0(\cdot)$, $c_1(\cdot)$ and $g^m(\cdot)$, under assumptions A2-A4 and B1-B3, the mapping Q is invertible.*

Proof: See Appendix

The invertibility of the mapping can be exploited to obtain an alternative representation of the value funcions in terms of choice probabilities. The Bellman's equation (17) can be written as:

$$V(x_t) = \sum_{m=1}^{M} P^m(x_t) \; \{ \pi^m(x_t, \theta) - g^m(x_t) E[\varepsilon_t | x_t, d_t^* = d^m] - c_1(x_t) E[\omega_t | x_t, d_t^* = d^m] + $$

$$+ \beta \sum_{x_{t+1}} V(x_{t+1}, \theta)\; p(x_{t+1} | x_t, d_t^m) \Big\} \qquad (24)$$

where the expectation of the unobservables conditional on the optimal choice of alternative d^m, that is, $E[\varepsilon_t | x_t, d_t^* = d^m]$ and $E[\omega_t | x_t, d_t^* = d^m]$, are functions of the elements of the vector of unknown functions $e(x_t)$. For $m > 0$:

$$E[\varepsilon_t | x_t, d_t^* = d^m] = \frac{1}{P^m(x_t)} \int_{e^{m+1}(x_t)}^{e^m(x_t)} \varepsilon \; F_\omega[D(x_t)] \;\; f(\varepsilon)\, d\varepsilon$$

$$E[\omega_t | x_t, d_t^* = d^m] = \frac{1}{P^m(x_t)} \int_{e^{m+1}(x_t)}^{e^m(x_t)} \left[\int_{-\infty}^{D(x_t)} \omega \; f_\omega(\omega)\, d\omega \right] f(\varepsilon)\, d\varepsilon$$

where

$$D(x_t) = \frac{1}{c_1(x_t)} \left(\sum_{k=1}^{m} \Delta g^k(x_t) e^k(x_t) - c_0(x_t, \theta) \right) - b^m(x_t)\varepsilon_t$$

Let us denote these expectations as $h_\varepsilon^m(P)$ and $h_\omega^m(P)$ respectively. Including these expressions into (24) and solving the fixed point equations in compact matrix notation, we can write the value function in terms of the conditional choice probabilities as:

$$V(x) = \left(I_H - \beta \sum_{m=1}^{M} P^m(x) * F^m(x) \right)^{-1} \left\{ \sum_{m=1}^{M} P^m(x) * [\pi^m(x) - g^m(x_t) h_\varepsilon^m(P) - c_1(x_t) h_\omega^m(P)] \right\}$$

where $*$ is the product element-by-element, H is the dimension of the discretized space of observable state variables and $F^m(x)$ is the $H \times H$ matrix of the conditional transitional probabilities.

From the computational point of view the main implication induced by the presence of fixed costs is that there are not closed form expressions of $e^m(x_t)$ in terms of the elements of $P(x_t)$. These expressions must be obtained numerically solving the highly nonlinear system of equations given by (23).

5 Estimation of the structural parameters

We follow Aguirregabiria and Mira (2002), who show that for any value of the structural parameters θ, the vector of conditional choice probabilities P_θ associated with the solution of the dynamic programming problem can be obtained as the unique fixed point of a mapping in probability space, $P_\theta = \Psi_\theta(P_\theta)$, the Policy Iteration mapping. They propose the Nested Pseudo-Likelihood algorithm (NPL): an inner algorithm maximizes in θ a pseudo-likelihood function based on choice probabilities,$\Psi_\theta(P)$, where P is an estimate of these probabilities; an outer algorithm computes $\Psi_\theta(P)$ at the current parameter estimates to update the estimate of P.

The keypoint ot the NPL algorithm is the Policy Iteration operator $\Psi(\cdot)$ in the space of conditional choice probabilities:

$$P = \Psi(P) = \Lambda(\varphi(P))$$

where $\varphi(\cdot)$ is a policy valuation operator which maps a vector of conditional choice probabilities into a vector in value function space using Hotz and Miller's representation, that is, the invertibility of the mapping from conditional choice probabilities to conditional value functions, and $\Lambda(\cdot)$ is a policy improvement operator which maps a vector in value function space into a vector of conditional choice probabilities.

Let us describe Aguirregabiria and Mira's NPL algorithm: first, we must obtain a consistent estimator of conditional transition probabilities, $\hat{F}^m$, for $m = 0, 1, ..., M$. Start with an initial guess of the parameter vector, $\hat{\theta}^{(0)}$, and an initial guess for the conditional choice probabilities, $\hat{P}^{(0)}$ (for example, a consistent nonparametric estimator of these probabilities). For iteration $K \geq 1$, do the following steps:

Step 1: Obtain a representation of conditional value functions in terms of conditional choice probabilities:

$$\hat{e}^{(K-1)} = Q^{-1}\left(\hat{P}^{(K-1)}\right)$$

Step 2: Obtain a new pseudo-likelihood estimator $\hat{\theta}^{(K)}$ as:

$$\hat{\theta}^{(K)} = \arg\max_{\theta\in\Theta} \sum_{n=1}^{N}\sum_{t=1}^{T}\sum_{m=0}^{M} 1\left(d_{nt} = d^m \middle| x_{nt}\right) \ln \Psi^m\left(\hat{P}^{(K-1)}\middle| x_{nt}\right)$$

where

$$\Psi^m\left(\hat{P}^{(K-1)}\middle| x_{nt}\right) = \Pr\left(d_{nt} = d^m \middle| x_{nt}\right) \equiv \Psi^m\left(\hat{\theta}^{(K-1)}, \hat{P}^{(K-1)}, \hat{F}^m\middle| x_{nt}\right)$$

Step 3: Update $\hat{P}$ using the estimator obtained in step 2:

$$\hat{P}^{(K)} = \Psi^m\left(\hat{\theta}^{(K)}, \hat{P}^{(K-1)}, \hat{F}^m\middle| x_{nt}\right)$$

Go to step 1 and iterate in K until convergence in $\hat{P}$ and $\hat{\theta}$ is reached.[5]

As it is stressed in Aguirregabiria and Mira (2002), when the NPL algorithm is initialized with consistent nonparametric estimates of conditional choice probabilities succesive iterations return a sequence of estimators of the structural parameters, which they call K-stage Policy Iteration estimators, which includes as extreme cases a Hotz-Miller CCP estimator (for $K = 1$) and Rust's Nested Fixed Point estimator (when $K \to \infty$). The K-stage Policy Iteration estimators are asymptotically equivalent to the maximum likelihood estimator.

6 Concluding remarks

In this paper we propose a new approach to estimate the structural parameters in dynamic programming models with censored decision variables. This approach overcome some of the drawbacks of the estimation tecniques proposed in the literature in

[5]See more details on the algorithm and the necessary assumptions in Aguirregabiria and Mira (2002).

this context. Instead of handling with the original model, we consider an approximate model in which the decision variable is discretized in a finite number of values. This discretization yields an ordered choice model, which seems a natural way to approximate a censored continuous decision model. We have considered two different sources of censoring, non-negativity constraints and lump-sum costs, which can appear in many economic problems. For each source of censoring considered, we have obtained the optimal decision rule of the corresponding ordered choice model and a representation of conditional choice value functions in terms of conditional choice probabilities. This approach provides the necessary tools to estimate the structural parameters by pseudo-maximum likelihood following Aguirregabiria and Mira's NPL algorithm. In this sense, our approach extends to the context of ordered choice models the field of application of the kind of Hotz and Miller estimators proposed in the literature for the estimation of binary or multinomial choice models with independent alternatives.

We have ilustrated our estimation approach with an investment model with irreversibility and fixed adjustment costs. However, other sources of censoring and many other economic problems can be considered. For example, models of labor demand where firms decide hirings and firings of workers. There is a discrete decision, about keep the employment constant vs. adjust employment, and a continuous decision on the amount of workers that the firm will hire or fire. If we assume that hiring costs and firing costs are different, this creates a kink in the one-period profit function at zero, which implies a positive probability of corner solutions. Therefore, the decision variable, the employment level, will be censored at zero due to partial irreversibility. Another examples in which our estimation approach can be applied are, among others, decision models of inventories or prices, in which the possible existence of lump sum costs associated with purchases or price changes respectively induce censoring in the decision variable.

Appendix

Proof of Lemma 1:

The proof of this lemma is straigtforward. The mapping in (9) is a nonlinear system of equations that can be solved very easily. Let us consider first the probability of choosing alternative $d^0 = 0$:

$$P^0(x_t) = 1 - F_\varepsilon(e^1(x_t))$$

From this equation we can obtain:

$$e^1(x_t) = F_\varepsilon^{-1}\left[1 - P^0(x_t)\right] = F_\varepsilon^{-1}\left[P^1(x_{nt}) + ... + P^M(x_t)\right]$$

Once $e^1(x_t)$ has been obtained, the mapping (9) is a triangular system that can be solved in a recursive way, equation by equation, resulting, for $m \geq 1$:

$$e^m(x_t) = F_\varepsilon^{-1}\left[P^m(x_{nt}) + P^{m+1}(x_{nt})... + P^M(x_t)\right]$$

That is, there exists the inverse mapping Q^{-1} from the conditional choice probabilities to the conditional choice value functions. More specifically, $e^m(x_t)$ depends on the vector of conditional choice probabilities only through the last $M - m + 1$ elements.

Proof of Lemma 2:

The conditional choice probabilities $P(x_t)$ are given in terms of the threshold functions $e(x_t)$, $P(x_t) = Q(e(x_t))$, where Q is an $M-$dimensional vector function, whose m-th element is given by (23).

Then, the conditional choice probability $P^m(x_t)$ depends on the vector $e(x_t)$ only through the first $m + 1$ components. That is, for $m = 1, ..., M$:

$$P^m(x_t) = Q^m(e(x_t)) = Q^m(e^1(x_t), ..., e^{m+1}(x_t))$$

where $e^{M+1}(x_t) = -\infty$.

Let G be the M-dimensional square matrix of partial derivatives of the mapping $Q(e(x_t))$, that is, $G_{hl} = \left(\frac{\partial Q^h(e(x_t))}{\partial e^l(x_t)}\right)$, $h, l = 1, ..., M$. In order to prove the invertibility of the mapping $Q(e(x_t))$, we need to prove that G is an invertible matrix.

The structure of the matrix G is as follows:

(1) Elements above the upper diagonal: G_{hl}, for $h = 1, ..., M-2$, $l = h+2, ..., M$.

It is clear that these elements are zero, since for $m = 1, ..., M$ the dependence of the probability $P^m(x_t)$ on the vector $e(x_t)$ is only through the first $m+1$ components of this vector.

(2) Elements in the upper diagonal: $G_{m,m+1}$,for $m = 1, ..., M-1$ is given by:

$$G_{m,m+1} = \frac{\partial Q^m\left(e(x_t)\right)}{\partial e^{m+1}(x_t)} = -F_\omega\left(w^m(x_t) - b^m(x_t)e^{m+1}(x_t)\right)\ f_\varepsilon\left(e^{m+1}(x_t)\right)$$

Then, $G_{m,m+1} < 0$ for $m = 1, ..., M-1$.

(3) Elements in the main diagonal: G_{mm}, $m = 1, ..., M$.

$$G_{mm} = \frac{\partial Q^m\left(e(x_t)\right)}{\partial e^m(x_t)} = F_\omega\left(w^m(x_t) - b^m(x_t)e^m(x_t)\right)\ f_\varepsilon\left(e^m(x_t)\right) +$$

$$+\frac{\Delta g^m(x_t)}{c_1(x_t)}\int_{e^{m+1}(x_t)}^{e^m(x_t)} f_\omega\left(w^m(x_t) - b^m(x_t)e^m(x_t)\right)\ f_\varepsilon(\varepsilon)\ d\varepsilon$$

Since $c_1(x_t)$ is strictly positive and $g^m(x_t)$ are positive and strictly increasing in d^m, it is clear that $G_{mm} > 0$ for all $m = 1, ..., M$.

(4) Elements below the main diagonal: G_{mh}, for $m = 2, ..., M$ and $h \leq m-1$:

$$G_{mh} = \frac{\partial Q^m\left(e(x_t)\right)}{\partial e^h(x_t)} = \frac{\Delta g^h(x_t)}{c_1(x_t)}\int_{e^{m+1}(x_t)}^{e^m(x_t)} f_\omega\left(w^m(x_t) - b^m(x_t)e^m(x_t)\right)\ f_\varepsilon(\varepsilon)\ d\varepsilon$$

Since $c_1(x_t)$ is strictly positive and $g^m(x_t)$ are positive and strictly increasing in d^m, it is clear that $G_{mh} > 0$.

Then, G is a square matrix whose elements are strictly negative in the upper diagonal, zero above and strictly positive below. It is straightforward to prove that a matrix with this structure is a positive definite matrix, and then, G^{-1} does exist.

Then, there exists a differentiable inverse mapping Q^{-1} such that $e(x_t) = Q^{-1}(P(x_t))$, that is, we can obtain the difference in conditional value functions, and thus, the elements of $e(x_t)$, as a computable function of the conditional choice probabilities in $P(x_t)$.

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