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Active redundancy allocation in systems

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Abstract

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Singh and Misra (1994) considered the criterion where C_1 is preferred to C_2 for redundancy allocation if $P(U_1 > U_2) \geq P(U_2 > U_1)$. In this paper we use the same criterion of Singh and Misra (1994) and we investigate the allocation of one active redundancy when it differs depending on the component with which it is to be allocated. We find sufficient conditions for the optimization which depend on the components and redundancies probability distributions. We also compare the allocation of two active redundancies (say S_1 and S_2) in two different ways, that is, S_1 with C_1 and S_2 with C_2 and viceversa. For this case the hazard rate order plays an important role. We obtain results for the allocation of more than two active redundancies to a *k*-out-of-*n* systems.

Keywords: active redundancy, stochastic order, hazard rate ordering.

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Active-Redundancy Allocation in Systems

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Abstract

An effective way of improving the reliability of a system is the allocation of active redundancy. Let X_1, X_2 be independent lifetimes of the components C_1 and C_2, respectively, which form a series system. Let denote $U_1 = min(max(X_1, X), X_2)$ and $U_2 = min(X_1, max(X_2, X))$, where X is the lifetime of a redundancy (say S) independent of X_1 and X_2. That is $U_1(U_2)$ denote the lifetime of a system obtained by allocating S to $C_1(C_2)$ as an active redundancy. Singh and Misra (1994) considered the criterion where C_1 is preferred to C_2 for redundancy allocation if $P(U_1 > U_2) \geq P(U_2 > U_1)$. In this paper we use the same criterion of Singh and Misra (1994) and we investigate the allocation of one active redundancy when it differs depending on the component with which it is to be allocated. We find sufficient conditions for the optimization which depend on the components and redundancies probability distributions. We also compare the allocation of two active redundancies (say S_1 and S_2) in two different ways, that is, S_1 with C_1 and S_2 with C_2 and viceversa. For this case the hazard rate order plays an important role. We obtain results for the allocation of more than two active redundancies to a k-out-of-n systems.

Keywords: active redundancy, stochastic order, hazard rate order

1 Introduction

An effective way of improving the reliability of a system is the allocation of active redundancies. This problem has been studied by different authors using different criteria (see [1], [2] and [3]). Singh and Misra [4] considered the following criterion. Let X_1, X_2 be independent lifetimes of components C_1 and C_2 which form a series system. Let U_1 and U_2 denote the lifetime of two systems such that $U_1 = \wedge(\vee(X_1, X), X_2)$ and $U_2 = \wedge(X_1, \vee(X_2, X))$, where X is the

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lifetime of a redundancy V, independent of X_1 and X_2 and the symbols $\vee$ and $\wedge$ denote the max and min, respectively. If we are going to compare the total lifetimes of these systems, then it is better to allocate V as an active redundancy with C_1 instead of with C_2 if the following inequality holds,

$$P\left(U_1 > U_2\right) \geq P\left(U_2 > U_1\right). \tag{1}$$

In some cases it is more realistic to consider the active redundancy differs depending on the component with which it could be allocated. Rade [5] obtains results in this regard for some series-parallel systems when the components are exponentially distributed.

Let Y_1 and Y_2 be independent lifetimes of spares V_1 and V_2. Let now $U_1 = \wedge\left(\vee\left(X_1, Y_1\right), X_2\right)$ and $U_2 = \wedge\left(X_1, \vee\left(X_2, Y_2\right)\right)$. Recall a random variable X is said to be stochastically greater than a random variable Y, written $X \geq_{st} Y$, if $P(X > t) \geq P(Y > t)$ for all real value t. As is pointed out by Singh and Misra, the condition $U_1 \geq_{st} U_2$ may not always imply (1) since U_1 and U_2 are dependent random variables. So it would be of interest to find out sufficient conditions for the lifetimes of components and redundancies such that (1) holds. For the case $Y_1 = Y_2$ in [4] it is shown that if $X_2 \geq_{st} X_1$ then (1) holds and this result is extended to k-out-of-n systems.

The structure of this paper is as follows. In section 2 we establish some results that will be used in the proofs of the next two sections. In section 3 we find sufficient conditions on the distribution functions of the lifetimes of components and redundancies for (1) to hold when it is allocated one active redundancy that differs depending on the component with which it could be allocated, extending in this way the results given in [4]. In section 4 we compare the allocation of two redundancies in two different ways, i.e, V_1 with C_1 and V_2 with C_2 and viceversa. We also give results on the allocation of more than two active redundancies. In sections 3 and 4 we consider in the analysis k-out-of-n systems.

Recall the following definitions we will use. Let X and Y be nonnegative random variables and $\overline{F}\left(t\right)$ and $\overline{G}\left(t\right)$ denote the respective survival functions of X and Y. X is said to be greater than Y in the hazard rate ordering, written $X \geq_{hr} Y$, if $\overline{F}\left(t\right)/\overline{G}\left(t\right)$ is non-decreasing for all $t \geq 0$ where this quotient is defined. If the density functions of X and Y, say $f(t)$ and $g(t)$, exist then the ordering $X \geq_{hr} Y$ can be equivalently expressed as

$$\frac{f\left(t\right)}{\overline{F}\left(t\right)} \leq \frac{g\left(t\right)}{\overline{G}\left(t\right)}.$$

Following [3] we will say that X is greater than Y in the probability order, written $X \geq_{pr} Y$, if $P\left(X > Y\right) \geq P\left(Y > X\right)$ holds. For a general reference in stochastic ordering see [6].

2 Preliminary results

Let denote by z a value of a real random variable Z. Given a real number t let define z_t as $z_t = 1$ if $z \geq t$, and $z_t = 0$ if $z < t$.

Consider now the values x and y of two random variables X and Y, respectively. Observe that the inequality $x > y$ is valid if and only if there exists a real number t such that $x_t > y_t$. This equivalence allows us to reduce the treatment of inequalities between real valued random variables to the treatment of Boolean inequalities. In the following in place of the variables z_t we will simply write z.

For a set of random variables $\{Z_1, Z_2, ..., Z_n\}$, let $(Z_1, Z_2, ..., Z_n)_{[k]}$ denote the kth largest order statistics, so that $(Z_1, Z_2, ..., Z_n)_{[1]} \geq ... \geq (Z_1, Z_2, ..., Z_n)_{[n]}$. Let consider the random variables $X_1, X_2..., X_n,\ Y_1, Y_2, n = 2, 3...$ and denote

$$\begin{aligned} U_1^{(k)} &= \left(\vee\left(X_1, Y_1\right), X_2, X_3, ..., X_n\right)_{[k]} \\ U_2^{(k)} &= \left(X_1, \vee\left(X_2, Y_2\right), X_3, ..., X_n\right)_{[k]}, \end{aligned} \tag{2}$$

$k = 3, ..., n$

Proposition 2.1 *The following equivalencies hold:*
a) $U_1 > U_2$ *if and only if* $X_1 < \wedge(X_2, Y_1)$.
b) For $n > 2$, $U_1^{(n)} > U_2^{(n)}$ *if and only if* $X_1 < \wedge(Y_1, X_2, X_3, ..., X_n)$.
c) For $n > 2$ *and* $1 < k < n$, $U_1^{(k)} > U_2^{(k)}$ *if and only if one of the following* $\begin{pmatrix} n-2 \\ k-2 \end{pmatrix} + \begin{pmatrix} n-2 \\ k-1 \end{pmatrix}$ *excluding inequalities is satisfied:*

$$\begin{aligned} &\vee\left(X_1, X_{j_1}, ..., X_{j_{n-k}}\right) < \wedge\left(X_2, Y_1, X_{i_1}, ..., X_{i_{k-2}}\right), \\ &\vee\left(X_1, X_2, Y_2, X_{r_1}, ..., X_{r_{n-k-1}}\right) < \wedge\left(Y_1, X_{v_1}, ..., X_{v_{k-1}}\right), \end{aligned}$$

where

$$\begin{aligned} \{i_1, ..., i_{k-2}\} &\subseteq \{3, ..., n\}, \\ \{j_1, ..., j_{n-k}\} &\subseteq \{3, ..., n\}, \\ \{r_1, ..., r_{n-k-1}\} &\subseteq \{3, ..., n\}, \\ \{v_1, ..., v_{k-1}\} &\subseteq \{3, ..., n\} \end{aligned}$$

and

$\{i_1, ..., i_{k-2}\} \bigcap \{j_1, ..., j_{n-k}\} = \emptyset$ *and* $\{v_1, ..., v_{k-1}\} \bigcap \{r_1, ..., r_{n-k-1}\} = \emptyset$.

Proof. We will only prove b) and c), since a) follows in a similar fashion. Inequality

$$U_1^{(k)} > U_2^{(k)} \tag{3}$$

holds if and only if the following system of Boolean inequalities is satisfied

$$\vee\left(x_1, y_1\right) + x_2 + x_3 + ... + x_n \geq k, \tag{4}$$

$$x_1 + \vee(x_2, y_2) + x_3 + ... + x_n \leq k - 1. \tag{5}$$

Suppose $x_1 = 1$. Then $\vee(x_1, y_1) = 1$ and it is easy to see that in this case (4) and (5) do not hold simultaneously, consequently, $x_1 = 0$. Subtracting now (5) from (4) we obtain $y_1 + x_2 \geq 1 + \vee(x_2, y_2)$ and this implies $y_1 = 1$, since $x_2 \leq \vee(x_2, y_2)$ and, therefore

$$\vee(x_2, y_2) = x_2. \tag{6}$$

Substituting this last equality and the values $x_1 = 0$ and $y_1 = 1$ in (4), (5) we obtain

$$x_2 + x_3 + ... + x_n = k - 1. \tag{7}$$

Then the system (4), (5) is satisfied only if $x_1 = 0$, $y_1 = 1$ and the system (6), (7) is satisfied. It is straightforward to verify that, conversely, if these conditions hold, the system (4), (5) is satisfied.

For $k = n$ the system (6), (7) has for all value of y_2 the unique solution $x_2 = x_3 = ... = x_n = 1$ and then (3) is equivalent to

$$X_1 < \wedge(Y_1, X_2, X_3, ..., X_n).$$

Consider now the case $k < n$. Observe that if $x_2 = 0$ the equation (7) has $\begin{pmatrix} n-2 \\ k-1 \end{pmatrix}$ solutions and, on the other hand, from (6) we obtain $y_2 = 0$. In the case $x_2 = 1$ the equation (7) has $\begin{pmatrix} n-2 \\ k-2 \end{pmatrix}$ solutions and the value of y_2 may be arbitrary.

Then it can be easily seen that the equivalent inequalities for (3) stated in the proposition are obtained. This completes the proof.

Let now denote

$$V_1 = \wedge(\vee(X_1, Y_2), \vee(X_2, Y_1)),$$

$$V_2 = \wedge(\vee(X_1, Y_1), \vee(X_2, Y_2)),$$

$$\begin{aligned} V_1^{(k)} &= (\vee(X_1, Y_2), \vee(X_2, Y_1), X_3, ..., X_n)_{[k]}, \\ V_2^{(k)} &= (\vee(X_1, Y_1), \vee(X_2, Y_2), X_3, ..., X_n)_{[k]}, \end{aligned} \tag{8}$$

$k = 3, ..., n$.

Proposition 2.2 *The following equivalences hold:*
a) $V_1 > V_2$ *if and only if one of the following two excluding inequalities is satisfied*

$$\vee(X_1, Y_1) < \wedge(X_2, Y_2), \quad \vee(X_2, Y_2) < \wedge(X_1, Y_1).$$

b) For $n > 2$, $V_1^{(n)} > V_2^{(n)}$ *if and only if one of the following two excluding inequalities is satisfied*

$$\vee(X_1, Y_1) < \wedge(X_2, Y_2, X_3, ..., X_n),$$

$$\vee(X_2, Y_2) < \wedge(X_1, Y_1, X_3, ..., X_n).$$

c) For $n > 2$ and $1 < k < n$, $V_1^{(k)} > V_2^{(k)}$ if and only if one of the following $2\left(\begin{array}{c} n-2 \\ k-2 \end{array}\right)$ excluding inequalities is satisfied

$$\vee\left(X_1, Y_1, X_{j_1}, ..., X_{j_{n-k}}\right) < \wedge\left(X_2, Y_2, X_{i_1}, ..., X_{i_{k-2}}\right),$$

$$\vee\left(X_2, Y_2, X_{j_1}, ..., X_{j_{n-k}}\right) < \wedge\left(X_1, Y_1, X_{i_1}, ..., X_{i_{k-2}}\right),$$

where

$$\{i_1, ..., i_{k-2}\} \subseteq \{3, ..., n\},$$
$$\{j_1, ..., j_{n-k}\} \subseteq \{3, ..., n\}$$

and

$$\{i_1, ..., i_{k-2}\} \bigcap \{j_1, ..., j_{n-k}\} = \emptyset.$$

Proof. We consider the cases b) and c), since a) follows in a similar manner. Inequality

$$V_1^{(k)} > V_2^{(k)} \tag{9}$$

holds if and only if the following system of inequalities holds

$$\vee(x_1, y_2) + \vee(x_2, y_1) + x_3 + ... + x_n \quad \geq \quad k. \tag{10}$$

$$\vee(x_1, y_1) + \vee(x_2, y_2) + x_3 + ... + x_n \quad \leq \quad k-1 \tag{11}$$

If $\vee(x_1, y_1) = \vee(x_2, y_2) = 1$ from (11) we have $x_3 + ... + x_n \leq k-3$, but then (10) is not satisfied. Let $\vee(x_1, y_1) = \vee(x_2, y_2) = 0$. In this case from (10) and (11) we obtain the contradictory inequalities $x_3 + ... + x_n \geq k$ and $x_3 + ... + x_n \leq k-1$.

Suppose now that $\vee(x_1, y_1) = 1$ and $\vee(x_2, y_2) = 0$, or $\vee(x_1, y_1) = 0$ and $\vee(x_2, y_2) = 1$. In these cases, from (11) we have $x_3 + ... + x_n \leq k-2$. Subtracting this inequality from (10) we obtain $\vee(x_1, y_2) + \vee(x_2, y_1) = 2$ and also from (10) we have $x_3 + ... + x_n \geq k-2$. Then the system (10), (11) is satisfied only if $x_3 + x_4 + ... + x_n = k-2$ and, $\vee(x_2, y_2) = 0$ and $\wedge(x_1, y_1) = 1$, or $\vee(x_1, y_1) = 0$ and $\wedge(x_2, y_2) = 1$. Conversely, if these conditions hold, the system (10), (11) is satisfied. This proves the proposition.

3 Allocation of an active redundancy

When assumed to exist we will denote the probability densities and hazard rates of X_1 and X_2 by $f_1(x)$, $f_2(x)$, $\lambda_1(x)$ and $\lambda_2(x)$, respectively. We will denote the distribution functions of $Y_1, Y_2, X_1, X_2, ..., X_n$ by $G_1(x), G_2(x)$, $F_1(x), F_2(x), ..., F_n(x)$, respectively. For any distribution function G we will denote $\bar{G}(x) = 1 - G(x)$.

Lemma 3.1 *Let X_1, X_2, Y_1, Y_2 and Z be nonnegative independent random variables. Suppose*

i) X_1 *and* X_2 *have probability densities and*

$$\lambda_1(x)\bar{G}_1(x) \geq \lambda_2(x)\bar{G}_2(x), \quad x \geq 0 \tag{12}$$

or

$$ii)\ X_1 \leq_{st} X_2,\ and\ \bar{F}_2(x)\bar{G}_1(x) \geq \bar{F}_1(x)\bar{G}_2(x), \quad x \geq 0. \tag{13}$$

Then

$$a)\ P\left(X_1 < \wedge\left(Y_1, X_2\right)\right) \geq P\left(X_2 < \wedge\left(X_1, Y_2\right)\right)$$

and

$$b)\ P\left(X_1 < \wedge\left(Y_1, X_2, Z\right)\right) \geq P\left(X_2 < \wedge\left(X_1, Y_2, Z\right)\right).$$

Proof. We only prove part b), since a) follows in a similar fashion. Let $H(x)$ denote the distribution function of Z.

$$\begin{aligned}\Delta &= P\left(X_1 < \wedge\left(Y_1, X_2, Z\right)\right) - P\left(X_2 < \wedge\left(X_1, Y_2, Z\right)\right) \\ &= \int_0^\infty \bar{F}_2(x)\,\bar{G}_1(x)\,\bar{H}(x)\,dF_1(x) - \int_0^\infty \bar{F}_1(x)\,\bar{G}_2(x)\,\bar{H}(x)\,dF_2(x).\end{aligned}$$

But from ii) follows

$$\Delta \geq \int_0^\infty \bar{F}_1(x)\,\bar{G}_2(x)\,\bar{H}(x)\,dF_1(x) - \int_0^\infty \bar{F}_1(x)\,\bar{G}_2(x)\,\bar{H}(x)\,dF_2(x) \geq 0$$

since $\bar{F}_1(x)\,\bar{G}_2(x)\,\bar{H}(x)$ is a non-increasing function and $F_1(x) \geq F_2(x)$ [6]. This prove *b*).
Observe now that if X_1 and X_2 have probability densities

$$\begin{aligned}\Delta = &\int_0^\infty \bar{F}_1(x)\,\bar{F}_2(x)\,\bar{G}_1(x)\,\lambda_1(x)\,\bar{H}(x)\,dx \\ &- \int_0^\infty \bar{F}_1(x)\,\bar{F}_2(x)\,\bar{G}_2(x)\,\lambda_2(x)\,\bar{H}(x)\,dx.\end{aligned}$$

Then *b*) follows from *i*).

Proposition 3.1 *Let X_1, X_2,...,X_n, Y_1 and Y_2 be independent lifetimes. Suppose*

i) X_1 *and* X_2 *have probabilities densities and*

$$\lambda_1(x)\bar{G}_1(x) \geq \lambda_2(x)\bar{G}_2(x), \quad x \geq 0, \tag{14}$$

or

$$ii)\ X_1 \leq_{st} X_2\ \ and\ \ \bar{F}_2(x)\bar{G}_1(x) \geq \bar{F}_1(x)\bar{G}_2(x), \quad x \geq 0. \tag{15}$$

Then

$$U_1 \geq_{pr} U_2 \quad and \quad U_1^{(n)} \geq_{pr} U_2^{(n)}. \tag{16}$$

Proof. Accordingly to Proposition 2.1, part b), $U_1^{(n)} \geq_{pr} U_2^{(n)}$ holds if and only if

$$P\left(X_1 < \wedge\left(Y_1, X_2, X_3, ..., X_n\right)\right) \geq P\left(X_2 < \wedge\left(X_1, Y_2, X_3, ..., X_n\right)\right).$$

Then the result straightforwardly follows from part b) of Lemma 3.1 taking $Z = \wedge\left(X_3, ..., X_n\right)$. It is obvious that the case $U_1 \geq_{pr} U_2$ follows in a similar way.

Conditions i) and ii) of Proposition 3.1 give us criteria for the optimal allocation in the sense of probability ordering of a redundancy which differs depending on the component with which it is allocated. Suppose, for example, that $Y_1 \geq_{st} Y_2$ and $X_1 \leq_{hr} X_2$ or $X_1 \leq_{st} X_2$, then it is optimal in probability order to allocate the stronger redundancy to the weaker component. If $G_1 = G_2$, condition $i)$ reduces to hazard rate order between lifetimes X_1 and X_2 and condition $ii)$ reduces to stochastic order between lifetimes X_1 and X_2. This case is covered in [4], where is also given a counterexample that in our case allows to show that condition $ii)$ it is not necessary for probability ordering (16) to hold.

Note that $\bar{F}_i(x)\bar{G}_j(x)$, $i, j = 1, 2$, is the survival function of a series system formed by components with lifetimes X_i and Y_j. Then condition $ii)$ can be stated in the following way. If the series system formed by component C_2 with the redundancy V_1 is stochastically greater than the series system formed by component C_1 with redundancy V_2, and $X_1 \leq_{st} X_2$, then it is better to allocate redundancy V_1 with component C_1 than to allocate redundancy V_2 with component C_2.

Remark 3.1 *If X_1, X_2, Y_1 and Y_2 are independent exponential random variables with means $1/\lambda_1, 1/\lambda_2, 1/\mu_1$ and $1/\mu_2$, respectively, then it is seen that*

$$P\left(\wedge\left\{\vee\left(X_1, Y_1\right), X_2\right\} > \wedge\left\{X_1, \vee\left(X_2, Y_2\right)\right\}\right) = \frac{\lambda_1}{\lambda_1 + \lambda_2 + \mu_1},$$

$$\begin{aligned} P\left(\wedge\left\{\vee\left(X_1, Y_1\right), X_2\right\} = \wedge\left\{X_1, \vee\left(X_2, Y_2\right)\right\}\right) \\ = \frac{\lambda_1\mu_1 + \lambda_2\mu_2 + \mu_1\mu_2}{\left(\lambda_1 + \lambda_2 + \mu_1\right)\left(\lambda_1 + \lambda_2 + \mu_2\right)}. \end{aligned} \tag{17}$$

Lemma 3.2 will be useful in extending the result of Proposition 3.1 to k-out-of-n systems. Result $b)$ in Lemma 3.2 is stated in Lemma 2.1 of [4].

Lemma 3.2 *Let X_1, X_2, Y_1, Y_2, Z_1 and Z_2 be nonnegative independent random variables and Z_3, Z_4 nonnegative random variables independent of Y_1 and Y_2. Suppose that $X_1 \leq_{st} X_2$ and $Y_1 \geq_{st} Y_2$. Then*

$$\begin{aligned} &a)\ P\left(\vee\left(X_1, Z_1\right) < \wedge\left(X_2, Y_1, Z_2\right)\right) \geq P\left(\vee\left(X_2, Z_1\right) < \wedge\left(X_1, Y_2, Z_2\right)\right). \\ &b)\ P\left(\vee\left(Y_2, Z_3\right) < \wedge\left(Y_1, Z_4\right)\right) \geq P\left(\vee\left(Y_1, Z_3\right) < \wedge\left(Y_2, Z_4\right)\right). \end{aligned} \tag{18}$$

Proof. Let $H_1(x)$ and $H_2(x)$ denote the distribution functions of Z_1 and Z_2, respectively.

$$\begin{aligned}\Delta &= P\left(\vee\left(X_1, Z_1\right) < \wedge\left(X_2, Y_1, Z_2\right)\right) - P\left(\vee\left(X_2, Z_1\right) < \wedge\left(X_1, Y_2, Z_2\right)\right) \\ &= \int_0^\infty \int_0^\infty \bar{F}_2\left(\vee(x,y)\right)\bar{G}_1\left(\vee(x,y)\right)\bar{H}_2\left(\vee(x,y)\right) dF_1(x)\, dH_1(y) \\ &- \int_0^\infty \int_0^\infty \bar{F}_1\left(\vee(x,y)\right)\bar{G}_2\left(\vee(x,y)\right)\bar{H}_2\left(\vee(x,y)\right) dF_2(x)\, dH_1(y).\end{aligned}$$

Since $\bar{G}_1(x) \geq \bar{G}_2(x)$ and $\bar{F}_2(x) \geq \bar{F}_1(x)$ then

$$\begin{aligned}\Delta &\geq \int_0^\infty \int_0^\infty \bar{F}_2\left(\vee(x,y)\right)\bar{G}_1\left(\vee(x,y)\right)\bar{H}_2\left(\vee(x,y)\right) dF_1(x)\, dH_1(y) \\ &- \int_0^\infty \int_0^\infty \bar{F}_2\left(\vee(x,y)\right)\bar{G}_1\left(\vee(x,y)\right)\bar{H}_2\left(\vee(x,y)\right) dF_2(x)\, dH_1(y).\end{aligned}$$

Using the same argument as in the proof of Lemma 3.1 it can be obtained that $\Delta \geq 0$ and then $a)$ follows.

Proposition 3.2 *Let $X_1, ..., X_n, Y_1$ and Y_2 be independent lifetimes. Suppose that $X_1 \leq_{st} X_2$ and $Y_1 \geq_{st} Y_2$. Then for $1 < k < n$, $n > 2$,*

$$U_1^{(k)} \geq_{pr} U_2^{(k)}. \tag{19}$$

Proof. It is sufficient to use part c) of Proposition 2.1 with the same notation and conditions stated there and to take

$$\begin{aligned}&Z_1 = \vee\left(X_{j_1}, ..., X_{j_{n-k}}\right), \quad Z_2 = \wedge\left(X_{i_1}, ..., X_{i_{k-2}}\right), \\ &Z_3 = \vee\left(X_1, X_2, X_{r_1}, ..., X_{r_{n-k-1}}\right), \quad Z_4 = \wedge\left(X_{v_1}, ..., X_{v_{k-1}}\right)\end{aligned}$$

in Lemma 3.2.

4 Allocation of more than one redundancy

We consider now the allocation of two active redundancies. In what follows we will denote the probability densities of Y_1 and Y_2 by $g_1(x)$ and $g_2(x)$, respectively.

Lemma 4.1 *Let X_1, Y_1, X_2, Y_2, Z_1 and Z_2 be independent random variables. Suppose X_1, Y_1, X_2 and Y_2 have probability densities. Let $X_1 \leq_{hr} X_2$ and $Y_1 \leq_{hr} Y_2$. Then*

$$\begin{aligned}&a)\ P\left(\wedge\left(X_2, Y_2\right) > \vee\left(X_1, Y_1\right)\ OR\ \wedge\left(X_1, Y_1\right) > \vee\left(X_2, Y_2\right)\right) \\ &\geq P\left(\wedge\left(X_2, Y_1\right) > \vee\left(X_1, Y_2\right)\ OR\ \wedge\left(X_1, Y_2\right) > \vee\left(X_2, Y_1\right)\right)\end{aligned}$$

and

$$\begin{aligned}&b)\ P\left(\wedge\left(X_2, Y_2, Z_2\right) > \vee\left(X_1, Y_1, Z_1\right)\ OR\ \wedge\left(X_1, Y_1, Z_2\right) > \vee\left(X_2, Y_2, Z_1\right)\right) \\ &\geq P\left(\wedge\left(X_2, Y_1, Z_2\right) > \vee\left(X_1, Y_2, Z_1\right)\ OR\ \wedge\left(X_1, Y_2, Z_2\right) > \vee\left(X_2, Y_1, Z_1\right)\right).\end{aligned}$$

Proof. We will only prove b) since a) follows in a similar way. It is sufficient to prove that

$$\begin{aligned}\Delta &= P\left(\wedge\left(X_2, Y_2, Z_2\right) > \vee\left(X_1, Y_1, Z_1\right)\right) + P\left(\wedge\left(X_1, Y_1, Z_2\right) > \vee\left(X_2, Y_2, Z_1\right)\right) \\ &- P\left(\wedge\left(X_2, Y_1, Z_2\right) > \vee\left(X_1, Y_2, Z_1\right)\right) - P\left(\wedge\left(X_1, Y_2, Z_2\right) > \vee\left(X_2, Y_1, Z_1\right)\right) \geq 0,\end{aligned}$$

But

$$\begin{aligned}\Delta = & \int_0^\infty \int_0^\infty \int_0^\infty \bar{F}_2 (\vee (x,y,z)) \bar{G}_2 (\vee (x,y,z)) \bar{H}_2 (\vee(x,y,z)) \, dG_1 (x) \, dF_1 (y) \, dH_1 (z) \\ & + \int_0^\infty \int_0^\infty \int_0^\infty \bar{F}_1 (\vee (x,y,z)) \bar{G}_1 (\vee (x,y,z)) \bar{H}_2 (\vee(x,y,z)) \, dG_2 (x) \, dF_2 (y) \, dH_1 (z) \\ & - \int_0^\infty \int_0^\infty \int_0^\infty \bar{F}_2 (\vee (x,y,z)) \bar{G}_1 (\vee (x,y,z)) \bar{H}_2 (\vee(x,y,z)) \, dG_2 (x) \, dF_1 (y) \, dH_1 (z) \\ & - \int_0^\infty \int_0^\infty \int_0^\infty \bar{F}_1 (\vee (x,y,z)) \bar{G}_2 (\vee (x,y,z)) \bar{H}_2 (\vee(x,y,z)) \, dG_1 (x) \, dF_2 (y) \, dH_1 (z),\end{aligned}$$

where $H_1(x)$ and $H_2(x)$ denote the distribution function of Z_1 and Z_2, respectively.

A sufficient condition for $\Delta \geq 0$ is

$$\begin{aligned} & \bar{F}_2 (\vee (x,y,z)) \bar{G}_2 (\vee (x,y,z)) g_1 (x) f_1 (y) \\ & +\bar{F}_1 (\vee (x,y,z)) \bar{G}_1 (\vee (x,y,z)) g_2 (x) f_2 (y) \\ & \geq \bar{F}_2 (\vee (x,y,z)) \bar{G}_1 (\vee (x,y,z)) g_2 (x) f_1 (y) \\ & +\bar{F}_1 (\vee (x,y,z)) \bar{G}_2(\vee (x,y,z)) g_1 (x) f_2 (y), \end{aligned} \tag{20}$$

which can be rewritten as

$$\begin{aligned} & g_1 (x) \bar{G}_2 (\vee (x,y,z)) \left[f_1 (y) \bar{F}_2 (\vee (x,y,z)) - f_2 (y) \bar{F}_1 (\vee (x,y,z)) \right] \\ & \geq g_2 (x) \bar{G}_1 (\vee (x,y,z)) \left[f_1 (y) \bar{F}_2 (\vee (x,y,z)) - f_2 (y) \bar{F}_1 (\vee (x,y,z)) \right]. \end{aligned} \tag{21}$$

Observe now that if $a \geq b \geq 0$, then

$$f_1 (b) \bar{F}_2 (a) - f_2 (b) \bar{F}_1 (a) \geq 0,$$

since from $X_1 \leq_{hr} X_2$ follows

$$f_1 (b) \geq f_2 (b) \frac{\bar{F}_1 (b)}{\bar{F}_2 (b)} \geq f_2 (b) \frac{\bar{F}_1 (a)}{\bar{F}_2 (a)}.$$

Likewise from $Y_1 \leq_{hr} Y_2$ follows

$$g_1 (b) \bar{G}_2 (a) - g_2 (b) \bar{G}_1 (a) \geq 0.$$

Then (21) holds and the proof is complete.

Proposition 4.1 *Let $X_1, ..., X_n$, Y_1 and Y_2 be independent lifetimes. Suppose X_1, Y_1, X_2 and Y_2 have probability densities. Let $X_1 \leq_{hr} X_2$ and $Y_1 \leq_{hr} Y_2$. Then*

$$V_1 \geq_{pr} V_2 \quad \text{and} \quad V_1^{(k)} \geq_{pr} V_2^{(k)}, \tag{22}$$

for $1 < k \leq n$.

Proof. We only consider the case $1 < k < n, n > 2$, since the remaining cases can be proved in a similar way. Then it is sufficient to use part c) of Proposition 2.2 with the same notation and conditions stated there and to take

$$Z_1 = \vee \left(X_{j_1}, ..., X_{j_{n-k}} \right), \quad Z_2 = \wedge \left(X_{i_1}, ..., X_{i_{k-2}} \right)$$

in Lemma 4.1.

Corollary 4.1 *Let $X_1, X_2, ..., X_n, Y_1, Y_2, ..., Y_n$ be independent lifetimes for which the probability densities exist. Suppose $X_1 \leq_{hr} X_2 \leq_{hr} ... \leq_{hr} X_n$ and $Y_1 \leq_{hr} Y_2 \leq_{hr} ... \leq_{hr} Y_n$. Then for $2 \leq k \leq n$, $n \geq 2$,*

$$\begin{aligned}&\left(\vee\left(X_1, Y_n\right), \vee\left(X_2, Y_{n-1}\right), ..., \vee\left(X_n, Y_1\right)\right)_{[k]}\\ &\geq_{pr} \left(\vee\left(X_1, Y_{\pi(1)}\right), \vee\left(X_2, Y_{\pi(2)}\right), ..., \vee\left(X_n, Y_{\pi(n)}\right)\right)_{[k]}\end{aligned}$$

for any permutation $\pi = (\pi(1), \pi(2), ..., \pi(n))$ of $\{1, 2, ..., n\}$.

Proof. For two arbitrary permutations π and δ of $\{1, 2, ..., n\}$ let

$$\begin{aligned}\varphi(\delta; \pi) = P\{&\left(\vee(X_1, Y_{\delta(1)}), \vee(X_2, Y_{\delta(2)}), ..., (X_n, Y_{\delta(n)})\right)_{[k]}\\ &> \left(\vee(X_1, Y_{\pi(1)}), \vee(X_2, Y_{\pi(2)}), ..., (X_n, Y_{\pi(n)})\right)_{[k]}\}.\end{aligned}$$

Denoting by π_0 the permutation $(n, n-1, ..., 1)$ we can rewrite the result that is required to prove as $\varphi(\pi_0; \pi) \geq \varphi(\pi; \pi_0)$.

Given a permutation $\pi = (\pi(1), \pi(2), ..., \pi(n))$ let consider the permutation $\pi_i = (\pi(1), \pi(2), ..., \pi(i-1), \pi(i+1), \pi(i), \pi(i+2), ..., \pi(n))$, $i = 1, 2, ..., n-1$. If $Y_{\pi(i+1)} \geq_{hr} Y_{\pi(i)}$, from Proposition 4.1 we obtain $\varphi(\pi_i; \pi) \geq \varphi(\pi; \pi_i)$ for all $i = 1, 2, ..., n-1$. But under the suppositions that are made this result implies $\varphi(\pi_0; \pi) \geq \varphi(\pi; \pi_0)$ for any permutation π. Consequently, the corollary is proved.

Corollary 4.1 means that if we have a k-out-of-n system formed by components $c_1, c_2, ..., c_n$ with respective failure rates $\lambda_1(t) \geq \lambda_2(t) \geq ... \geq \lambda_n(t)$, and r $(r \leq n)$ redundancies $c'_1, c'_2, ..., c'_r$ with respective failure rates $\mu_1(t) \geq \mu_2(t) \geq ... \geq \mu_r(t)$, then if we are going to allocate each redundancy to a component as an active redundancy, the optimal allocation regarding the probability ordering is to allocate c'_r with c_1, c'_{r-1} with c_2, and so on.

In [7] is considered the decision between to expand a k-out-of-n system and improving the already existing system by means of redundancy. In the following proposition we consider this situation.

Proposition 4.2 *Let $X_1, ..., X_n$ and $Y_1, Y_2, ..., Y_r$ $(r \leq n)$ be lifetimes. Then the following inequality always holds*

$$\begin{aligned}&(X_1, X_2, ..., X_n, Y_1, Y_2, ..., Y_r)_{[k]}\\ &\geq (\vee(X_1, Y_1), \vee(X_2, Y_2), ..., \vee(X_r, Y_r), X_{r+1}, ..., X_n)_{[k]}.\end{aligned} \tag{23}$$

Proof. Let suppose, on the contrary, that (23) does not hold. In this case using the notation of section 2 it is no hard to see that the system

$$\begin{aligned}\vee(x_1, y_1) + \vee(x_2, y_2) + ... + \vee(x_r, y_r) + x_{r+1} + ... + x_n &\geq k\\ x_1 + x_2 + ... + x_n + y_1 + y_2 + ... + y_r &\leq k-1\end{aligned}$$

must be satisfied. Nevertheless this system has not solution and, consequently, (23) always holds.

It is obvious that the result obtained in Proposition 4.2 implies that substituting in (23) the symbol $\geq$ by the symbols $\geq_{st}$ and $\geq_{pr}$ the inequality also holds.

We now examine the following problem. Suppose we have a k-out-of-n system and there are R spares to be allocated in parallel with its components. Suppose further that lifetimes of all the components and the spares are independent, the lifetimes of the components are identically distributed and the lifetimes of the redundancies are identically distributed. We are interested in determining the optimal allocation of the spares. For a series system this problem has been considered in [8] and [9] from the point of view of stochastic and failure rate ordering, respectively. In those works it has been found that in order to optimize the lifetime of the system, the allocation of spares must be balanced among the components as much as possible. We will show that it is also the optimal allocation in the sense of the probability ordering for a k-out-of-n system.

Let $(x_1, x_2, ..., x_n)$ be a nonnegative n-dimensional vector and let $x_{[1]}$, $x_{[2]}$, ..., $x_{[n]}$ denote the coordinates of the vector arranged in decreasing order. For a nonnegative vector $(y_1, y_2, ..., y_n)$, similarly define $y_{[i]}, i = 1, 2, ..., n$. Recall that x majorizes y ($x >^m y$) if $\sum_{i=1}^{j} x_{[i]} \geq \sum_{i=1}^{j} y_{[i]}$ holds for all $j = 1, ..., n-1$ and moreover $\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}$ [10].

We will denote by $r = (r_1, r_2, ..., r_n)$ and $l = (l_1, l_2, ..., l_n)$ two possible arrangements of spares to be placed in parallel with the components of the system such that r_i (respectively l_i) spares are allocated with the i^{th} component, where $r_i, l_i \in \{0, 1, 2, ..., R\}$. Of course $\sum_{i=1}^{n} r_i = \sum_{i=1}^{n} l_i = R$.

Let X_1,X_2,...,X_n denote the lifetimes of the components and Y_1,Y_2,...,Y_R denote the lifetimes of the spares. Let consider the sets of lifetimes $\{Y_1^{(i)}, Y_2^{(i)}, ..., Y_{r_i}^{(i)}\}$, $i = 1, 2, ..., n$, which constitute a partition of the set $\{Y_1, Y_2, ..., Y_R\}$ corresponding to the arrangement r. That is, $Y_j^{(i)}$ denotes the lifetime of the j^{th} active redundancy allocated to the i^{th} component, $j = 1, ..., r_i$, $i = 1, ..., n$. Similarly, we consider the partition $\{Z_1^{(i)}, Z_2^{(i)}, ..., Z_{l_i}^{(i)}\}$, $i = 1, 2, ..., n$, of the set $\{Y_1, Y_2, ..., Y_R\}$ corresponding to the arrangement l.

Let

$$\begin{aligned} w(r,l;k) = P\Bigg(&\left(X_1 \vee \left\{\vee_{i=1}^{r_1} Y_i^{(1)}\right\}, X_2 \vee \left\{\vee_{i=1}^{r_2} Y_i^{(2)}\right\}, ..., X_n \vee \left\{\vee_{i=1}^{r_n} Y_i^{(n)}\right\}\right)_{[k]} \\ &> \left(X_1 \vee \left\{\vee_{i=1}^{l_1} Z_i^{(1)}\right\}, X_2 \vee \left\{\vee_{i=1}^{l_2} Z_i^{(2)}\right\}, ..., X_n \vee \left\{\vee_{i=1}^{l_n} Z_i^{(n)}\right\}\right)_{[k]} \Bigg), \\ &k = 2, ..., n \end{aligned}$$

Proposition 4.3 *Suppose $X_1, X_2, ..., X_n, Y_1, Y_2, ..., Y_R$ are independent lifetimes such that $X_1, X_2, ..., X_n$ are identically distributed and $Y_1, Y_2, ..., Y_R$ are identically distributed. Let $r >^m l$. Then*

$$w(r,l;k) \leq w(l,r;k).$$

Proof. Let consider an arrangement of spares $r = (r_1, r_2, ..., r_n)$ and denote by $r(i) = (r_1, r_2, ..., r_{i-1}, r_i + 1, r_{i+1} - 1, r_{i+2}, ..., r_n)$, $i = 1, 2, ..., n-1$, where $r_{i+1} > 0$, the arrangement obtained from r changing the spare with lifetime

$Y^{(i+1)}_{r_{i+1}}$ from the $(i+1)^{th}$ to the i^{th} component. By the nature of majorization it is sufficient to show that if $r_i + 1 \leq r_{i+1} - 1$ then $w(r, r(i); k) \leq w(r(i), r; k)$. Then the problem is reduced to analyze the allocation of the redundancy with lifetime $Y^{(i+1)}_{r_{i+1}}$ when the arrangement of spares is r, between the i^{th} and the $(i+1)^{th}$ components. That is, we must compare the lifetime

$$\Big(X_1 \vee \left\{\vee_{j=1}^{r_1} Y_j^{(1)}\right\}, X_2 \vee \left\{\vee_{j=1}^{r_2} Y_j^{(2)}\right\}, ..., \\ X_i \vee \left\{\vee_{j=1}^{r_i} Y_j^{(i)}\right\}, X_{i+1} \vee \left\{\vee_{j=1}^{r_{i+1}} Y_j^{(i+1)}\right\}, ..., X_n \vee \left\{\vee_{j=1}^{r_n} Y_j^{(n)}\right\}\Big)_{[k]}$$

which is obtained when the arrangement of the spares is r versus the lifetime

$$\Big(X_1 \vee \left\{\vee_{j=1}^{r_1} Y_j^{(1)}\right\}, X_2 \vee \left\{\vee_{j=1}^{r_2} Y_i^{(2)}\right\}, ..., \\ \vee(X_i, \vee_{j=1}^{r_i} Y_j^{(i)}, Y^{(i+1)}_{r_{i+1}}), X_{i+1} \vee \left\{\vee_{j=1}^{r_{i+1}-1} Y_j^{(i+1)}\right\}, ..., X_n \vee \left\{\vee_{j=1}^{r_n} Y_i^{(n)}\right\}\Big)_{[k]}$$

which is obtained when the arrangement of the spares is $r(i)$.

Since $r_i < r_{i+1} - 1$

$$\vee \left\{X_i, Y_1^{(i)}, Y_2^{(i)}, ..., Y_{r_i}^{(i)}\right\} \leq_{st} \vee \left\{X_{i+1}, Y_1^{(i+1)}, Y_2^{(i+1)}, ..., Y^{(i+1)}_{r_{i+1}-1}\right\},$$

then the result follows from Propositions 3.1 and 3.2.

Proposition 4.3 gives us a criteria for the allocation of active redundancies to k-out-of-n systems, regarding probability ordering. If we have $m = r_1 + r_2 + ... + r_n = pn$ redundancies to be allocated as active redundancies to a k-out-of-n system, that is to allocate r_i redundancies to the i^{th} component, $i = 1, ...n$, then the better allocation regarding probability ordering is to take $r_i = p$, $i = 1, ..., n$. For an arbitrary number of redundancies the best choice is to allocate the redundancies the most uniformly as possible among the components.

5 Conclusions

In this paper we have discussed on the allocation of one active redundancy which differs depending on the component with it is to be allocated and we have analyzed the allocation of more than one redundancy. In the one redundancy case, stochastic ordering together with restrictions on the distribution functions of the components and the redundancy, are found as sufficient conditions for the probability ordering to hold. In the case of more than one redundancy allocation, the sufficient conditions are expressed through the hazard rate order. Finally we have obtained results on the allocation of more than two redundancies.

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References

[1] P. Boland, E. El-Neweihi and F. Proschan, "Stochastic Order for Redundancy allocations in Series and Parallel Systems," *Adv. Appl. Prob,* 24, pp. 161-171, 1992.

[2] Jie Mi, "Bolstering Components for Maximizing System Lifetime," *Naval Research Logistics,* 45, pp 497-509, 1998.

[3] Jie Mi, "Optimal Active Redundancy Allocation in k-out-of-n system," *J. Appl. Prob,* 36, pp927-933, 1999.

[4] H. Singh and N. Misra, "On redundancy allocation in systems," *J. Appl. Prob,* 31, pp 1004-1014, 1994.

[5] L. Rade, "Expected Time to Failure of Reliability Systems," *Math. Scientist,* 14, pp 24-37, 1989.

[6] M. Shaked and J. George Shanthikumar *Stochastic Orders and their Applications.* Academic Press, 1994.

[7] P. Boland, E. El-Neweihi and F. Proschan, "Redundancy Importance and Allocation of Spares in Coherent Systems," *Journal of Statistical Planning and Inference,* 29, pp 55-66, 1991.

[8] M. Shaked and J.G. Shanthikumar, "Optimal Allocation of Resources to Nodes of Series and Parallel Systems," *Advances in Applied Probability,* 24, pp 894-914, 1992.

[9] H. Singh and R.S. Singh, "Note: Optimal Allocation of Resources to Nodes of Series Systems with Respect to Failure-Rate Ordering," *Naval Research Logistic,* 44, pp 147-152, 1997.

[10] A. Marshall and I. Olkin (1979). Inequalities: Theory of Majorization and its applications. New York: Academic Press.