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Departamento de Economía de la Empresa
Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (34-91) 6249608

COMPUTING CONTINUOUS-TIME GROWTH MODELS WITH BOUNDARY CONDITIONS VIA WAVELETS*

Mercedes Esteban-Bravo[1] and Jose M. Vidal-Sanz[2]

Abstract

This paper presents an algorithm for approximating the solution of deterministic/stochastic continuous-time growth models based on the Euler's equation and the transversality conditions. The main issue for computing these models is to deal efficiently with the boundary conditions associated. This approach is a wavelets-collocation method derived from the finite-iterative trapezoidal approach. Illustrative examples are given.

JEL classification codes: C63.
Key words: Wavelets, continuous-time growth models, boundary value problems.

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[1] Dept. of Business Economics, Universidad Carlos III de Madrid, C/ Madrid, 126. 28903 Getafe, Madrid, Spain. E-mail: mesteban@emp.uc3m.es

[2] Dept. of Business Economics, Universidad Carlos III de Madrid, C/ Madrid, 126. 28903 Getafe, Madrid, Spain. E-mail: jvidal@emp.uc3m.es

1 Introduction

Boundary value problems (BVP) typically arise from the application of the Pontryagin's maximum principle to control optimum problems with finite time horizon, a common way to study the dynamics of growth models. Unfortunately, BVP usually cannot be solved analytically and the use of some numerical method is required. Despite the rapid growth on numerical methods for approximating solutions to continuous-time models (for recent survey see Rust (1996), Santos (1999), the text by Judd (1998) and the collection of essays edited by Marimon and Scott (1999)), few algorithms have been developed to cope with boundary conditions.

We present a wavelets-collocation method for solving BVP derived from the recursive rules of the trapezoidal approach. The first part of the paper is devoted to the computation of deterministic BVP, whilst the second part extends these results to the stochastic problem.

Given a continuous function $f : \mathbb{R}^{R+1} \to \mathbb{R}^R$, and a vector of continuous linear functionals $\alpha = (\alpha_1, ..., \alpha_R)'$, with $\alpha_j : C^1([a,b])^R \to \mathbb{R}$ linearly independent, suppose that we are interested in solving the following deterministic boundary value problem:

$$\begin{aligned} Dy(t) &= f(t, y), \\ \alpha(y) &= c, \end{aligned}$$

where $y = (y_1, .., y_R)'$, $y_j \in C^1([a,b])$ for $j = 1, ..., R$. This boundary condition specification $\alpha(y) = c$ includes most of the initial and boundary value problems considered in macroeconomic analysis. In particular, we are concerned with the boundary conditions of the form $y(t) = 0$ for some $t \in [a,b]$, or other more general specifications such as $\alpha(y) = \sum_{k=1}^{K} A_k y(t_k)$, where A_k is an $R-$dimensional square matrix and $t_k \in [a,b]$.

Shooting methods are probably the most popular numerical method for solving BVP. A shooting method is a successive substitution method based on the idea of guessing the initial condition which associate solution satisfies the desired boundary condition. Then, any finite-difference algorithm can be considered to solve this "new" initial value problem. For details see Ascher et al (1995), Roberts and Shipman (1972) and Keller (1976). Unfortunately, these methods can be quite inefficient as they may often converge quite slowly, or not at all, and a wrong guess could substantially increases the computer time. Furthermore, the numerical errors can be magnified. The possible difficulties with shooting methods are frequently discussed in the literature, see Conte (1966), Keller (1968, 1976), and Osborne (1969) for example.

As an alternative, boundary value problems can be solved using some projection-based method, such as Galerkin or collocation techniques. In particular, those based on splines are commonly used, see Varga (1971), Russell and Shampine (1972), Lucas and Reddien (1972), de Boor and Schwartz (1973), and Prenter (1975), for example. In this context collocation methods often have better performance than Galerkin methods, but the choice of the collocation points greatly influences the effectiveness of the method. Furthermore, if the solution path exhibit some abrupt changes, the approximation could be inaccurate.

In numerical analysis, the discovery of compactly supported wavelets has proven to be a useful tool for the approximation of functions, differential and integral operators. The use of wavelets based algorithms is superficially similar to other projection methods, but these algorithms are more efficient because of the localization of wavelet bases in both space and frequency domain. Therefore, the approximation of a function using wavelets bases may be advantageous when it exhibits abrupt changes. Wavelets have been applied to a wide range of problems such as signal processing, image analysis, data compression and time series econometrics.

The proposed method exploits the good approximation properties of wavelets. Moreover, being a collocation-based approach, this algorithm is flexible enough to deal with complex boundary conditions. Furthermore, the use of trapezoidal discretization avoids the numerical instabilities often observed in many algorithms for solving differential equations, and also offers a great advantage in terms of cost as it does not require the computation of wavelets derivatives. However, others finite-iterative methods, such as high-order Runge-Kutta approximations, could be considered under some high order differentiability requirements.

Often, associated with the Euler's equation and the transversality conditions there are additional inequality constraints on the states or/and controls. For example, many economic models with borrowing constraints have been considered extensively in growth theory. Using the wavelets-collocation method proposed, we also present an algorithm to deal with inequality constraints based on interior-point algorithms.

The rest of the paper is organized as follows. Section 2 is devoted to present the method for a deterministic BVP and some examples that illustrate the good performance of the algorithm (Annexo A presents a MATLAB code for solving a simple example). Section 3 presents the theoretical convergence analysis. In Section 3 we study the extension to the stochastic BVP. All the proofs can be found in Annexo B.

2 Algorithm for deterministic BVP

The main idea of the proposed method is the use of a wavelet-collocation approach for the solution of the finite-difference approximation of the BVP. With this end in view, we first introduce the main concepts of wavelet approximation.

Let $L_2(\mathbb{R})$ denote the vector space of all classes of Lebesgue-measurable functions y defined almost everywhere on $\mathbb{R}$ (we identify functions that are equal almost everywhere) such that $\int |y(t)|^2 \, dt < \infty$.

Let consider a sequence of closed subspaces $\{V_n\}_{n\in\mathbb{Z}}$ which is monotonously increasing $V_n \subset V_{n+1}$, for all $n \in \mathbb{Z}$, and satisfies $\bigcap_{n\in\mathbb{Z}} V_n = \{0\}$, and $\bigcup_{n\in\mathbb{Z}} V_n$ is dense in $L_2(\mathbb{R})$. In particular we say that $\{V_n\}_{n\in\mathbb{Z}}$ is a multiresolution if each subspace V_n is the span of an orthonormal basis $\left\{\phi_{n,k}\right\}_{k\in\mathbb{Z}}$, with $\phi_{n,k}(t) =$

$2^{n/2}\phi(2^n t - k)$ and $\phi \in L_2(\mathbb{R})$, is known as the father wavelet. This idea was introduced by Mallat (1989).

As $\{\phi_{n,k}\}_{k\in\mathbb{Z}}$ are orthonormal, if $\Pi_{V_n}(y)$ denote the orthogonal projection of an arbitrary $y \in L_2(\mathbb{R})$ into V_n, then

$$\Pi_{V_n}(y)(t) = \sum_{k\in\mathbb{Z}} \langle y, \phi_{n,k} \rangle_{L_2} \phi_{n,k}(t). \tag{1}$$

Whenever ϕ has compact support, for each $t \in \mathbb{R}$ the summation in (1) contains a finite number of non null terms. Otherwise it should be truncated for practical applications. If $\{V_n\}_{n\in\mathbb{Z}}$ is a multiresolution, then

$$y(t) = \lim_{n\to\infty} \Pi_{V_n}(y)(t) = \lim_{n\to\infty} \sum_{k\in\mathbb{Z}} \langle y, \phi_{n,k} \rangle_{L_2} \phi_{n,k}(t).$$

in L_2 sense, which means that $\|y - \Pi_{V_n}(y)\|_{L_2} \to 0$ as $n \to \infty$. Under appropriate conditions, the approximation property holds in the supremum norm, for $y(t)$ continuous with compact support. The wavelet analysis can be analogously established for $L_2([a,b])$, taking a wavelet multiresolution $\{V_n\}_{n=1}^{\infty}$, where ϕ is supported on $[a,b]$.

In practice, one of the most popular wavelets systems in $L_2(\mathbb{R})$ is the compact-valued wavelet proposed by Daubechies (1992). Similar wavelets can be established for $L_2([a,b])$, see Daubechies (1994) for a detailed review. Some wavelets basis, for example the Daubechies basis of order N, also approximate the derivatives of smooth functions y. Let $W_2^q(\mathbb{R})$ the Sobolev space of functions weakly differentiable up to order q with derivatives square integrable. Then, it can be established that for any $y \in W_2^q(\mathbb{R})$ with $q \leq N$

$$D^m y(t) = \lim_{n\to\infty} \sum_{k\in\mathbb{Z}} \langle y, \phi_{n,k} \rangle_{L_2} D^m \phi_{n,k}(t)$$

for any derivative of order $m \leq q$. Furthermore, when $y \in C^q(\mathbb{R})$ and have compact support the convergence is uniform.

Whenever ϕ has compact support as Daubechies wavelets, a finite set of functions $\phi_{n,k}$ is just needed to approximate $y(t)$ for any t. In particular, we only consider those functions $\phi_{n,k}$ which support contains t. Therefore, as we only use local information at each point t, wavelets provide particularly good performance when non smooth functions $y(t)$ are approximated. However, when a smooth path is considered, splines and other polynomial basis could be recommendable.

The first wavelet basis can be at least traced to the Haar (1910) work, but the theoretical foundations of wavelets have been established by physicians and mathematicians from the early 30's to the 80's. The interest on wavelets has increased since Mallat (1989) and Meyer (1992) introduced the use of multiresolution as a framework to study wavelets expansions. A historical perspective can be found in Daubechies (1992) and Meyer (1993). Excellent monographs in wavelets are Chui (1992), Daubechies (1992), Meyer (1992, 1993) and Walnut (2001).

In order to consider the wavelet approximation of a vector of R functions $y(t) = (y_1(t), ..., y_R(t))'$, we will use the following notation throughout the remainder of the paper

$$\Pi_{V_n}(y)(t) = \sum_{k\in\mathbb{Z}} \theta^R_{n,k}\ \phi_{n,k}(t),$$

where $\theta^R_{n,k} \in \mathbb{R}^R$ is a vector of coefficients.

The second step is the choice of an appropriate finite-difference approximation for the BVP. We will consider the trapezoidal approach

$$\begin{aligned} y_n(t_i) - y_n(t_{i-1}) &= \frac{h_n}{2}\left(f(t_i, y_n(t_i)) + f(t_{i-1}, y_n(t_{i-1}))\right), \\ \alpha(y_n) &= c, \end{aligned}$$

where $h_n = (t_i - t_{i-1})$, due to its good stability properties and its applicability to systems without high order differentiability requirements. Nevertheless, under higher order differentiability, we could consider other high-order approximations such as Runge-Kutta approaches.

Thus, the third and final step consists of applying the wavelet-collocation method to the trapezoidal approximation $y_n \in V_n$, and the problem to find the solution of the BVP is reduced to solve the following system of equations in $\theta^R_{n,k} \in \mathbb{R}^R$,

$$\begin{aligned} & \sum_{k\in\mathbb{Z}} \theta^R_{n,k}\ \left(\phi_{n,k}(t_{i,n}) - \phi_{n,k}(t_{i-1,n})\right) \\ = & \frac{h_n}{2}\left(f\left(t_{i,n}, \sum_{k\in\mathbb{Z}} \theta^R_{n,k}\ \phi_{n,k}(t_{i,n})\right) + f\left(t_{i-1,n}, \sum_{k\in\mathbb{Z}} \theta^R_{n,k}\ \phi_{n,k}(t_{i-1,n})\right)\right), \\ & \sum_{k\in\mathbb{Z}} \theta^R_{n,k}\ \alpha\left(\phi_{n,k}\right) = c, \end{aligned}$$

at the points $t_{i,n} = 2^{-n}i$, $i \in \mathbb{Z}$, taking values in $[a, b]$. Let $H(\theta) = 0$ denote this system of equations. Whenever f is linear, this system is solved analytically in $\left\{\theta^R_{n,k}\right\}$. Otherwise, it is solved numerically by Newton methods. An additional advantage of the method, is that it does not require the computation of the derivatives $D\Pi_{V_n}(y_n) = \sum_{k\in\mathbb{Z}} \theta^R_{n,k}\ D\phi_{n,k}$, i.e. the computation of $D\phi_{n,k}$.

Note that the order N of the Daubechies wavelets determines the support of ϕ, and as a consequence, it could be necessary to consider a change of scale such that the support of function y is similar to the support of ϕ (see Example 2).

In Annexo A we present a MATLAB code to solve the stiff problem $\dot{y}+y = 0$, $y(0) = 1$, that illustrates how to compute the solution of a simple differential problem using the proposed method. Next we present some BVP examples to illustrate the proposed approach. The algorithm has been implemented, and the tests have been carried out, on MATLAB 6.0.

Example 1 *A two-body problem.*

Consider the periodic problem:

$$\begin{aligned} \ddot{y} &= \frac{-y}{\sqrt{y^2+z^2}},\ y(0)=1,\ \dot{y}(0)=0, \\ \ddot{z} &= \frac{-z}{\sqrt{y^2+z^2}},\ z(0)=0,\ \dot{z}(0)=1, \end{aligned} \tag{2}$$

whose analytical solution is given by $y(t) = \cos t,\ z(t) = \sin t$. We rewrite Problem (2) as

$$\begin{aligned} \dot{y} &= u,\ \dot{u} = \frac{-y}{\sqrt{y^2+z^2}},\ y(0)=1, u(0)=0, \\ \dot{z} &= v,\ \dot{v} = \frac{-z}{\sqrt{y^2+z^2}},\ z(0)=1, v(0)=0. \end{aligned}$$

For a step size $h = 2^{-2}$ over $[0,1]$, $N = 3$, $n = 2$, the proposed method obtains the approximation to its solution with an error of 10^{-3}. In Figure 1, we present a detail of the results:

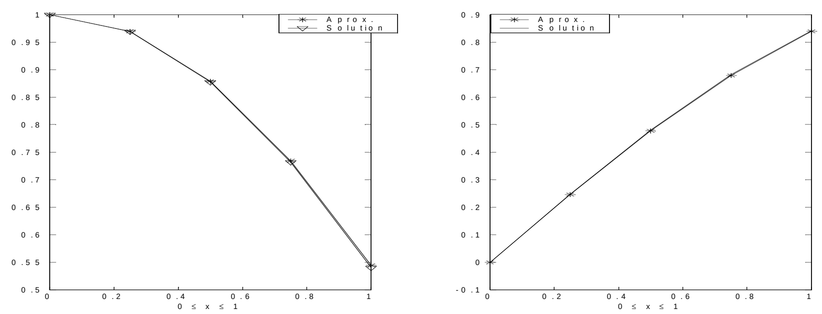

Fig. 1. Numerical resolution of Example 1, with $N = 3$, $n = 2$.

Previously, we have presented a methodology to solve *BVP*. But, as we mentioned before, often variables $y(t)$ must satisfy inequality constraints. Using the method describe above, we also propose an algorithm to deal with these inequalities based on interior-point methods.

Consider the *BVP* with $y(t) \geq 0$, for example. As $y_n \in V_n$, it is satisfied $\Pi_{V_n}(y)(t_{i,n}) = \sum_{k\in\mathbb{Z}} \theta^R_{n,k}\phi^R_{n,k}(t_{i,n}) \geq 0$ at the points $t_{i,n} = 2^{-n}i,\ i \in \mathbb{Z}$. These inequalities can be transformed into equations by adding nonnegative *slack variables*, $s_{i,n}$ as*:* $\Pi_{V_n}(y)(t_{i,n}) - s_{i,n} = \sum_{k\in\mathbb{Z}} \theta^R_{n,k}\phi^R_{n,k}(t_{i,n}) - s_{i,n} = 0$. Therefore, we will formulate the problem as a bound-constrained least squares problem:

$$\min\left\{\frac{1}{2}\left\|\begin{pmatrix} H(\theta) \\ \sum_{k\in\mathbb{Z}} \theta^R_{n,k}\phi^R_{n,k}(t_{i,n}) - s_{i,n}\end{pmatrix}\right\|_2^2,\ s\geq 0\right\}$$

and solve it as a nonlinear constrained problem. Although this problem can be solved using any standard programming packages, we propose the use of the interior-point algorithm presented in Esteban-Bravo (2003). The use of interior-point methods avoids one of the weaknesses of the least-squares approach, namely, the ill-conditioning problem often observed. An additional major advantage in terms of cost is that this algorithm exploits the special structure of the problem, omitting the second order information of the system of equations to solve as in the Gauss-Newton method.

2.1 Illustrative economic examples

In this section we present numerical results of two continuous-time life-cycle models to illustrate the performance of the proposed method in economic applications.

Example 2 *Continuous-time life-cycle model for an economy with one good and one capital stock per capita:*

$$\begin{array}{ll} \max & \int_0^T e^{-\rho t} u(c_t)\, dt \\ \text{s.t.} & \dot{A}_t = f(A_t) + w_t - c_t \\ & A(0) = a_0,\ A(T) = a_T. \end{array}$$

Assume that the asset return function is given by $f(A) = rA$, and that $u(c) = c^{1+\gamma}/(1+\gamma)$ with $\rho = 0.05$, $r = 0.10$, $\gamma = -2$, $w(t) = 0.5 + (t/10) - 4(t/50)^2$, $T = 50$ and $A(0) = A(T) = 0$ (see Judd (1998), p. 389). Then, the solutions $(c(t), A(t))$ of this problem satisfy the following system of differentials equations:

$$\begin{gathered} \dot{c} - 0.025\, c = 0, \\ \dot{A} - 0.1\ A - 0.5 - (t/10) + 4(t/50)^2 + c = 0; \end{gathered} \tag{3}$$

with the boundary conditions $A(0) = 0$ and $A(T) = 0$. And, by applying the proposed method with $N = 3$, $n = 2$ and changing the time scale to $[0, 5]$, the equilibrium path is obtained as illustrated by Figure 2.

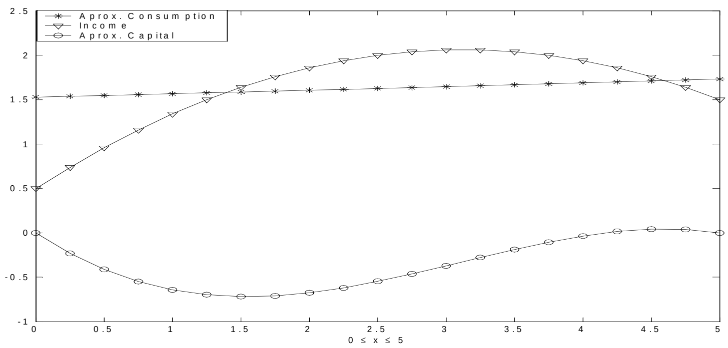

Fig. 2. Numerical resolution of Example 2, with $N = 3$, $n = 2$.

Example 3 *Continuous-time life-cycle model of Example 2 with borrowing constraints:*

$$\begin{array}{ll} \max & \int_0^T e^{-\rho t} u\left(c_t\right) dt \\ \text{s.t.} & \overset{\bullet}{A}_t = f\left(A_t\right) + w_t - c_t, \\ & A_t \geq 0, \\ & A\left(0\right) = a_0,\ A\left(T\right) = a_T. \end{array}$$

The solutions of this problem (c_t, A_t) are the solution of the System (3) with the boundary conditions $A(0) = a_0$ and $A(T) = a_T$, and the inequality constraints $A_t \geq 0$ for all $t \in [0, T]$. We will consider the equivalent constraints $A_t - s_t = 0$ for all $t \in [0, T]$, where $s_t \geq 0$ are nonnegative slack variables. Thus, approximating (c_t, A_t) with a Daubechies wavelets $N = 3$, $n = 2$ as $c_t = \sum_{k \in \mathbb{Z}} a_k \phi_{n,k}(t)$ and $A_t = \sum_{k \in \mathbb{Z}} b_k \phi_{n,k}(t)$, the problem to be solved is

$$\begin{array}{ll} \min & \frac{1}{2} \left\| \begin{pmatrix} H\left(\{a_k\}_k, \{b_k\}_k\right) \\ \sum_{k \in \mathbb{Z}} b_k \phi_{n,k}\left(t_{i,n}\right) - s_{i,n} \end{pmatrix} \right\|_2^2 \\ s.t. & s_{i,n} \geq 0,\ \forall n, i \text{ such that } t_{i,n} = 2^{-n} i \in [0, T], \\ & A_0 = \sum_{k \in \mathbb{Z}} b_k \phi_{n,k}(0) = 0, \\ & A_T = \sum_{k \in \mathbb{Z}} b_k \phi_{n,k}(T) = 0. \end{array}$$

And, by changing the time scale to $[0, 5]$, Figure 3 contains the equilibrium path obtained:

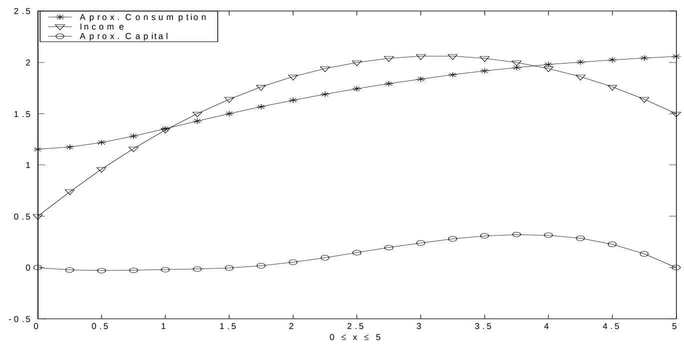

Fig. 3. Numerical resolution of Example 3, with $N = 3$, $n = 2$.

3 Convergence analysis

In this section, we will prove the convergence of the method. First, we will establish an interpolative property for wavelets. We will formulate the following assumption, which is satisfied for the commonly used wavelets:

A.1. Let $\{V_n\}$ be a multiresolution in $L_2(\mathbb{R})$, with compactly supported father wavelet ϕ and assume for all $y \in W_2^r(\mathbb{R})$ with $1 \leq r \leq q$, $q \geq 1$, and all integer vector ν, $0 \leq \|\nu\|_1 \leq r - 1$, it is satisfied

$$\|D^\nu y - D^\nu \Pi_{V_n}(y)\|_{L_2} = O\left(2^{-(r - \|\nu\|_1)n}\right),$$

Whenever $y \in C^r(\mathbb{R})$ with compact support the same rates are satisfied replacing the L_2 norm by the supremum norm. In spaces $L_2([a, b])$, an analogous behavior is assumed.

Several sufficient conditions for this result can be found in the literature, often based on the regularity of order q assumption. The father wavelet ϕ is said to be *regular of order* $q \in \mathbb{N}$, if ϕ has a version q times continuously differentiable and for $0 \leq \|\nu\|_1 \leq q$, and any positive integer $p \in \mathbb{N}$, there exists a constant $C_p > 0$ such that

$$|D^\nu \phi(x)| < (1 + \|x\|)^{-p} C_p, \qquad \forall x \in \mathbb{R}.$$

See Meyer (1992) for details.

In order to prove the convergence of the proposed method, we first provide a result on interpolation which will play a crucial role to prove the wavelet-collocation convergence, then we prove the convergence of the wavelet-Galerkin and finally, the convergence of the wavelet-collocation methods. All the proofs can be found in Annexo B.

Theorem 4 *Consider a multiresolution $\{V_n\}$ in $L_2(\mathbb{R})$ satisfying A.1. For each $y \in L_2(\mathbb{R})$ with an a.e. continuous version with compact support, we define $\Gamma_{V_n}(y)$ as the function $g_n \in V_n$ such that $g_n(x_{n,i}) = y(x_{n,i})$, for all $\{x_{n,i} = 2^{-n}i\}_{i\in\mathbb{Z}}$, that is, the function $g_n(x) = \sum_{k\in\mathbb{Z}} \theta_{n,k}\phi_{n,k}(x)$ satisfies*

$$\sum_{k\in\mathbb{Z}} \theta_{n,k}\phi_{n,k}(x_{n,i}) = y(x_{n,i}).$$

Then, there exists a unique $\Gamma_{V_n}(y)$.

Furthermore, assuming

1. *ϕ is regular of order $q \geq 1$, and*

2. *the Poisson summa $\sum_{k\in\mathbb{Z}} \Phi(\omega + 2\pi k) > 0$, for almost every $\omega \in [0, 2\pi]$, being $\Phi(\omega) = \int_{\mathbb{R}} \phi(x) e^{ix\omega} dx$ the Fourier transformed of ϕ;*

for all $y \in W_2^q(\mathbb{R})$ with compact support, there exist $K > 0$ and n_0 such that, $\forall n > n_0$,

$$\|\Gamma_{V_n}(y) - y\|_{L_2} \leq K \|\Pi_{V_n}(y) - y\|_{W_2^q}.$$

The same result trivially holds for multiresolutions in $L_2([a, b])$.

Next, we prove the convergence of the wavelet-Galerkin and wavelet-collocation methods. Note that there is a unique solution associated to the homogeneous problem $Dy(x) = 0$ with $\alpha(y) = c$ since α are linearly independent (at least over $Ker\{D\}$). Moreover, let define a Green's matrix of functions $G(x, z)$ such that any g continuous in $[a, b]$ with Dg integrable can be expressed as follows

$$g(x) = P_0(g)(x) + \int_a^b G(x, z)\, Dg(z)\, dz,$$

where $P_0(g)$ is the unique element in $Ker\{D\}$ which agrees with $\alpha(g)$. Furthermore,

$$Dg(z) = D_x P_0(g)(x) + \int_a^b D_x G(x, z)\, g(z)\, dz.$$

As a consequence, the following property can be used to express the BVP in a more convenient way: Let define $Dy = u$, thus $u = G[y]$ and $G^{-1}[u] = y$, reciprocally, with

$$\begin{aligned} G[y](x) &= P_0(y)(x) + \int_a^b G(x, z)\, y(z)\, dz, \qquad (4) \\ G^{-1}[u](x) &= D_x\{P_0(y)(x)\} + \int_a^b D_x G(x, z)\, y(z)\, dz. \end{aligned}$$

Therefore, rewriting the BVP as $u = f\left(x, G^{-1}[u]\right)$ and defining $T(u) := f\left(x, G^{-1}[u]\right)$, we can guarantee the existence of solution in BVP by proving the existence of a fixed point u for T, $u = Tu$. It is sufficient to prove that T is a continuous retraction on the Banach space $C([a,b])^R$, and a unique solution u_0 exists, so that $y_0 = G^{-1}(u_0)$ is the unique solution of BVP.

Now, given the multiresolution $\{V_n\}$, let $y_n \in V_n$ be the wavelet-Galerkin solution to the BVP, and therefore y_n satisfies

$$\Pi_{V_n}\{Dy_n - f(x, y_n)\} = 0, \qquad \alpha(y_n) = c.$$

The next result establishes the rate of approximation of the wavelet-Galerkin method.

Theorem 5 *Let consider the problem BVP with solution* $y_0(x)$, *and a multiresolution sequence* $\{V_n\}$ *in* $L_2([a,b])$ *such that* $\|y - \Pi_{V_n}(y)\|_{L_\infty} \to 0$, *for all* $y \in C([a,b])$. *Let define the curve* $\mathcal{C} = \left\{\left(x, y_0(x)'\right)' : x \in [a,b]\right\}$. *Assume that,* $f \in C^2(\mathcal{N})$ *where* $\mathcal{N} \subset \mathbb{R}^{R+1}$ *is an* ε*-neighborhood of* $\mathcal{C}$ *in the* L_∞ *norm, and it is satisfied that* $\det\{(I - D_y f(x, y_0(x)))\} \neq 0$, *for all* $x \in [a,b]$. *Then there exist* $\delta > 0$ *and an integer* M *such that* y_0 *is unique in* $B(y_0, \delta) = \left\{y : \|y - y_0\|_{L_\infty} \leq \delta\right\}$, *and the projected system*

$$\Pi_{V_n}\{Dy_n - f(x, y_n)\} = 0,$$

has a unique solution $y_n \in V_n \cap B(y_0, \delta)$. *Furthermore, there exists* $c > 0$ *such that*

$$\max\left\{\|y_n - y_0\|_{L_\infty}, \|Dy_n - Dy_0\|_{L_\infty}\right\} \leq c\,\|Dy_0 - \Pi_{V_n}(Dy_0)\|_{L_\infty}.$$

If $\{V_n\}$ *satisfies assumption A.*1. *and* $y \in C^1([a,b])$, *then*

$$\max\left\{\|y_n - y_0\|_{L_\infty}, \|Dy_n - Dy_0\|_{L_\infty}\right\} = O\left(2^{-n}\right).$$

Given the multiresolution $\{V_n\}$, let $y_n \in V_n$ denote the wavelet-collocation solution to the BVP, and therefore satisfying

$$\Pi_{V_n}\{Dy_n - f(x, y_n)\} = 0, \qquad \alpha(y_n) = c.$$

The following result is an immediate consequence of Theorems 4 and 5.

Corollary 6 *Under the assumptions of Theorems 4 and 5, the wavelet-collocation method satisfies the approximation property at rate* $O(2^{-n})$.

Therefore, we only need to prove the consistence of the proposed method based on the trapezoidal rule:

$$\widetilde{y}_n(x_{i,n}) - \widetilde{y}_n(x_{i-1,n}) = \frac{h_n}{2}\left(f(x_{i,n}, \widetilde{y}_n(x_{i,n})) + f(x_{i-1,n}, \widetilde{y}_n(x_{i-1,n}))\right), \quad (5)$$

for $\widetilde{y}_n \in V_n$.

Proposition 7 *Under the assumptions of Theorems 5 and 4. Let* $\widetilde{y}_n \in V_n$ *denote the approximation generated by the proposed method and* y_n *the solution of the wavelet-collocation method. Then, it is satisfied* $\|\widetilde{y}_n - y_n\| = O(2^{-n})$.

4 Extension to Stochastic BVP

Finally, we consider the extension of the proposed method to the numerical resolution of stochastic differential equations with functional boundary conditions. Such systems are the natural extension of deterministic BVP, when the uncertainty is included in terms of a Brownian motion.

There is an increasing literature on boundary value stochastic differential equations, see e.g. Huang (1984), Ocone and Pardoux (1998), Nualart and Pardoux (1991a), and Alabert and Ferrante (2002). Typical examples are $\dot{Y}(t) = f(Y(t)) + \dot{W}(t)$, for $t \in [0,T]$ with $a_1 Y(0) + a_2 Y(T) = \rho$, or the second order integral $\ddot{Y}(t) = f\left(Y(t), \dot{Y}(t)\right) + \dot{W}(t)$ with $Y(0) = \rho_1$, $Y(T) = \rho_2$. As second order systems can be expressed as a first order system in the space of states, see e.g. Nualart and Pardoux (1991b), we will just focus on the first order case. In this context, the solution will not be Markovian, though some weak analogous properties can be considered see Alabert et al (1995) and Alabert and Ferrante (2002).

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space on which a standard $R-$dimensional Brownian motion $\{W_t : t \in [0,T]\}$ with $W_0 = 0$ a.e. is defined, and $\{\mathcal{F}_t\}$ is a left continuous filtration, where $\mathcal{F}_t$ is the completion of the σ-algebra generated by $\{W_s : 0 \leq s \leq t\}$. The stochastic BVP_is defined as:

$$\begin{gathered} dY_t = b(t, Y_t)\, dt + \sigma dW_t, \\ \alpha(Y) = c, \end{gathered}$$

for $t \in [0,T]$, where $\sigma \in \mathbb{R}^{R\times R}$, $\|b(t,y)\| \leq C_1(1 + \|y\|)$, and $\|b(t,y) - b(t,z)\| \leq C_2 \|y - z\|$, for all $y, z \in \mathbb{R}$, uniformly in $t \in [0,T]$, and $c \in \mathbb{R}^R$ and $\alpha = (\alpha_1, ..., \alpha_R)'$ is a vector of R-linearly independent continuous linear functionals $\alpha_j : C([0,T]) \to \mathbb{R}$, which can be expressed as $\alpha_j(Y) = \int_0^T Y_t \nu_j(dt)$ where ν is a bounded signed Borel measure on $[0,T]$.

The existence of solution of the stochastic boundary value problem can be studied analogously to deterministic systems. Define $Dy = u$, thus $u = G[y]$ and $G^{-1}[u] = y$; with operator G defined as in (4). Therefore, defining $U = G(Y)$, and the nonlinear operator

$$T[U](t) := b\left(t, G^{-1}[U](t)\right) + \sigma\left(t, G^{-1}[U](t)\right)\dot{W}_t,$$

we can express the stochastic BVP as $U = T[U]$. Therefore, we can guarantee the existence of solution in BVP by proving the existence of a fixed point U^0 for T, and the stochastic BVP has a unique solution which can be expressed as $Y^0 = G^{-1}\left(U^0\right)$. However the solution of a stochastic BVP is not necessarily $\mathcal{F}_t$ adapted nor Markovian.

For the sake of simplicity we present the method for autonomous differential equation systems $dY_t = b(Y_t)\, dt + \sigma dW_t$, with $\alpha(Y) = c$. The extension to non homogenous equations is straightforward.

As in the deterministic context, the wavelet method can be used to approximate the solution of the stochastic BVP. Instead of the trapezoidal rule, we consider the Euler-Maruyama approach (for an introduction to stochastic finite-difference approximation methods, see Kloden and Platen, 1999),

$$\begin{aligned} Y_n(t_{i,n}) - Y_n(t_{i-1,n}) &= (t_{i,n} - t_{i-1,n})\, b(Y_n(t_{i-1,n})) + \sigma\left(W_{t_{i,n}} - W_{t_{i-1,n}}\right), \\ \alpha(Y_n) &= c. \end{aligned}$$

Applying the wavelet-collocation method to the Euler-Maruyama approximation, the computation of the approximate solution of the stochastic BVP is reduced to solve the following system of equations in $\theta^R_{n,k} \in \mathbb{R}^R$,

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \theta^R_{n,k}\ \left(\phi_{n,k}(t_{i,n}) - \phi_{n,k}(t_{i-1,n})\right) &= h_n b\left(\sum_{k\in\mathbb{Z}} \theta^R_{n,k}\ \phi_{n,k}(t_{i-1,n})\right) \\ &\quad +\sigma\left(W_{t_{i,n}} - W_{t_{i-1,n}}\right), \\ \sum_{k\in\mathbb{Z}} \theta^R_{n,k}\ \alpha\left(\phi_{n,k}\right) &= c, \end{aligned}$$

where $t_{i,n} = 2^{-n} i \in [0,T]$ with $i \in \mathbb{Z}$, and $h_n = 2^{-n}$.

The same methodology can be applied to the non autonomous differential equation systems, considering an approximation for non homogeneous stochastic differential equations, see e.g. Kloeden and Platen (1999, Chap 5, Sect. 5).

The following example illustrate the good performance of the method using Daubechies wavelets.

Example 8 *Stochastic BVP with integral boundary condition.*

Consider the problem

$$dY_t = dW_t,\ t \in [0,1],\ \text{with}\ \int_0^1 Y_t = 0.$$

The solution of this problem has the general form $Y_t = -\int_0^1 W_t dt + W_t$. Using the step length $h = 2^{-6}$, Figure 4 shows the computed approximation by the proposed approach with $\|Y^*(t_i) - Y_{\theta^*,n}(t_i)\|_2 \leq 10^{-4}$,where $t_i = 2^{-6} i$, for all $i = 0, 1, ..., 2^n(2N-1)$, with $N = 3$, $n = 6$.

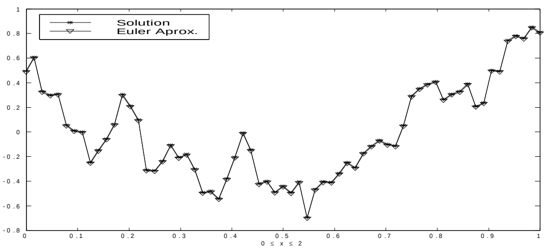

Fig 4. Numerical resolution of Example 4 with $N = 3$, $n = 6$.

The wavelet-collocation method is also useful in order to solve stochastic differential equations with inequality constraints, such as $dY_t = b(Y_t)\,dt + \sigma(Y_t)\,dW_t$, with $Y_0 = c$, and $\delta(Y_t) \geq 0$ for all $t \in [0, T]$, where δ is a continuous linear functional. In this context, if σ is differentiable we can use the Milstein finite-difference approach instead of the classical Euler-Maruyama approach,

$$\begin{aligned}
&Y_n(t_{i,n}) - Y_n(t_{i-1,n}) = h_n b(Y_n(t_{i-1,n})) + \sigma\left(W_{t_{i,n}} - W_{t_{i-1,n}}\right) \\
&+\sigma(Y_n(t_{i-1,n}))\frac{\partial \sigma}{\partial x}(Y_n(t_{i-1,n}))\left[\int_{t_{i-1,n}}^{t_{i,n}}\int_{t_{i-1,n}}^{s_1} dW_{s_1}dW_{s_2}\right], \\
&\delta(Y_n(t_{i,n})) - s_{i,n} = 0, \\
&Y_n(0) = c,
\end{aligned}$$

where $s_{i,n} \geq 0$ are slack variables. The double stochastic integral can be readily computed, e.g. in the scalar case

$$\int_{t_{i-1,n}}^{t_{i,n}}\int_{t_{i-1,n}}^{s_1} dW_{s_1}dW_{s_2} = \frac{1}{2}\left(\left(W_{t_{i,n}} - W_{t_{i-1,n}}\right)^2 - h_n\right).$$

For the multivariate case see Kloeden and Platen (1999, Chap 5, Sec. 8).

Thus, applying the wavelet-collocation method to this approximation, the computation of the stochastic BVP is reduced to solve the above system of equations in $\theta_{n,k}^R \in \mathbb{R}^R$, via interior point methods as in the deterministic case. To illustrate the approach, consider the following economic example.

4.1 Stochastic Solow-Swan Model

Consider the neoclassic growth model introduced by Solow (1956) and Swan (1956), with a two factors technology $Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$, as function of capital K_t and work L_t, where A_t is the exogenous technological component. Assume a constant rate of saving $s \in (0,1)$, and an exogenous rate for the population $\dot{L}_t = nL_t$, in equilibrium the stock of capital per capita $k_t = K_t/L_t$, follows the *fundamental equation of the Solow-Swan model*,

$$\dot{k}_t = sA_t k_t^{\alpha} - (\delta + n)\, k_t.$$

There are many variations of the Solow-Swan model, for details see e.g. Barro and Sala-i-Martin (1995). The classical model considers $A_t = A > 0$, but Romer (1986) introduced external effects due to the learning by doing effect, and the knowledge spillovers. Considering $A_t = Ak_t^{\eta}$ we obtain $\dot{k}_t = sAk_t^{\alpha+\eta} - (\delta + n)\, k_t$.

However, there are random and unknown effects affecting the technology evolution. Assume that $A_t = A_0 k_t^{\eta} + \sigma \overset{\bullet}{W}_t$, with $A_0 > 0$, where $\overset{\bullet}{W}_t$ denotes Gaussian white noise generalized stochastic process. Then, we get the stochastic equation

$$dk_t = \left(A_0 k_t^{\alpha+\eta} - (\delta + n)\, k_t\right) dt + \left(s k_t^{\alpha} \sigma\right) dW_t,$$

with $k_0 > 0$.

In order to obtain a meaningful solution it is necessary to set $A_t \geq 0$, for all $t \in [0, T]$. This constraint is equivalent to $\int_0^{t_1} A_s ds \leq \int_0^{t_2} A_s ds$ for all $t_1 \leq t_2$, where $\int_0^{t} A_s ds = A \int_0^{t} k_s^{\eta} ds + \sigma W_t$. In other words, for all $t_1 \leq t_2$, $A \int_{t_1}^{t_2} k_s^{\eta} ds + \sigma \left(W_{t_2} - W_{t_1}\right) \geq 0$. Imposing these constraints at consecutive points $t_{i,n}$, $t_{i-1,n}$ where $t_i = 2^{-n} i$, for all $i = 0, 1, ..., 2^j\,(2N-1)$, and approximating $\int_{t_{i-1,n}}^{t_{i,n}} k_s^{\eta} ds \approx k_{t_{i-1,n}}^{\eta} \left(t_{i,n} - t_{i-1,n}\right)$, the inequality constraints to be satisfied are:

$$k_{t_{i-1,n}}^{\eta} \left(t_{i,n} - t_{i-1,n}\right) + \sigma \left(W_{t_{i,n}} - W_{t_{i-1,n}}\right) \geq 0.$$

Assuming $A = 1$, $\alpha = 0.33$, $\eta = 0.03$, $\delta = 0.03$, $n = 0.01$, $s = 0.25$ and $\sigma = 3$, we compute its numerical solution by the presented method using Euler-Maruyama and Milstein finite-differences approaches (see Fig. 5).

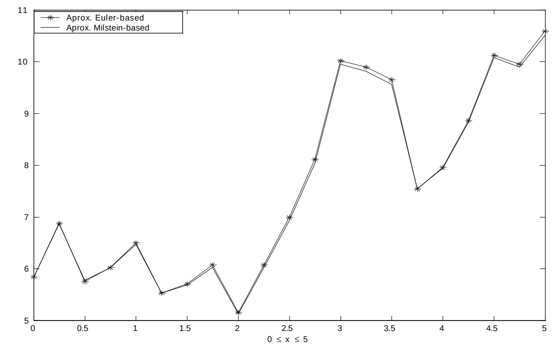

Fig 5. Numerical resolution of Stochastic Solow-Swan model with $N = 3$, $n = 6$.

5 Conclusions

In this paper we present a projection-based method derived from the recursive rules of the finite-iterative methods for computing deterministic or stochastic differential equations with boundary conditions. We also consider the numerical solution of BVP with inequality constraints. This type of problems are common in the economics literature. The results confirm the well-performance of the proposed approach, achieving a high level of accuracy, and its practicability. Therefore, this approach is an efficient alternative to the existing methods for solving boundary-value problems. Given the local adaptability of wavelets, this method is especially useful when the solution path is nonsmooth.

6 Annexo A: MATLAB code example

% This MATLAB program computes the solution of the problem $dy\,(x)\,/dx + Ay\,(x) = 0$, and the initial condition $y(0) = 1$, for all x, $T1 <= x <= T2$, by the proposed wavelet-based method using the implicit trapezoidal method. Solution $y = exp(-x)$.

```
y0=1;
N=3; % N order of the Daubechies wavelets.
n=2; % n level of resolution of the wavelets
T1=0; % [T1,T2] interval
T2=6;
wname='db3'; % wavelet function: Daubechies, dbN with N=3
h=2^(-n); % distant between points where the wavelet function is defined
```

```
% definition of the wavelet and scaling function given by the Wavelet toolbox
[phi,psi,x] = WAVEFUN(wname,n);
dimension=length(x);
% number of points where the wavelet and scaling function is defined
n_vars=T2*2^n-1-(T1*2^n+2-2*N)+1; % number of coefficients
theta=zeros(n_vars,1); % initialization of the coefficients to zero
% Computing the value of sum(phi(x(i,n))) in k
for i=T1*2^n+2-2*N:T2*2^n-1,
summ(:,i-(T1*2^n+2-2*N)+1)=zeros(dimension,1);
for l=1:length(x),
phi_i(l)=0;
if ( ((2^n)*x(l)-i>=0) & (((2^n))*x(l)-i<=2*N-1) ),
phi_i(l) = (2^(n/2)) * phi(((2^n)*x(l)-i)*2^n+1);
end
end
summ(:,i-(T1*2^n+2-2*N)+1)=phi_i(:);
end
% Compute sum(phi(x0(i,n))) in k
aux_K=zeros(n_vars,1);
for j=T1*2^n+2-2*N:T2*2^n-1,
if ((-j>=0) & (-j<=2*N-1)),
aux_K(j-(T1*2^n+2-2*N)+1)=(2^(n/2))*phi((-j)*2^n+1); end
end
% Compute K and b such that K*theta=b.
K=[
(1+(h/2)).*summ(2:dimension,:)+(-1+(h/2)).*summ(1:dimension-1,:);
aux_K'];
b=[zeros(dimension-1,1);y0];
% Solve the system K*theta=b.
theta=K\b;
% Then, compute the approximation
x=0:h:T2;
aprox=zeros(length(x),1);
for k = T1*2^n+2-2*N:T2*2^n-1,
for i=1:dimension,
phijk(i,k-(T1*2^n+2-2*N)+1)=0;
if (((2^n)*x(i)-k>=0) & (((2^n))*x(i)-k<=2*N-1)),
phijk(i,k-(T1*2^n+2-2*N)+1)=(2^(n/2))*phi(((2^n)*x(i)-k)*2^n+1);
end
aprox(x(i)*2^n+1)=aprox(x(i)*2^n+1)+theta(k-(T1*2^n+2-2*N)+1).*phijk(i,k-
(T1*2^n+2-2*N)+1);
end
end
% Plot of the approximation and the exact solution.
f=(y0).*exp(-x);
figure(1)
```

```
plot(x,f,'-*',x,aprox,'s-')
h =title('Exact and Approximate solution to dy/dx+y=0');
h = legend('Solution','Approx.',2);
xlabel('0 \leq x \leq 6')
disp('||Solution-Aprox||')
disp(norm((y0).*exp(-x)'-aprox))
disp('x, Solution, Approximation, Error')
disp([x',f',aprox,f'-aprox])
```

7 Annexo B: Proofs

A) Proof of Theorem 4

The problem of interpolation in V_n at points $x_{n,i} = 2^{-n}i$ can be reduced to solve the problem $g_0(i) = y(x_{n,i})$ in $g_0 \in V_0$ and then take $g_n(x) = 2^{n/2} g_0(2^n x)$. Therefore, assume that $g_0(x) = \sum_{k\in\mathbb{Z}} \theta_k \phi(x-k)$ solves this problem, i.e.

$$\sum_{k\in\mathbb{Z}} \theta_k \phi(i-k) = y(x_{n,i}).$$

Clearly, a unique solution exists since $\{\phi(x-k)\}_{k\in\mathbb{Z}}$ are linearly independent functions. To simplify the notation, we denote $y_i = y(x_{n,i})$, hence $\sum_{k\in\mathbb{Z}} \theta_k \phi(i-k) = y_i$. This is a convolution equation that we will solve in the spectral domain. Let define the discrete Fourier transform of ϕ by

$$\widetilde{\Phi}(\omega) = \sum_{k\in\mathbb{Z}} \phi(k) e^{-ik\omega}.$$

The Poisson formula states that $\widetilde{\Phi}(\omega) = \sum_{k\in\mathbb{Z}} \Phi(\omega + 2\pi k)$. If ϕ is regular of at least order 1, this series converges uniformly on compact sets. Furthermore, as $\widetilde{\Phi}(\omega) > 0$ a.e. for $\omega \in [0, 2\pi]$, the inverse has a Fourier expansion $\left(1/\widetilde{\Phi}(\omega)\right) = \sum_{k\in\mathbb{Z}} \beta_k e^{-ik\omega}$ where $b := \sum_{k\in\mathbb{Z}} |\beta_k| < \infty$, by the Wiener-Lévy theorem. Thus, we can explicitly evaluate the coefficients $\{\theta_k\}$ as,

$$\theta_k = \sum_{k\in\mathbb{Z}} \beta_{k-i} y_i.$$

Obviously,

$$\sum_{k\in\mathbb{Z}} |\theta_k|^2 = \sum_{k\in\mathbb{Z}} \left| \sum_{l\in\mathbb{Z}} \beta_{l-i} y_i \right|^2 \leq b^2 \sum_{k\in\mathbb{Z}} |y_i|^2 = b^2 \sum_{i\in\mathbb{Z}} |y(x_{n,i})|^2,$$

with $\sup_{n>1} \sum_{i\in\mathbb{Z}} |y(x_{n,i})|^2 < \infty$ as y is continuous with compact support.

Next, we will prove that

$$\|\Gamma_{V_n}(y)\|_{L_2} \leq b \|y\|_n,$$

where $\|y\|_n = \left(2^{-n}\sum_{i\in\mathbb{Z}}|y(x_{n,i})|\right)^{1/2}.$

Notice that $\|g_n\|_{L_2} = 2^{-n}\|g_0\|_{L_2} = 2^{-n}\|\mathcal{F}(g_0)\|_{L_2}$, where $\mathcal{F}(g_0)(\omega)$ is the continuous Fourier transformed of g_0. We will prove that $\|\mathcal{F}(g_0)\|_{L_2}^2 = \sum_{k\in\mathbb{Z}}|\theta_k|^2$ and the result follows. Let define $\widetilde{c}(\omega) = \sum_{k\in\mathbb{Z}}\theta_k e^{-ik\omega}$, then

$$\begin{aligned}
\|\mathcal{F}(g_0)\|_{L_2}^2 &= \int_{\mathbb{R}}\left|\mathcal{F}\left(\sum_{k\in\mathbb{Z}}\theta_k\phi_{0,k}\right)(\omega)\right|^2 d\omega = \int_{\mathbb{R}}\left|\Phi(\omega)\left(\sum_{k\in\mathbb{Z}}\theta_k e^{-ik\omega}\right)\right|^2 d\omega \\
&= \int_{\mathbb{R}}|\Phi(\omega)\widetilde{c}(\omega)|^2 d\omega = \sum_{k\in\mathbb{Z}}\int_{2k\pi}^{2(k+1)\pi}|\Phi(\omega)\widetilde{c}(\omega)|^2 d\omega \\
&= \int_0^{2\pi}|\widetilde{c}(\omega)|^2\left|\sum_{k\in\mathbb{Z}}\Phi(\omega+2\pi k)\right|^2 d\omega \\
&= \int_0^{2\pi}|\widetilde{c}(\omega)|^2 d\omega = \sum_{k\in\mathbb{Z}}|\theta_k|^2,
\end{aligned}$$

as $\{\phi(x-k)\}_{k\in\mathbb{Z}}$ is orthonormal if and only if $\sum_{k\in\mathbb{Z}}|\Phi(\omega+2\pi k)|^2 = 1$ a.e., for details see Daubechies (1992). Hence, we have that $\|\Gamma_{V_n}(y)\|_{L_2}^2 \leq b^2\|y\|_n^2.$

Defining $y_n = \Pi_{V_n}(y)$, we have that $\Gamma_{V_n}(y_n) = y_n$ since $y_n \in V_n$ and has compact support. And as a consequence,

$$\begin{aligned}
\|\Gamma_{V_n}(y) - y\|_{L_2} &= \|\Gamma_{V_n}(y_n - y) + y_n - y\|_{L_2} \leq b^2\|y_n - y\|_n + \|y_n - y\|_{L_2} \\
&= b^2\|\Pi_{V_n}(y) - y\|_n + \|\Pi_{V_n}(y) - y\|_{L_2}.
\end{aligned}$$

Moreover, as for all $y \in W_2^r(\mathbb{R})$, with $r \geq 1$,

$$\|y\|_n^2 \leq C\left\{\int_{-2^n\pi}^{2^n\pi}|\mathcal{F}(y)(\omega)|^2 d\omega + 2^{-nr}\|y\|_{W_2^r}^2\right\}$$

see Thomée (1973, Lemma 4.4.), the result follows applying the same bound to $\|\Pi_{V_n}(y) - y\|_n^2.$

B) Proof of Theorem 5

We will use the following Theorem,

Theorem 9 *Let B a Banach space, $\{V_n\} \subset B$ a sequence of increasing linear subspaces, and Π_{V_n} a sequence of continuous projections converging pointwise to the identity operator on B. Let T a (non linear) operator in B. If $(1-T)u = 0$ has a solution u_0, T is continuously Frechet differentiable at u_0 and $\left(1-T'_{u_0}\right)u = 0$ has only the trivial solution in B, then u_0 is unique in some sphere $B(u_0,\delta) = \{u \in B : \|u-u_0\| \leq \delta\}$ for some $\delta > 0$, and there exists an integer M such that for all $n > M$ the equation $\Pi_{V_n}\{(1-T)u\} = 0$ has a unique solution $u_n \in V_n \cap B(u_0,\delta)$. Moreover, $\exists K > 0$ such that*

$$\|u_n - u_0\| \leq K\|\Pi_{V_n}u_0 - u_0\|.$$

Proof. See e.g. Vainikko (1967, Th. 5). ■

Using the properties of the Green function and the continuity of f, the functional T is continuous relative to the uniform norm on a neighborhood of $u_0 = G(y_0)$. Since $(1-T)u = 0$ can be seen as an equation in $C([a,b])$, we will consider the equation $\Pi_n (I-T) u_n = 0$ in V_n.

First, we check the continuously Frechet differentiability of T. For any $u \in B(u_0, \delta)$ define $h = u - u_0$. Notice that $\mathcal{N}$ contains all line segments in $\mathbb{R}^{R+1}$ such as $\{u_0 + \theta h : \theta \in [0,1]\}$, since

$$y(x) - y_0(x) = \int_a^b D_x G(x,z) h(z) \, dz,$$

with $\|y - y_0\|_{L_\infty} < \varepsilon$ whenever δ is small enough, using that

$$\chi := ess \sup_{x \in [a,b]} \int_a^b |D_x G(x,z)| \, dz < \infty.$$

Notice that $u_0 = G(y_0)$. We will see that the Frechet derivative T at $u_0(x) = Dy_0(x)$ respect to the direction $h = (u - u_0)$ is given by

$$T'_{u_0}(h)(x) = D_u f(x, u_0) \int_a^b D_x G(x,z) h(z) \, dz,$$

and the error term is given by

$$\epsilon_{u_0}(u)(x) = |||h|||^2 \int_a^b (1-\theta) f''(x, u_0(x) + \theta h(x)) \, dz,$$

being f'' the second directional derivative of $f(x, \cdot)$ in the direction $h/|||h|||$, and $|||h|||^2 = \sum_{r=1}^R \|h_r\|^2$. Clearly $\|\epsilon_{u_0}(u)\|_{L_\infty} \le c_1 \|u - u_0\|_{L_\infty}^2$, where c_1 is the maximum between the bound on f'' over all directions on $adh(\mathcal{N})$ and χ.

Notice also that $T'_{u_0}(h)(x)$ can be expressed in the original domain as the operator $T'_{y_0}(y) = D_y f(x, y_0(x)) Dy$. Since $\det\{(I - D_y f(x, y_0(x)))\} \neq 0$, for all $x \in [a,b]$, there exists a unique trivial solution for $(I - D_y f(x, y_0(x))) Dy = 0$ with $\alpha(y) = c$. This implies the same result for $\left(I - T'_{u_0}\right) u = 0$, hence assumptions of Theorem 9 are satisfied.

Thus, there exists an integer $M > 0$ such that, for all $n > M$ a solution $u_n \in V_n$ exists and is unique in the same sphere. Moreover, there exists a constant $c > 0$ such that $u_n = Dy_n$, $u_0 = Dy_0$ and

$$\|u_n - u_0\|_{L_\infty} \le c \|\Pi_n u_0 - u_0\|_{L_\infty}.$$

By the Banach-Steinhaus theorem,

$$\begin{aligned} \|\Pi_n u_0 - u_0\|_{L_\infty} &= \|\Pi_n (u_0 - u) - (u - u_0)\|_{L_\infty} = \|(1 - \Pi_n)(u_0 - u)\|_{L_\infty} \\ &\le c' \inf\left\{\|u_0 - u\|_{L_\infty} : u \in V_n\right\}, \end{aligned}$$

therefore $\exists K > 0$ such that

$$\|Dy_n - Dy_0\|_{L_\infty} = \|u_n - u_0\|_{L_\infty} \leq K \inf \left\{ \|u_0 - u\|_{L_\infty} : u \in V_n \right\}.$$

Finally, using that $y_n - y_0 = G^{-1}(u_n - u_0)$, we have

$$\|y_n - y_0\|_{L_\infty} \leq \left\|G^{-1}\right\|_{L_\infty} \|u_n - u_0\|_{L_\infty},$$

Finally, the rate $O(2^{-n})$ follows from Assumption A.1.

C) Proof of Proposition 7

By assumption $f \in C^2(\mathcal{N})$. The collocation solution satisfies the system $y_n(x_{i,n}) - y_n(x_{i-1,n}) = \int_{x_{i-1,n}}^{x_{i,n}} f(x, y_n(x))\, dx$, and we have proved that $y_n \to y_0$ uniformly. Using the trapezoidal integration rule it can be expressed as,

$$\begin{aligned} y_n(x_{i,n}) - y_n(x_{i-1,n}) &= \frac{h_n}{2}\left[f(x_{i,n}, y_n(x_i)) + f(x_{i-1,n}, y_n(x_{i-1,n}))\right] \\ &\quad - \frac{h_n^3}{12} f''\left(\xi_{i,n}, y_n\left(\xi_{i,n}\right)\right), \end{aligned}$$

with $\xi_{i,n} \in [a,b]$. Let $A_n(y_n) = b_n(y_n)$ denote this system of nonlinear equations, where $\|b_n(y_n)\| \leq h_n^3 M/12$.

Let $\widetilde{y}_n(x)$ denote the solution of the proposed method that satisfies (5). Let $A_n(\widetilde{y}_n) = 0$ denote this system of nonlinear equations. Then, applying the mean value theorem, we have

$$b_n(y_n) = A_n(y_n) = A_n(\widetilde{y}_n) + DA_{\varphi_n}(y_n - \widetilde{y}_n) = DA_{\xi_n}(y_n - \widetilde{y}_n),$$

where DA_{φ_n} is the Frechet derivative at some intermediate point φ_n. Since $DA_{\varphi_n}(\cdot)$ has uniformly continuous inverse, it is satisfied

$$\|y_n - \widetilde{y}_n\| \leq c h_n^3 M/12,$$

and the result follows.

8 References

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