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Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax 34 91 624 9607

MARKET IMPERFECTIONS, DISCOUNT FACTORS AND STOCHASTIC DOMINANCE: AN EMPIRICAL ANALYSIS WITH OIL-LINKED DERIVATIVES*

Alejandro Balbás, Anna Downarowicz and Javier Gil-Bazo [1]

Abstract Oil-linked derivatives are becoming very important in Modern Investment Theory. Accordingly, the analysis of Pricing Techniques and Portfolio Choice Problems involving these securities is a major topic for both managers and researchers. We focus on both the No-Arbitrage Approach and Stochastic Discount Factor (SDF) based methods in order to study oil-linked derivatives available at The New York Mercantile Exchange, Inc, one of the world's largest markets in energy and precious metals. First, we generalize some theoretical properties of the SDF in order to capture the effects induced by the bid-ask spread when analyzing dominated/efficient portfolios. Secondly, we apply our findings and empirically analyze the existence of dominated assets and portfolios in the oil derivatives market. Our results reveal the systematic presence of dominated prices, which should be taken into account by traders when composing their portfolios. Additionally, the test yields pricing and portfolio choice methods as well as new strategies that may allow brokers to outperform their service for their clients. It is worth to point out that the conclusions of the test have two important characteristics: On the one hand, they are very precise since we draw on perfectly synchronized bid/ask prices, as provided by Reuters. On the other hand, they are robust in the sense that they do not depend on any assumption about the underlying asset price dynamics. Finally, despite the empirical test focuses on oil derivatives, the methodology is general enough to apply to a broad range of markets.

Keywords: Stochastic Discount Factor, Portfolio Choice Problem, Stochastic Dominance, Oil-Linked Derivative.

JEL Classification: G11, G13, M21, C61.

[1]Business Department, Carlos III, C/ Madrid 126 (Getafe), 28903; E-mail: alejandro.balbas@uc3m.es, anna.downarowicz@uc3m.es, javier.gil.bazo@uc3m.es.

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Market Imperfections, Discount Factors and Stochastic Dominance: An Empirical Analysis with Oil-Linked Derivatives [1]

Alejandro Balbás [2] **Anna Downarowicz** [3] **Javier Gil-Bazo** [4]

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[2] Department of Business Economics. University Carlos III of Madrid.
C/ Madrid, 126. 28903 Getafe (Madrid, Spain).
Telephone number: +34 91 624 9636, fax number: +34 91 624 9607, e-mail: alejandro.balbas@uc3m.es

[3] Department of Business Economics. University Carlos III of Madrid.
C/ Madrid, 126. 28903 Getafe (Madrid, Spain).
Telephone number: +34 91 624 9892, fax number: +34 91 624 9607, e-mail: anna.downarowicz@uc3m.es

[4] Department of Business Economics. University Carlos III of Madrid.
C/ Madrid, 126. 28903 Getafe (Madrid, Spain).
Telephone number: +34 91 624 5884, fax number: +34 91 624 9607, e-mail: javier.gil.bazo@uc3m.es

Abstract

Commodity-linked and oil-linked derivatives are becoming more and more important in Modern Investment Theory since they provide a useful tool when diversifying the levels and sources of risk. Accordingly, the analysis of Pricing Techniques and Portfolio Choice Problems involving these securities is a major topic for both managers and researchers. In this paper we focus on both the No-Arbitrage Approach (NAA) and Stochastic Discount Factor (SDF) based methods in order to study oil-linked derivatives available at The New York Mercantile Exchange, Inc ($NYMEX$), one of the world's largest markets in energy and precious metals. First, we generalize some theoretical properties of the SDF in order to capture the effects induced by the bid-ask spread when analyzing dominated/efficient portfolios. More accurately, we construct a reachable payoff that provides the link between payoffs and prices, since it yields a theoretical price process lying under the real price process and matching those prices of efficient portfolios. Secondly, we apply our findings and empirically analyze the existence of efficient/dominated assets and portfolios in the oil derivatives market. Our results reveal the systematic presence of dominated prices, which should be taken into account by traders when composing their portfolios. Additionally, the test yields pricing and portfolio choice methods as well as new strategies that may allow brokers to outperform their service for their clients. It is worth to point out that the conclusions of the test have two important characteristics: On the one hand, they are very precise since we draw on perfectly synchronized bid/ask prices, as provided by Reuters. On the other hand, they are robust in the sense that they do not depend on any assumption about the underlying asset price dynamics. Finally, despite the empirical test focuses on oil derivatives, the methodology is general enough to apply to a broad range of markets.

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1 Introduction

The absence of arbitrage (AA) is a basic assumption when dealing with Pricing and/or Equilibrium Models. It has been characterized by the existence of state prices or risk neutral probability measures under which the price process becomes a martingale (Harrison and Kreps, 1979, Dalang *et al.* 1990, Rubinstein, 1994, Jackwerth and Rubinstein, 1996). Furthermore, it implies the fulfillment of the Law of One Price (LOP) that leads to existence of Stochastic Discount Factors (SDF) providing us with pricing rules and, as long as risk levels are measured by standard deviations, also with optimal portfolios (see, for instance, Chamberlain and Rothschild, 1983, Hansen and Richard, 1987, Hansen and Jagannathan, 1997). In particular, if one considers the classical $CAPM$, there exists a close relationship between the Market Portfolio and the SDF. Finally, if the market is complete, the SDF may be understood as the density function between the risk neutral probability and the "real" probability, which may be used, for instance, for recovering the degree of risk aversion (Jackwerth, 2000).

Market imperfections are receiving increasing attention from researchers since they capture better many properties of some important financial markets. When frictions are considered, the martingale property is satisfied by a price process lying within both the bid and the ask process (Jouini and Kallal, 1995, Schachermayer, 2004, among others). However, as far as we know, the notion of SDF has not been so clearly extended yet, in the sense that there are no results guaranteeing the existence of an attainable payoff minimizing the return variance and simultaneously providing the pricing rules for computing bid or ask prices.

The first contribution of this paper is the introduction of some kind of SDF when transaction costs are incorporated. As stated above, the role of the SDF may be crucial when dealing with Portfolio Choice Problems in the imperfect market case, which is becoming the object of growing concern to researchers and practitioners. [1]

Another important feature of imperfect markets is the existence of dominated portfolios even if the AA is fulfilled. This point is clearly made in Jouini and Kallal (2001) where a very general concept of efficiency is characterized. Nevertheless, it is worthwhile to indicate that our notions of dominance and/or efficiency are much weaker than those of Jouini and Kallal. Their efficiency implies ours but the converse does not hold in general. A second contribution of this paper is the analysis of a new weak and general definition of efficiency by the existence of appropriate SDF in the imperfect market case.

Once the SDF have been introduced, one can deal with markets with significant imperfections and look for the existence of dominated portfolios as well as for optimal portfolios in a mean-variance context. Since our empirical analysis has focused on the oil-derivatives market, due to the cumulative distribution of the real returns it is not so clear that the variance may measure the risk level. Thus, we have decided to concentrate our test on the existence of dominated portfolios.

Oil-linked derivatives are becoming very important for many traders and managers because they can be an interesting risk-diversification alternative, mainly when negative correlations with equity markets are observed. Moreover, the oil market seems to exert an increasing influence on other financial markets, which further stimulates empirical research on its behavior.

Our empirical analysis focuses on some securities available at $NYMEX$, one of the world's largest markets for trading in futures and options contracts on the energy products such as crude oil, gasoline, heating oil, natural gas, electricity, etc. The study has been done with the highest possible precision since we draw on perfectly synchronized bid/ask prices as yielded by Reuters, and frictions have been discounted. We apply our $SDF-$liked findings and follow the static point

[1] There are interesting, different and alternative studies concerning Portfolio Choice Problems with frictions. See, for instance, Longstaff (2001).

of view so that we can guarantee the robustness of our conclusions regardless of any dynamic assumption about the behavior of the spot or the future oil price. We consider only positions in options on the futures contracts, but not positions in the futures themselves. By excluding the futures contracts we avoid identifying as inefficiencies possible mispricings due to risks associated with the physical delivery of the crude oil.

Despite the level of generality of the analysis, our findings seem to reveal the existence of clear inefficiencies in the market. The paper presents several specific examples and reports are provided throughout the paper along with a summary of general results for the whole sample. This empirical evidence is a major contribution for a number of reasons. Firstly, it may be useful to provide further evidence about the degree of efficiency of non-equity-linked financial markets. Secondly, many brokers and managers may be interested since, as mentioned above, they are showing an increasing reliance on oil derivatives in order to construct optimal portfolios. Thirdly, the existence of inefficiencies provides brokers with new practical pricing and investment methods that may outperform their clients' portfolios. So, they can extend the pricing method proposed in Balbás *et al.* (1999) for catastrophe-linked derivatives that were available at $CBOT$, [2] and, bearing in mind that by aggregating efficient strategies the final result may be dominated, they can also check for the efficiency of their clients' global position. If it is dominated, they can build and purchase the dominating portfolio–rather than the initial one–in order to generate additional earnings. [3]

Our paper is related to Constantinides *et al.* (2004) who also find some kind of inefficiencies when considering transaction costs and dealing with the $S\&P500$ Index-options. This interesting empirical paper emphasizes that the analysis assumes much weaker hypotheses than those associated with the $CAPM$. We should indicate that our assumptions are even weaker than those of Constantinides *et al.* (2004). There are also deep differences between their approach and ours, since both methodologies are fairly distinct. Furthermore, they use a far larger sample and are not so interested in perfectly synchronized data since they are not so important in a market with a longer history and in a test more closely related with some classical equilibrium models. On the contrary, our sample is not big since we prefer to introduce precision in a less studied market with far larger transaction costs.

Finally, the methodology proposed in this paper is general enough to apply in every financial market, and therefore it may enable us to study emerging and maybe illiquid markets that are becoming very interesting for traders trying to correctly price and/or diversify their risk. It should be borne in mind that imperfections and other drawbacks may make it rather difficult to apply the classical pricing methods of frictionless markets.

The outline of the paper is as follows. Section 2 extends the notion of SDF for imperfect markets and presents the methodology applied in the empirical study. Section 3 summarizes the characteristics of the market and the database we employ. Section 4 reports the empirical results and describes several interesting strategies that could have been implemented in the market, and, finally, Section 5 concludes the article. The proofs of those theoretical results of the second section are presented in the appendix.

[2] *i.e.*, if the purchase (sale) of a given security or portfolio is dominated then they can ask (bid) this security/portfolio and simultaneously improve its market quote. If a new agent accepts and buys (sells) then the position may be hedged by an arbitrage. Concrete examples found in the market are presented (see Example 2 below).

[3] Interesting methods for pricing oil-linked derivatives may be found in Black (1976), Gibson and Schwartz (1990), Schwartz (1997), Clewlow and Strickland (2000), etc. Nevertheless it may be worth to remark that the proposed pricing technique above is not affected for the significant frictions empirically reflected by this market. On the contrary, it allows traders to improve the quotes and provide liquidity.

2 Generalized Stochastic Discount Factors

In this section we will introduce those theoretical notions and results that will apply in the empirical test. As already said in the introduction, they are related to the existence of SDF in imperfect markets. The empirical analysis will focus on a static approach in order to prevent the sensitivity of the conclusions with regard to dynamic assumptions. However, since any dynamic approach contains the static one as a particular case, it is worthwhile to develope the theory in a dynamic setting.

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space composed of the set Ω, the $\sigma-$algebra $\mathcal{F}$ and the probability measure μ. Suppose that $[0, T]$ represents a time interval and $\mathcal{T} \subset [0, T]$ is a set (finite or infinite) of trading dates such that $\{0, T\} \subset \mathcal{T}$. As usual, the arrival of information will be generated by the increasing family $[\mathcal{F}_t]_{t \in \mathcal{T}}$ of $\sigma-$algebras of Ω (filtration) such that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$.

Prices, transaction costs and investment strategies will be introduced by following the approach of Jouini and Kallal (1995), since it incorporates those frictions created by the bid/ask spread and simultaneously generates a linear relationship between each portfolio and its current price. [4] Thus we will consider $n+1$ different securities, denoted by $S_0, S_1, ..., S_n$, whose bid and ask prices will be represented by two $\mathbb{R}^{n+1}-$valued adapted processes

$$\{b(\omega, t); \omega \in \Omega, t \in \mathcal{T}\}$$

and

$$\{a(\omega, t); \omega \in \Omega, t \in \mathcal{T}\}$$

respectively. In order to simplify the notation, the previous pair of processes may be also denoted by (b, a). If

$$b(\omega, t) = (b_0(\omega, t), b_1(\omega, t), ..., b_n(\omega, t))$$

where $b_j(\omega, t) \in \mathbb{R}$ for $\omega \in \Omega$, $t \in \mathcal{T}$ and $j = 0, 1, ..., n$, then $b_j(\omega, t)$ represents the bid price of S_j at t under the state ω. An analogous notation will be used for ask prices and, more generally, for any $\mathbb{R}^{n+1}-$valued adapted stochastic process.

The first asset S_0 will play the role of a numeraire, and therefore we will impose the existence of $v_0 \in \mathbb{R}$ such that the inequalities

$$0 < v_0 \leq b_0(\omega, t) \leq a_0(\omega, t) \tag{0.1}$$

hold for every $\omega \in \Omega$ and $t \in \mathcal{T}$.

Obviously, we will require the assumption

$$b(\omega, t) \leq a(\omega, t)$$

for every $\omega \in \Omega$ and $t \in \mathcal{T}$. For a fixed $\omega \in \Omega$ the corresponding path or trajectory of the bid (ask) price will be denoted by $b(\omega, -)$ $(a(\omega, -))$, while for any fixed trading date $t \in \mathcal{T}$ the symbol $b(-, t)$ $(a(-, t))$ yields the random variable providing us with the bid (ask) price at t. Similar notations will be used in similar situations.

A feasible portfolio (x, y) is a couple of $\mathbb{R}^{n+1}-$valued adapted stochastic processes

$$\{x(\omega, t); \omega \in \Omega, t \in \mathcal{T}\}$$

and

$$\{y(\omega, t); \omega \in \Omega, t \in \mathcal{T}\}$$

[4] Actually this approach enables us to represent many types of frictions, and not only those generated by the bid/ask spread. See Jouini and Kallal (1995) for a further discussion.

such that its trajectories $x(\omega,-)$ and $y(\omega,-)$ are increasing functions of $t \in \mathcal{T}$ and there exist a finite number $r \in \mathbb{N}, r \geq 1$, and a finite strictly increasing sequence $\{t_0 = 0, t_1, t_2, ..., t_r = T\} \subset \mathcal{T}$ (that depend on (x,y) but do not depend on ω) for which $x(\omega,-)$ and $y(\omega,-)$ remain constant on each $\mathcal{T} \cap [t_{i-1}, t_i)$, $i = 1, 2, ..., r$.

In addition, if

$$[x(\omega,t_i) - x(\omega,t_{i-1})]\, a(\omega,t_i) - [y(\omega,t_i) - y(\omega,t_{i-1})]\, b(\omega,t_i) = 0 \qquad (0.2)$$

$i = 1, 2, ..., r-1$, then (x,y) will be called self-financing. [5] The set of self-financing portfolios will be denoted by $\mathcal{S}$. It may be easily proved that $\alpha_1(x^1,y^1) + \alpha_2(x^2,y^2) \in \mathcal{S}$ if $(x^i,y^i) \in \mathcal{S}$ and $\alpha_i \geq 0$ in $\mathbb{R}$, $i = 1, 2$.

If $(x,y) \in \mathcal{S}$ then

$$\lambda(x,y) = x(\omega,0)a(\omega,0) - y(\omega,0)b(\omega,0) \in \mathbb{R} \qquad (0.3)$$

will be the price of (x,y). It immediately follows that $\lambda\left(\alpha_1(x^1,y^1) + \alpha_2(x^2,y^2)\right) = \alpha_1\lambda(x^1,y^1) + \alpha_2\lambda(x^2,y^2)$ if $(x^i,y^i) \in \mathcal{S}$ and $\alpha_i \geq 0$ in $\mathbb{R}$, $i = 1, 2$.

Let $(x,y) \in \mathcal{S}$. The $\mathcal{F}_T-$measurable random variable

$$\Lambda(x,y)(\omega) = [x(\omega,t_{r-1}) - y(\omega,t_{r-1})]^+\, b(\omega,T) - [y(\omega,t_{r-1}) - x(\omega,t_{r-1})]^+\, a(\omega,T) \qquad (0.4)$$

will be the final payoff of (x,y), and will be denoted by $\Lambda(x,y)$ or, if necessary, by $\Lambda(x,y)(\omega)$. [6] Once again, $\Lambda\left(\alpha_1(x^1,y^1) + \alpha_2(x^2,y^2)\right) = \alpha_1\Lambda(x^1,y^1) + \alpha_2\Lambda(x^2,y^2)$ if $(x^i,y^i) \in \mathcal{S}$ and $\alpha_i \geq 0$ in $\mathbb{R}$, $i = 1, 2$.

We follow usual conventions in order to introduce the concept of arbitrage.

Definition 1 *Let $(x,y) \in \mathcal{S}$. (x,y) is said to be an arbitrage if*
a) $\lambda(x,y) \leq 0$
b) $\Lambda(x,y)(\omega) \geq 0$, $\mu - a.s.$
c) $\mu\left(\Lambda(x,y) - \lambda(x,y) > 0\right) > 0$. □

Hereafter we will assume that the market is arbitrage free.

Definition 2 *Let (x,y), $(x',y') \in \mathcal{S}$. We will say that (x,y) dominates (x',y') if*
a) $\lambda(x,y) \leq \lambda(x',y')$
b) $\Lambda(x,y)(\omega) \geq \Lambda(x',y')(\omega)$, $\mu - a.s.$
c) $\mu\left[(\Lambda(x,y) - \Lambda(x',y')) + (\lambda(x',y') - \lambda(x,y)) > 0\right] > 0$. □

Henceforth non-dominated strategies will be also called efficient.

Notice that Condition *c*) above is fulfilled if *a*) holds in terms of strict inequality. Arbitrage and dominance are closely related crucial notions. In particular an arbitrage is a portfolio dominating the null strategy. It is well known that the AA is equivalent to the absence of dominated strategies in the frictionless case ($b = a$). However, under our present approach, the existence of transaction costs may lead to the existence of dominated portfolios in arbitrage free markets (our empirical results will be a clear proof of that).

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[5] Notice that products in (0.2) are scalar (or inner) products of $\mathbb{R}^{n+1}$.

[6] As usual, $\alpha^+ = Max\{\alpha, 0\}$ and $\alpha^- = Max\{-\alpha, 0\}$ if $\alpha \in \mathbb{R}$, and, for every $m \in \mathbb{N}$ and $\alpha \in \mathbb{R}^m$, $\alpha^+ = (\alpha_1^+, \alpha_2^+, ..., \alpha_m^+)$ and $\alpha^- = (\alpha_1^-, \alpha_2^-, ..., \alpha_m^-)$.

As usual, the set M of marketed claims is composed of those $\mathcal{F}_T-$measurable $\mathbb{R}-$valued random variables m such that there exists $(x,y) \in \mathcal{S}$ with $m \leq \Lambda(x,y)$, $\mu - a.s.$ If so, (x,y) will be called a super-replication of m, and the set of super-replications of m will be represented by $\mathcal{S}_m$. It may be very easily proved that M is a convex cone (*i.e.*, $\alpha_1 m_1 + \alpha_2 m_2 \in M$ if $m_i \in M$ and $\alpha_i \geq 0$ in $\mathbb{R}$, $i = 1,2$) and (0.1) shows that $L^\infty(\mathcal{F}_T) \subset M$, $L^\infty(\mathcal{F}_T)$ being the space of $\mathbb{R}-$valued $\mathcal{F}_T-$measurable essentially bounded random variables. More generally, hereafter we will fix $1 \leq p \leq \infty$ and a closed linear manifold V of $L^p(\mathcal{F}_T)$ such that $V \subset M$, where for $p < \infty$ the space $L^p(\mathcal{F}_T)$ is composed of the $\mathbb{R}-$valued $\mathcal{F}_T-$measurable random variables ξ such that the expectation of $\mid \xi \mid^p$ is bounded.

We will define the function of minimum cost by

$$M \ni m \to \pi(m) = Inf\{\lambda(x,y); (x,y) \in \mathcal{S}_m\} \in \mathbb{R} \cup \{-\infty\} \tag{0.5}$$

It is easy to show that π is increasing and sublinear on M, *i.e.*,

$$\pi(m_1) \geq \pi(m_2) \tag{0.6}$$

whenever $m_i \in M$, $i = 1,2$, $m_1 \geq m_2$ $\mu - a.s.$,

$$\pi(m_1 + m_2) \leq \pi(m_1) + \pi(m_2) \tag{0.7}$$

if $m_i \in M$, $i = 1,2$, and

$$\pi(\alpha m) = \alpha \pi(m) \tag{0.8}$$

if $m \in M$ and $\alpha > 0$ in $\mathbb{R}$.

The positiveness of the bid and ask prices of S_0 and the AA easily lead to $\pi(0) = 0$ and $\pi(m) > -\infty$ for every $m \in L^\infty(\mathcal{F}_T)$. We will also assume that $\pi(m) > -\infty$ for every $m \in V \subset L^p(\mathcal{F}_T)$. [7]

If $p < \infty$ then $q \in (1,\infty]$ will denote its conjugate value, *i.e.*, $(1/p) + (1/q) = 1$ holds, and, according to the Riesz Representation Theorem, $L^q(\mathcal{F}_T)$ is the dual space of $L^p(\mathcal{F}_T)$. If $p = \infty$ then the dual space of $L^\infty(\mathcal{F}_T)$ is composed of those finitely-additive measures

$$\mathcal{F}_T \ni A \longmapsto \delta(A) \in \mathbb{R}$$

$\mu-$continuous (*i.e.*, $\mu(A) = 0 \Longrightarrow \delta(A) = 0$) and with finite variation (see, for instance, Diestel and Uhl, 1977). This space will be denoted by $\mathcal{M}(\mathcal{F}_T)$.

Next we will establish the tie between the absence of dominance and the existence of SDF. Proofs are presented in the appendix.

Lemma 1 *Let $l \in \mathbb{N}$ and $(x^1,y^1),(x^2,y^2),...,(x^l,y^l) \in \mathcal{S}$. The following assertions are equivalent:*
1.1. *$\sum_{i=1}^{l} \beta_i (x^i, y^i)$ is non-dominated for every $\beta_1, \beta_2, ..., \beta_l \geq 0$.*
1.2. *There exist $\alpha_1, \alpha_2, ..., \alpha_l > 0$ such that $\sum_{i=1}^{l} \alpha_i (x^i, y^i)$ is non-dominated.* □

Henceforth the operator $\mathcal{E}$ will be used to represent the expectation of any random variable.

Theorems 2 and 3 below are established under the assumption that V satisfies the properties of a Banach Lattice. For instance, this assumption is fulfilled if $V = L^p(\mathcal{F}_T)$. The properties of a Banach Lattice may be found in Schaeffer (1974).

[7] Being V a vector space we have that $a_0(-,T) \in V$ is a sufficient condition. Indeed, suppose that $v_0 \in V$ and $\pi(v_0) = -\infty$. Then for every $v \in V \subset M$ is $v - v_0 \in V \subset M$ and (0.7) gives

$$\pi(v) \leq \pi(v_0) + \pi(v - v_0) = -\infty,$$

which is a contradiction with the AA if one takes, for instance, $v = a_0(-,T)$.

Theorem 2 *Let $l \in \mathbb{N}$ and $(x^i, y^i) \in \mathcal{S}$, $i = 1, 2, ..., l$. Suppose that* 1.1 *or* 1.2 *hold. Suppose finally that $\Lambda(x^i, y^i) \in V$, $i = 1, 2, ..., l$. Then:*

2.1. *If $p < \infty$ then there exists $\delta \in L^q(\mathcal{F}_T)$, $\delta \geq 0$ $\mu - a.s.$, such that*

$$\mathcal{E}\left(m\delta\right) \leq \pi\left(m\right) \tag{2.9}$$

for every $m \in V$, and

$$\mathcal{E}\left(\Lambda(x^i, y^i)\delta\right) = \lambda(x^i, y^i) \tag{2.10}$$

$i = 1, 2, ..., l.$

2.2. *If $p = \infty$ then there exists a non-negative $\delta \in \mathcal{M}(\mathcal{F}_T)$ such that*

$$\int_\Omega m(\omega) d\delta(\omega) \leq \pi\left(m\right)$$

for every $m \in V$, and

$$\int_\Omega \Lambda(x^i, y^i)(\omega) d\delta(\omega) = \lambda(x^i, y^i)$$

$i = 1, 2, ..., l.$ □

Expression (2.10) allows us to establish the link between prices and payoffs of efficient portfolios. Prices are given by the mathematical expectation of distorted payoffs. This is in the line of the usual conditions for the arbitrage absence in perfect markets, but there is a major difference since we are dealing with efficiency rather than arbitrage. As already mentioned, the absence of arbitrage in the presence of frictions is studied by the existence of price processes lying within the bid/ask spread and satisfying the martingale property with respect to some distorted probability measure. This is closely related to Expression (2.9), [8] so we can conclude that the combination of (2.9) and (2.10) reflects the existence of linear expressions capturing both properties since they are lying under the price process and matching those prices of efficient portfolios.

The distortion random variable δ is related to both risk neutral measures and SDF. In this paper we will focus on the second concept since it is more closely related to Portfolio Choice Problems. Therefore we will consider random variables with bounded expectation and variance ($p = 2$).

Theorem 3 *Let $l \in \mathbb{N}$ and $(x^i, y^i) \in \mathcal{S}$, $i = 1, 2, ..., l$. Suppose that* 1.1 *or* 1.2 *hold. Suppose finally that $p = 2$ and $\Lambda(x^i, y^i) \in V$, $i = 1, 2, ..., l$. Then there exists a unique $\delta \in V \subset L^2(\mathcal{F}_T)$ such that (2.9) and (2.10) hold.* □

There are two important differences with respect to Theorem 2, since δ is unique and may be super-replicated ($\delta \in V \subset M$). δ will be called SDF and is a genuine extension of the SDF of a perfect market (see, for instance, Chamberlain and Rothschild, 1983 or Hansen and Jagannathan, 1997). As long as a riskless asset is available, if δ is a reachable payoff and is not risk-free it may be proved that every optimal portfolio in a mean/variance context is composed of the riskless asset minus $k\delta$ ($k > 0$), although we do not address this question that is beyond our scope.

A important particular case arises if we consider a static approach ($\mathcal{T} = \{0, T\}$) and transaction costs are paid at the initial date, *i.e.*, $a(\omega, T) = b(\omega, T)$ for $\omega \in \Omega$. Both properties will be

[8] Notice that if $(x, y) \in \mathcal{S}$ and $\Lambda(x, y) \in V$ then (2.9) gives $\mathcal{E}\left(\Lambda(x, y)\delta\right) \leq \lambda(x, y)$.

assumed in the rest of this section. We will also assume that $a_j(-,T)$ has bounded variance (*i.e.*, is in $L^2(\mathcal{F}_T)$) for $j = 0, 1, ..., n$. We will take as V the linear manifold of $L^2(\mathcal{F}_T)$ generated by $\{a_j(-,T)\}_{j=0}^n$. It is obvious that δ is reachable in this framework and the discussion above applies.

Consider partitions (J_1, J_2, J_3) of $J = \{0, 1, ..., n\}$. For instance, if $n = 3$, a feasible partition is $(J_1 = \{0, 2\}, J_2 = \emptyset, J_3 = \{1, 3\})$. Portfolio (x, y) is said to be associated with the partition (J_1, J_2, J_3) if $x_j = 0$ and $y_j > 0$ whenever $j \in J_1$ and $x_j > 0$, $y_j = 0$ for $j \in J_2$. Then Theorem 3 implies:

Theorem 4 *Consider an arbitrary partition (J_1, J_2, J_3). If portfolios associated with this partition are efficient* [9] *then there exists a unique reachable δ such that*

$$b_j(-,0) \leq \mathcal{E}\left(a_j(-,T)\delta\right) \leq a_j(-,0) \tag{4.11}$$

for $j = 0, 1, ..., n$,

$$b_j(-,0) = \mathcal{E}\left(a_j(-,T)\delta\right) \tag{4.12}$$

for $j \in J_1$ and

$$\mathcal{E}\left(a_j(-,T)\delta\right) = a_j(-,0) \tag{4.13}$$

for $j \in J_2$. □

According to Chamberlain and Rothschild (1983) the SDF δ may be computed by fixing expected returns and minimizing variances. Moreover the SDF provides us with the family of optimal portfolios if investors measure the risk level by means of standard deviations. [10] Thus Theorem 4 may be quite useful when testing for the existence of inefficient portfolios in imperfect markets. Indeed, in a first step one can check the existence of SDF for some feasible partitions in order to guarantee that their associated portfolios are non-dominated. [11] This step is general and does not depend on the applied risk measurement methods, since every Portfolio Choice Problem rejects those strategies that are dominated in the sense of Definition 2. The second step applies when the degree of risk is measured by means of standard deviations. In such a case the payoff δ of Theorem 4 indicates those portfolios that are optimal, since we can draw on the findings of Chamberlain and Rothschild (1983) for perfect markets. As mentioned above, the computation in practice of the SDF may be addressed by several optimization methods (see also Balbás and Mayoral, 2004).

Our test in the oil-linked derivatives market draws on the first step above. Hence, we analyze the existence of dominated securities by considering partitions and testing the existence of associated SDF. The constraints (4.11), (4.12) and (4.13) of the SDF become more restrictive as sets J_1 and J_2 increase. It is coherent with Lemma 1, which guarantees that if the sum of self-financing portfolios is efficient then all the terms are efficient, but the converse may fail. Therefore, in order to detect the degree of fulfillment of (4.11) , (4.12) and (4.13), first we have decided to look for dominated portfolios composed of a single security and later, if necessary, we prevent the existence of single dominated securities by modifying their quotes and then look for more complex dominated portfolios.

[9] Recall that Lemma 1 implies that all the strategies associated with the partition are efficient if at least there exists one efficient associated portfolio.

[10] It depends on their utility function and the cumulative distribution of each $a_j(-,T)$. Although the standard deviation often applies as a risk measure, many alternative ways have been provided in order to compute the level of risk (see for instance Artzner *et al* (1999) for a general approach or Rockafellar and Uryasev (2002) for problems dealing with credit risk).

[11] One could check the whole set of partitions but this set will be extremely large unless n is very small.

3 Market and Data

The New York Mercantile Exchange, Inc. ($NYMEX$) is one of the world's largest markets for trading in futures and options contracts on the energy products such as crude oil, gasoline, heating oil, natural gas, electricity, etc. Among other products, $NYMEX$ Division [12] offers trading in a light sweet crude oil futures contract used as a principal international pricing benchmark, options on the futures contract, calendar spread options, crack spread options and average price options.

Light, sweet crude oil futures contract trades in units of $1,000$ barrels of West Texas Intermediate (WTI) crude oil with delivery in Cushing, Oklahoma. Trade is conducted through an open outcry and the after-hours electronic trading system $NYMEX$ $ACCESS$. The Exchange offers contracts with physical settlement for thirty consecutive months and long-dated contracts for 36, 48, 60, 72 and 84 months. Trading of the contract ceases on the third business day prior to the $25th$ calendar day of the month preceding the delivery month or the third business day prior to the business day preceding the $25th$ of the month if this is a non-business day. Delivery is carried out over the course of the month and has to start on or after the first calendar day and be completed by the last calendar day of the delivery month.

The light, sweet crude oil options are standardized, floor-traded, American-style contracts on the $NYMEX$ Division light, sweet crude oil futures contract. There are options with maturity in thirty consecutive months and long-dated options for 36, 48, 60, 72 and 84 months. Prices are quoted in $U.S.$ dollars and cents per barrel. Options are available with at least sixty one strike prices which comprise twenty strike prices in increments of 50 cents per barrel above and below the at-the-money strike price and ten strike prices in increments of 2.50 dollars above the highest and below the lowest existing strike prices. Trading ends three business days before the last trading day of the underlying futures contract.

Our analysis mainly focuses on the period from November 3 to November 16, 2004, although, as will be shown in next section, we also tested some dates of 2005 in order to guarantee the robustness of the conclusions.

During November, 2004, the underlying crude oil price declined steadily by 11%. Data were provided by Reuters and correspond to the synchronized bid and ask prices for call and put options on the light sweet crude oil $NYMEX$ futures contract, and $U.S.$ Treasury bills rates with maturities close to the maturity of the contracts. Figure 1 shows the evolution of futures prices in the sample period. [13]

In total, we analyze the data corresponding to 10 days within the considered period (the sample period contains two weekends). We define a time-point as an exact instant of time when we observe the market (e.g. $3/11/2004$ $15:30$ defines a time-point). Since data were sampled at thirty-minute intervals, and the options market remains open from 10 am to $14:30$ $p.m.$, each day contains 10 time-points, so the total number of time-points is 100. For each time-point we analyze available bid and ask prices for call and put options maturing in December 2004 and January 2005, as well as one-month and three-months $T-$bills interest rates. The maturities are fixed on $20th$ calendar day of each month. [14]

The specification of the options contained in our data set are presented in Table 1.

[12] Trading on $NYMEX$ is conducted through two divisions: the $NYMEX$ Division for the energy, platinum, and palladium and the $COMEX$ Division on which all other metals are traded.

[13] In the analysis on the existence of dominated portfolios we consider only positions in options on the futures contracts, but not positions in the futures themselves. By excluding the futures contracts we avoid identifying as inefficiencies possible mispricings due to risks associated with the physical delivery of the crude oil.

[14] The expiration dates of the futures contracts with maturity in December 2004 and January 2005 are November 19, 2004 and December 20, 2004 respectively. For options these dates are: November 16, 2004 and December 15, 2004 respectively.

TABLE 1

Option	Maturity	Strike Range in $ (every $.50)	Number of options at each time-point
Calls	Dec 04	32 – 63	63
	Jan 05	32 – 63	63
Puts	Dec 04	32 – 63	63
	Jan 05	32 – 58	53
		Total	242

Analyzing the properties of the data we observe that, in general, during the study period there were more options with non-null ask than with non-null bid prices, *i.e.*, dealers quoted more sale than purchase orders to investors. Only on November 8 and 9 more long than short positions in call options were quoted. [15] Put option bids were the least frequent quotes during the analyzed period. We can further observe that during this period options with shorter time to maturity were quoted more often, especially, in long positions. Table 2 presents the average minimum, mean and maximum number of options with non-null ask or non-null bid quotes for the analyzed period.

TABLE 2

Option	Maturity	Options with non-null ask			Options with non-null bid		
		Average Min [16]	Mean	Average Max	Average Min	Mean	Average Max
Calls	Dec 04	1.6	3.77	5.3	0.3	1.49	2.6
	Jan 05	0.1	1	2.1	0.2	1.53	2.4
	All calls	1.8	4.77	7.1	0.8	3.02	4.6
Puts	Dec 04	1.2	3.35	5.1	0.1	0.88	1.7
	Jan 05	1.1	2.22	3.4	0.1	0.67	1.5
	All puts	2.2	5.57	7.9	0.2	1.55	3.0

Figures 2 and 3 show the number of options with quoted prices available in each time-point. The minimum number of securities (including the two risk-free assets) is four, and the maximum number of available securities is 32, which accounts roughly for 13% of the total number of possible securities. Interestingly, the highest number of assets with quoted asks or bids is found when time-to-maturity is highest. Note, finally, that on November 16 the number of securities decreases dramatically, which is explained by the fact that options maturing in December stop trading on that day.

To give a sense of the evolution of quotes with time, Figures 4 and 5 display asks and bids of the three most quoted calls and puts during the study period, respectively. As can be seen, quotes are rare even for the most quoted assets.

4 Empirical Results

The objective of the empirical analysis of this paper is to test for the existence of dominated single securities and dominated portfolios in the *NYMEX* market and check if it is possible to

[15] On these days the average prices of the underlying futures contracts deacreased by more than 2% (see Figure 1).

[16] Daily minimum, mean and maximum values are averaged over the total number of trading dates during the analyzed period.

improve the market prices of the available options on the futures contracts. Next, we specify the methodology employed in the empirical study and present the obtained results.

First, we group the available securities according to maturity. On each time-point we consider only those securities for which at least one price (either bid or ask) is available. We also require at least two assets with available prices on each time-point.

In order to be able to calculate the future payoffs of the derivatives we determine the set of the future states of the world in such a way that each state corresponds to one of the possible strike prices of the options available on a given time-point. We also add two extreme events and two intermediate states: one state below the lowest strike, one intermediate state and two states above the highest strike prices. [17] We capitalize the payoffs of the assets maturing in December 2004 to January 2005. Therefore, although the derivatives in the sample have two different maturities, our model can be considered in a static, one-period framework in which t_0 is always the considered time-point (*i.e.* the point of time when we observe the market) and the future date T is the expiration date of the contracts with maturity in January 2005. The total number of states of the world at time T is equal to the number of strike prices of the options expiring in December 2004 multiplied by the number of strike prices of the options with maturity in January 2005 plus four additional states.

Payoffs of the call and put options on the underlying futures contracts are calculated as the usual differences between the possible futures contracts prices (*i.e.* strike prices corresponding to each state of the world) and the strike price of the option. Although, the light, sweet crude oil options traded on $NYMEX$ are American options, we can assume they would only be exercised at maturity. Since the underlying does not pay dividends, call options will never be early exercised and also put options are hardly exercised before the maturity due to the low interest rates prevailing in the analyzed period.

4.1 Dominated Securities

In order to test for the existence of dominated securities in the $NYMEX$ market, we employ Theorem 4.

Since some of the considered assets lack either bid or ask prices, first, we provide these securities with the artificial prices (0 for the bid and 1000 for the ask prices). Next, we check for the existence of arbitrage opportunities during the analyzed period. Considering the assets of both maturities, we detect those time-points for which there exist portfolios with a nonnegative payoff and a negative price. Arbitrage opportunities have been detected on a single occasion in the $2:00$ *p.m.* $2:30$ *p.m.* interval of November 5. Example 1 below describes the arbitrage opportunity.

Example 1. Arbitrage Strategy

On November 5, at the end of the trading session, the following market data were available:

Asset Type	Maturity	Strike	Ask	Bid
Call	Dec. 04	47	0.55	–
Call	Dec. 04	51	–	0.55
Call	Dec. 04	56	0.1	–

Consider the following portfolio:

[17] These states correspond to the futures contract payoffs equal to \$0, \$50, \$70 and \$1000.

Asset Type	Maturity	Strike	Position
Call	Dec. 04	47	1.25
Call	Dec. 04	51	−2.25
Call	Dec. 04	56	1

This strategy has a cost of $-.45$ dollars, but its payoff at maturity is nonnegative for every state as shown on Figure 6. □

Next for all arbitrage-free time-points we determine the minimum cost functions (π, see Section 2) for all the assets in the sample. The shadow ask price of a considered asset is equal to the minimum cost of super-replicating its payoff and the shadow bid price to the maximum possible income from any portfolio dominated by the considered asset. The price improvement is therefore the difference between the original price of the considered asset and its shadow (improved) price.

Results obtained at this stage (see Table 3 below) show that the possibility to improve prices in the market has been quite often present during the period of the analysis. Inefficient ask prices have arisen on the market on 46% and inefficient bid prices on 16% of the time-points during the considered period. This reveals clear inefficiencies in the prices of the analyzed oil derivatives, in particular, in the case of the ask quotes. On average, 15.33% of the ask quotes and 13.82% of the bid quotes could have been improved. It seems that this improvement could have been quite significant. An average mean improvement of the quotes could have been 25.49% for the ask and 929% for the bid prices. If we compare the average maximum improvement, this could have been 32.38% for the ask and 929% for the bid quotes. The possibility of obtaining such large price improvements shows that the bid-ask spreads of the oil derivatives could be significantly reduced by using arbitrage methods. For illustration purposes Example 2 below presents an inefficiency that arose on the market during the considered period and a possible price improvement of the derivative involved in the arbitrage strategy.

TABLE 3. Dominated Securities (Summary)	Ask	Bid
Time-points with improved quotes (for at least.1 cent)	46%	16%
Average fraction of assets with improved quote	15.33%	13.82%
Average mean improvement [18]	25.49%	929.00%
Average maximum improvement [19]	32.38%	929.00%

Example 2. Dominated Security

On November 11, at 1 : 30 *p.m.*, a call option with strike \$40 and maturity in December 2004 had a bid price of 35 cents. A short position in this call option could be super-replicated by a long position in the put option with strike \$47, maturity in December 2004 and ask price 80 cents and 7 short positions in December 2004 $T-$bills with interest rate equal to 1.875%. The revenue of the super-replicating portfolio was \$6.2479, so the bid price of the call option could have been improved by as much as \$5.8979. The payoffs of both the short position in the call option and the dominating portfolio are shown in the top Panel of Figure 7. The stochastic discount factor that would price the call option efficiently is displayed in the bottom Panel. □

[18] The mean improvement for all the assets with the improved quotes is averaged over the total number of dates in the sample.

[19] The maximum improvement for all the assets with the improved quotes is averaged over the total number of dates in the sample.

4.2 Dominated Portfolios

Once we have improved the prices of some of the derivatives, we would like to check if further improvement of the available quotes is possible by considering combinations of assets rather than single securities. For that purpose, we test for the existence of dominated portfolios composed of more than one asset.

We search for the dominating strategies considering different combinations of derivatives available at each time-point during the studied period. We consider partitions of the set of the available assets into three disjoint sets of securities J_1, J_2, J_3. Each such partition of the set of assets corresponds to a portfolio formed from available securities. Since the number of assets in each $J_i,\ i = 1, 2, 3$ can vary from 0 to the maximum number of assets available at each time-point, we divide our analysis in several steps. For all available assets at each date, we form partitions with two, three and four securities in $\{J_1, J_2\}$ and search for SDF with two, three and four assets. [20]

First we consider all possible partitions of the whole set of assets by taking pairs of securities. For each pair of assets $\{a, b\}$ we have four possible partitions:

$\{\emptyset, \{a, b\}\}$	a long position in asset a and b
$\{\{a, b\}, \emptyset\}$	a short position in asset a and b
$\{\{a\}, \{b\}\}$	a short position in asset a and a long position in asset b
$\{\{b\}, \{a\}\}$	a short position in asset b and a long position in asset a

with the rest of the assets in J_3.

At each time-point we go across all possible partitions with two assets in $\{J_1, J_2\}$ and search for the dominating strategies. We do not consider those partitions in which one of the assets in J_2 originally had no ask price or one of the assets in J_1 had no bid price. Analogously to the stage with the single securities, we calculate the minimum cost functions of the combinations of two assets and check if it is possible to improve their prices. As before the price improvement is the difference between the original cost of the portfolio and the cost of the dominating portfolio. By usual methods we obtain the SDF that eliminates the inefficiency in the assets prices and the new improved prices of the securities. Table 4 below summarizes the main results for the existence of dominated portfolios composed of two securities.

[20] We limit the maximum number of assets in $\{J_1, J_2\}$ to four due to extremely high computational cost of the analysis.

TABLE 4. Dominated Portfolios with Two Assets	
Long Position in both Assets	
Time-points with improved ask price (at least of .1 cents)	9%
Average fraction of portfolios with improved ask price	1.19%
Average mean improvement in ask prices [21]	1.99%
Average maximum improvement in ask prices [22]	2.22%
Short Position in both Assets	
Time-points with improved bid prices (at least of .1 cents)	8%
Average fraction of portfolios with improved bid price	2.76%
Average mean improvement in bid prices	4.90%
Average maximum improvement in bid prices	5.89%
One Short and one Long Position	
Time-points with improved prices (at least of .1 cents)	41%
Average fraction of portfolios with improved price	3%
Average mean improvement in prices	11.92%
Average maximum improvement in prices	16.34%

Analogously to the previous step, we carry out our analysis by considering all possible partitions of the sets of assets available at each time-point with three and four assets in $\{J_1, J_2\}$. The results are presented in Table 5 and Table 6 respectively.

TABLE 5. Dominated Portfolios with Three Assets	
Long Position in Three Assets	
Time-points with improved ask price (at least of .1 cents)	6%
Average fraction of portfolios with improved ask price	2.42%
Average mean improvement in ask prices	1.63%
Average maximum improvement in ask prices	2.17%
Short Position in Three Assets	
Time-points with improved bid prices (at least of .1 cents)	8%
Average fraction of portfolios with improved bid price	8%
Average mean improvement in bid prices	2.84%
Average maximum improvement in bid prices	5.64%
One Short and Two Long Positions	
Time-points with improved prices (at least of .1 cents)	40%
Average fraction of portfolios with improved price	5.56%
Average improvement in prices	5.95%
Average maximum improvement in prices	12.92%
One Long and Two Short Positions	
Time-points with improved prices (at least of .1 cents)	49%
Average fraction of portfolios with improved price	5.36%
Average mean improvement in prices	3.39%
Average maximum improvement in prices	6.74%

[21] The mean difference between the price of the dominating portfolio and the dominated one for all the dominated combinations involving long position in two assets, averaged over the total number of dates in the sample.

[22] The maximum difference between the price of the dominating portfolio and the dominated one for all the dominated combinations involving long position in two assets, averaged over the total number of dates in the sample.

TABLE 6. Dominated Portfolios with Four Assets

Long Position in Four Assets	
Time-points with improved ask price (at least of .1 cents)	5%
Average fraction of portfolios with improved ask price	2.33%
Average mean improvement in ask prices	0.75%
Average maximum improvement in ask prices	1.13%
Short Position in Four Assets	
Time-points with improved bid prices (at least of .1 cents)	9%
Average fraction of portfolios with improved bid price	1.57%
Average mean improvement in bid prices	1.85%
Average maximum improvement in bid prices	4.73%
One Short and Three Long Positions	
Time-points with improved prices (at least of .1 cents)	39%
Average fraction of portfolios with improved price	1.45%
Average mean improvement in prices	4.16%
Average maximum improvement in prices	14.27%
One Long and Three Short Positions	
Time-points with improved prices (at least of .1 cents)	49%
Average fraction of portfolios with improved price	0.51%
Average mean improvement in prices	2.88%
Average maximum improvement in prices	7.80%
Two Long and Two Short Positions	
Time-points with improved prices (at least of .1 cents)	49%
Average fraction of portfolios with improved price	0.89%
Average mean improvement in prices	3.41%
Average maximum improvement in prices	11.96%

Examples $3-5$ below illustrate possible price improvements of the considered dominated strategies that arose on the market during the analyzed period.

Example 3. Dominance with Two Assets.

On November 11, at $11:30$ *a.m.*, a call option with expiration in December and strike price of 40 dollars could be sold for 35 cents. The analysis from the previous stage, however, suggests that this position could be super-replicated synthetically with a revenue of 88.33 cents, which we take as the option's artificial bid. At the same time, a call option with the same maturity and strike 51 dollars could be bought for 20 cents. Selling the former and buying the latter would therefore provide a revenue of 68.33 cents. At the same time, the following quotes were available:

Asset Type	Maturity	Strike	Ask	Bid
Call	Dec. 04	50	0.30	0.25
Call	Dec. 04	53	0.06	–

The two asset portfolio revenue could therefore be improved to 69.67 cents at no extra risk by taking the following positions:

Asset Type	Maturity	Strike	Position
Call	Dec. 04	40	-0.0062
Call	Dec. 04	50	-3.6397
Call	Dec. 04	53	3.6460

Figure 8 (top panel) shows the original portfolio's payoff function in December (thick line) as well as that corresponding to the dominating strategy (thin line). The SDF gives a new shadow ask price for the 51 call option of 18.67 cents. The bottom panel displays the stochastic discount factor that eliminates the inefficiency. □

Example 4. Dominance with Three Assets

On November 15 at 11 : 30 *a.m.*, a put option with expiration in December and a strike price of 46 dollars, could be bought for 8 cents. At that time, there were two put options with strikes equal to 45.5 and 46.5 dollars, that could be sold for 2 cents and 10 cents respectively. Buying the first option and selling the other two would have given a revenue of 4 cents. An investor seeking a dominating strategy would sell 1/2 units of the 45.5−dollar put and an equal amount of the 46.5−dollar put. This strategy gives a revenue of 6 cents. Figure 9 (top panel) displays the original portfolio's payoff function (thick line) as well as the dominating strategy's payoff function (thin line). Note that in this example there is no inefficient portfolio if assets are taken two by two. The bottom panel of Figure 9 displays the stochastic discount factor that eliminates the inefficiency. Using this state price vector, the efficient price for the 46−dollar option is 6 cents. □

Example 5. Dominance with Four Assets.

On November 15 we had the following market data:

Asset Type	Maturity	Strike	Ask [23]	Bid
Put	Dec. 04	37	0.0086047*	–
Put	Dec. 04	38	0.0088372*	–
Put	Dec. 04	39	0.0090698*	–
Put	Dec. 04	40.5	0.0094186*	–
Put	Dec. 04	43	0.01	–
Put	Dec. 04	45.5	0.068333*	0.02
Put	Dec. 04	46	0.08	0.05
Put	Dec. 04	46.5	0.58	0.1

Consider taking two long positions in the 40.5 and 46 puts, and two short positions in the 45.5 and 46.5 puts. This strategy has a revenue of:

$$0.02 + 0.1 - 0.0094186 - 0.08 = 0.0305814$$

Consider next the following strategy:

Asset Type	Maturity	Strike	Position
Put	Dec. 04	37	0.012607
Put	Dec. 04	38	0.013359
Put	Dec. 04	39	0.015285
Put	Dec. 04	40.5	0.024807
Put	Dec. 04	43	0.88198
Put	Dec. 04	45.5	−0.5
Put	Dec. 04	46	0
Put	Dec. 04	46.5	−0.5

[23] Prices marked with * are not original prices, but the minimum cost (the maximum revenue) from buying (selling) portfolios that dominate those assets.

Its total revenue is 0.050581 dollars, 2 cents greater, and its payoff dominates that of the original strategy as can been seen on Figure 10. □

Results reveal that the existence of dominance has been quite often present in the market during the analyzed period. This shows that the assets prices, although efficient if one consider a single asset, can be still inefficient if one combines different derivatives into portfolios and explores dominance opportunities. For portfolios involving only long positions in two, three and four assets, the prices could have been further improved respectively, in 9, 6 and 5 time-points during the analyzed period. If we consider portfolios with only short positions in two, three or four assets, the possibility of further improvement in the prices has been detected in about 8 time-points. Moreover, in mean terms, this improvement seems to be higher than in the case of portfolios with only long positions, both if we compare the average fraction of portfolios with the improved prices and the average size of improvement. The most frequent dominance of complex portfolios has been detected in case of the strategies consisting of both long and short positions in the available derivatives. Depending on the number of the involved securities, some kind of dominated portfolios were available in between 39 and 49 out of 100 time-points. Also the average size of the possible improvement and the average fraction of portfolios for which such improvement was possible seems to be more significant compared to the portfolios with only long or only short positions. These results show that further quotes improvement and spread reduction can be possible by analyzing the prices of combinations of assets available on the market.

Finally, because bid improvements on November 11, 2004, were so economically significant, there exists the possibility that data from that specific day may bias the results from the whole sample. For this reason we eliminate time-points corresponding to that day, and repeat the analysis for the remaining sample. Although results are not reported for the sake of brevity, we find that single asset bid price improvements become less frequent (6% of time-points as opposed to 16% in the whole sample), and the mean improvement on bid prices also reduces dramatically from 929% to 10%. However, when portfolios rather than individual assets are considered, results remain largely unaltered, which implies that results from Table 4 are not driven by this specific day.

4.3 Alternative Samples

In order to verify the robustness of the reported results we decided to check two additional samples.

Because bid-ask quotes obtained from the trading floor might be subject to errors, in our second sample we also examined prices from an alternative source. More specifically, an authorized broker at $NYMEX$ recorded quotes for us for six options on the futures contract with maturity in May 2005: Three call options with strike prices equal to 50, 56.50 and 60 and three put options with the same strikes. Data were collected at the end of the following trading sessions: April,1; April, 7; April, 11; and April, 12.

We found that ask prices and bid prices could be improved three days and four of those days, respectively. Improvements in both ask and bid prices were below 1%. The maximum improvements were 12.72% and 2.57% respectively. Results are summarized in Table 7. Interestingly, once original prices were replaced with improved quotes, it was not possible to improve the price of any portfolio consisting of positions in two or more assets.

TABLE 7. Dominated Portfolios (Second period)	Ask	Bid
No. of time-points with improved quotes (for at least.1 cent)	75%	100%
Average fraction of assets with improved quote	16.66%	16.51%
Average mean improvement	2.87%	0.78%
Average maximum improvement	4.70%	0.97%

Our last sample is quite similar to the first one and draws on those prices available on July, 11, 14 and 20, 2005, as provided by Reuters. We have considered nine synchronized sets of data per date. As in the previous analyses we did not use spot or future prices and therefore only the riskless security and call and put options were involved. Quotes of July 11 and 14 correspond to options with expiration date in August and September 2005, whereas all the options available on July 14 matured in September.

Tables 8 and 9 below report and summarize the empirical results and, as can be seen, there are no significant differences with regard to the remainder tested periods.

TABLE 8. Dominated Securities (Third Period)

Dominated Securities	Ask	Bid
No.of dates with improved quotes (for at least.1 cent)	62.96&%	14.81%
Average fraction of assets with improved quote	13.92%	15.82%
Average mean improvement	22.22%	54.17%
Average maximum improvement	26.16%	54.17%

TABLE 9. Dominated Portfolios (Third Period)

Dominated Portfolios with Two Assets	
Short Position in both Assets	
No.of dates with improved bid price (at least of .1 cents)	0
Long Position in both Assets	
No. of dates with improved ask prices (at least of .1 cents)	0
One Long and one Short Position	
No. of dates with improved prices (at least of .1 cents)	55.56%
Average % of portfolios with improved price	4.39%
Average mean improvement in prices	10.28%
Average maximum improvement in prices	11.22%
Dominated Portfolios with Three Assets	
Short position in Three Assets	
No.of dates with improved bid price (at least of .1 cents)	0
Long position in Three Assets	
No. of dates with improved ask price (at least of .1 cents)	0
One Long and two Short Positions	
No. of dates with improved prices (at least of .1 cents)	55.56%
Average % of portfolios with improved price	9.09%
Average mean improvement in prices	1.15%
Average maximum improvement in prices	1.47%
One Short and two Long Positions	
No. of dates with improved prices (at least of .1 cents)	55.56%
Average % of portfolios with improved price	8.97%
Average improvement in individual prices	5.07%
Average maximum improvement in prices	10.40%

Dominated Portfolios with Four Assets	
Short Position in Four Assets	
No.of dates with improved bid price(at least of .1 cents)	0
Long Position in Four Assets	
No. of dates with improved bid prices (at least of .1 cents)	0
One Long and Three Short Positions	
No. of dates with improved prices (at least of .1 cents)	59.26%
Average % of portfolios with improved price	13.75%
Average mean improvement in prices	3.31%
Average maximum improvement in prices	11.60%
One Short and Three Long Positions	
No. of dates with improved prices (at least of .1 cents)	59.26%
Average % of portfolios with improved price	17.21%
Average mean improvement in prices	3.76%
Average maximum improvement in prices	14.65%
Two Short and Two Long Positions	
No. of dates with improved prices (at least of .1 cents)	59.26%
Average % of portfolios with improved price	13.41%
Average mean improvement in prices	3.97%
Average maximum improvement in prices	13.86%

5 Conclusions

The paper has addressed the problem of extending the existence of SDF in imperfect markets. It has been shown that they can apply in order to analyze the existence of dominated portfolios, as well as to compute optimal strategies in a mean/variance setting. They provide a useful link between payoffs and prices/quotes, since they yield a theoretical price process lying under the real price process and matching those quotes of the efficient portfolios.

We have used the SDF approach in order to empirically test the existence of inefficiencies when trading with oil linked derivatives available at $NYMEX$. The test has been done with the highest possible precision since we draw on perfectly synchronized bid/ask prices and frictions have been discounted. We follow the static point of view in the test so that we can guarantee the robustness of our conclusions regardless any dynamic assumption about the price behavior.

Despite the level of generality, our findings seem to reveal the existence of clear inefficiencies in the market. Several specific examples have been provided and the general results have been reported. This empirical evidence seems to be a major contribution. Indeed, firstly it may be useful to provide further evidence about the degree of efficiency of non-equity-linked financial markets. Secondly, many brokers and managers may be interested because they are incorporating oil derivatives in order to construct optimal and well diversified portfolios. Thirdly, the existence of inefficiencies provides brokers with new practical pricing and investment methods that may outperform their clients' portfolios.

Finally, the methodology is general enough to apply in any financial market, and therefore it may enable us to study emerging and maybe illiquid markets that are becoming very interesting for traders trying to correctly diversify their risk. It is worth to bear in mind that imperfections and other drawbacks may make it rather difficult to apply the classical pricing methods of frictionless markets.

6 Appendix

Before providing the proofs of the theoretical results of Section 2 we will present a simple lemma whose proof is immediate and therefore omitted.

Lemma 5 *Suppose that (x,y) and (x',y') are self-financing portfolios such that (x,y) dominates (x',y'). Then $\theta(x,y)$ dominates $\theta(x',y')$ for every $\theta > 0$ and $(x,y)+(x'',y'')$ dominates $(x',y')+(x'',y'')$ for every $(x'',y'') \in \mathcal{S}$.* $\square$

Proof of Lemma 1. Suppose that $\sum_{i=1}^{l} \alpha_i (x^i, y^i)$ is non-dominated and take $\beta_1, \beta_2, ..., \beta_l \geq 0$. Consider $\theta > 0$ with $\theta\beta_i < \alpha_i$ for $i = 1,2,...,l$. If we prove that $\sum_{i=1}^{l} \theta\beta_i (x^i, y^i)$ is non-dominated then we can multiply by $1/\theta > 0$ and the efficiency of $\sum_{i \in I} \beta_i (x^i, y^i)$ follows from the previous lemma. Suppose that (x,y) dominates $\sum_{i=1}^{l} \theta\beta_i (x^i, y^i)$. Then

$$(x,y) + \sum_{i=1}^{l} (\alpha_i - \theta\beta_i)(x^i, y^i)$$

dominates

$$\sum_{i=1}^{l} \theta\beta_i (x^i, y^i) + \sum_{i=1}^{l} (\alpha_i - \theta\beta_i)(x^i, y^i) = \sum_{i=1}^{l} \alpha_i (x^i, y^i)$$

against the assumption. $\square$

In order to prove the remainder results we provide without proof a lemma of Convex Analysis.

Lemma 6 *Let E be a vector space and $\Gamma : E \longrightarrow \mathbb{R}$ an arbitrary function such that $\Gamma(m_1 + m_2) \leq \Gamma(m_1) + \Gamma(m_2)$ whenever m_1 and m_2 belong to E and $\Gamma(\alpha m) = \alpha\Gamma(m)$ whenever $\alpha \geq 0$ and m belongs to E. If $m_0 \in E$ then there exists a linear function $\phi : E \longrightarrow \mathbb{R}$ such that $\phi(m) \leq \Gamma(m)$ for every $m \in E$ and $\phi(m_0) = \Gamma(m_0)$. If E is ordered and Γ is (strictly) increasing then ϕ is non-negative (positive).*

Proof of Theorem 2. Since $\sum_{i=1}^{l} (x^i, y^i)$ is efficient one has that

$$\pi\left(\Lambda\left(\sum_{i=1}^{l}(x^i, y^i)\right)\right) = \lambda\left(\sum_{i=1}^{l}(x^i, y^i)\right) = \sum_{i=1}^{l} \lambda(x^i, y^i).$$

Take $E = V$, $\Gamma = \pi$ and $m_0 = \sum_{i=1}^{l} \Lambda(x^i, y^i)$ and apply the previous lemma. There exists $\phi : V \longrightarrow \mathbb{R}$ such that

$$\phi(m) \leq \pi(m)$$

for every $m \in V$ and

$$\phi\left(\sum_{i=1}^{l} \Lambda(x^i, y^i)\right) = \sum_{i=1}^{l} \lambda(x^i, y^i).$$

The first inequality leads to $\phi\left(\Lambda(x^i, y^i)\right) \leq \lambda(x^i, y^i)$, $i = 1,2,...,l$, and consequently, the equality gives $\phi\left(\Lambda(x^i, y^i)\right) = \lambda(x^i, y^i)$, $i = 1,2,...,l$. Furthermore, Sine π is increasing ϕ is non-negative. Hence the continuity of ϕ is guaranteed because every linear and real valued positive operator on a Banach Lattice is continuous (Schaeffer, 1974). According to the Hanh-Banach Theorem

(Schaeffer, 1974), there exists a linear extension φ of ϕ to the whole space $L^p(\mathcal{F}_T)$. Whence the Riesz Representation Theorem implies the existence of δ. $\square$

Proof of Theorem 3. The existence of a continuous ϕ satisfying the conditions of the previous proof is absolutely similar. Being V a Hilbert Space the Riesz Representation Theorem leads to the existence of a unique $\delta \in V$ representing ϕ. $\square$

Proof of Theorem 4. First of all take the set of strategies

$$\{((x_J = 0), (y_{J_1} = (0, ..., 1, ...0), y_{J_2} = 0, y_{J_3} = 0))\}$$

and

$$\{((x_{J_1} = 0, x_{J_2} = (0, ..., 1, ...0), x_{J_3} = 0), (y_J = 0))\}$$

As in the previous proof one can construct the function ϕ. Its continuity holds even if V is not a Banach Lattice because this space has a finite dimension. Thus, the existence of a unique $\delta \in V$ satisfying (2.9) and (2.10) is ensured. In particular, Expression (2.10) applied on the strategies above leads to (4.12) and (4.13), and Expression (2.9) applied on the whole set of available securities in long and short position, respectively, leads to (4.11). $\square$

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Figures

Figure 1. Average futures prices during the analyzed period.

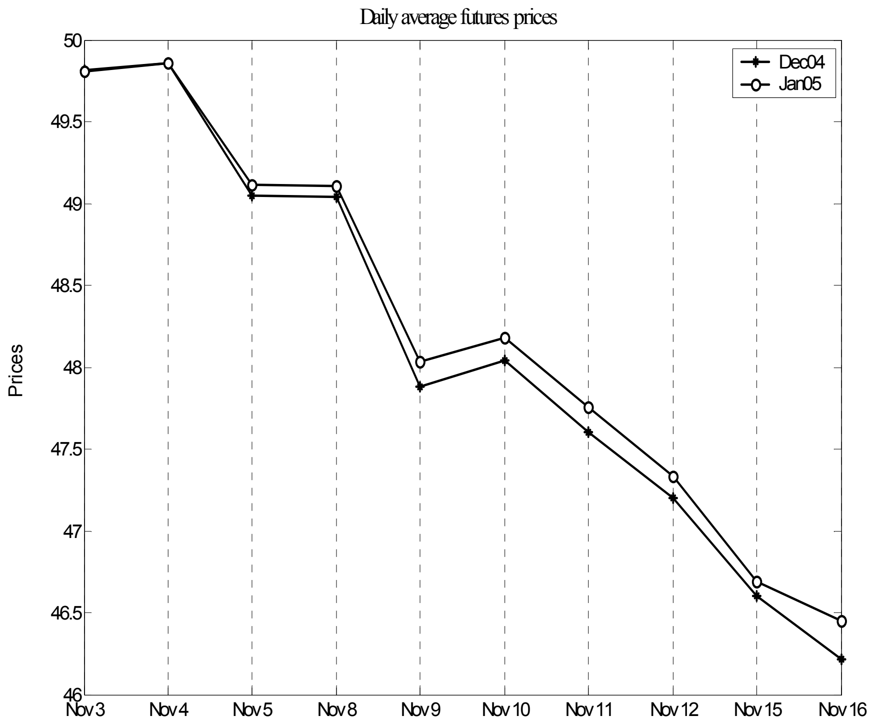


Figure 2. Number of options maturing in December 2004 with quotes available in each time-point. For each time-point in the sample, the graph shows the number of options for which either an ask price, a bid price, of both, were available in our data set.

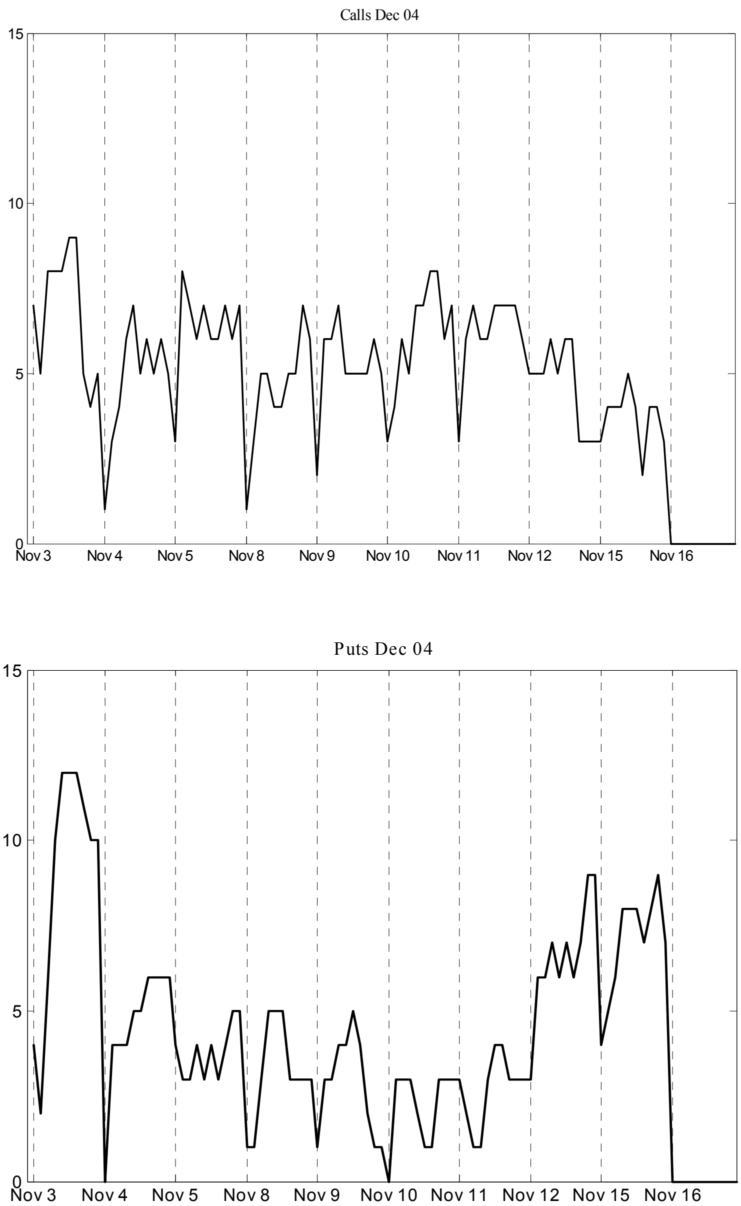
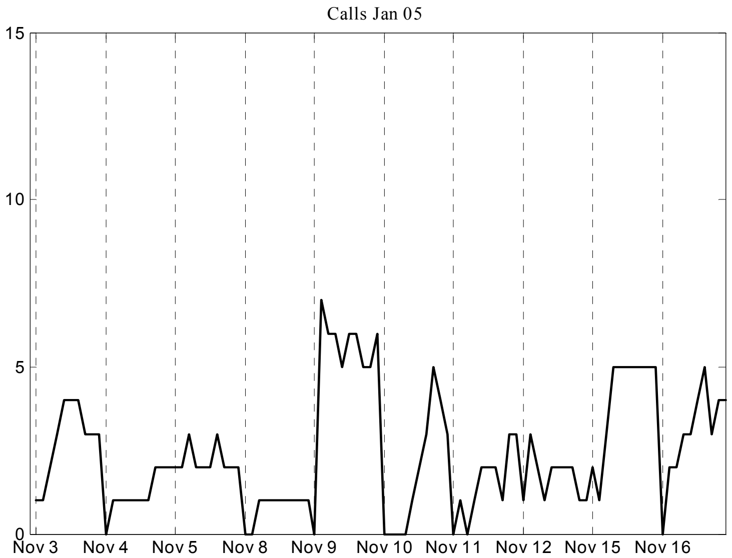


Figure 3. Number of options maturing in January 2005 with quotes available in each time-point. For each time-point in the sample, the graph shows the number of options for which either an ask price, a bid price, of both, were available in our data set.



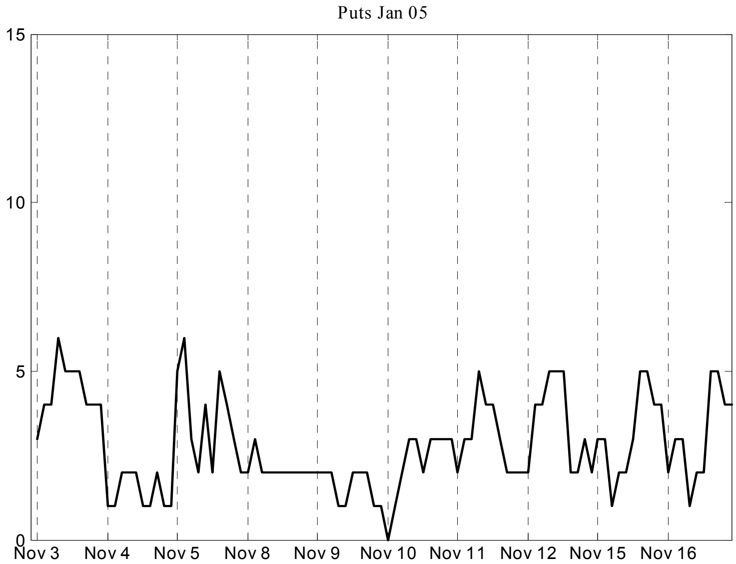


Figure 4. Ask and bid prices of the three most quoted calls during the analyzed period.

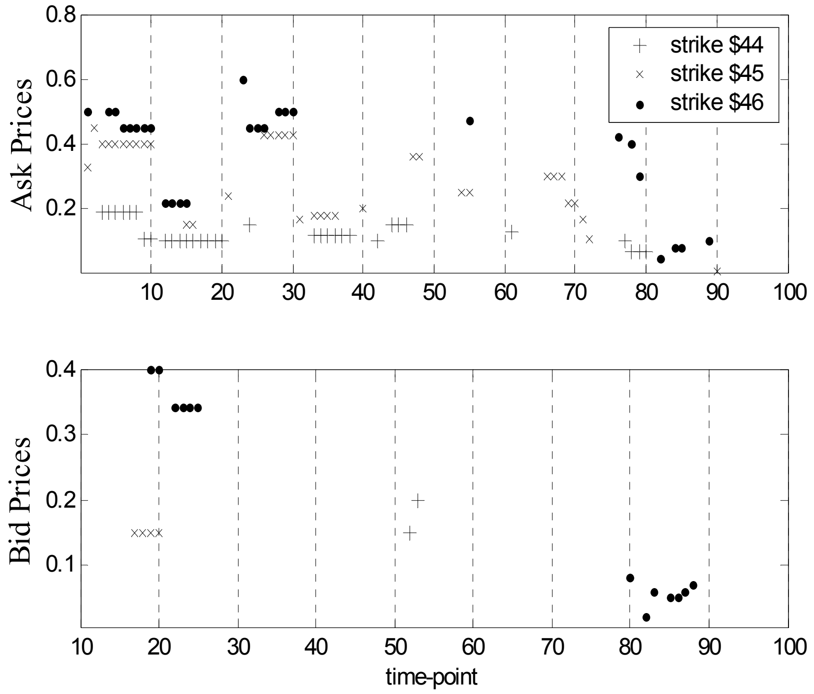


Figure 5. Ask and bid prices of the three most quoted puts during the analyzed period.

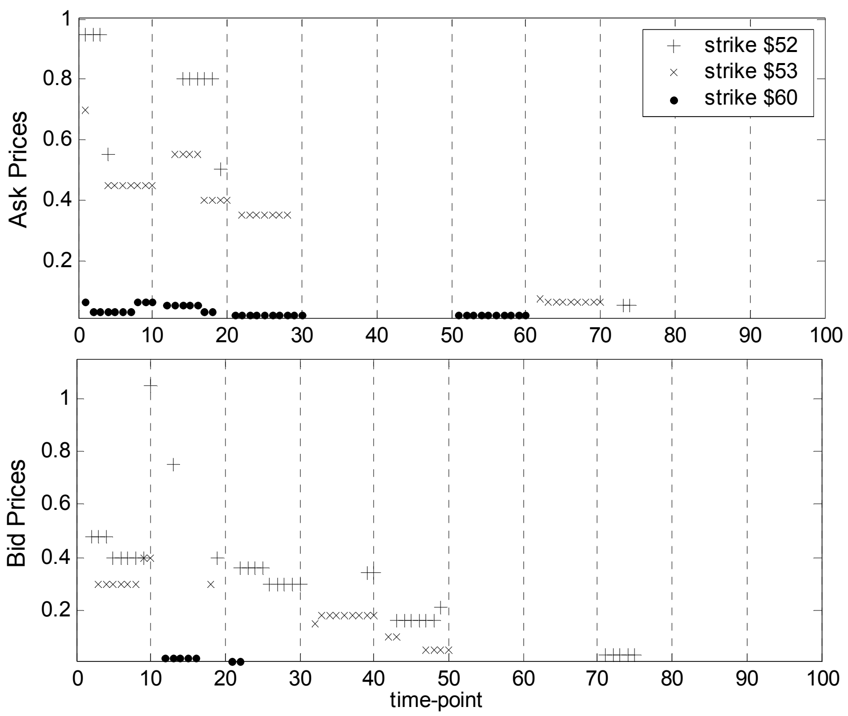


Figure 6. Example 1. The Figure displays the payoff function of an arbitrage strategy on November 5.

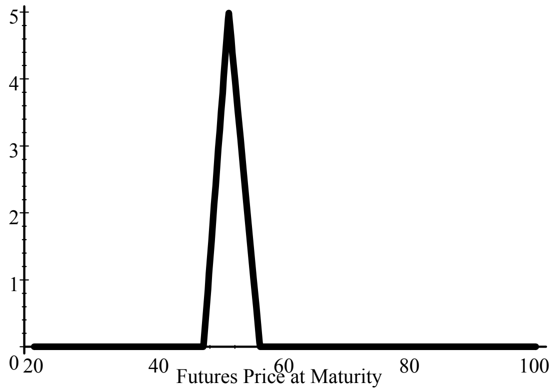


Figure 7. Example 2. The top panel displays the payoff functions of the dominated asset (thick line) and the dominant portfolio, whereas the bottom panel shows the stochastic discount factor that would eliminate the inefficiency.

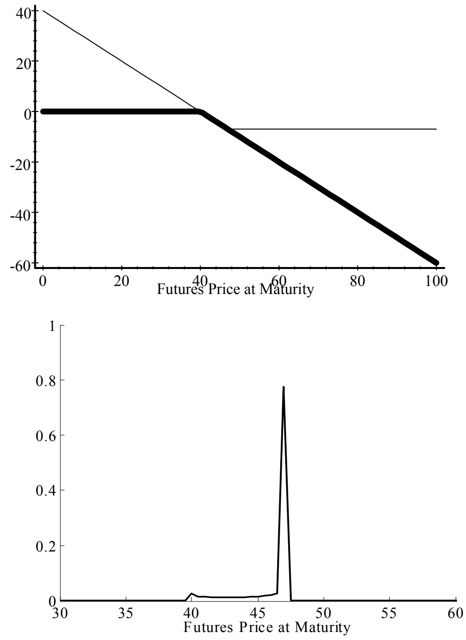


Figure 8. Example 3. The top panel displays the payoff functions of the dominated portfolio (thick line) and the dominant portfolio, whereas the bottom panel shows the stochastic discount factor that would eliminate the inefficiency.

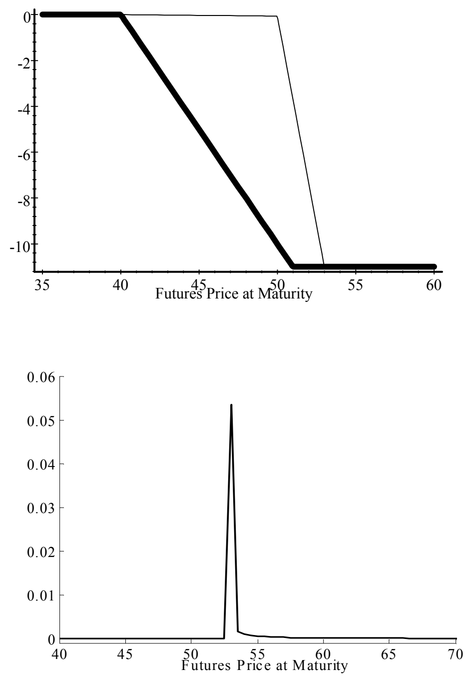


Figure 9. Example 4. The top panel displays the payoff functions of the dominated portfolio (thick line) and the dominant portfolio, whereas the bottom panel shows the stochastic discount factor that would eliminate the inefficiency.

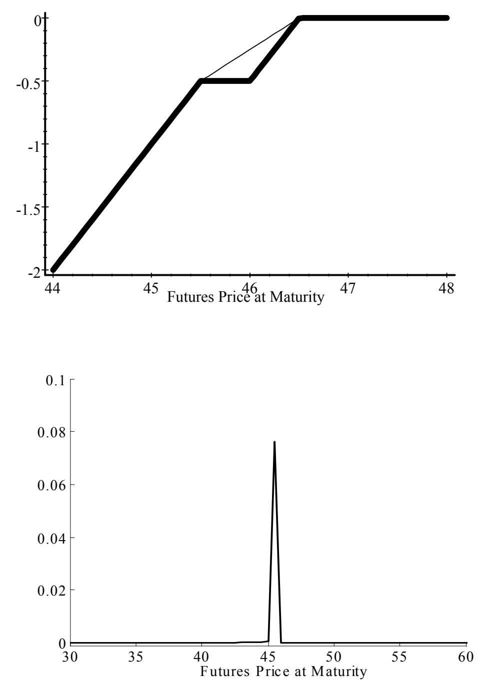


Figure 10. Example 5. The top panel displays the payoff functions of the dominated portfolio (thick line) and the dominant portfolio, whereas the bottom panel shows the stochastic discount factor that would eliminate the inefficiency.