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ON DOUBLE PERIODIC NON-HOMOGENEOUS POISSON PROCESSES *

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Here periodicity does not repeat the exact same short term pattern every year, but lets its peak intensity vary over a longer period. This model reflects periodic environments like those forming hurricanes, in alternating El Niño/La Niña years. The properties of the model are discussed in detail.

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Abstract

Non–homogenous Poisson processes with periodic claim intensity rate are proposed as the claim counting process of risk theory. We introduce a doubly periodic Poisson model with short and long term trends, illustrated by a double–beta intensity function.

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1 Introduction

Homogeneous Poisson processes are commonly used in risk theory to model claim frequency. These sometimes give a crude representation since their claim intensity rate λ is constant. A more general time–dependent model is obtained with non–homogeneous Poisson (NHP) processes, as their intensity rate $\lambda(t)$ is a function of time.

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Many natural phenomena evolve in a periodic environment or under seasonal conditions. In turn, these events generate insurance claims. For example, weather factors are known to affect automobile or fire insurance claims, while seasonal snow storms in the north and hurricanes or floods in the south affect property insurance. A periodic time–dependent intensity rate is a reasonable model for the claim frequency in such situations. We show that it can also be tractable, even for the corresponding aggregate claim process.

The similarities between intensity and failure rate functions, used in reliability models, help exploring different applications of NHP process. Some characterization properties of the NHP process with (single) periodic failure rate are derived in Chukova et al. (1993) and Dimitrov et al. (1997). These properties are exploited in a risk model by Garrido et al. (1996). Berg and Haberman (1994) use a non–homogeneous Markov birth process, of which the NHP is a special case, to predict trends in life insurance claim occurences. Some ruin problems in a periodic environment are also considered by Asmussen and Rolski (1994) and by Rolski et al. (1999).

A more practical case is when the periodic environment does not repeat itself exactly from year to year, but the short term peak changes over a relatively long period, with different levels in each year. This defines a double periodic environment, especially appropriate to model natural catastrophes, such as hurricanes, which have a peak season in the middle of the year, but with an intensity level also depending on long term climatological effects like La Niña or El Niño. A corresponding Poisson process model with double periodicity is introduced in here.

Section 2 discusses the periodicity of the NHP and related characteristics. Section 3 presents some practical forms for the claim intensity periodicity. The corresponding compound NHP sums are also studied.

2 The NHP Process and Preliminary Results

Let λ be a non–negative (measurable and locally integrable) deterministic function. Consider the number of claims in the time interval $[s,\ t)$, denoted $N_{[s,\ t)}$ for $0 \leq s < t$ (and N_t when $s = 0$). An NHP process is defined as follows.

Definition 1 A counting process $\{N_t;\ t \geq 0\}$ is said to be non–homogeneous Poisson (NHP) with intensity function λ, where $\lambda(t) \geq 0$, for $t \geq 0$, if it satisfies:

(a) $N_t = 0$ at $t = 0$;

(b) $\{N_t;\ t \geq 0\}$ has independent increments;

(c) $P\{N_{t+h} - N_t = 1\} = \lambda(t)h + o(h)$, for all $t,\ h \geq 0$;

(d) $P\{N_{t+h} - N_t \geq 2\} = o(h)$, for all $t,\ h \geq 0$,

The function Λ defined by

$$\Lambda(t) = \int_0^t \lambda(v)dv\ , \quad \text{for } t \geq 0\ , \tag{1}$$

is called the hazard function or the cumulative intensity function of the process.

Consider the number, $N_{[\tau,\ \tau+t)}$, in an interval of the form $[\tau,\ \tau+t)$, where $\tau, t \geq 0$. The time parameter τ, called the initial age of the process, marks the beginning of the time observation period when claims start to be counted. It is well known that for a NHP process the probability of n claims occurring in a time interval of duration t starting at time τ is given by

$$P\{N_{[\tau,\ \tau+t)} = n\} = \frac{e^{-[\Lambda(\tau+t)-\Lambda(\tau)]}[\Lambda(\tau+t) - \Lambda(\tau)]^n}{n!}\ , \quad n \in \mathbb{N}\ . \tag{2}$$

That is, for a NHP process with intensity function λ, $N_{[\tau,\ \tau+t)}$ has a Poisson distribution with mean $\Lambda(\tau+t) - \Lambda(\tau) = \int_\tau^{\tau+t} \lambda(v)dv$.

It is clear that a NHP process becomes a homogeneous Poisson process when its intensity function λ does not depend on time, i.e. $\lambda(t) = \lambda$, for all $t \geq 0$, and therefore $\Lambda(t) = \lambda t$ is linear.

Now, we consider the case where the risk process evolves in a periodic environment, as when the claim arrival rate may depend on the seasons. Then the intensity function of a NHP claim counting process $\{N_t;\ t \geq 0\}$, is a periodic function, say with a period of $c > 0$ years. Consequently $t - \lfloor \frac{t}{c} \rfloor c \in [0, c)$ is the time of the season, where $\lfloor t \rfloor$ is the integer part of $t \in \mathbb{R}$. A model with double periodicity is introduced in the next section where it is illustrated by a double–beta function.

Referring to Dimitrov et al. (1997) for proofs, we list the following properties for the NHP process $\{N_t;\ \geq 0\}$ with periodic intensity function.

Theorem 1 Suppose that the intensity function λ is periodic with period c, then

(a) The hazard function Λ has the almost linear property

$$\Lambda(t) = \lfloor \frac{t}{c} \rfloor \Lambda(c) + \Lambda(t - \lfloor \frac{t}{c} \rfloor c) \ , \quad t \geq 0 \ .$$

(b) For any integer $n \geq 0$ and $t \geq 0$

$$P\{N_{[nc,\, nc+t)} = k\} = P\{N_t = k\} \ , \quad k = 0, 1, \dots \ .$$

Moreover, the random variables N_{nc} and $N_{[nc, nc+t)}$ are mutually independent.

(c) The NHP process has a periodic intensity function λ with period $c > 0$ if and only if the random variables $N_{[0,\, c)}$ and $N_{[c,\, c+t)}$ are mutually independent and distributed as N_c and N_t, respectively.

(d) For any $t \geq 0$ the random variable N_t can be decomposed in the form

$$N_t = \begin{cases} N_{[0,\, t)} \ , & \text{if } t \leq c \\ M_1 + M_2 + \cdots + M_{\lfloor \frac{t}{c} \rfloor} + N_{[0,\, t - \lfloor \frac{t}{c} \rfloor c)} \ , & \text{if } t > c \end{cases} \ ,$$

where $\{M_i\}_{i \geq 1}$ are i.i.d. Poisson random variables distributed as $N_{[0,\, c)}$ and independent of $N_{[0,\, t - \lfloor \frac{t}{c} \rfloor c)}$, the latter being a Poisson r.v. distributed as $N_{t - \lfloor \frac{t}{c} \rfloor c}$, for $t - \lfloor \frac{t}{c} \rfloor c \in [0,\ c)$.

3 A Double–Beta Periodic Intensity Model

We define here some simple and practical forms of double–periodic claim intensity models.

First, assume that the intensity function in each short term, say each year and denoted by $\lambda_1(t)$, has a beta–shape. Further assume that the peak value in each year follows another beta function $\lambda_c(t)$, the long term intensity function. That is

$$\lambda_1(t) = \frac{\left(\frac{s - \lfloor s \rfloor - m_1}{d} \right)^{p_1 - 1} \left(1 - \frac{s - \lfloor s \rfloor - m_1}{d} \right)^{q_1 - 1}}{\alpha_1^*} \ , \quad t \geq 0 \ , \tag{3}$$

for $s = t - \lfloor \frac{t}{c} \rfloor c$, $m_1 \leq s - \lfloor s \rfloor < m_2$, $p_1, q_1 \geq 0$ and where

$$\alpha_1^* = \left(\frac{t_1^* - m_1}{d} \right)^{p_1 - 1} \left(1 - \frac{t_1^* - m_1}{d} \right)^{q_1 - 1} \ , \tag{4}$$

is the adjusted factor, while $d = m_2 - m_1$ and $0 \leq m_1 < m_2 \leq 1$ represent the starting and ending point of the annual interval, respectively, for which the intensity function does not equal to zero. Finally

$$t_1^* = m_1 + d\frac{p_1 - 1}{p_1 + q_1 - 2} , \tag{5}$$

denotes the mode of $\lambda_1(t)$, while

$$\lambda_c(t) = h(\lfloor t \rfloor + t_1^*) , \quad t \geq 0 , \tag{6}$$

where

$$\begin{aligned} h(t) &= a + \frac{b-a}{\alpha_c^*}\left(\frac{t-m_c}{c} - \lfloor\frac{t-m_c}{c}\rfloor\right)^{p_c-1} \\ &\quad \left[1-\left(\frac{t-m_c}{c} - \lfloor\frac{t-m_c}{c}\rfloor\right)\right]^{q_c-1} , \qquad (7) \\ \alpha_c^* &= \left(\frac{t_c^*-m_c}{c}\right)^{p_c-1}\left(1-\frac{t_c^*-m_c}{c}\right)^{q_c-1} , \qquad (8) \end{aligned}$$

is an adjusted beta–type factor. Here a, b denote the minimum and maximum amplitude of the peak values, respectively, m_c is the starting point of the complete cycle of the second beta function, and

$$t_c^* = m_c + c\left(\frac{p_c - 1}{p_c + q_c - 2}\right)$$

denotes the mode of $\lambda_c(t)$.

Then the double beta intensity function is given by

$$\lambda(t) = \lambda_c(t)\lambda_1(t) , \quad \text{for } t \geq 0 , \tag{9}$$

where λ_1 and λ_c are given in (3) and (6), respectively.

Figure 1 illustrates a possible shape of $\lambda(t)$ when $p_1 = 3$, $q_1 = 2$, $m_1 = \frac{5}{12}$, $d = \frac{6}{12}$, $c = 5$, $p_c = 2$, $q_c = 1.5$, $m_c = 3.75$, $a = 3$ and $b = 7$. The dotted line represents the base beta function (long term) that explains the fluctuations in the peak values of the short term beta periodicity.

By Theorem 1, we can obtain an explicit expression for the hazard function Λ, defined by (1) in the double–beta periodic case. The corresponding claim counting process $\{N_t,\ t \geq 0\}$ is also decomposed in i.i.d. components.

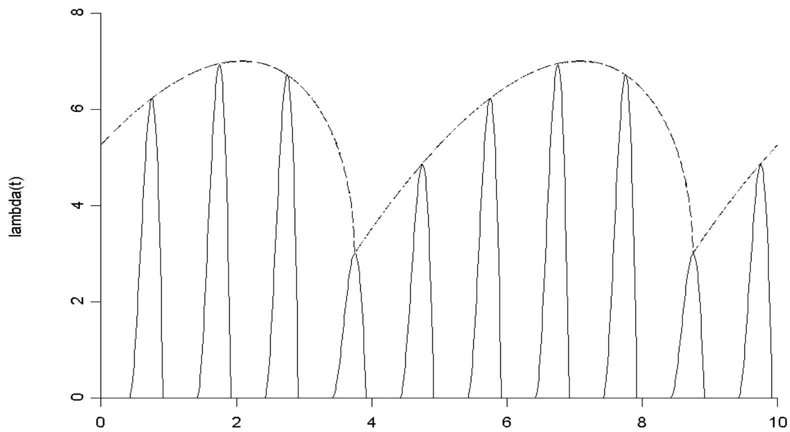

Figure 1: Double–beta intensity function $\lambda(t)$

Theorem 2 Assume that the intensity function λ is given by (9), then

(a) The hazard function Λ has the almost linear property, given by

$$\begin{aligned}\Lambda(t) \;&=\; \lfloor \frac{t}{c} \rfloor d\, B(p_1, q_1) \sum_{j=0}^{c-1} \frac{h(j+t_1^*)}{\alpha_1^*} + d\, B(p_1, q_1) \sum_{j=0}^{\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor - 1} \frac{h(j+t_1^*)}{\alpha_1^*} \\ &\quad + d\, B\left(p_1, q_1, \frac{t - \lfloor t \rfloor - m_1}{d}\right) \frac{h(\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor + t_1^*)}{\alpha_1^*} \,, \qquad (10)\end{aligned}$$

for $t \geq m_1$, where $h(t)$ has the form in (7) and

$$B(p,\ q) = \int_0^1 v^{p-1}(1-v)^{q-1} dv = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

is the beta function at $p,\ q > 0$, while

$$B(p,\ q;\ t) = \int_0^t v^{p-1}(1-v)^{q-1} dv \ , \quad t \in (0,\ 1) \ ,$$

is the usual incomplete beta function (with $B(p,\ q;\ t) = 0$ if $t \leq 0$).

(b) For any $t \geq 0$, the random variable N_t is decomposed as

$$N_t = M_1 + \cdots + M_{\lfloor \frac{t}{c} \rfloor} + N^*_{\frac{t - \lfloor t \rfloor - m_1}{d}} \ , \qquad (11)$$

where

$$N^*_{\frac{t-\lfloor t\rfloor-m_1}{d}} = \sum_{j=0}^{\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor-1} N_c^{(j)} + N^{(\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor)}_{\frac{t-\lfloor t\rfloor-m_1}{d}} \ , \tag{12}$$

and the $\{M_i\}_{i\geq 1}$ are i.i.d. Poisson with mean $d\,B(p_1,q_1)\ \sum_{j=0}^{c-1}\frac{h(j+t_1^*)}{\alpha_1^*}$, independent of $N_c^{(j)}$, for $j=0,1,\ldots,\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor-1$, and $N^{(\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor)}_{\frac{t-\lfloor t\rfloor-m_1}{d}}$, which are all Poisson random variables with mean $d\,B(p_1,q_1)\frac{h(j+t_1^*)}{\alpha_1^*}$, where $j=0,1,2,\ldots,\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor-1$, and $d\,B\left(p_1,q_1,\frac{t-\lfloor t\rfloor-m_1}{d}\right)\frac{h(\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor)}{\alpha_1^*}$, respectively.

Proof:

(a) By (1) and the periodicity of the intensity function λ,

$$\begin{aligned}
\Lambda(t) &= \int_0^t \frac{h(\lfloor v\rfloor+t_1^*)}{\alpha_1^*}\left(\frac{v-\lfloor\frac{v}{c}\rfloor c-\lfloor v-\lfloor\frac{v}{c}\rfloor c\rfloor-m_1}{d}\right)^{p_1-1} \\
&\qquad \left(1-\frac{v-\lfloor\frac{v}{c}\rfloor c-\lfloor v-\lfloor\frac{v}{c}\rfloor c\rfloor-m_1}{d}\right)^{q_1-1} dv \\
&= \lfloor\frac{t}{c}\rfloor\int_0^c \frac{h(\lfloor v\rfloor+t_1^*)}{\alpha_1^*}\left(\frac{v-\lfloor v\rfloor-m_1}{d}\right)^{p_1-1}\left(1-\frac{v-\lfloor v\rfloor-m_1}{d}\right)^{q_1-1} dv \\
&\quad + \int_0^{t-\lfloor\frac{t}{c}\rfloor c}\frac{h(\lfloor v\rfloor+t_1^*)}{\alpha_1^*}\left(\frac{v-\lfloor v\rfloor-m_1}{d}\right)^{p_1-1}\left(1-\frac{v-\lfloor v\rfloor-m_1}{d}\right)^{q_1-1} dv \\
&= \lfloor\frac{t}{c}\rfloor\sum_{j=0}^{c-1}\frac{h(j+t_1^*)}{\alpha_1^*}\int_{j+m_1}^{j+m_1+d}\left(\frac{v-j-m_1}{d}\right)^{p_1-1}\left(1-\frac{v-j-m_1}{d}\right)^{q_1-1} dv \\
&\quad + \sum_{j=0}^{\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor-1}\frac{h(j+t_1^*)}{\alpha_1^*}\int_{j+m_1}^{j+m_1+d}\left(\frac{v-j-m_1}{d}\right)^{p_1-1}\left(1-\frac{v-j-m_1}{d}\right)^{q_1-1} dv \\
&\quad + \frac{h(\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor+t_1^*)}{\alpha_1^*}\int_{\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor+m_1}^{t-\lfloor\frac{t}{c}\rfloor c}\left(\frac{v-\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor-m_1}{d}\right)^{p_1-1} \\
&\qquad \left(1-\frac{v-\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor-m_1}{d}\right)^{q_1-1} dv\ .
\end{aligned} \tag{13}$$

Letting $s=\frac{v-j-m_1}{d}$ in the first two integrals and $\frac{v-\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor-m_1}{d}$ in the last integral in (13) gives

$$\begin{aligned}
\Lambda(t) &= \lfloor \frac{t}{c} \rfloor \sum_{j=0}^{c-1} d\, \frac{h(j+t_1^*)}{\alpha_1^*} \int_0^1 s^{p_1-1}(1-s)^{q_1-1} ds \\
&\quad + \sum_{j=0}^{\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor -1} d\, \frac{h(j+t_1^*)}{\alpha_1^*} \int_0^1 s^{p_1-1}(1-s)^{q_1-1} ds \\
&\quad + d\, \frac{h(\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor + t_1^*)}{\alpha_1^*} \int_0^{\frac{t - \lfloor \frac{t}{c} \rfloor c - \lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor - m_1}{d}} s^{p_1-1}(1-s)^{q_1-1} ds \, .
\end{aligned}$$

Then (10) follows by definition of the beta and incomplete beta functions.

(b) By Theorem 1–(d), N_t can be decomposed as follows

$$N_t = \sum_{i=1}^{\lfloor \frac{t}{c} \rfloor} \sum_{j=0}^{c-1} N^{(j)}_{[(i-1)c,\, ic)} + \sum_{j=0}^{\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor -1} N^{(j)}_{[\lfloor \frac{t}{c} \rfloor c,\, (\lfloor \frac{t}{c} \rfloor +1)c)} + N^{(\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor)}_{[\lfloor \frac{t}{c} \rfloor c,\, (\lfloor \frac{t}{c} \rfloor +1)c)} \,, \tag{14}$$

where

$$M_i = \sum_{j=0}^{c-1} N^{(j)}_{[(i-1)c,\, ic)} \,, \quad i = 1, 2, \dots, \lfloor \frac{t}{c} \rfloor \tag{15}$$

is the i-th complete period subsum, while the second summation in (14) is the decomposition of the complete years included in the last (incomplete) period. Finally the last term represents the claim count for the last (incomplete) year of the last (incomplete) period.

By periodicity of the function λ and Theorem 1–(b), it is clear that $N^{(j)}_{[(i-1)c,\, ic)}$, $i = 1, 2, \dots, \lfloor \frac{t}{c} \rfloor + 1$, are mutually independent and Poisson distributed random variables with mean $d\, B(p_1, q_1) \frac{h(j+t_1^*)}{\alpha_1^*}$, just as $N_c^{(j)}$, for $j = 0, 1, \dots, c-1$.

As the additive property of the NHP processes, we consequently get that M_i, given in (15), $i = 1, 2, \dots, \lfloor \frac{t}{c} \rfloor$ are i.i.d. Poisson random variables distributed as N_c, with mean $d\, B(p_1, q_1) \sum_{j=0}^{c-1} \frac{h(j+t_1^*)}{\alpha_1^*}$. Similarly, $N^{(\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor)}_{[\lfloor \frac{t}{c} \rfloor c,\, (\lfloor \frac{t}{c} \rfloor +1)c)}$ is Poisson with mean $d\, B\left(p_1, q_1, \frac{t - \lfloor t \rfloor - m_1}{d}\right) \frac{h(\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor + t_1^*)}{\alpha_1^*}$, like $N^{(\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor)}_{\frac{t - \lfloor \frac{t}{c} \rfloor c - \lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor - m_1}{d}}$ or $N^{(\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor)}_{\frac{t - \lfloor t \rfloor - m_1}{d}}$.

Now, setting

$$N^*_{\frac{t-\lfloor t\rfloor-m_1}{d}} = \sum_{j=0}^{\lfloor t-\lfloor \frac{t}{c}\rfloor c\rfloor-1} N^{(j)}_{[\lfloor \frac{t}{c}\rfloor c,\ (\lfloor \frac{t}{c}\rfloor+1)c)} + N^{(\lfloor t-\lfloor \frac{t}{c}\rfloor c\rfloor)}_{[\lfloor \frac{t}{c}\rfloor c,\ (\lfloor \frac{t}{c}\rfloor+1)c)}$$

$$= \sum_{j=0}^{\lfloor t-\lfloor \frac{t}{c}\rfloor c\rfloor-1} N^{(j)}_c + N^{(\lfloor t-\lfloor \frac{t}{c}\rfloor c\rfloor)}_{\frac{t-\lfloor t\rfloor-m_1}{d}}\ , \qquad (16)$$

gives (12). Combining with (15), (14) leads to (11) and hence (b) holds. □

Now consider $N_{[\tau,\ \tau+t)}$, the number of claims in the time interval $[\tau,\ \tau+t)$. It is assumed to follow a NHP process with parameter $\lambda(t)$ given by (9). From Theorem 2, the probability of $n \in \mathbb{N}$ claims in the time interval $[\tau,\ \tau+t)$ is:

$$P\{N_{[\tau,\ \tau+t)} = n\} = \frac{[\Lambda(\tau+t)-\Lambda(\tau)]^n}{n!} e^{-[\Lambda(\tau+t)-\Lambda(\tau)]}\ ,$$

where $\Lambda(\tau+t)-\Lambda(\tau) = \int_{\tau}^{\tau+t} \lambda(v)dv$ can be derived from (10).

The moment generating function (m.g.f.) of $N_{[\tau,\ \tau+t)}$ is given by

$$E(e^{rN_{[\tau,\ \tau+t)}}) = e^{[\Lambda(\tau+t)-\Lambda(\tau)](e^r-1)}\ ,$$

and the expected number of claims over this time interval equals its variance and is given by

$$E(N_{[\tau,\ \tau+t)}) = V(N_{[\tau,\ \tau+t)}) = \Lambda(\tau+t)-\Lambda(\tau)\ .$$

In particular, the m.g.f. of the number of claims over one period of length c, with an initial age of τ (that we will denote α) equals

$$\alpha = E(e^{N_{[\tau,\ \tau+c)}}) = e^{-\Lambda(c)} = e^{-d\,B(p_1,q_1)\sum_{j=0}^{c-1}\frac{h(j+t_1^*)}{\alpha_1^*}}\ , \qquad (17)$$

where $h(j+t_1^*)$ can be derived from (7) for $j = 1, 2, \cdots, c-1$.

Moreover, the probability to survive the time interval $[\tau,\ \tau+t)$ without a claim is

$$P\{N_{[\tau,\ \tau+t)} = 0\} = e^{-[\Lambda(\tau+t)-\Lambda(\tau)]}\ ,$$

while the waiting time T_1 for the first claim in $[0,\ t)$ has an almost–lack–of–memory distribution [see Dimitrov et al. (1997)] and is given by

$$P\{T_1 \le t\} = 1 - P\{N_t = 0\} = 1 - e^{-\Lambda(t)}$$
$$= 1 - \alpha^{\lfloor \frac{t}{c}\rfloor} e^{-d\,B(p_1,q_1)\sum_{j=0}^{\lfloor t-\lfloor \frac{t}{c}\rfloor c\rfloor-1}\frac{h(j+t_1^*)}{\alpha_1^*}} e^{-d\,B\left(p_1,q_1,\frac{t-\lfloor t\rfloor-m_1}{d}\right)\frac{h(\lfloor t-\lfloor \frac{t}{c}\rfloor c\rfloor+t_1^*)}{\alpha_1^*}}\ ,$$

where α is given by (17). The corresponding p.d.f. is

$$
\begin{aligned}
f_{T_1}(t) \;=\; & \alpha^{\lfloor \frac{t}{c} \rfloor} e^{-d\,B(p_1,q_1) \sum_{j=0}^{\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor - 1} \frac{h(j+t_1^*)}{\alpha_1^*}} e^{-d\,B\left(p_1,q_1,\frac{t-\lfloor t \rfloor - m_1}{d}\right) \frac{h(\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor + t_1^*)}{\alpha_1^*}} \\
& h\left(\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor + t_1^* \right) \lambda_1 \left(t - \lfloor \frac{t}{c} \rfloor c \right) ,
\end{aligned}
$$

while the expectation of T_1 is given by

$$
\begin{aligned}
E(T_1) \;=\; & m_1 + \frac{c\alpha + \sum_{j=1}^{c-1} j e^{-dB(p_1,q_1)\frac{h(j-1+t_1^*)}{\alpha_1^*}} \left(1 - e^{dB(p_1,q_1)\frac{h(j+t_1^*)}{\alpha_1^*}} \right)}{1-\alpha} \\
& + \frac{d^2 \int_0^1 \sum_{j=1}^{c-1} j e^{-dB(p_1,q_1)\frac{h(j+t_1^*)}{\alpha_1^*}} \frac{h(j+t_1^*)}{\alpha_1^*} v^{p_1} (1-v)^{q_1-1} dv}{1-\alpha} .
\end{aligned}
$$

Finally, at time t, the excess–life until the next claim, $T_{N_t+1} - t$ is distributed as

$$
P\{T_{N_t+1} - t \leq s\} = 1 - e^{-[\Lambda(t+s) - \Lambda(t)]} , \quad s \geq 0 .
$$

The flexibility of the beta family of intensity functions, which depends on the value of the shape parameters p and q, provides many possible forms of short and long term seasonal claim intensities. Other shapes, like periodic trigonometric functions can also be considered to model the long term periodicity. For example

$$
h(t) = a + b \sin 2\pi \left(\frac{t - m_c}{c} - \lfloor \frac{t - m_c}{c} \rfloor \right) ,
$$

where $a \geq b$ and $a+b$, $a-b$ represent, respectively, the maximum and minimum amplitude of the peak values for the long term periodicity, while m_c is the starting point of the periodic sine function.

Figure 2 illustrates the shape of $\lambda(t)$ for $p_1 = q_1 = 2$, $m_1 = 0$, $d = 1$, $c = 4$, $m_c = \frac{3}{2}$ $a = \frac{5}{4}$ and $b = 1$. Here the short–term beta peak values vary according to the sine function (dotted line). The properties for the corresponding hazard function Λ and claim counting process $\{N_t,\ t \geq 0\}$ can be derived analogously.

3.1 The Aggregate Claims Process

The decompositions of Theorem 2 for the NHP process can be extended to compound NHP sums.

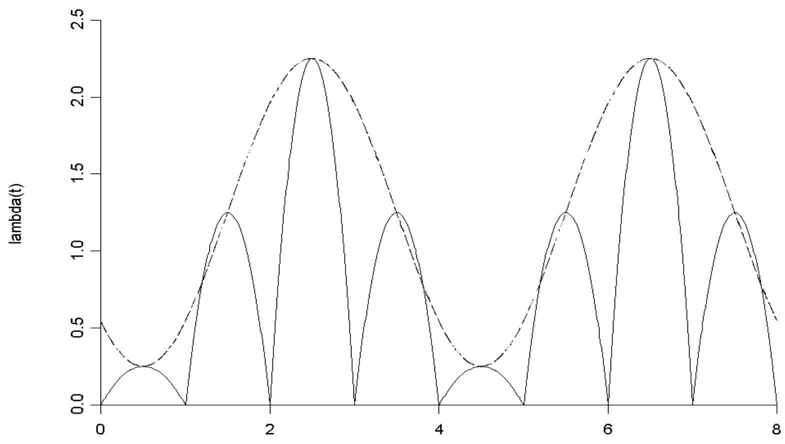

Figure 2: Sine-beta intensity function $\lambda(t)$

Again consider a NHP claim counting process $\{N_t;\ t \geq 0\}$. Then the corresponding aggregate claims process

$$S_t = \begin{cases} \sum_{j=1}^{N_t} X_j & \text{if } N_t > 0 \\ 0 & \text{if } N_t = 0 \end{cases} ,$$

is called a compound NHP process and is denoted as $S_t \sim$ C.P.$[\Lambda;\ F_X]$, for $x \geq 0$. The X_j are i.i.d. claim severities, with common c.d.f. F_X and finite mean μ, independent of N_t.

Consider the claim counting process $\{N_{[\tau,\,\tau+t)},\ t \geq 0\}$, for a fixed initial age τ and periodic intensity function λ. Its corresponding hazard function has the following structure:

(i) If both τ and t are integers then

$$\Lambda(\tau + t) - \Lambda(\tau) = \Lambda(c)\lfloor \frac{t}{c} \rfloor + \sum_{j=\tau+\lfloor \frac{t}{c} \rfloor c}^{\tau+t-1} \frac{h(j + t_1^*)}{\alpha_1^*} ,$$

where $\Lambda(c) = d\,B(p_1, q_1)\sum_{j=0}^{c-1} \frac{h(j+t_1^*)}{\alpha_1^*}$ and α_1^*, t_1^*, h are given in (4) to (7).

(ii) If τ is an integer but t is not, then

$$\Lambda(\tau+t)-\Lambda(\tau)=\Lambda(c)\lfloor\frac{t}{c}\rfloor+d\,B(p_1,q_1)\sum_{j=\tau+\lfloor\frac{t}{c}\rfloor c}^{\lfloor\tau+t\rfloor-1}\frac{h(j+t_1^*)}{\alpha_1^*}$$
$$+d\,B\left(p_1,q_1,\frac{t-\lfloor t\rfloor-m_1}{d}\right)\frac{h(\lfloor\tau+t\rfloor+t_1^*)}{\alpha_1^*}\ .$$

(iii) If τ is not an integer but t is, then

$$\Lambda(\tau+t)-\Lambda(\tau)=\Lambda(c)\lfloor\frac{t}{c}\rfloor+d\,B\left(p_1,q_1,\frac{\tau-\lfloor\tau\rfloor-m_1}{d},1\right)$$
$$\frac{h(\lfloor\tau+\lfloor\frac{t}{c}\rfloor c\rfloor+t_1^*)}{\alpha_1^*}+d\,B(p_1,q_1)\sum_{j=\lfloor\tau+\lfloor\frac{t}{c}\rfloor c\rfloor+1}^{\lfloor\tau\rfloor+t-1}\frac{h(j+t_1^*)}{\alpha_1^*}$$
$$+d\,B\left(p_1,q_1,\frac{\tau-\lfloor\tau\rfloor-m_1}{d}\right)\frac{h(\lfloor\tau\rfloor+t+t_1^*)}{\alpha_1^*}\ ,$$

where $B(p,q,t,1)=B(p,q)-B(p,q,t)$, for any $p,q>0$.

(iv) If neither τ nor t are integer, then

$$\Lambda(\tau+t)-\Lambda(\tau)=\Lambda(c)\lfloor\frac{t}{c}\rfloor+d\,B\left(p_1,q_1,\frac{\tau-\lfloor\tau\rfloor-m_1}{d},1\right)$$
$$\frac{h(\lfloor\tau+\lfloor\frac{t}{c}\rfloor c\rfloor+t_1^*)}{\alpha_1^*}+d\,B(p_1,q_1)\sum_{j=\lfloor\tau+\lfloor\frac{t}{c}\rfloor c\rfloor+1}^{\lfloor\tau+t\rfloor-1}\frac{h(j+t_1^*)}{\alpha_1^*}$$
$$+d\,B\left(p_1,q_1,\frac{\tau+t-\lfloor\tau+t\rfloor-m_1}{d}\right)\frac{h(\lfloor\tau+t\rfloor+t_1^*)}{\alpha_1^*}\ .$$

The aggregate claims over $[\tau,\ \tau+t)$ is then given by $S_{[\tau,\,\tau+t)}=\sum_{n=1}^{N_{[\tau,\,\tau+t)}}X_n$, where $N_{[\tau,\,\tau+t)}$ is a NHP process with periodic intensity function λ as in (9) and $S_{[\tau,\,\tau+t)}=0$ if $N_{[\tau,\,\tau+t)}=0$. Theorem 1 implies the following decomposition result.

Corollary 1 (a) If τ is an integer but t is not (note that case (i) above is a special case of (ii)), then the claim counting process $N_{[\tau,\,\tau+t)}$ is equivalent to the process which has the same time period but starts from $\tau \mod (c)$. Then $S_{[\tau,\,\tau+t)}$ can be decomposed as

$$S_{[\tau,\,\tau+t)}=S_1+\cdots+S_{\lfloor\frac{t}{c}\rfloor}+S^*_{\tau+\lfloor\frac{t}{c}\rfloor c}+\cdots+S^*_{\lfloor\tau+t\rfloor-1}+S^*_{[\lfloor\tau+t\rfloor,\tau+t)}\ , \quad (18)$$

where the S_i's are i.i.d. random variables distributed as $S_1 = \sum_{n=1}^{N_c} X_n$, and N_c is a Poisson r.v. with parameter $\Lambda(c)$. Then $S^*_{\tau+\lfloor\frac{t}{c}\rfloor c}, \cdots, S^*_{\lfloor\tau+t\rfloor-1}$ are also compound Poisson representing claims for complete years in the last (incomplete) period. Their respective parameters are $d\,B(p_1,q_1)\frac{h(\tau+\lfloor\frac{t}{c}\rfloor c+t_1^*)}{\alpha_1^*}, \cdots,$ $d\,B(p_1,q_1)\frac{h(\lfloor\tau+t\rfloor-1+t_1^*)}{\alpha_1^*}$. While the term $S^*_{[\lfloor\tau+t\rfloor,\tau+t)}$ is the compound Poisson sum representing the last (incomplete) year in the last period. It is independent of other aggregate claims S_i and has a mean of $d\,B(p_1,q_1,\frac{t-\lfloor t\rfloor-m_1}{d})$ $\frac{h(\lfloor\tau+t\rfloor+t_1^*)}{\alpha_1^*}$.

(b) If neither τ nor t are integer (note that case (iii) above is a special case of (iv)), then $S_{[\tau\ \tau+t)}$ contains incomplete terms and can be decomposed as

$$\begin{aligned} S_{[\tau,\,\tau+t)} &= S_1 + S_2 + \cdots + S_{\lfloor\frac{t}{c}\rfloor} + S^*_{[\tau+\lfloor\frac{t}{c}\rfloor c,\lfloor\tau+\lfloor\frac{t}{c}\rfloor c\rfloor+1)} \\ &\quad + S^*_{\lfloor\tau+\lfloor\frac{t}{c}\rfloor c\rfloor+1} + \cdots + S^*_{\lfloor\tau+t\rfloor-1} + S^*_{[\lfloor\tau+t\rfloor,\tau+t)}\ , \end{aligned}$$

where the complete sums S_i are compound Poisson with parameter $\Lambda(c)$, mutually independent from the two terms $S^*_{[\tau+\lfloor\frac{t}{c}\rfloor c,\lfloor\tau+\lfloor\frac{t}{c}\rfloor c\rfloor+1)}$ and $S^*_{[\lfloor\tau+t\rfloor,\tau+t)}$. The latter are also independent compound Poisson sums for two different incomplete years, with parameters $d\,B(p_1,q_1,\frac{\tau-\lfloor\tau\rfloor-m_1}{d},1)\frac{h(\lfloor\tau+\lfloor\frac{t}{c}\rfloor c\rfloor+t_1^*)}{\alpha_1^*}$, respectively, and $d\,B(p_1,q_1,\frac{\tau+t-\lfloor\tau+t\rfloor-m_1}{d})\frac{h(\lfloor\tau+t\rfloor+t_1^*)}{\alpha_1^*}$. Finally, the sums $S^*_{\lfloor\tau+\lfloor\frac{t}{c}\rfloor c\rfloor+1}+$ $\ldots + S^*_{\lfloor\tau+t\rfloor-1}$ are also compound Poisson in an incomplete period, with parameters $d\,B(p_1,q_1)\frac{h(\lfloor\tau+\lfloor\frac{t}{c}\rfloor c\rfloor+1+t_1^*)}{\alpha_1^*}, \ldots, d\,B(p_1,q_1)\frac{h(\lfloor\tau+t\rfloor-1+t_1^*)}{\alpha_1^*}$, respectively.

Moreover, the moment generating function of $S_{[\tau,\,\tau+t)}$ is obtained as

$$E(e^{rS_{[\tau,\,\tau+t)}}) = e^{[\Lambda(\tau+t)-\Lambda(\tau)][M_X(r)-1]}\ , \tag{19}$$

where M_X is the m.g.f. of the claims severity distribution. Moments of $S_{[\tau,\,\tau+t)}$ are easily obtained from (19). For instance, the total initial premium is given by

$$E(S_{[\tau,\,\tau+t)}) = [\Lambda(\tau+t) - \Lambda(\tau)]E(X_1)\ .$$

Conclusion

Non–homogeneous Poisson processes with periodic claim intensity rate are useful in modeling risk processes under periodic environments. A double–beta periodic claim intensity model is proposed as a generalization of the classical risk model. It also serves as a more realistic alternative to periodic models with only short term (single) periodic intensity functions.

The flexible shapes of the beta function and the explicit results obtained for the risk process should make these double–periodic models as practical as the classical one. In addition, statistical methods to estimate the beta parameters of the model from real data sets are readily available and shall be illustrated in subsequent work.

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