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Insurance Contracts**

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# Quantile hedging for telegraph markets and its applications to a pricing of equity-linked life insurance contracts

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## Abstract

*In this paper we develop a financial market model based on continuous time random motions with alternating constant velocities and with jumps occurring when the velocity switches. If jump directions are in the certain correspondence with the velocity directions of the underlying random motion with respect to the interest rate, the model is free of arbitrage and complete. Closed form formulas for the option prices and perfect hedging strategies are obtained.*

*The quantile hedging strategies for options are constructed. This methodology is applied to the pricing and risk control of insurance instruments.*

**JEL Classification:** G10, G12, D81

**Keywords:** jump telegraph model, perfect hedging, quantile hedging, pure endowment, equity-linked life insurance

## Resumen

*En este documento está desarrollado un modelo de mercado financiero basado en movimientos aleatorios con tiempo continuo, con velocidades constantes alternantes y saltos cuando hay cambios en la velocidad. Si los saltos en la dirección tienen correspondencia con la dirección de la velocidad del comportamiento aleatorio subyacente, con respecto a la tasa de interés, el modelo no presenta arbitraje y es completo. Se construye en detalle las estrategias replicables para opciones, y se obtiene una presentación cerrada para el precio de las opciones.*

*Las estrategias de cubrimiento quantile para opciones son construidas. Esta metodología es aplicada al control de riesgo y fijación de precios de instrumentos de seguros.*

**Clasificación JEL:** G10, G12, D81

**Palabras clave:** modelo telegráfico con saltos, cubrimiento perfecto, cubrimiento quantile, contribución pura, seguro de vida unido por equidad

## 1. Introduction

Equity-linked life insurance contracts are rather new insurance derivative securities that combine elements of insurance and financial risks. In traditional life insurance future liabilities are fixed. Hence, the corresponding risk (under stable interest rates) could be reduced completely by investments in bonds of the net present value of the fixed amount. By contrast, the payoffs in equity-linked life insurance contracts depend on the evolution of the risky asset during a contract period. In these circumstances an insurance company should try to hedge the corresponding contingent claim working on an incomplete market.

The problem of premium calculations for the equity-linked insurance contracts have been investigated first by Brennan and Schwartz [6]-[7], Boyle and Schwartz [5]. These and more recent papers (see e.g., [10], [19]) are based on traditional Black-Scholes and binomial models. These papers also discussed an imperfect hedging approach, i. e. mean variance or quantile (efficient) hedging (see [11]-[12]).

The present paper recasts in this financial framework the model of the price process proposed in [23]-[24]. This model is based on (inhomogeneous) telegraph process [14], which is a continuous time random motion with constant velocities alternating at independent and exponentially distributed time intervals. We assume the log-price of risky asset follows this process with jumps at the times of trend changes. This approach looks rather natural. Moreover the underlying process converges to Brownian motion under suitable rescaling. However this process is not a Lévy process, so the general option pricing theory does not work.

The rest of the paper proceeds as follows. Section 2 presents the inhomogeneous telegraph processes and martingales related to the telegraph evolutions and to the driving inhomogeneous Poisson process. The jump telegraph model of financial market is described. Exploiting Girsanov theorem for the telegraph processes with jumps we construct the martingale measure. A fundamental equation for the strategy value is obtained and the strategy is derived. In Section 3 we derive perfect hedging strategies for standard call options. The closed formulas for its price are presented. These formulas are analytic tractable and combine the outlines of the Black-Scholes and Merton formulas. Section 4 describes the basic ideas of quantile hedging. In Section 5 applies these ideas to a pricing of equity-linked life insurance contracts. Appendix contains the exact formulas for the distributions of the underlying processes, which are necessary for the call option pricing.

This paper exploits the ideas presented by the author on the 2nd Nordic-Russian Symposium on Stochastic Analysis [22] and continues the author's previous papers devoted to the jump telegraph model [23]-[24].

## 2. No homogeneous telegraph processes and martingales. Dynamics of the risky asset and the martingale measure

### 2.1. Telegraph and Poisson martingales. Measure change

Consider the process  $\sigma = \sigma(t)$ ,  $t \geq 0$  with values  $\pm 1$  such that

$$P(\sigma(t + \Delta t) = +1 \mid \sigma(t) = -1) = \lambda_- \Delta t + o(\Delta t),$$

$$P(\sigma(t + \Delta t) = -1 \mid \sigma(t) = +1) = \lambda_+ \Delta t + o(\Delta t), \quad \Delta t \rightarrow 0.$$

Here  $\lambda_-$ ,  $\lambda_+ > 0$ , and  $\sigma(0) = \xi$ , where  $\xi$  is a random variable with two values  $\pm 1$ . The time intervals  $\tau_j - \tau_{j-1}$ ,  $j = 1, 2, \dots$  ( $\tau_0 = 0$ ), separated by instants  $\tau_j$ ,  $j = 1, 2, \dots$  of value changes are independent and independent of  $\xi$  random variables. Denote by  $N(t)$  the number of value changes of  $\sigma$  in time  $t$ , i. e.  $\sigma(t) = \xi(-1)^{N(t)}$ .

Let  $c_- < c_+$ ,  $h_-$ ,  $h_+$  be real numbers. We denote

$$V(t) = c_{\sigma(t)}, \quad X(t) = \int_0^t V(s) ds \quad (2.1)$$

and

$$J(t) = \sum_{j=1}^{N(t)} h_{\sigma(\tau_j-)}, \quad t \geq 0. \quad (2.2)$$

The process  $N = N(t)$ ,  $t \geq 0$  is an inhomogeneous Poisson process with alternating parameters  $\lambda_{\pm}$ . The process  $(X, V)$  is called the (inhomogeneous) telegraph process with states  $(c_-, \lambda_-)$  and  $(c_+, \lambda_+)$ . The process  $J = J(t)$ ,  $t \geq 0$  is a pure jump process with jumps at the Poisson times  $\tau_j$ ,  $j = 1, 2, \dots$ . For  $\lambda_- = \lambda_+$  and  $-c_- = c_+ = c$  the processes  $V = \xi c(-1)^{N(t)}$  and  $X = \xi c \int_0^t (-1)^{N(s)} ds$ ,  $t \geq 0$  are well known [13], [14]-[15] and they are called the telegraph and integrated telegraph processes respectively. It is known also, that if  $\lambda, c \rightarrow \infty$  and  $c^2/\lambda \rightarrow 1$ , then the process  $X(t)$  converges to the standard Brownian motion. The inhomogeneous process is less known (see [3], where the exact distributions of inhomogeneous  $X(t)$  are calculated).

**Remark 2.1.** Let  $X = X(t)$  and  $\tilde{X} = \tilde{X}(t)$ ,  $t \geq 0$  be telegraph processes with states  $(c_{\pm}, \lambda_{\pm})$  and  $(\tilde{c}_{\pm}, \lambda_{\pm})$  respectively, governed by the common Poisson process  $N = N(t)$ . Then

$$\tilde{X}(t) = aX(t) + bt \quad (2.3)$$

with

$$a = a_{\tilde{c}} = \frac{\tilde{c}_+ - \tilde{c}_-}{c_+ - c_-}, \quad b = b_{\tilde{c}} = \frac{c_+ \tilde{c}_- - c_- \tilde{c}_+}{c_+ - c_-}. \quad (2.4)$$

Notice that  $c_{\sigma} a_{\tilde{c}} + b_{\tilde{c}} \equiv \tilde{c}_{\sigma}$ ,  $\sigma = \pm 1$ .

To construct related martingales we have the following theorem.

**Theorem 2.1.** *Let  $(X, V)$  be the telegraph process with states  $(c_+, \lambda_+)$  and  $(c_-, \lambda_-)$ , defined in (2.1), and  $J$  be the pure jump process, defined in (2.2). Then  $X + J$  is a martingale if and only if  $\lambda_\sigma h_\sigma = -c_\sigma$ ,  $\sigma = \pm 1$ .*

In particular case  $\lambda_\pm = \lambda$ ,  $h_\pm = h$ ,  $c_\pm = c$  the theorem evidently follows from the martingale property of  $N(t) - \lambda t$ ,  $t \geq 0$ . The general proof follows from the exact representation of expectations  $\mathbb{E}(J(t) | \mathcal{F}_s)$  and  $\mathbb{E} X(t) | \mathcal{F}_s$ :

$$\mathbb{E}(J(t) | \mathcal{F}_s) = J(s) + \gamma H(t-s) + \lambda_\sigma a_\sigma \frac{1 - e^{-\Lambda(t-s)}}{\Lambda},$$

$$\mathbb{E}(X(t) | \mathcal{F}_s) = X(s) + g(t-s) + \lambda_\sigma d_\sigma \frac{1 - e^{-\Lambda(t-s)}}{\Lambda}, \quad \sigma = \pm 1.$$

Here  $H = h_- + h_+$ ,  $\Lambda = \lambda_- + \lambda_+$ ,  $\gamma = \frac{\lambda_- \lambda_+}{\Lambda}$ ,  $g = \frac{c_+ \lambda_- + c_- \lambda_+}{\Lambda}$ ,  $a_\sigma = \frac{\lambda_\sigma h_\sigma - \lambda_{-\sigma} h_{-\sigma}}{\Lambda}$ ,  $d_\sigma = \frac{c_\sigma - c_{-\sigma}}{\Lambda}$ ,  $\sigma = \pm 1$ . See details in [24].

Fix time horizon  $T$ . Let

$$Z(t) = \frac{d\mathbf{P}_t^*}{d\mathbf{P}_t} = \mathcal{E}_t(X^* + J^*), \quad 0 \leq t \leq T \quad (2.5)$$

be the density of new measure  $\mathbf{P}^*$  relative to  $\mathbf{P}$ . Here  $X^*$  is the telegraph process with the states  $(c_\pm^*, \lambda_\pm)$  and  $J^* = -\sum_{j=1}^{N(t)} c_{\sigma(\tau_j^-)}^* / \lambda_{\sigma(\tau_j^-)}$  is the pure jump process with the jump values  $h_\sigma^* = -c_\sigma^* / \lambda_\sigma > -1$ ,  $\sigma = \pm 1$ . Both processes are driven by the same inhomogeneous Poisson process  $N$ .  $\mathcal{E}_t(\cdot)$  denotes the stochastic exponential.

From (2.5) we obtain

$$Z(t) = e^{X^*(t)} \kappa^*(t), \quad (2.6)$$

where  $\kappa^*(t) = \prod_{s \leq t} (1 + \Delta J^*(s))$  and  $\Delta J^*(s) = J^*(s) - J^*(s-)$ .

Let us consider the sequence  $\kappa_n^{*,\sigma}$ , which is defined as follows

$$\kappa_n^{*,\sigma} = \kappa_{n-1}^{*,\sigma} (1 + h_\sigma^*), \quad n \geq 1, \quad \kappa_0^{*,\sigma} \equiv 1, \quad \sigma = \pm 1. \quad (2.7)$$

Thus if  $n = 2k$ ,

$$\kappa_n^{*,\sigma} = (1 + h_\sigma^*)^k (1 + h_{-\sigma}^*)^k, \quad (2.8)$$

and if  $n = 2k + 1$ ,

$$\kappa_n^{*,\sigma} = (1 + h_\sigma^*)^{k+1} (1 + h_{-\sigma}^*)^k, \quad (2.9)$$

Therefore  $\kappa^*(t) = \kappa_{N(t)}^{*,\sigma}$ , where  $\sigma = \pm 1$  indicates the initial direction.

The following theorem replaces the Girsanov theorem in this framework.

**Theorem 2.2.** [24] *Under the probability  $\mathbf{P}^*$  with density  $Z(t)$  relative to  $\mathbf{P}$ , process  $N = N(t)$ ,  $t \geq 0$  is again the Poisson process with intensities  $\lambda_-^* = \lambda_- - c_-^* = \lambda_- (1 + h_-^*)$  and  $\lambda_+^* = \lambda_+ - c_+^* = \lambda_+ (1 + h_+^*)$ .*

*Under the probability  $\mathbf{P}^*$  process  $X = X(t)$ ,  $0 \leq t \leq T$  is the telegraph process with the states  $(c_-, \lambda_-^*)$  and  $(c_+, \lambda_+^*)$ .*

## 2.2. Dynamics of the risky asset and the martingale measure

We assume the bond price

$$B(t) = e^{Y(t)}, \quad Y(t) = \int_0^t r_{\sigma(s)} ds, \quad r_-, r_+ > 0. \quad (2.10)$$

To introduce the price process for a risky asset let  $X = X(t)$ ,  $t \geq 0$  be the telegraph process with the states  $(c_-, \lambda_-)$  and  $(c_+, \lambda_+)$ ,  $c_+ > c_-$  and  $J = J(t) = \sum_{j=1}^{N(t)} h_{\sigma(\tau_{j-})}$ ,  $h_{\pm} > -1$ .

We assume the price of risky asset follows the equation

$$dS(t) = S(t-)d(X(t) + J(t)), \quad t > 0. \quad (2.11)$$

Here the process  $S(t)$ ,  $t \geq 0$  is right-continuous.

Integrating (2.11) we obtain

$$S(t) = S_0 \mathcal{E}_t(X + J) = S_0 e^{X(t)} \kappa(t), \quad (2.12)$$

where  $S_0 = S(0)$  and

$$\kappa(t) = \prod_{s \leq t} (1 + \Delta J(s)) = \kappa_{N(t)}^{\sigma}.$$

The sequence  $\kappa_n^{\sigma}$ ,  $n \geq 0$  is defined in (2.7)-(2.9) (with  $h_{\pm}$  instead of  $h_{\pm}^*$ ).

We assume the following restrictions to the parameters of the model

$$\frac{r_{\sigma} - c_{\sigma}}{h_{\sigma}} > 0, \quad \sigma = \pm 1. \quad (2.13)$$

Since the process  $N$  is the unique source of randomness, it is possible the only one martingale measure.

**Theorem 2.3.** *Let  $Z(t) = \mathcal{E}_t(X^* + J^*)$ ,  $t \geq 0$  with  $h_{\sigma}^* = -c_{\sigma}^*/\lambda_{\sigma}$  be the density of probability  $\mathbf{P}^*$  relative to  $\mathbf{P}$ .*

*The process  $(B(t))^{-1}S(t)_{t \geq 0}$  is the  $\mathbf{P}^*$ -martingale if and only if*

$$c_{\sigma}^* = \lambda_{\sigma} + \frac{c_{\sigma} - r_{\sigma}}{h_{\sigma}}, \quad \sigma = \pm 1.$$

*Under the probability  $\mathbf{P}^*$  the Poisson process  $N$  is driven by the parameters*

$$\lambda_{\sigma}^* = \frac{r_{\sigma} - c_{\sigma}}{h_{\sigma}} > 0, \quad \sigma = \pm 1.$$

**Proof.** First notice that by Theorem 2.2  $X(t) - Y(t)$  is the telegraph process (with respect to  $\mathbf{P}^*$ ) with the states  $(c_{\sigma} - r_{\sigma}, \lambda_{\sigma} - c_{\sigma}^*)$ ,  $\sigma = \pm 1$ . From Theorem 2.1 it follows that  $X(t) - Y(t) + J(t)$ ,  $t \geq 0$  is the  $\mathbf{P}^*$ -martingale, if and only if

$$(\lambda_{\sigma} - c_{\sigma}^*)h_{\sigma} = -(c_{\sigma} - r_{\sigma}).$$

Hence  $c_{\sigma}^* = \lambda_{\sigma} + (c_{\sigma} - r_{\sigma})/h_{\sigma}$  and  $h_{\sigma}^* = -c_{\sigma}^*/\lambda_{\sigma} = -1 + (r_{\sigma} - c_{\sigma})/\lambda_{\sigma}h_{\sigma}$ . Theorem is proved.

**Remark 2.2.** From condition (2.13) it follows  $h_\sigma^* > -1$  and  $\lambda_\sigma^* = \lambda_\sigma - c_\sigma^* = (r_\sigma - c_\sigma)/h_\sigma > 0$ ,  $\sigma = \pm 1$ . Therefore  $Z = Z(t) = \mathcal{E}_t(X^* + J^*)$  really defines the density of new probability measure.

### 2.3. Fundamental equation. Predictability of the strategy

Consider the function

$$F(t, x, \sigma) = \mathbf{E}^* \left[ e^{-Y(T-t)} f(xe^{X(T-t)} \kappa(T-t)) | \xi = \sigma \right],$$

$$\sigma = \pm 1, 0 \leq t \leq T,$$

where  $\mathbf{E}^*$  denotes the expectation with respect to martingale measure  $\mathbf{P}^*$ , which is defined in Theorem 2.3. The density  $Z(t)$  of  $\mathbf{P}^*$  relative to  $\mathbf{P}$  is defined in (2.6)-(2.9). Function  $F_t = F(t, S(t), \sigma(t)) = \varphi_t S(t) + \psi_t B(t)$  is the strategy value at time  $t$  of the option with the claim  $f(S_T)$  at the maturity time  $T$ .

Notice that  $Y(t) = a_r X(t) + b_r t$  with  $a_r = \frac{r_+ - r_-}{c_+ - c_-}$ ,  $b_r = \frac{c_+ r_- - c_- r_+}{c_+ - c_-}$  (see Remark 2.1). Conditioning on the number of jumps we can write

$$F(t, x, \sigma) = e^{-b_r(T-t)} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} e^{-a_r y} f(xe^{y\kappa_n^\sigma}) p_{*,n}^{(\sigma)}(y, T-t) dy, \quad (2.14)$$

where  $p_{*,n}^{(\sigma)}$ ,  $n \geq 0$  are the probability densities of telegraph process  $X = X(t)$ ,  $0 \leq t \leq T$ , which commences  $n$  turns, with respect to martingale measure  $\mathbf{P}^*$ , i. e. for any measurable set  $\Delta$

$$\mathbf{P}^*(X(t) \in \Delta, N(t) = n | \xi = \sigma) = \int_{\Delta} p_{*,n}^{(\sigma)}(x, t) dx.$$

First notice that functions  $p_{*,n}^{(\sigma)}(x, t)$ ,  $\sigma = \pm 1$  satisfy the equations [24]

$$\frac{\partial p_{*,n}^{(\sigma)}}{\partial t} + c_\sigma \frac{\partial p_{*,n}^{(\sigma)}}{\partial x} = -\lambda_\sigma^* p_{*,n}^{(\sigma)} + \lambda_\sigma^* p_{*,n-1}^{(-\sigma)}, \quad n \geq 1 \quad (2.15)$$

with zero initial conditions:  $p_{*,n}^{(\sigma)}|_{t=0} = 0$ ,  $n \geq 1$ . Moreover  $p_{*,0}^{(\sigma)}(x, t) = e^{-\lambda_\sigma^* t} \delta(x - c_\sigma t)$ .

Hence function  $F$  solves the following difference-differential equation, which plays the same role as the fundamental equation in the Black-Scholes model. Exploiting equation (2.15) and the identity  $c_\sigma a_r + b_r = r_\sigma$ ,  $\sigma = \pm 1$  (see Remark 2.1) from (2.14) we obtain

$$\begin{aligned} & \frac{\partial F}{\partial t}(t, x, \sigma) + c_\sigma x \frac{\partial F}{\partial x}(t, x, \sigma) \\ &= (r_\sigma + \lambda_\sigma^*) F(t, x, \sigma) - \lambda_\sigma^* e^{-b_r(T-t)} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} e^{-a_r y} f(xe^{y\kappa_n^\sigma}) p_{*,n-1}^{(-\sigma)}(y, T-t) dy. \end{aligned}$$

By equalities (2.7) and  $\lambda_\sigma^* = \frac{r_\sigma - c_\sigma}{h_\sigma}$  the latter equation takes the form



$$\begin{aligned} & \frac{\partial F}{\partial t}(t, x, \sigma) + c_\sigma x \frac{\partial F}{\partial x}(t, x, \sigma) \\ &= (r_\sigma + \frac{r_\sigma - c_\sigma}{h_\sigma})F(t, x, \sigma) - \frac{r_\sigma - c_\sigma}{h_\sigma}F(t, x(1 + h_\sigma), -\sigma), \quad \sigma = \pm 1 \end{aligned} \quad (2.16)$$

with the terminal condition  $F_{t \uparrow T} = f(x)$ .

**Remark 2.3.** *Note that the above equations do not depend on  $\lambda_\pm$  as the respective equation in the Black-Scholes model does not depend on the drift parameter.*

To identify the self-financing strategy  $(\varphi_t, \psi_t)$ , such that  $F_t = \varphi_t S(t) + \psi_t B(t)$ ,  $0 \leq t \leq T$  we have  $dF_t = dF(t, S(t), \sigma(t)) = \varphi_t dS(t) + \psi_t dB(t)$ . Hence

$$F_t = F_0 + \int_0^t \varphi_s S(s) dX(s) + \int_0^t \psi_s dB(s) + \sum_{j=1}^{N(t)} \varphi_{\tau_j} h_{\sigma(\tau_j-)} S(\tau_j-).$$

From the identity  $\psi_t = B(t)^{-1}(F_t - \varphi_t S(t))$  we obtain

$$\begin{aligned} F_t &= F_0 + \int_0^t r_{\sigma(s)} F_s ds \\ &+ \int_0^t \varphi_s S(s) (c_{\sigma(s)} - r_{\sigma(s)}) ds + \sum_{j=1}^{N(t)} \varphi_{\tau_j} h_{\sigma(\tau_j-)} S(\tau_j-). \end{aligned} \quad (2.17)$$

On the other hand

$$\begin{aligned} F_t &= F_0 + \int_0^t \frac{\partial F}{\partial s}(s, S(s), \sigma(s)) ds + \int_0^t \frac{\partial F}{\partial x}(s, S(s), \sigma(s)) S(s) c_{\sigma(s)} ds \\ &+ \sum_{j=1}^{N(t)} (F_{\tau_j} - F_{\tau_j-}). \end{aligned} \quad (2.18)$$

Comparing the latter two equations we have between jumps

$$\varphi_t = \frac{S(t) c_{\sigma(t)} \frac{\partial F}{\partial x} + \frac{\partial F}{\partial t} - r_{\sigma(s)} F}{S(t) (c_{\sigma(t)} - r_{\sigma(s)})}.$$

From the fundamental equation (2.16) it follows that (between the jumps)

$$\varphi_t = \frac{F(t, S(t)(1 + h_{\sigma(t)}), -\sigma(t)) - F(t, S(t), \sigma(t))}{S(t) h_{\sigma(t)}}. \quad (2.19)$$

Moreover, from (2.17) and (2.18) we obtain the jump values of  $\varphi$

$$\varphi_{\tau_j} = \frac{F_{\tau_j} - F_{\tau_j-}}{S(\tau_j-) h_{\sigma(\tau_j-)}} = \frac{F(\tau_j, S(\tau_j), \sigma(\tau_j)) - F(\tau_j, S(\tau_j-), -\sigma(\tau_j))}{S(\tau_j-) h_{\sigma(\tau_j-)}}. \quad (2.20)$$

Formulas (2.19)-(2.20) remind the CRR and BS-formulas for the amounts of risky asset held over the time.

**Lemma 2.1.** *The strategy  $\varphi_t$ ,  $0 \leq t < T$  is left-continuous.*

**Proof.** To prove  $\varphi_{\tau_j-} = \varphi_{\tau_j}$  first notice that by (2.12)

$$S(\tau_j-)(1 + h_{\sigma(\tau_j-)}) = S(\tau_j). \quad (2.21)$$

Applying (2.21) to (2.19)-(2.20) it is easy to finish the proof.

### 3. Perfect hedging. Pricing a standard call

In the framework of the market model (2.10), (2.11)-(2.12)

the price of the option with contingent claim  $f$  can be expressed as follows

$$\mathbf{c} = \mathbf{c}^\sigma = \mathbf{E}_\sigma^*(B(T)^{-1}f) = \sum_{n=0}^{\infty} \mathbf{E}_\sigma^*(B(T)^{-1}f \mid N(T) = n) \pi_{*,n}^{(\sigma)}(T), \quad (3.1)$$

$$\sigma = \pm 1.$$

Here  $\mathbf{E}_\sigma^*(\cdot)$  is the expectation with respect to the martingale measure  $\mathbf{P}^*$ ,  $\pi_{*,n}^{(\sigma)}(T) = \mathbf{P}_\sigma^*(N(T) = n)$ ,  $n \geq 0$  and  $\sigma$  indicates the initial state. If  $\lambda_-^* = \lambda_+^* := \lambda$ , then  $\pi_{*,n}^{(\sigma)}(T) = \frac{(\lambda T)^n}{n!} e^{-\lambda T}$ . In general case  $\lambda_-^* \neq \lambda_+^*$  probabilities  $\pi_{*,n}^{(\sigma)}(T)$ ,  $\sigma = \pm 1$ ,  $n \geq 0$  are calculated in Appendix.

For the standard call option with contingent claim  $f = (S(T) - K)^+$  we rewrite (3.1) in the form

$$\mathbf{c} = \sum_{n=0}^{\infty} \mathbf{c}_n(K, T) \quad (3.2)$$

with

$$\mathbf{c}_n(K, T) = S_0 U_n^{(\sigma)}(y - b_n^{(\sigma)}, T) - K u_n^{(\sigma)}(y - b_n^{(\sigma)}, T), \quad (3.3)$$

where  $y = \ln K/S_0$  and  $b_n^{(\sigma)} = \ln \kappa_n^{*,\sigma}$ . Here functions  $u_n^{(\sigma)}$  and  $U_n^{(\sigma)}$ ,  $n \geq 0$  are defined as follows:

$$u_n^{(\sigma)}(y, t) = u_n^{(\sigma)}(y, t; \lambda_\pm^*, c_\pm, r_\pm) = \mathbf{E}_\sigma^* \left[ B(t)^{-1} \mathbf{1}_{\{X(t) > y, N(t)=n\}} \right] \quad (3.4)$$

$$= e^{-b_r t} \int_y^\infty e^{-a_r x} p_{*,n}^{(\sigma)}(x, t) dx$$

with  $a_r = \frac{r_+ - r_-}{c_+ - c_-}$  and  $b_r = \frac{c_+ r_- - c_- r_+}{c_+ - c_-}$ ;

$$\begin{aligned} U_n^{(\sigma)}(y, t) &= U_n^{(\sigma)}(y, t; \lambda_\pm^*, c_\pm, r_\pm) \\ &= \mathbf{E}_\sigma^* \left( B(t)^{-1} \mathcal{E}_t(X + J) \mathbf{1}_{\{X(t) > y\}} \mid N(t) = n \right) \pi_{*,n}^{(\sigma)}(t) \end{aligned} \quad (3.5)$$

$$= \kappa_n^{*,\sigma} e^{-b_r t} \int_y^\infty e^{-a_r x + x} p_{*,n}^{(\sigma)}(x, t) dx$$

Functions  $u_n^{(\sigma)}(y, t)$ ,  $n \geq 1$  satisfy the equation (see (2.15))

$$\frac{\partial u_n^{(\sigma)}}{\partial t}(y, t) + c_\sigma \frac{\partial u_n^{(\sigma)}}{\partial y}(y, t) = -(\lambda_\sigma^* + r_\sigma)u_n^{(\sigma)}(y, t) + \lambda_\sigma^* u_{n-1}^{(-\sigma)}(y, t) \quad (3.6)$$

with initial conditions  $u_n^{(\sigma)}|_{t=0} = 0$ ,  $n \geq 1$ . Functions  $u_n^{(\sigma)}$ ,  $n \geq 1$  are assumed to be continuous and piece-wise continuously differentiable.

It is plain, that  $u_0^{(\sigma)}(y, t) = e^{-(\lambda_\sigma^* + r_\sigma)t} \theta(c_\sigma t - y)$ ,  $\sigma = \pm 1$ . Moreover  $u_n^{(\sigma)} \equiv 0$ , if  $y > c_+ t$ , and for  $y < c_- t$ ,

$$u_n^{(\sigma)}(y, t) \equiv \rho_n^{(\sigma)}(t) = e^{-b_r t} \int_{-\infty}^{\infty} e^{-a_r x} p_{n,*}^{(\sigma)}(x, t) dx. \quad (3.7)$$

In the latter case system (3.6) has the form

$$\frac{d\rho_n^{(\sigma)}}{dt} = (-\lambda_\sigma^* + r_\sigma)\rho_n^{(\sigma)} + \lambda_\sigma^* \rho_{n-1}^{(-\sigma)}, \quad n \geq 1, \quad (3.8)$$

$\rho_0^{(\sigma)} = e^{-(\lambda_\sigma^* + r_\sigma)t}$  and  $\rho_n^{(\sigma)}|_{t=0} = 0$ ,  $n \geq 1$ ,  $\sigma = \pm 1$ .

As it is demonstrated in Appendix the solution of (3.8) can be written in the form

$$\rho_n^{(\sigma)}(t) = e^{-(\lambda_-^* + r_-)t} \Lambda_n^{(\sigma)} P_n^{(\sigma)}(t), \quad \sigma = \pm 1, \quad n \geq 0,$$

where  $\Lambda_n^{(\sigma)} = \lambda_{\sigma}^{[(n+1)/2]} \lambda_{-\sigma}^{[n/2]}$  and functions  $P_n^{(\sigma)}$  are defined as follows:

$$\begin{aligned} P_0^{(+)} &= e^{-at}, & P_0^{(-)} &\equiv 1, \\ P_n^{(\sigma)} &= P_n^{(\sigma)}(t) = \frac{t^n}{n!} \left[ 1 + \sum_{k=1}^{\infty} \frac{(m_n^{(\sigma)} + 1)_k}{(n+1)_k} \cdot \frac{(-at)^k}{k!} \right], & \sigma &= \pm 1, \quad n \geq 1. \end{aligned} \quad (3.9)$$

Here

$$m_n^{(+)} = [n/2], \quad m_n^{(-)} = [(n-1)/2],$$

$$(m)_k = m(m+1) \dots (m+k-1), \quad a = \lambda_+^* - \lambda_-^* + r_+ - r_-.$$

Notice that  $\Lambda_{2n}^{(+)} = \Lambda_{2n}^{(-)} \equiv \Lambda_{2n}$ ,  $P_{2n+1}^{(+)} \equiv P_{2n+1}^{(-)} \equiv P_{2n+1}$ .

To write down  $u_n^{(\sigma)} = u_n^{(\sigma)}(y, t)$  for  $c_- t < y < c_+ t$  let us define coefficients  $\beta_{k,j}$ ,  $j < k$ :  $\beta_{k,0} = \beta_{k,1} = \beta_{k,k-2} = \beta_{k,k-1} = 1$ ,

$$\beta_{k,j} = \frac{(k-j)_{[j/2]}}{[j/2]!}. \quad (3.10)$$

Let functions  $\varphi_{k,n}$  are defined as follows:  $\varphi_{0,n} = P_{2n+1}$  and

$$\varphi_{k,n} = \sum_{j=0}^{k-1} a^{k-j-1} \beta_{k,j} P_{2n-j}^-, \quad 1 \leq k \leq n. \quad (3.11)$$

For  $p, q > 0$  we denote  $v_0^{(-)} \equiv 0$ ,  $v_0^{(+)} = e^{-ap}$ ,  $v_1^{(\sigma)} = P_1(p)$ ,  $\sigma = \pm 1$  and for  $n \geq 1$

$$\begin{aligned} v_{2n+1}^{(\pm)} &= v_{2n+1}^{(\pm)}(p, q) = P_{2n+1}(p) + \sum_{k=1}^n \frac{q^k}{k!} \varphi_{k,n}(p), \\ v_{2n}^{(-)} &= v_{2n}^{(-)}(p, q) = P_{2n}^{(-)}(p) + \sum_{k=1}^{n-1} \frac{q^k}{k!} \varphi_{k+1,n}(p), \\ v_{2n}^{(+)} &= v_{2n}^{(+)}(p, q) = P_{2n}^{(+)}(p) + \sum_{k=1}^n \frac{q^k}{k!} \varphi_{k-1,n-1}(p), \end{aligned} \quad (3.12)$$

**Theorem 3.1.** *The solution of system (3.6) has the form*

$$u_n^{(\sigma)} = \begin{cases} 0, & y > c_1 t, \\ w_n^{(\sigma)}(p, q), & c-t \leq y \leq c+t, \quad \sigma = \pm 1, \\ \rho_n^{(\sigma)}(t), & y < c-t, \end{cases} \quad (3.13)$$

where  $w_n^{(\sigma)} = e^{-(\lambda_+^* + r_+)q - (\lambda_-^* + r_-)p} \Lambda_n^{(\sigma)} v_n^{(\sigma)}(p, q)$ ,  $p = \frac{c+t-y}{c_1 - c_-}$ ,  $q = \frac{y-c-t}{c_+ - c_-}$ . This solution is unique.

The proof see in Appendix.

**Remark 3.1.** *If  $\lambda_-^* = \lambda_+^* = \lambda$ ,  $r_+ = r_- = r$ , then  $P_n^{(\sigma)} = \frac{t^n}{n!}$ ,  $\pi_n^{(\sigma)} \equiv \pi_n = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$ ,  $\rho_n^{(\sigma)} = e^{-rt} \pi_n(t)$  and  $\varphi_{k,n} = P_{2n-k+1}^{(\sigma)}$ . Moreover*

$$v_n^{(\sigma)} = \frac{1}{n!} \sum_{k=0}^{m_n^{(\sigma)}} \binom{n}{k} q^k p^{n-k}.$$

**Remark 3.2.** *By definition function  $u_0^{(-)}$  is discontinuous at  $q = 0$  and  $u_0^{(+)}$  has the discontinuity at  $p = 0$ . It is easy to see that functions  $u_n^{(\sigma)}$ ,  $n \geq 1$ , defined in (3.13), are continuous. We can show that  $u_n^{(\sigma)}$ ,  $n \geq 2$  are continuously differentiable, but it is a bit tricky. The points of possible discontinuity of derivatives are concentrated on the lines  $p = 0$  and  $q = 0$ . For example for  $u_1^{(\sigma)}$ ,  $\sigma = \pm 1$  we have*

$$\frac{\partial u_1^{(\sigma)}}{\partial q} \Big|_{q=+0} - \frac{\partial u_1^{(\sigma)}}{\partial q} \Big|_{q=-0} = \lambda_\sigma^* e^{-(\lambda_+^* + r_+)p}$$

and

$$\frac{\partial u_1^{(\sigma)}}{\partial p} \Big|_{p=+0} - \frac{\partial u_1^{(\sigma)}}{\partial p} \Big|_{p=-0} = \lambda_\sigma^* e^{-(\lambda_+^* + r_+)q}.$$

Moreover, using (3.13) it is possible to proof that  $u_n^{(\sigma)} \in \mathcal{C}^{n-1}$ .

Similarly, functions  $U_n^{(\sigma)} = U_n^{(\sigma)}(y, t)$ ,  $n \geq 1$  fit the equation

$$\frac{\partial U_n^{(\sigma)}}{\partial t} + c_\sigma \frac{\partial U_n^{(\sigma)}}{\partial y} = -(\lambda_\sigma^* + r_\sigma - c_\sigma) U_n^{(\sigma)} + \lambda_\sigma^* (1 + h_\sigma) U_{n-1}^{(-\sigma)}. \quad (3.14)$$

For  $\lambda_\sigma^* = \frac{r_\sigma - c_\sigma}{h_\sigma}$  (see Theorem 2.3) it follows that  $\lambda_\sigma^*(1 + h_\sigma) = \lambda_\sigma^* + r_\sigma - c_\sigma := \bar{\lambda}_\sigma$ . Therefore equation (3.14) the same form as (3.6) with  $\bar{\lambda}_\sigma$  instead of  $\lambda_\sigma^*$ ,  $r_\pm = 0$  and  $U_0^{(\sigma)} = e^{-\bar{\lambda}_\sigma t} \theta(c_\sigma t - y)$ .

Hence

$$U_n^{(\sigma)}(y, t; \lambda_\pm^*, c_\pm, r_\pm) = u_n^{(\sigma)}(y, t; \bar{\lambda}_\pm, c_\pm, 0). \quad (3.15)$$

Exploiting (3.2)-(3.3) we can consider the following particular cases in details.

### 1. Merton model <sup>1</sup>.

Assume that  $r_- = r_+ = r$ ,  $c_- = c_+ = c$ ,  $h_- = h_+ = -h$ ,  $\lambda_- = \lambda_+ = \lambda$ . Then equation (2.11) has the form

$$dS(t) = S(t-)(cdt - hdN(t)),$$

where  $N = N(t)$ ,  $t \geq 0$  is the (homogeneous) Poisson process with parameter  $\lambda > 0$ . From call option pricing formula (3.2)-(3.3) we obtain

$$c = S_0 U(\ln K/S_0, T) - K u(\ln K/S_0, T). \quad (3.16)$$

If  $0 < h < 1$  and  $c > r$ , then  $b_n^{(\sigma)} \equiv b_n = n \ln(1 - h) \downarrow -\infty$  and

$$\begin{aligned} u &= u(\ln K/S_0, T) = e^{-rT} \sum_{n=0}^{n_0} u_n^{(\sigma)}(\ln(K/S_0) - b_n, T) \\ &= e^{-rT} P_\sigma(N(T) \leq n_0) = e^{-rT} \Psi_{n_0}(\lambda^* T). \end{aligned}$$

Here  $\lambda^* = (c - r)/h > 0$  and  $\Psi_{n_0}(z) = e^{-z} \sum_{n=0}^{n_0} \frac{z^n}{n!}$ . Function  $U$  has the form

$$U(y, T) = \Psi_{n_0}(\lambda^*(1 - h)T).$$

For  $h < 0$  and  $c < r$ , i. e.  $b_n^{(\sigma)} = n \ln(1 - h) \uparrow +\infty$ , we have

$$u(y, T) = e^{-rT} (1 - \Psi_{n_0}(\lambda^* T)),$$

$$U(y, T) = 1 - \Psi_{n_0}(\lambda^*(1 - h)T).$$

By  $n_0$  we denote

$$n_0 = \inf \left\{ n : S_0 e^{n \ln(1-h) + (c-r)T} > B(T)^{-1} K \right\} = \left\lceil \frac{\ln(K/S_0) - cT}{\ln(1-h)} \right\rceil.$$

<sup>1</sup>This model is called the Merton model (see [17], [18]), but [18] contains the reference to [8]. See also [9].

2. If  $(1 + h_-)(1 + h_+) < 1$ , then  $\ln(1 + h_-) + \ln(1 + h_+) < 0$  and  $b_n^{(\sigma)} \rightarrow -\infty$ . The call option price is given by the same formula (3.16) with

$$u^{(\sigma)}(y, T) = \sum_{k=0}^{n_-^{(\sigma)}} \rho_k^{(\sigma)}(T) + \sum_{k=n_-^{(\sigma)+1}^{n_+^{(\sigma)}}} u_k^{(\sigma)}(y - b_k^{(\sigma)}, T; \lambda_{\pm}^*, c_{\pm}, r_{\pm}),$$

and from (3.15) it follows

$$U^{(\sigma)}(y, T) = u^{(\sigma)}(y, T; \bar{\lambda}_{\pm}, c_{\pm}, 0), \quad (3.17)$$

$$y = \ln K/S_0.$$

Here

$$n_-^{(\sigma)} = \min \{n : y - b_n^{(\sigma)} > c_- T\},$$

$$n_+^{(\sigma)} = \min \{n : y - b_n^{(\sigma)} > c_+ T\}.$$

3. If  $(1 + h_-)(1 + h_+) > 1$ , then  $\ln(1 + h_-) + \ln(1 + h_+) > 0$  and  $b_n^{(\sigma)} \rightarrow +\infty$ . Denoting

$$m_-^{(\sigma)} = \max \{n : y - b_n^{(\sigma)} > c_- T\},$$

$$m_+^{(\sigma)} = \max \{n : y - b_n^{(\sigma)} > c_+ T\},$$

we obtain the call option price formula of the form (3.16) with

$$u^{(\sigma)}(y, T) = \sum_{k=m_+^{(\sigma)}}^{m_-^{(\sigma)}} u_k^{(\sigma)}(y - b_k^{(\sigma)}, T; \lambda_{\pm}^*, c_{\pm}, r_{\pm}) + \sum_{k=m_-^{(\sigma)+1}^{\infty} \rho_k^{(\sigma)}(T).$$

For  $U^{(\sigma)}(y, T)$  we again apply (3.17).

## 4. Quantile hedging

The strategy  $(\varphi_t, \psi_t)$  is called admissible, if  $F_t = \varphi_t S(t) + \psi_t B(t) \geq 0$  for all  $t \in [0, T]$ . The set of successful hedging for the claim  $f$  and the admissible strategy  $(\varphi_t, \psi)$  with the initial capital  $v$  is

$$A = A(v, \varphi, f) = \{\omega : B(T)^{-1} F_T \geq f\}.$$

In the case of the perfect hedging  $P(A) = 1$ , which requires the initial capital  $V_0 = E^* B(T)^{-1} f$ . The problem of quantile hedging maximizes probability of  $A$  under the budget restriction, i. e.

$$\begin{cases} P(A(v, \varphi, f)) \rightarrow \max \\ v \leq v_0 < E_{\sigma}^* (B(T)^{-1} f) = \mathbf{c}^{\sigma}, \end{cases} \quad (4.1)$$

where  $v_0$  is the initial capital of the investor. It is known (see [11]) that (4.1) is equivalent to the following optimization problem

$$\begin{cases} P(A) \rightarrow \max, \\ \mathbf{E}_\sigma^*(B(T)^{-1}f \cdot \mathbf{1}_A) \leq v. \end{cases} \quad (4.2)$$

Let  $\tilde{A} = \tilde{A}_\sigma$  is the solution of (4.2). The perfect hedge  $\tilde{\varphi}$  with initial capital  $v$  for the claim  $\tilde{f} = f \cdot \mathbf{1}_{\tilde{A}}$  is the solution of (4.1) and its set of successful hedging  $A = A(v, \tilde{\varphi}, f)$  coincides with  $\tilde{A}$ .

Moreover the structure of the set  $\tilde{A}$  is

$$\tilde{A} = \left\{ \frac{dP_T}{dP_T^*} \geq \gamma \cdot f \right\}, \quad \gamma = \text{const}, \quad \gamma > 0. \quad (4.3)$$

Using (2.5)-(2.6) and (2.3)-(2.4) we have

$$\frac{dP_T^*}{dP_T} = \mathcal{E}_T(X^* + J^*) = e^{X^*(T)} \kappa^*(T) = e^{aX(T)+bT} \kappa^*(T),$$

where  $a = \frac{c_+^* - c_-^*}{c_+ - c_-}$  and  $b = \frac{c_+ c_-^* - c_- c_+^*}{c_+ - c_-}$ . Hence the set of successful hedging  $\tilde{A}$  can be represented as

$$\tilde{A} = \tilde{A}^\gamma = \left\{ e^{-aX(T)} \geq \gamma e^{bT} \kappa^*(T) \cdot f \right\}.$$

For the standard call option with  $f = (S(T) - K)^+ = (S_0 e^{X(T)} \kappa(T) - K)^+$  the set  $\tilde{A}$  has the form

$$\tilde{A} = \left\{ e^{-aX(T)} > \gamma e^{bT} \kappa^*(T) (S_0 e^{X(T)} \kappa(T) - K)^+ \right\} = \bigsqcup_{n=0}^{\infty} A_n,$$

where

$$A_n = \left\{ e^{-aX(T)} \geq \gamma \kappa_n^{*,\sigma} e^{bT} (S_0 \kappa_n^\sigma e^{X(T)} - K)^+, \quad N(T) = n \right\}.$$

In the case  $-a \leq 1$  the sets  $A_n$  have the form

$$A_n = \{X(T) \leq y_n, \quad N(T) = n\}.$$

Here  $y_n = y_n(\gamma) = \ln z_n - b_n^{(\sigma)}$ , where  $b_n^{(\sigma)} = \ln \kappa_n^{(\sigma)} = z_n = z_n(\gamma)$  is the unique solution of algebraic equation

$$z^{-a} = \gamma \kappa_n^{*,\sigma} (\kappa_n^\sigma)^{-a} e^{bT} (S_0 z - K)^+. \quad (4.4)$$

It is clear that  $y_n = y_n(\gamma)$  decreases in  $\gamma$  and  $y_n \geq \ln K/S_0 - b_n^{(\sigma)}$ .

To find the constant  $\gamma$  we have the equation

$$\begin{aligned} v = S_0 \sum_{n=0}^{\infty} & \left[ U_n^{(\sigma)}(y - b_n^{(\sigma)}, T) - U_n^{(\sigma)}(y_n, T) \right] \\ & - K \sum_{n=0}^{\infty} \left[ u_n^{(\sigma)}(y - b_n^{(\sigma)}, T) - u_n^{(\sigma)}(y_n, T) \right], \end{aligned} \quad (4.5)$$

where  $u_n^{(\sigma)}$  and  $U_n^{(\sigma)}$ ,  $n \geq 0$ ,  $\sigma = \pm 1$  are defined in (3.3)-(3.4),  $y = \ln K/S_0$ . For  $\forall v \in (0, \mathbf{c}(K, T))$  this equation has the unique solution  $\gamma = \gamma(v)$ , because of monotonicity of  $y_n = y_n(\gamma)$ .

The solution of the quantile hedging problem is

$$\mathbf{P}(\tilde{A}) = \sum_{n=0}^{\infty} \mathbf{P}(A_n) = 1 - \sum_{n=0}^{\infty} u_n^{(\sigma)}(y_n(\gamma), T; \lambda_{\pm}, c_{\pm}, 0). \quad (4.6)$$

### Example

Let  $\lambda_+ = \lambda_- = \frac{r_+ - c_+}{h_+} = \frac{r_- - c_-}{h_-}$ . It means that initially the distribution of discounted asset price is the martingale with homogeneous governing Poisson process. Hence  $y_n \equiv \ln \frac{K+1/\gamma}{S_0} - b_n^{(\sigma)}$ ,  $a = b = 0$  and equation (4.5) for  $\gamma = \gamma(v)$  has the form

$$v = \mathbf{c}(K, T) - \mathbf{c}(K + 1/\gamma, T) - \frac{1}{\gamma} u^{(\sigma)}(K + 1/\gamma, T),$$

where  $u^{(\sigma)}(z, T) = \sum_{n=0}^{\infty} u_n^{(\sigma)}(z - y_n^{(\sigma)}, T)$ ,  $u_n^{(\sigma)}$ ,  $n \geq 0$  are defined in (3.13). The probability of successful hedging equals to

$$\begin{aligned} \mathbf{P}(\tilde{A}) &= \mathbf{P}_{\sigma}(S(T) < K + 1/\gamma) \\ &= 1 - \sum_{n=0}^{\infty} u_n^{(\sigma)} \left( \ln \frac{K + 1/\gamma}{S_0}, T; \lambda_{\pm}, c_{\pm}, 0 \right), \quad \gamma = \gamma(v), \end{aligned}$$

where  $u_n^{(\sigma)}$ ,  $n \geq 0$  are defined in (3.13) with  $\lambda_{\pm}^* = \lambda_{\pm}$  and  $r_{\pm} = 0$ .

Let  $-a > 1$ . Then we have

$$A_n = \left\{ X(T) \leq y_n^{(1)}, N(T) = n \right\} \cup \left\{ X(T) \geq y_n^{(2)}, N(T) = n \right\}.$$

Here  $y_n^{(1)} = \ln z_n^{(1)} - b_n^{(\sigma)}$  and  $y_n^{(2)} = \ln z_n^{(2)} - b_n^{(\sigma)}$ , where  $z_n^{(1)}$  and  $z_n^{(2)}$  are the solutions of (4.4).

The equation for  $\gamma$  has the form

$$\begin{aligned} v &= S_0 \sum_{n=0}^{\infty} \left[ U_n(y - b_n^{(\sigma)}, T) - U_n(y_n^{(1)}, T) + U_n(y_n^{(2)}, T) \right] \\ &\quad - K \sum_{n=0}^{\infty} \left[ u_n(y - b_n^{(\sigma)}, T) - u_n(y_n^{(1)}, T) + u_n(y_n^{(2)}, T) \right] \end{aligned}$$

and the solution of quantile hedging problem is

$$\mathbf{P}(\tilde{A}) = 1 - \sum_{n=0}^{\infty} \left[ u_n(y_n^{(1)}, T; \lambda_{\pm}, c_{\pm}, 0) - u_n(y_n^{(2)}, T; \lambda_{\pm}, c_{\pm}, 0) \right]. \quad (4.7)$$



The dual problem

$$\begin{cases} v \rightarrow \min \\ \mathbf{P}(A(v, \varphi, f)) \geq 1 - \varepsilon \end{cases} \quad (4.8)$$

minimizes the initial capital under fixed risk level. It can be solved as follows. Using (4.6) and (4.7) we can find  $\gamma$  from the equation  $\mathbf{P}(\tilde{A}^\gamma) = 1 - \varepsilon$ , i. e.

$$\sum_{n=0}^{\infty} u_n^{(\sigma)}(y_n(\gamma), T; \lambda_{\pm}, c_{\pm}, 0) = \varepsilon \quad (\text{for } -a \leq 1), \quad (4.9)$$

$$\sum_{n=0}^{\infty} \left[ u_n^{(\sigma)}(y_n^{(1)}(\gamma), T; \lambda_{\pm}, c_{\pm}, 0) - u_n^{(\sigma)}(y_n^{(2)}(\gamma), T; \lambda_{\pm}, c_{\pm}, 0) \right] = \varepsilon \quad (4.10)$$

(for  $-a > 1$ ),

where  $y_n = \ln z_n - b_n^{(\sigma)}$  and  $z_n = z_n(\gamma)$ ,  $n \geq 0$  solve equation (4.4). The set of successful hedging  $\tilde{A}$  is now defined and the optimal strategy is the perfect hedge of the claim  $f \cdot \mathbf{1}_{\tilde{A}}$ .

## 5. Application to equity-linked insurance contracts

Insurance company supplies a life insurance contract with future payment  $f$ . The size of payment depends on the evolution of risky asset during the contract period  $[0, T]$ . In the “pure endowment” framework the payment is exercised when the client is still alive an time  $T$ .

Denote by  $T(x)$  the remaining life time of a policy holder, who is currently of age  $x$ . Then the future payment is  $f \cdot \mathbf{1}_{\{T(x) > T\}}$ . We can put  ${}_T p_x = \mathbf{P}(T(x) > T)$ . Hence the premium

$${}_T c_x = \mathbf{E}^* \left[ B(T)^{-1} f \cdot \mathbf{1}_{\{T(x) > T\}} \right] = {}_T p_x \cdot \mathbf{E}^* \left[ B(T)^{-1} f \right], \quad (5.1)$$

where for the standard call option  $f = (S(T) - K)^+$  or the for standard pure endowment with guarantee life insurance contract  $f = \max(S(T), K) = K + (S(T) - K)^+$ .

This premium  ${}_T c_x$  is less than corresponding fair option price  $\mathbf{E}^* [B(T)^{-1} f]$ . Hence, the perfect hedge is impossible, but we can apply the quantile hedging. For a call option with  $f = (S(T) - K)^+$  the initial capital is  ${}_T c_x = {}_T p_x \cdot c(K, T)$ .

The maximal set of successful hedging  $\tilde{A}$  with initial capital  $v_0 = {}_T c_x < c(K, T)$  can be constructed as the solution of problem (4.2).

Determining the actuarial parameter  ${}_T p_x$  from suitable life table [4] we can construct the corresponding maximal set of successful hedging  $\tilde{A}$  and the strategy  $\tilde{\varphi}$  as the perfect hedge for the contingent claim  $f \cdot \mathbf{1}_{\tilde{A}}$ .

On the other hand, with the certain risk level  $\varepsilon$ ,  $0 < \varepsilon < 1$

$$1 - \varepsilon = \mathbf{P}(\tilde{A}^\gamma).$$

From (4.9)-(4.10) it yields to the certain value of  $\gamma$  and so, to the certain value of  ${}_T p_x$  and thus to initial capital  ${}_T c_x = {}_T c_x(\gamma)$ .

So a risk manager has a choice. Using a life table he (she) can choose an appropriated initial capital  ${}_T c_x$  or in accordance with a given risk level she (he) can choose an appropriate age  $x$  and a contract period  $T$ .

## 6. Appendix

As it follows from (2.15), functions

$$\begin{aligned} \rho_n^{(\sigma)} &= \rho_n^{(\sigma)}(t) = \mathbf{E}_\sigma^* \left( B(t)^{-1} \mathbf{1}_{\{N(t)=n\}} \right) \\ &= e^{-b_{\sigma} t} \int_{-\infty}^{\infty} e^{-a_{\sigma} x} p_{n,*}^{(\sigma)}(x, t) dx, \quad t \geq 0, \quad \sigma = \pm 1, \quad n \geq 1 \end{aligned}$$

satisfy the system

$$\begin{cases} \dot{\rho}_n^{(+)} = \lambda_+^* \rho_{n-1}^{(-)} - (\lambda_+^* + r_+) \rho_n^{(+)} \\ \dot{\rho}_n^{(-)} = \lambda_-^* \rho_{n-1}^{(+)} - (\lambda_-^* + r_-) \rho_n^{(-)} \end{cases} \quad (6.1)$$

with  $\rho_0^{(\sigma)}(t) = e^{-(\lambda_\sigma^* + r_\sigma)t}$ ,  $t \geq 0$ ,  $t \geq 0$ ,  $\sigma = \pm 1$  and  $\rho_n^{(\pm)}|_{t=0} = 0$ ,  $n \geq 1$ . Here  $\dot{\rho}_n^{(\pm)} = \frac{d\rho_n^{(\pm)}}{dt}$ . For  $\lambda_+^* = \lambda_-^* = \lambda$  and  $r_\pm = 0$  the solution is well known:  $\rho_n^{(\pm)}(t) = \pi_n(t) = \mathbf{P}(N(t) = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$ .

Generally, we imply the following change of variables

$$\rho_n^{(\sigma)}(t) = e^{-(\lambda_-^* + r_-)t} \Lambda_n^{(\sigma)} P_n^{(\sigma)}(t)$$

with  $\Lambda_n^{(\sigma)} = (\lambda_\sigma)^{[(n+1)/2]} (\lambda_{-\sigma})^{[n/2]}$ . In these notations we have  $P_0^{(+)}(t) = e^{-at}$ ,  $a = (\lambda_+^* + r_+) - (\lambda_-^* + r_-)$ ;  $P_0^{(-)}(t) = 1$ ;  $P_n^{(\pm)}|_{t=0} = 0$ ,  $n \geq 1$  and the system

$$\begin{cases} \dot{P}_n^{(+)} + a P_n^{(+)} = P_{n-1}^{(-)}, & n \geq 1, \\ \dot{P}_n^{(-)} = P_{n-1}^{(+)} \end{cases} \quad (6.2)$$

$$\dot{P}_n^{(\pm)} = \frac{dP_n^{(\pm)}}{dt}.$$

The latter system has the following solution

$$\begin{aligned} P_{2n+1} &\equiv P_{2n+1}^{(\pm)} = \frac{t^{2n+1}}{(2n+1)!} \left[ 1 + \sum_{k=1}^{\infty} \frac{(n+1)\dots(n+k)}{(2n+2)\dots(2n+k+1)} \cdot \frac{(-at)^k}{k!} \right], \\ P_{2n}^{(-)} &= \frac{t^{2n}}{(2n)!} \left[ 1 + \sum_{k=1}^{\infty} \frac{n(n+1)\dots(n+k-1)}{(2n+1)\dots(2n+k)} \cdot \frac{(-at)^k}{k!} \right], \\ P_{2n}^{(+)} &= \frac{t^{2n}}{(2n)!} \left[ 1 + \sum_{k=1}^{\infty} \frac{(n+1)\dots(n+k)}{(2n+1)\dots(2n+k)} \cdot \frac{(-at)^k}{k!} \right]. \end{aligned} \quad (6.3)$$

**Remark 6.1.** Formulas (6.3) can be expressed by hypergeometric function [1]:

$$P_n^{(\sigma)}(t) = \frac{t^n}{n!} \cdot {}_1F_1(m_n^{(\sigma)} + 1; n + 1; -at), \quad m_n^{(+)} = [n/2], \quad m_n^{(-)} = [(n - 1)/2].$$

Hypergeometric function  ${}_1F_1(\alpha; \beta; z)$  is defined as follows (see e. g. [2], formula (1.6))

$${}_1F_1(\alpha; \beta; z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha + 1) \dots (\alpha + n - 1)}{n! \beta(\beta + 1) \dots (\beta + n - 1)} z^n = 1 + \sum_{n=1}^{\infty} \frac{(\alpha)_n}{n! (\beta)_n} z^n.$$

As well, using (6.3) it easy to check that  $P_{2n}^{(-)} - P_{2n}^{(+)} = aP_{2n+1}$ ,  $n \geq 0$ .

To obtain

$$u_n^{(\sigma)}(y, t) = \mathbf{E}_\sigma^* \left( B(t)^{-1} \mathbf{1}_{\{X(t) > y, N(t)=n\}} \right)$$

we apply the change of variables  $p = \frac{c_+t-y}{c_+-c_-}$ ,  $q = \frac{y-c_-t}{c_+-c_-}$  and

$$u_n^{(\sigma)} = e^{-(\lambda_+^*+r_+)q-(\lambda_-^*+r_-)p} \Lambda_n^{(\sigma)} v_n^{(\sigma)}(p, q)$$

to equation (3.6).

Evidently,  $u_n^{(\sigma)}(y, t) \equiv 0$ , if  $p < 0$ , and  $u_n^{(\sigma)}(y, t) \equiv \rho_n^{(\sigma)}(t)$ , if  $q < 0$ . For  $p, q > 0$  we have the system

$$\begin{cases} \frac{\partial v_n^{(+1)}}{\partial q} = v_{n-1}^{(-1)}, \\ \frac{\partial v_n^{(-1)}}{\partial p} = v_{n-1}^{(+1)}, \end{cases} \quad n \geq 1 \tag{6.4}$$

with

$$v_0^{(+)} = e^{-ap}\theta(p), \quad v_0^{(-)} = e^{aq}\theta(-q), \quad v_n^{(\pm)}|_{p<0} \equiv 0$$

and

$$v_n^{(\sigma)}|_{q<0} = e^{aq} P_n^{(\sigma)}(p + q). \tag{6.5}$$

Here  $a = (\lambda_+^* + r_+) - (\lambda_-^* + r_-)$  and  $P_n^{(\sigma)}$ ,  $n \geq 0$ ,  $\sigma = \pm 1$  are defined in (6.3).

It is plain to check that the exact representation of the solution of (6.4) for  $p, q > 0$  has the form

$$\begin{aligned} v_{2n+1}^{(\pm)} &= v_{2n+1} = P_{2n+1}(p) + \sum_{k=1}^n \frac{q^k}{k!} \varphi_{k,n}(p), \\ v_{2n}^{(+)} &= P_{2n}^{(+)}(p) + \sum_{k=1}^n \frac{q^k}{k!} \varphi_{k-1,n-1}(p), \\ v_{2n}^{(-)} &= P_{2n}^{(-)}(p) + \sum_{k=1}^{n-1} \frac{q^k}{k!} \varphi_{k+1,n}(p), \end{aligned}$$

where  $\varphi_{0,n} = P_{2n+1}$ ,  $\varphi_{1,n} = P_{2n}^{(-)}$  and

$$\varphi'_{k,n} = \varphi_{k-1,n-1}, \quad 1 \leq k \leq n. \tag{6.6}$$

**Proposition 6.1.** The solution of system (6.6) has the form (3.11):

$$\varphi_{k,n} = \sum_{j=0}^{k-1} a^{k-j-1} \beta_{k,j} P_{2n-j}^-.$$

**Proof.** Indeed, from (3.11) and (6.2) it follows

$$\varphi'_{k,n} = \sum_{j=0}^{k-1} a^{k-j-1} \beta_{k,j} P_{2n-j-1}^{(+)}$$

By the identities  $P_{2n+1}^{(+)} = P_{2n+1}^{(-)}$  and  $P_{2n}^{(-)} - P_{2n}^{(+)} = aP_{2n+1}$ ,  $n \geq 0$  (see Remark 6.1) we have

$$\begin{aligned} \varphi'_{k,n} &= \sum_{j \geq 0, j \text{ is even}} a^{k-j-1} \beta_{k,j} P_{2n-j-1} \\ &+ \sum_{j \geq 0, j \text{ is odd}} a^{k-j-1} \beta_{k,j} P_{2n-j-1}^{(-)} - \sum_{j \geq 0, j \text{ is odd}} a^{k-j} \beta_{k,j} P_{2n-j}. \end{aligned}$$

To complete the proof it is sufficient to apply the following identities  $\beta_{k,2m+1} = \beta_{k-1,2m}$ ,  $\beta_{k,2m} - \beta_{k,2m+1} = \beta_{k-1,2m-1}$ , which are evident from the definition of  $\beta_{k,n}$  (see (3.10)).

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