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# Labor's Shares – Aggregate and Industry: Accounting for Both in a Model of Unbalanced Growth with Induced Innovation<sup>†</sup>

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# Labor's Shares – Aggregate and Industry: Accounting for Both in a Model of Unbalanced Growth with Induced Innovation

**Abstract:** The relative stability of aggregate labor's share constitutes one of the great macroeconomic ratios. However, relative stability at the aggregate level masks the unbalanced nature of industry labor's shares – the Kuznets stylized facts underlie those of Kaldor. We present a two-sector – one labor-only and the other using both capital and labor – model of unbalanced economic development with induced innovation that can rationalize these phenomena as well as several other empirical regularities of actual economies. Specifically, the model features (i) one sector ("goods" production) becoming increasingly capital-intensive over time; (ii) an increasing relative price and share in total output of the labor-only sector ("services"); and (iii) diverging sectoral labor's shares despite (iii) an aggregate labor's share that converges from above to a value between 0 and unity. Furthermore, the model (iv) supports either a neoclassical steady-state or long-run endogenous growth, giving it the potential to account for a wide range of real world development experiences.

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## I. INTRODUCTION

Aggregate labor's share displays no upward or downward trend over time. This is one of Nicholas Kaldor's [1961] stylized facts of economic growth and has endured across time and economies. However, underlying this Kaldor fact are trends in broad, industry-level labor's shares that are part of the Simon Kuznets [1965] stylized facts of economic development. In general, balanced growth in the aggregate masks unbalanced growth at the industry-level.

For example, **Figure 1** displays U.S. aggregate labor's share from 1958 through 1996. Labor's share remained between 65 and 70 percent of the entire period. However, **Figure 2** displays major industry labor's shares for agriculture, manufacturing and services over the same period; and **Figure 3** displays the same trends for those industries' shares in total value-added. (**Table 1** also presents summary statistics associated with the data used in **Figures 2 & 3**.) Manufacturing and agriculture labor's shares have been decreasing, while services labor's share remains stable or increases slightly. At the same time, manufacturing and agriculture value-added shares have decreased as services value-added share has increased markedly.<sup>1</sup>

Considering cross-sections of both developing and developed economies, Echevarria [1997] summarized a set of useful stylized facts (partially enumerated below).

1. The value-added share of agriculture is inversely related to total value-added; the services share of value-added is positively related to total output.
2. The relative price of services is positively related to total output.
3. The employment share of agriculture decreases as output increases; the employment share of services increases as output increases

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<sup>1</sup> Similar stylized facts for the U.S. were reported by Kongsamut, Rebelo and Xie [2001].

4. Labor's share's relation to total output appears to be positive.

The belief that labor's share is generally higher in richer countries has been recently called into question. Gollin [2002] demonstrated that, once labor income for the self-employed is treated properly, labor's shares appear approximately the same across economies – "nothing . . . to suggest that there are systematic differences between rich and poor countries in factor shares" [p. 471]. Furthermore, even not-adjusting the employee compensation data, Echevarria [1997, p. 435] suggested that the positive relationship between labor's share and total output may have diminished over time (e.g., consider the recent U.S. experience). Thus a plausible alternative to "4." above is:

4\*. Labor's share appears to have no relationship to total output; or a weak inverse relationship at most.

In other words, unbalanced growth stylized facts "1.", "2.", and "3." underlie the balanced growth fact "4\*.". **Table 2** presents a more detailed perspective of this summarization, including at the 2-digit SIC level U.S. industries' changes in labor's shares and value-added shares, and their contributions to changes in aggregate labor's share, for the 1958 to 1996 period.

In this paper we present a model that begins to account for the stylized facts of both balanced and unbalanced growth. The model is a two-sector – one sector that is labor-only; the other sector uses both labor and capital – model with induced innovation that features (i) one sector ("goods" production) becoming increasingly capital-intensive over time; (ii) an increase in the relative price and share in total output of a labor-only sector ("services" production); and (iii) diverging sectoral labor's shares despite (iii) a relatively stable aggregate labor's share that converges from above to a value between 0

and unity. Furthermore, the model (iv) supports either a neoclassical steady-state or long-run endogenous growth.

To our knowledge this is the first model of induced innovation with these features, though other types of growth models have also attempted to account for both Kuznets and Kaldor facts. Therefore Section II positions this paper in terms of the existing literature. The model is presented in Section III. Sections IV and V discuss long-run growth paths and transitional dynamics respectively. Section VI concludes.

## II. MODELS OF BOTH BALANCED AND UNBALANCED GROWTH

Recent years witnessed a resurgence of growth models seeking to account for the unbalanced nature of development at the industry-level while remaining consistent with balanced growth in the aggregate. For example, Kongsamut, Rebelo and Xie [2001] focused on changes in the marginal rate of substitution in consumption between different sectors' outputs.<sup>2</sup> They posited a representative agent with preferences,

$$U = \int_0^{\infty} e^{-\rho t} \frac{\left[ (A_t - \bar{A})^\beta M_t^\gamma (S_t - \bar{S})^\theta \right]^{1-\sigma}}{1-\sigma} dt,$$

where  $A$ ,  $M$ , and  $S$  are interpreted as agricultural goods, manufactured goods, and services respectively;  $\bar{A} > 0$  and  $\bar{S} > 0$  are subsistence consumption of food and home production of services; parameters  $\rho$ ,  $\sigma$ ,  $\gamma$ ,  $\beta$ ,  $\theta$  are strictly positive and  $\beta + \gamma + \theta = 1$ .

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<sup>2</sup> Other examples include Murphy, Shleifer and Vishny [1989], Matsuyama [1992], Echevarria [1997], Laitner [2000], Caseli and Coleman [2000] and Gollin, Parente and Rogerson [2002].

With these preferences, the income elasticity of substitution is less than unity for  $A$ , unity for  $M$ , and greater than unity for  $S$ . As the economy grows, output and employment shares of  $A$ ,  $M$ , and  $S$  decrease, remain constant, and increase respectively.<sup>3</sup>

Acemoglu and Guerrieri [2006], alternatively, demonstrated that, given different capital intensities in different sectors, unbalanced growth accompanies capital deepening if the sectors' outputs are gross complements in consumption. Specifically, outputs from two sectors enter a consumption aggregate,

$$Y = \left[ \gamma Y_1^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma) Y_2^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}},$$

where  $\varepsilon < 1$  and  $0 < \gamma < 1$ ; and  $Y_1$  and  $Y_2$  are sectoral outputs produced according to,

$$Y_1 = B_1 L_1^{\alpha_1} K_1^{1-\alpha_1} \quad \text{and} \quad Y_2 = B_2 L_2^{\alpha_2} K_2^{1-\alpha_2},$$

where the  $B_i$ 's are positive;  $L_i$  and  $K_i$  are labor and capital in sector  $i$ ; and  $\alpha_1 > \alpha_2$ .

As capital accumulates, the relative price of the more capital-intensive sector's good falls.

As this relative price falls, both the other sector's capital stock and employment shares

both converge towards unity. Aggregate labor's share converges to a constant from

below. (However, as each sector is Cobb-Douglas the sectoral labor's shares are

constant.) Furthermore, in experiments with the calibrated model, even after 500 years

aggregate labor's share only increases from 62.5 to 65 percent.

Finally, Ngai and Pissarides [2006] focused on different exogenous total factor productivity (TFP) growth rates across sectors. Specifically, outputs from  $m$  sectors enter a consumption aggregate,

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<sup>3</sup> Though the model achieves balanced growth, the evolution of aggregate labor's share need not be (approximately) balanced depending on the range of values covered during the transition.

$$\phi(c_1, \dots, c_m) = \left( \sum_{i=1}^m \omega_i c_i^{(\varepsilon-1)/\varepsilon} \right)^{\varepsilon/(\varepsilon-1)}$$

where  $\theta = 1$ ,  $\varepsilon < 1$ , and  $\sum \omega_i = 1$  and outputs are produced according to sectoral production functions,

$$c_i = A_i F(n_i k_i, n_i) \quad \forall i \neq m \quad \text{and} \quad \dot{k}_i = A_m F(n_m k_m, n_m) - c_m,$$

where the  $n_i$ s and  $k_i$ s are employment shares and capital to labor ratios;  $F$  is neoclassical and  $\gamma_i = \dot{A}/A$  is exogenous for each  $i$  and not necessarily identical across sectors. In this model, since goods are gross complements in consumption, employment shares and relative prices are inversely-related to TFP growth rates while growth in the aggregate can be balanced.

To our knowledge, the model presented in Section III below is the only model of induced innovation that accounts for a wide range of both Kuznets and Kaldor facts.<sup>4, 5</sup> We interpret labor-only and capital-using sectors as, respectively, services and goods industries. Increasing labor productivity in goods production due to labor-saving innovations is accompanied by an increasing share of labor employed in the services industries. If endogenous growth is achieved, this uneven productivity and labor supply growth across industries leads towards a zero manufacturing labor's share and a services labor's share of unity: "deindustrialization". Furthermore, the physical output of goods

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<sup>4</sup> Acemoglu [2003] presented an induced innovation model where capital- or labor-augmenting technical change is available at the firm level and firms use capital or labor in production. This model can support (net) labor-augmentation of technology at the aggregate level and balanced growth with constant aggregate labor's share. Our induced innovation model can be contrasted to Acemoglu's in that, while still accounting for a stable aggregate labor's share, labor-augmenting technical change is not required at any level. As well, Acemoglu's model is not designed to account for Kuznet's facts in general or the evolution of industry labor's shares specifically.

<sup>5</sup> Studies of labor-saving innovations in relation to growth and development generally constitute a substantial literature. Early examples are Kennedy [1964], Samuelson [1965], Drandakis and Phelps [1966]; more recent examples include Acemoglu [2002], Boldrin and Levine [2002], Hornstein et al [2004] and Zeira [1998].



grows perpetually while the services output remains constant. *But*, due to diminishing marginal utility and the ever-increasing relative scarcity of services, the relative price of services is ever-increasing (the celebrated Baumol-Bowen [1966] effect). These offsetting effects allow for *aggregate* labor's share to converge towards a constant, positive value.<sup>6</sup>

### III. A TWO-SECTOR MODEL OF UNBALANCED GROWTH WITH INDUCED INNOVATION

Assume an economy with two sectors – one with a constant technology using only labor and one using both labor and capital.<sup>7, 8</sup> Labor-saving innovations can be pursued in the capital-using sector.

We assume that there exists a set of technologies, differentiated by the elasticity of output with respect to capital, on the interval  $(0, 1)$ .<sup>9</sup> At any instant, every technology is available but the adoption of a technology (i.e., innovation) is costly. The cost to innovation is increasing in its capital intensity. The productivity of an innovation depends on the accumulated capital stock and, likewise, the productivity of capital

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<sup>6</sup> Hawtrey [1931, pp. 55-56] describes an uncannily similar story: "There may be a general over-production of *factory* products. Modern methods of mass production tend to produce this result. Satiation of demand for such products might be reached, and the result might be the displacement of a large amount of redundant labor. [ . . . ] It has been happening visibly in the [U.S.] ever since [WWI]. The numbers employed in factories have been shrinking [while] the numbers employed in distribution and in rendering all the multifarious individual services [ . . . ] have been growing. It may be mentioned that in this division of tendencies agriculture is to be classed with manufacturing. Labor in agriculture is being displaced by machinery [ . . . ] We may be approaching a state of society in which the mere production of any desired commodity becomes almost as easy and cheap as picking it up from the ground, and all the hard work will be put into the business of discovering the needs of consumers, specifying the appropriate products, and then [ . . . ] making them available for sale."

<sup>7</sup> Capital and labor are broad categories meant to encompass reproducible (e.g., both physical and human capital) and non-reproducible (e.g., raw labor and land) inputs.

<sup>8</sup> The model below builds off of Zuleta [2005]. Other ways of modeling factor saving innovations can be found in Zeira [2006] and Peretto and Seater [2005].

<sup>9</sup> Assuming that all technologies "exist" at all times is for simplicity and does not matter for our results if we interpret the required investments for innovation as providing for the *discovery* of more capital intensive-methods. See below.

depends on the capital-intensity of the technology. This creates a tradeoff between investment in capital and capital-intensity.

Assume many identical agents and no population growth. There are no externalities in the model so we can speak of either a social planner or a representative agent (RA) solving the problem<sup>10</sup>,

$$(1) \quad \max \int_0^{\infty} e^{-\rho t} \log(C),$$

where  $\rho > 0$  and  $C$  is consumption.<sup>11</sup> Consumption is a Cobb-Douglas aggregate of two types of consumption goods,

$$(2) \quad C = C_Y^\lambda C_X^{1-\lambda} \quad 0 < \lambda < 1.$$

The RA is endowed with a single unit of labor at every instant.

Production in the labor-only sector is,

$$(3) \quad X = C_X = BL_X,$$

where  $B$  is an efficiency parameter and  $L_X$  is the sector's employment share. We think of this sector as the *services* industry.

The second sector uses both labor and capital ( $K$ ) in production:

$$(4) \quad Y = C_Y + I = AK^\alpha L_Y^{1-\alpha},$$

where  $I$  is investment and  $L_Y = (1 - L_X)$ . The output produced by the  $Y$  sector can be consumed or invested. We think of this capital-intensive sector as a *goods* industry.<sup>12</sup>

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<sup>10</sup> For a general model of endogenous growth under perfect competition see Boldrin and Levine [2005].

<sup>11</sup> Time arguments are omitted for ease of exposition.

<sup>12</sup> One was to think about the distinction is articulated by Baumol [1967, pp. 415]: one type of production in which "labor is primarily an instrument" (towards *goods*) and one in which "labor is an end in itself" (as *services*).

The investment from the  $Y$  sector can be devoted to capital deepening or to adopting more capital intensive production methods. Intuitively, to undertake labor-saving innovations some fraction of investment,  $0 < (1 - \xi) < 1$ , must be allocated towards the installation of new production methods, reorganization of existing productive structures, and replacement/refurbishing of obsolete capital.<sup>13</sup> Considering  $K$  broadly, the increment  $(1 - \xi)$  can also be thought of in terms of training for and adjustment to previously unused production methods.

The evolution of  $K$  as a function of the remaining fraction of investment,  $\xi$ , is,

$$(5) \quad \dot{K} = \xi I .$$

The entire spectrum of technologies,  $\alpha = [0,1]$ , is available at every instant. However, labor-saving innovations are costly in terms of foregone  $K$ . Specifically,

$$(6) \quad \dot{\alpha} = (1 - \alpha)(1 - \xi)I .$$

Equation (6) embodies several desirable properties. First,

$$(1 - \alpha) + \alpha = 1$$

so  $(\dot{\alpha} + \alpha)$ 's maximum value is unity (consistent with constant returns to scale). Second,

$$\frac{\partial \dot{\alpha}}{\partial \alpha} = -(1 - \xi) < 0 ,$$

so it becomes increasingly costly to increase  $\alpha$  as it approaches unity. Finally,

$$\frac{\partial \dot{\alpha}}{\partial (1 - \xi)} = (1 - \alpha) \geq 0 ;$$

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<sup>13</sup> As opposed to the representative agent, competitive equilibrium framework, we could have, at the expense of significantly more complexity, modeled monopolistic firms trying to innovate under uncertainty. We do not suspect that the main results, as far as Kuznets and Kaldor facts, would be affected, and it seems desirable to focus on the simplest case at first.

so positive investment in installation/reorganization/replacement is never counterproductive.<sup>14</sup>

Of note, Klyuev [2005] presented a model where relatively high capital intensity in a manufacturing sector yields both the Baumol-Bowen effect and an increasing employment share in services. Based on the greater capital intensity, capital accumulation alone drives the results. Klyuev's noted that his motivation for focusing on capital accumulation alone was in part that models assuming faster TFP growth in manufacturing than in services counterfactually predict a *decreasing* employment share for services. By incorporating labor-saving innovations, our two sector model reconciles the idea of faster technical change *and* a decreasing employment share in manufacturing (i.e., "deindustrialization" – see Baumol et al [1989] and Rowthorn and Ramaswamy [1999]).

At each instant the RA is confronted by the state of the economy ( $\alpha$  and  $K$ ) and makes choices ( $L_Y$ ,  $C_Y$ ,  $I$ , and  $\xi$ ).<sup>15</sup> The current-value Hamiltonian<sup>16</sup> is,

$$(7) \quad \begin{aligned} H = & \lambda \log(C_Y) + (1 - \lambda) \log(B(1 - L_Y)) \\ & \theta_1 \left[ (AK^\alpha L_Y^{1-\alpha} - C_Y) \xi \right] + \\ & \theta_2 \left[ (1 - \alpha)(1 - \xi)(AK^\alpha L_Y^{1-\alpha} - C_Y) \right] \end{aligned}$$

where  $\theta_1 = \pi_1 e^{\rho t}$  and  $\theta_2 = \pi_2 e^{\rho t}$  and  $\pi_1$  and  $\pi_2$  are the shadow prices of capital and "capital-intensity." The first-order conditions for maximization are,

$$(8) \quad \frac{\partial H}{\partial C_Y} = \frac{\lambda}{C_Y} - \theta_1 \xi - \theta_2 (1 - \xi)(1 - \alpha) = 0,$$

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<sup>14</sup> Seater [2005] presented a growth model that similarly has a Cobb-Douglas specification with an evolving parameter. However, the parameter evolution in Seater's model is exogenous; also his model is a one-sector model. Young [2004] considered parameter changes in the Cobb-Douglas specification of a real business cycle model.

<sup>15</sup> Because  $L_X$  is simply whatever labor remains after the allocation to  $L_Y$ , and because  $L_X$  determines  $C_X$  entirely, the representative agent's problem can be phrased entirely in terms of production and consumption of  $Y$ .

<sup>16</sup> We ignore corner solutions for ease of exposition.

$$(9) \quad \frac{\partial H}{\partial L_Y} = \theta_1 \xi (1-\alpha) A \left( \frac{K}{L_Y} \right)^\alpha + \theta_2 (1-\alpha)^2 (1-\xi) A \left( \frac{K}{L_Y} \right)^\alpha - \left( \frac{1-\lambda}{1-L_Y} \right) = 0'$$

$$(10) \quad \frac{\partial H}{\partial \xi} = \theta_1 (AK^\alpha L_Y^{1-\alpha} - C_Y) - \theta_2 (1-\alpha) (AK^\alpha L_Y^{1-\alpha} - C_Y) = 0,$$

$$(11) \quad \frac{\partial H}{\partial K} + \dot{\theta}_1 = \theta_1 \xi \alpha AK^{\alpha-1} L_Y^{1-\alpha} + \theta_2 (1-\xi) (1-\alpha) \alpha AK^{\alpha-1} L_Y^{1-\alpha} + \dot{\theta}_1 = \rho \theta_1$$

$$(12) \quad \frac{\partial H}{\partial \alpha} + \dot{\theta}_2 = [\theta_1 \xi + \theta_2 (1-\xi) (1-\alpha)] AK^\alpha L_Y^{1-\alpha} \ln \left( \frac{K}{L_Y} \right) - \theta_2 (1-\xi) (AK^\alpha L_Y^{1-\alpha} - C_Y) + \dot{\theta}_2 = \rho \theta_2$$

and we immediately note, from (10), that,

$$(13) \quad \theta_1 = \theta_2 (1-\alpha).$$

#### IV. LONG-RUN GROWTH PATHS

A defining property of this model is the value  $\alpha$  which converges to.

With the Cobb-Douglas production function,

$$(14) \quad \alpha = (1-\alpha) K \ln \left( \frac{K}{L_Y} \right) \quad \text{or} \quad \alpha = \frac{K \ln \left( \frac{K}{L_Y} \right)}{1 + K \ln \left( \frac{K}{L_Y} \right)}.$$

Totally differentiating (14) and manipulating leads to the expression,

$$(15) \quad \dot{\alpha} = K(1-\alpha)^2 \left[ \left( \ln \left( \frac{K}{L_Y} \right) + 1 \right) \frac{\dot{K}}{K} - \frac{\dot{L}_Y}{L_Y} \right].$$

The equations (14) and (15) give us insights into the model's steady-state properties.

PROPOSITION 1. The model can support either a *neoclassical steady-state* where

$$\frac{\dot{L}_Y}{L_Y} = \frac{\dot{\alpha}}{\alpha} = \frac{\dot{C}_X}{C_X} = \frac{\dot{C}_Y}{C_Y} = \frac{\dot{K}}{K} = 0, \quad 0 \leq \alpha < 1, \quad \text{and } 0 < L_Y < 1; \quad \text{or } \textit{endogenous growth}$$

$$\text{where } \frac{\dot{L}_Y}{L_Y} = \frac{\dot{\alpha}}{\alpha} = \frac{\dot{C}_X}{C_X} = 0 \quad \text{and} \quad \frac{\dot{C}_Y}{C_Y} = \frac{\dot{K}}{K} > 0, \quad \alpha = 1, \quad \text{and } L_Y = 0.$$

By (15), if  $\dot{\alpha} = 0$  and  $0 \leq \alpha < 1$ , then it must be the case that,

$$(16) \quad 0 = \left( \ln \left( \frac{K}{L_Y} \right) + 1 \right) \frac{\dot{K}}{K} - \frac{\dot{L}_Y}{L_Y} \quad \text{or} \quad K = 0.$$

A steady-state with  $K = 0$  is possible and, if  $\dot{L}_Y = 0$  in the long-run (which must be the case for  $0 \leq L_Y \leq 1$ ), then it must also be the case that in general  $\dot{K} = 0$ .

On the other hand, if we allow that  $\alpha$  converges to unity then (15) is still valid when  $\dot{K} > 0$ . Specifically, the optimal capital growth condition for this model is,<sup>17</sup>

$$(17) \quad \frac{\dot{K}}{K} = \xi \left( AK^{\alpha-1} L_Y^{1-\alpha} - \frac{C_Y}{K} \right).$$

It can be demonstrated that the ratio of  $C_Y$  to  $K$  goes to  $\rho$ , so with  $\alpha = 1$ ,  $\dot{K} > 0$  as long as  $\rho < A$ .

So there are two basic types of long-run equilibrium that the model can support. In the former case, a true steady-state (in all the variables' levels) is achieved; in the later

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<sup>17</sup> Detailed derivations of the optimal dynamic equations are provided in Appendix A.

case, the production of  $Y$  in the economy converges to "AK" and endogenous growth is achieved.<sup>18</sup>

What do these two long-run possibilities imply for aggregate labor's share ( $LSH$ )?<sup>19</sup>

$$(18) \quad LSH = \frac{(1-\alpha)}{(1-\alpha) + \alpha L_y}.$$

PROPOSITION 2. Aggregate labor's share,  $LSH$ , achieves a steady-state value greater than or equal to 0 and less than unity; if endogenous growth is achieved then  $0 < LSH < 1$ .

In a neoclassical steady-state there are two determinants of  $LSH$ :  $\alpha$  and  $L_y$ .  $L_y$  determines how labor is split between the  $Y$  sector (where labor's share is  $1 - \alpha$ ) and the  $X$  sector (where labor's share is always unity). The partial derivatives are,

$$(19) \quad \frac{\partial LSH}{\partial \alpha} = -\frac{L_y}{[(1-\alpha) + \alpha L_y]^2} \quad \frac{\partial LSH}{\partial L_y} = -\frac{\alpha(1-\alpha)}{[(1-\alpha) + \alpha L_y]^2},$$

which are both negative.

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<sup>18</sup> The baseline endogenous growth models of this type were provided by Jones and Manuelli [1990] and Rebelo [1991].

<sup>19</sup> For calculating aggregate labor income and output for this economy, we must consider sectoral products valued in terms of their marginal utilities. So labor income is

$$\frac{\lambda}{C_y} (1-\alpha) K^\alpha A^{1-\alpha} L_y^{-\alpha} \bullet L_y + \frac{1-\lambda}{B(1-L_y)} B \bullet (1-L_y)$$

and total income is  $\frac{\lambda}{C_y} K^\alpha (AL_y)^{1-\alpha} + \frac{(1-\lambda)}{B(1-L_y)} B(1-L_y)$ . Expression (5.5) is a simplification of the ratio of the two magnitudes.

When endogenous growth results, (18) cannot be evaluated at  $\alpha = 1$  and  $L_Y = 0$  and, instead, the limiting value must be considered. We can do this by exploiting the fact that Cobb-Douglas preferences imply,

$$(20) \quad \frac{P_Y C_Y}{P_X C_X} = \frac{\lambda}{1 - \lambda},$$

where  $P_Y$  and  $P_X$  are the prices of  $Y$  and  $X$  in terms of marginal utilities. As  $L_Y$  approaches 0, labor income approaches  $P_X C_X$ .  $LSH$  then becomes,

$$LSH = \frac{P_X C_X}{P_X C_X + P_Y (C_Y + I)} = \frac{1}{1 + \frac{P_Y C_Y}{P_X C_X} + \frac{P_Y I}{P_X C_X}} = \frac{1}{1 + \frac{\lambda}{1 - \lambda} + \frac{\lambda}{1 - \lambda} \frac{I}{C_Y}}.$$

which simplifies to,

$$(21) \quad LSH = \frac{1 - \lambda}{1 + \lambda \frac{I}{C_Y}}.$$

Given that  $I/C_Y$  is constant in the long-run<sup>20</sup>, (21) is a constant between 0 and 1 despite the growing capital to labor ratio in endogenous growth.

PROPOSITION 3. In endogenous growth the relative price of services,  $P_X/P_Y$ , increases while its relative share in value-added equals aggregate labor's share.

As well, because  $\dot{C}_Y > 0$  and  $\dot{C}_X = 0$ , by (20)  $P_X/P_Y$  must grow at the same rate as  $C_Y$ .

So the economy displays an ever-increasing relative price of services – a widely-

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<sup>20</sup> In the long run  $Y = AK$  and  $\frac{K}{C_Y} = \frac{1}{\rho}$  so  $\frac{I}{C_Y} = \frac{A - \rho}{\rho}$



recognized feature of many real economies (e.g. De Gregorio et al. [1994] and Baumol and Bowen [1966]). This ever-increasing relative price of services, along with the ever-decreasing share of services in "physical" output, results in a constant long-run value-added share for services (*SSH*).<sup>21</sup> Specifically, in the long-run

$$(22) \quad SSH = \frac{1}{1 + \frac{\lambda}{1 - \lambda} \frac{A}{\rho}} = LSH.$$

This connection between *SSH* and *LSH* is tied to share of total labor employed in the services sector going to unity.

We also note that, during endogenous growth, the growth rates of  $C_Y$  and  $K$  are identical while  $\frac{\dot{C}}{C} = \lambda \frac{\dot{C}_Y}{C_Y}$ . Evaluating (5.4) at  $\alpha = 1$  then implies that,

$$(23) \quad \frac{\dot{C}_Y}{C_Y} = \frac{\dot{K}}{K} = A - \rho \quad \text{and} \quad \frac{\dot{C}}{C} = \lambda(A - \rho).$$

Aggregate consumption grows more slowly than the capital stock, broadly conceived.

Finally, we call attention to an interesting counterfactual implication of the model and argue that it is understandable given the highly stylized framework. The marginal product of capital converges to a constant, but the marginal *value* product of capital goes to zero as the relative price of  $Y$  sector output goes to zero; the rate of *physical* investment becomes constant while the *value* of that investment goes to zero. We suspect that this is an artifact of production in the  $X$  sector (with its increasing relative price) being entirely void of capital and productivity growth. Relaxing the restrictions on the  $X$  sector and deriving new implications of the marginal value product of capital (and, ergo,

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<sup>21</sup> "Physical" here – though perhaps vague in terms of services – is meant to distinguish between shares in the total  $X + Y$  as opposed to shares in value as determined by relative prices.

the interest rate) is called for in future work but goes beyond the issues of factor shares and relative prices that we wanted to account for in the present work.

## V. TRANSITIONAL DYNAMICS

In this section we elaborate on the transition of the model economy to either a neoclassical steady-state or endogenous growth path. Recall the expression for changes in  $\alpha$  is given by (15). The relationship between capital accumulation and the sectoral allocation of labor is fundamental to the dynamics of  $\alpha$ .

The expression for capital accumulation,(17), can be set against the expression for optimal sectoral labor growth,

$$(24) \quad \frac{\dot{L}_Y}{L_Y} = \frac{I(1-L_Y)}{K(1-\alpha)} \left\{ \frac{\left( \frac{\alpha}{K} + 2 - \alpha \right) \left( \alpha - K - \frac{K}{I} (\alpha AK^{\alpha-1} L_Y^{1-\alpha} - \rho) \right) + K}{\left\{ \left( \frac{\alpha}{K} + 2 - \alpha \right) \left( L_Y + \frac{\alpha}{1-\alpha} \right) - (1-L_Y)K \right\}} \right\}.$$

Starting from any initial, positive  $(\alpha AK^{\alpha-1} L_Y^{1-\alpha} - \rho)$ ,  $K$  grows and  $L_Y$  falls, both changes exerting negative influences on the marginal product of capital. On the other hand,  $\alpha$  increases and exerts a positive influence on the marginal product of capital.

Like a standard growth model, diminishing returns imply that the incentives for investments (in both capital and capital intensity) vanish as  $(\alpha AK^{\alpha-1} L_Y^{1-\alpha} - \rho)$  approaches zero. Whether the economy settles into a neoclassical steady-state or achieves endogenous growth depends on whether  $\alpha$  converges to unity before  $(\alpha AK^{\alpha-1} L_Y^{1-\alpha} - \rho)$  converges to 0.

We now employ the above dynamics to describe the evolution of  $LSH$ .

PROPOSITION 4. Aggregate labor's share,  $LSH$ , converges to its steady-state value from above.

Equation (18) can be rewritten as,

$$(25) \quad LSH = \frac{1}{1 + \frac{\alpha}{1-\alpha} L_Y},$$

such that the dynamics of  $LSH$  depend on the sign of  $\frac{\dot{\alpha}}{\alpha} + \frac{\dot{\alpha}}{1-\alpha} + \frac{\dot{L}_Y}{L_Y}$  which is always

non-negative.<sup>22</sup> Starting from below the economy's steady-state/endogenous growth path, aggregate labor's share converges to its long-run, constant value from above. This is not inconsistent with the pattern of U.S. labor's share pictured in **Figure 1**.<sup>23</sup>

Furthermore, during the transition labor's share in the  $Y$  sector falls while it remains constant (at unity) in the  $X$  sector. This is the case despite the fundamental role that the  $X$  sector, with its time-invariant labor's share, plays in preventing aggregate labor's share from going to zero.

Despite  $LSH$ 's transitional decrease, the  $X$  sector's share of the economy's output (in terms of value) increases.

PROPOSITION 5. Services share in value-added,  $SSH$ , converges from below to its steady-state value.

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<sup>22</sup> The proof of this claim is in Appendix C. Some claims below that are also left unproved in the text are demonstrated in previous and subsequent Appendices.

<sup>23</sup> Beyond the U.S. evidence is mixed on this point. (See the discussion in Section I. Gollin [2002] suggested that there is no relationship between the level of labor's share and the level of economic development.) Also, Torrini [2005] reported that Italy's labor's share declined from the mid-1970s through the mid-1990s; and Garrido Ruiz [2005] reported that Spain's labor's share increased from 1955 through 2005.

The share of services is,

$$(26) \quad SSH = \frac{P_X C_X}{P_X C_X + P_Y (C_Y + I)},$$

which is notably identical to the expression for  $LSH$  during long-run endogenous growth (but *not* for  $LSH$  in general). Expression (26) can, as in Section IV, be manipulated into,

$$(27) \quad SSH = \frac{1 - \lambda}{1 + \lambda \frac{I}{C_Y}}.$$

$I/C_Y$  decreases during the transition, so  $SSH$  increases.

One additional implication of interest is that the growth rate of consumption may increase during the transition to endogenous growth. Given that  $\frac{\dot{C}_Y}{C_Y} = \alpha AK^{\alpha-1} L_Y^{1-\alpha} - \rho$ ,

it can be shown that the growth rate of  $\alpha AK^{\alpha-1} L_Y^{1-\alpha}$  is,

$$(28) \quad (1 - \alpha) \left\{ \left( \frac{1}{\ln(K/L_Y)} + \alpha \right) \ln(K/L_Y) \frac{\dot{K}}{K} + \frac{1}{\ln(K/L_Y)} \left[ \frac{\dot{K}}{K} - \frac{\dot{L}_Y}{L_Y} \right] \left( 1 - \frac{\alpha}{K} \right) \right\},$$

which is non-negative as long as  $K > L_Y$ . Furthermore, from (20) we know that,

$$(29) \quad \frac{\dot{C}_X}{C_X} = \frac{\dot{C}_Y}{C_Y} - \frac{\dot{P}}{P},$$

where  $P = P_X/P_Y$ . Given the Cobb-Douglas preferences,

$$(30) \quad \frac{\dot{C}}{C} = \frac{\dot{C}_Y}{C_Y} - (1 - \lambda) \frac{\dot{P}}{P}.$$

Since the relative price of  $X$  is increasing, it follows that  $\frac{\dot{C}}{C} > \lambda \frac{\dot{C}_Y}{C_Y}$ . Also,

$$(31) \quad \frac{\dot{C}}{C} = \lambda \frac{\dot{C}_Y}{C_Y} - (1-\lambda) \frac{\dot{L}_Y}{(1-L_Y)}.$$

The growth rate of  $C_Y$  is increasing during the transition; the time derivative of the second component is,

$$(32) \quad \frac{d \left[ (1-\lambda) \frac{\dot{L}_Y}{(1-L_Y)} \right]}{dt} = \frac{\ddot{L}_Y}{(1-L_Y)} - \left[ \frac{\dot{L}_Y}{(1-L_Y)} \right]^2,$$

which is a term of second-order importance.

Furthermore, it can be demonstrated that (i) for any  $\lambda > 0$  there exists a stock of capital  $\tilde{K}$  such that if  $K > \tilde{K}$  then  $\frac{\dot{C}}{C} \leq \lambda(A - \rho)$ ; and (ii) if  $\lambda \geq \frac{1}{2}$  then

$\frac{\dot{C}}{C} \leq \lambda(A - \rho)$ . Since (31) holds and the growth rate of  $C_Y$  is increasing towards its

long-run rate of  $(A - \rho)$  in endogenous growth, (i) and (ii) represent conditions where

$$-\frac{\dot{L}_Y}{1-L_Y} \leq \lambda(A - \rho).$$

Is an increasing, transitional growth rate of consumption counterfactual? While not consumption precisely, **Table 3** presents average growth rates of per capita GDP for various (now-developed) countries for various sub-periods from 1700 to 2000. There are several countries (including the U.S.) for which the average growth rate is monotonically increasing from earlier to later sub-periods. Overall there is little to indicate that an increasing growth rate is in stark contrast to the real-world experiences.

## VI. CONCLUSIONS

The process of economic development is famously characterized by certain great macroeconomic ratios, e.g. capital to output ratios, the return to capital, and labor's share. These ratios display surprising stability that transcends both time and economies. In the case of aggregate labor's share, this is all the more surprising given the trends of industry labor's shares and industry shares in aggregate output. Kuznets facts underlie Kaldor facts; balanced growth in the aggregate masks unbalanced growth at the industry level.

In this paper we develop a two-sector model of unbalanced economic development in the spirit of the induced innovation literature. One sector allows for innovations that increase the capital intensity of production, naturally raising capital's share in that sector's physical product. However, the second (labor-intensive) sector maintains a constant marginal physical product of labor, attracting an increasing portion of the available labor supply. Because the labor-intensive sector can have no long-run growth in physical product, the relative price of its output increases over time, so the marginal value product of labor increases. This effect maintains a non-zero labor's share even when the innovative sector achieves long-run, "AK"-type growth in physical product.

Our model provides a framework for interpreting several empirical regularities of real economies: (i) manufacturing industries becoming increasingly capital-intensive over time despite (ii) an increase in the relative price and share in total value-added of service industries; (iii) aggregate labor's share displaying a horizontal trend despite (iv) individual industry labor's shares that seem to evolve independently of one another. Furthermore, because the model can attain a neoclassical steady-state or long-run,

endogenous growth, it has the potential to account for a wide range of real world development experiences.

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## FIGURES

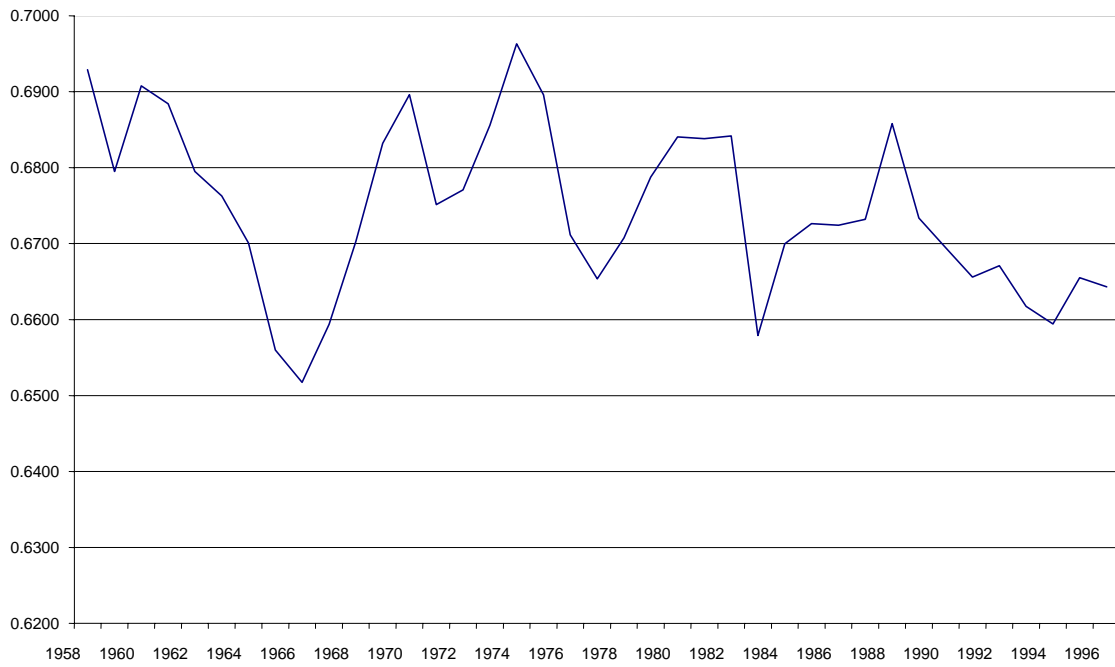
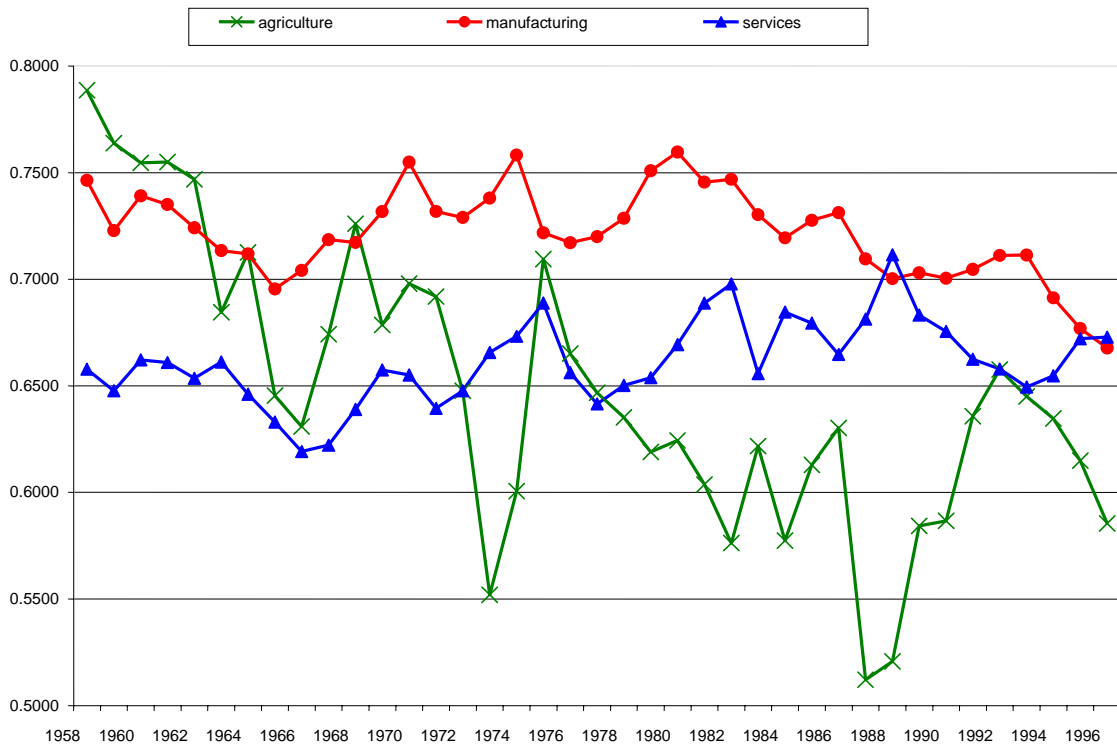


FIGURE 1. US AGGREGATE LABOR'S SHARE: 1958 - 1996

*Notes:* Calculated from aggregation of 35 industries' data. At the industry level, calculations are of labor's share of value added. At the aggregate level, industries weighted by their share of total value added.

**FIGURES (CONT.)**



**FIGURE 2. SELECT MAJOR U.S. INDUSTRY LABOR'S SHARES**

*Notes:* Data from 35-KLEM database. Methodology described in Jorgenson et. al. [1987]; recreated from Young [2006]. Agriculture is "Agriculture" industry. Manufacturing includes "Food and Kindred Products," Tobacco," "Textile Mill Products," "Apparel," "Lumber and Wood," "Furniture and Fixtures," "Paper and Allied," "Chemicals," "Petroleum and Coal Products," "Rubber and Miscellaneous Products," "Leather," "Stone, Clay and Glass," "Primary Metal," "Fabricated Metal," "Non-electrical," "Motor Vehicle," "Transportation Equipment and Ordinance," "Instruments," and "Miscellaneous Manufacturing" industries. Services include "Services," "Trade," and "Finance, Insurance and Real Estate" industries.

FIGURES (CONT.)

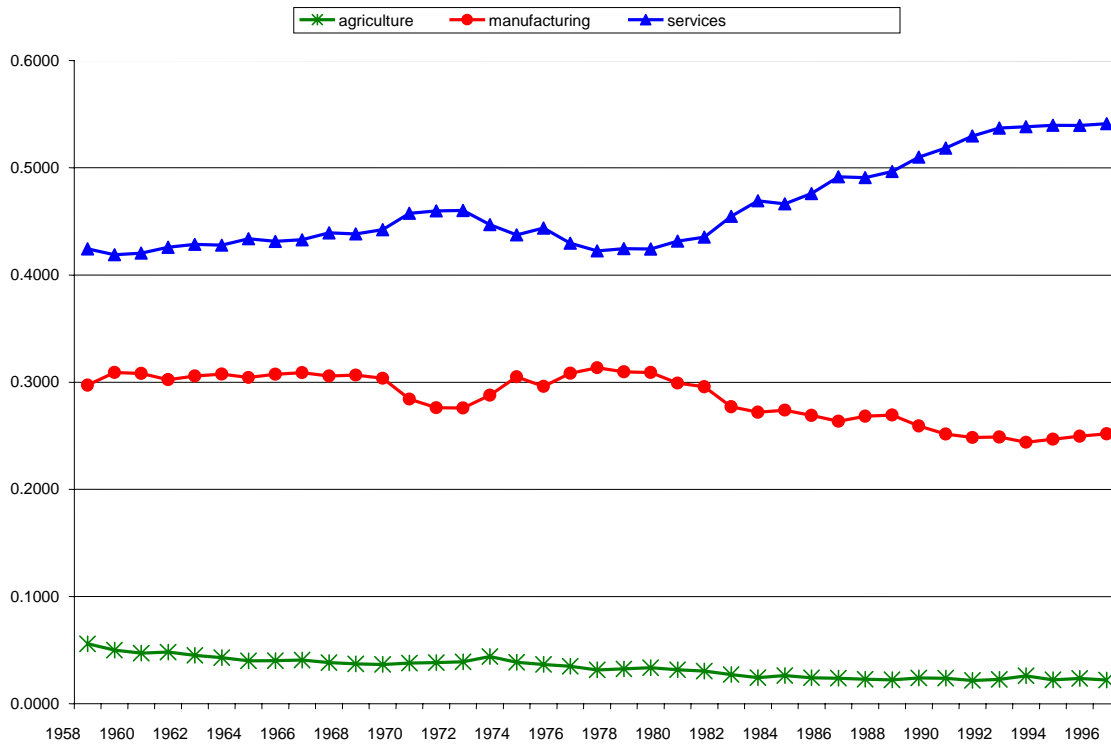


FIGURE 2. SELECT MAJOR U.S. INDUSTRY VALUE-ADDED SHARES

*Notes:* Data from 35-KLEM database. Methodology described in Jorgenson et. al. [1987]; recreated from Young [2006]. Agriculture is "Agriculture" industry. Manufacturing includes "Food and Kindred Products," "Tobacco," "Textile Mill Products," "Apparel," "Lumber and Wood," "Furniture and Fixtures," "Paper and Allied," "Print, Publishing & Allied," "Chemicals," "Petroleum and Coal Products," "Rubber and Miscellaneous Products," "Leather," "Stone, Clay and Glass," "Primary Metal," "Fabricated Metal," "Non-electrical Industry," "Electrical Industry," "Motor Vehicle," "Transportation Equipment and Ordinance," "Instruments," and "Miscellaneous Manufacturing" industries. Services include "Services," "Trade," and "Finance, Insurance and Real Estate" industries.

**TABLES**

TABLE 1 – SUMMARY STATISTICS FOR THREE U.S. INDUSTRY GROUPS

| Statistic for            | Agriculture | Manufacturing | Services |
|--------------------------|-------------|---------------|----------|
| <b>Labor's Share</b>     |             |               |          |
| Mean                     | 0.645       | 0.722         | 0.661    |
| $\sigma$                 | 0.066       | 0.021         | 0.020    |
| $\rho_{x,Agriculture}$   | 1.000       | 0.235         | -0.510   |
| $\rho_{x,Manufacturing}$ | 0.235       | 1.000         | 0.037    |
| $\rho_{x,Services}$      | -0.510      | 0.037         | 1.000    |
| $\Delta_{1958,1996}$     | -0.203      | -0.079        | 0.015    |
| <b>Value-Added Share</b> |             |               |          |
| Mean                     | 0.034       | 0.285         | 0.463    |
| $\sigma$                 | 0.009       | 0.023         | 0.041    |
| $\rho_{x,Agriculture}$   | 1.000       | 0.758         | -0.781   |
| $\rho_{x,Manufacturing}$ | 0.758       | 1.000         | -0.964   |
| $\rho_{x,Services}$      | -0.781      | -0.964        | 1.000    |
| $\Delta_{1958,1996}$     | -0.034      | -0.045        | 0.117    |

*Notes:* Data from 35-KLEM database. Methodology described in Jorgenson et al (1987). Agriculture is "Agriculture" industry. Manufacturing includes "Food and Kindred Products," "Tobacco," "Textile Mill Products," "Apparel," "Lumber and Wood," "Furniture and Fixtures," "Paper and Allied," "Print, Publishing & Allied," "Chemicals," "Petroleum and Coal Products," "Rubber and Miscellaneous Products," "Leather," "Stone, Clay and Glass," "Primary Metal," "Fabricated Metal," "Non-electrical Industry," "Electrical Industry," "Motor Vehicle," "Transportation Equipment and Ordinance," "Instruments," and "Miscellaneous Manufacturing" industries. Services include "Services," "Trade," and "Finance, Insurance and Real Estate" industries.



TABLE 2—CHANGE IN INDUSTRY LABOR'S AND VALUE-ADDED SHARES AND INDUSTRY CONTRIBUTIONS TO AGGREGATE LABOR'S SHARE CHANGE: 1958 – 1996

| Industry | Description                 | Change in Labor's Share | Change in Value-Added Share | Contribution to Agg. Labor's Share Change |
|----------|-----------------------------|-------------------------|-----------------------------|---|
| 1        | Agriculture                 | -0.203                  | -0.034                      | -0.031                                    |
| 2        | Metal Mining                | -0.027                  | -0.001                      | -0.001                                    |
| 3        | Coal Mining                 | -0.095                  | -0.002                      | -0.002                                    |
| 4        | Oil and Gas Extraction      | -0.032                  | -0.007                      | -0.003                                    |
| 5        | Non-metallic Mining         | -0.124                  | -0.001                      | -0.001                                    |
| 6        | Construction                | 0.022                   | -0.020                      | -0.016                                    |
| 7        | Food & Kindred Products     | -0.187                  | -0.006                      | -0.008                                    |
| 8        | Tobacco                     | -0.160                  | 0.001                       | 0.000                                     |
| 9        | Textile Mill Products       | -0.066                  | -0.003                      | -0.003                                    |
| 10       | Apparel                     | -0.093                  | -0.008                      | -0.008                                    |
| 11       | Lumber and Wood             | -0.069                  | -0.002                      | -0.002                                    |
| 12       | Furniture and Fixtures      | -0.064                  | -0.001                      | -0.001                                    |
| 13       | Paper and Allied            | -0.046                  | -0.002                      | -0.002                                    |
| 14       | Print., Publishing & Allied | -0.042                  | 0.002                       | 0.001                                     |
| 15       | Chemicals                   | -0.034                  | 0.005                       | 0.002                                     |
| 16       | Petroleum & Coal Products   | -0.206                  | 0.002                       | 0.000                                     |
| 17       | Rubber & Misc. Prod.        | 0.001                   | 0.003                       | 0.002                                     |
| 18       | Leather                     | -0.364                  | -0.003                      | -0.003                                    |
| 19       | Stone, Clay, Glass          | 0.097                   | -0.006                      | -0.003                                    |
| 20       | Primary Metal               | 0.104                   | -0.016                      | -0.010                                    |
| 21       | Fabricated Metal            | -0.177                  | -0.008                      | -0.009                                    |
| 22       | Non-electrical Industry     | -0.040                  | 0.001                       | -0.001                                    |
| 23       | Electrical Industry         | -0.179                  | 0.004                       | -0.001                                    |
| 24       | Motor Vehicles              | 0.018                   | -0.001                      | -0.001                                    |
| 25       | Transp. Equip & Ord.        | 0.023                   | -0.008                      | -0.007                                    |
| 26       | Instruments                 | 0.023                   | 0.004                       | 0.004                                     |
| 27       | Misc. Manufacturing         | -0.216                  | -0.002                      | -0.002                                    |
| 28       | Transportation              | 0.064                   | -0.019                      | -0.010                                    |
| 29       | Communications              | -0.062                  | 0.003                       | 0.000                                     |
| 30       | Electrical Utilities        | 0.010                   | -0.002                      | -0.001                                    |
| 31       | Gas Utilities               | -0.061                  | -0.002                      | -0.001                                    |
| 32       | Trade                       | 0.000                   | -0.044                      | -0.034                                    |
| 33       | Fin., Ins. & Real Estate    | 0.006                   | 0.033                       | 0.015                                     |
| 34       | Services                    | 0.074                   | 0.128                       | 0.104                                     |
| 35       | Government Enterprises      | -0.158                  | 0.014                       | 0.005                                     |

Notes: Calculated from 35 annual industries' data, 1958 – 1996. Labor's share is that of annual value added. Aggregate labor's share is calculated as a weighted average of industry labor's shares with industry shares in total value-added as weights. From top to bottom, shaded regions indicate agriculture, manufacturing, and services industries.

TABLE 3—LONG HISTORICAL AVERAGE GROWTH RATES OF PER CAPITA GDPs (PERCENTS)

| Country                 | 1700-1820 | 1820-1850 | 1850-1900 | 1900-1950 | 1950-2000 |
|-------------------------|-----------|-----------|-----------|-----------|-----------|
| Austria                 | 0.17      | 1.02      | 1.12      | 0.50      | 3.50      |
| Belgium                 | 0.12      | 1.13      | 1.42      | 0.77      | 2.69      |
| Denmark                 | 0.17      | 1.10      | 1.08      | 1.68      | 2.42      |
| Finland                 | 0.17      | 0.51      | 1.22      | 1.89      | 3.10      |
| France                  | 0.18      | 1.14      | 1.18      | 1.22      | 2.83      |
| Germany                 | 0.14      | 0.94      | 1.49      | 0.53      | 3.23      |
| Italy                   | 0.01      | 0.63      | 0.56      | 1.36      | 3.42      |
| Netherlands             | -0.12     | 0.85      | 0.74      | 1.13      | 2.60      |
| Norway                  | 0.09      | 0.59      | 1.36      | 2.15      | 3.11      |
| Sweden                  | 0.17      | 0.24      | 1.38      | 1.95      | 2.27      |
| Switzerland             | 0.17      | 1.04      | 1.91      | 1.74      | 1.83      |
| United Kingdom          | 0.26      | 1.05      | 1.32      | 0.87      | 2.16      |
| Total 12 Western Europe | 0.16      | 0.97      | 1.24      | 0.98      | 2.81      |
| Australia               | 0.22      | 4.56      | 1.43      | 1.23      | 2.16      |
| New Zealand             | 0.00      | 3.57      | 2.68      | 1.36      | 1.31      |
| Canada                  | 0.62      | 1.29      | 1.58      | 1.85      | 2.27      |
| United States           | 0.73      | 1.21      | 1.65      | 1.71      | 2.20      |
| Total Western Offshoots | 0.77      | 1.29      | 1.66      | 1.69      | 2.19      |

*Notes:* Numbers recreated from Madison (2003). All numbers are percentages. Shaded boxes indicate countries and country groups where the average growth rate is increasing monotonically from earlier to later sub-periods.

## Appendix A: Model Derivations

The basic framework of the model consists of,

$$(A.1) \quad \max \int_0^{\infty} e^{-\rho t} \log(C) \quad \rho > 0,$$

$$(A.2) \quad C = C_Y^\lambda C_X^{1-\lambda} \quad 0 < \lambda < 1,$$

$$(A.3) \quad X = C_X = BL_X,$$

$$(A.4) \quad Y = C_Y + I = K^\alpha (AL_Y)^{1-\alpha} \quad (1 - L_Y) = L_X,$$

$$(A.5) \quad \dot{K} = \xi I \quad 0 < \xi < 1, \quad \text{and}$$

$$(A.6) \quad \dot{\alpha} = (1 - \alpha)(1 - \xi).$$

The current-value Hamiltonian is,

$$(A.7) \quad H = \lambda \log(C_Y) + (1 - \lambda) \log(B(1 - L_Y)) \\ \theta_1 \left[ K^\alpha (AL_Y)^{1-\alpha} - C_Y \right] \xi + \theta_2 \left[ (1 - \alpha)(1 - \xi) \left( K^\alpha (AL_Y)^{1-\alpha} - C_Y \right) \right].$$

The necessary condition derived from the Hamiltonian are,

$$(A.8) \quad \frac{\partial H}{\partial C_Y} = \frac{\lambda}{C_Y} - \theta_1 \xi - \theta_2 (1 - \alpha)(1 - \xi) = 0,$$

$$(A.9) \quad \frac{\partial H}{\partial L_Y} = \theta_1 \xi (1 - \alpha) \left( \frac{K}{L_Y} \right) A^{1-\alpha} + \theta_2 (1 - \xi) (1 - \alpha)^2 \left( \frac{K}{L_Y} \right)^\alpha A^{1-\alpha} - \left( \frac{1 - \lambda}{1 - L_Y} \right) = 0,$$

$$(A.10) \quad \frac{\partial H}{\partial \xi} = \theta_1 \left[ K^\alpha (AL_Y)^{1-\alpha} - C_Y \right] - \theta_2 (1 - \alpha) \left[ K^\alpha (AL_Y)^{1-\alpha} - C_Y \right] = 0,$$

$$(A.11) \quad \frac{\partial H}{\partial K} + \dot{\theta}_1 = \theta_1 \xi \alpha K^{\alpha-1} (AL_Y)^{1-\alpha} + \theta_2 (1 - \alpha)(1 - \xi) \alpha K^{\alpha-1} (AL_Y)^{1-\alpha} + \dot{\theta}_1 = \rho \theta_1, \quad \text{and}$$

$$\begin{aligned}
& \frac{\partial H}{\partial \alpha} + \dot{\theta}_2 = \\
\text{(A.12)} \quad & \theta_1 \xi K^\alpha (AL_Y)^{1-\alpha} \log\left(\frac{K}{AL_Y}\right) + \theta_2 (1-\alpha)(1-\xi) K^\alpha (AL_Y)^{1-\alpha} \log\left(\frac{K}{AL_Y}\right) - \\
& \theta_2 (1-\xi) [K^\alpha (AL_Y)^{1-\alpha} - C_Y] + \dot{\theta}_2 = \rho \theta_2
\end{aligned}$$

From (A.10) we have  $\theta_1 = \theta_2(1-\alpha)$  so that (A.8) can be rewritten,

$$\frac{\lambda}{C_Y} = \theta_1 = \theta_2(1-\alpha).$$

Differentiating with respect to time:

$$\frac{\dot{C}_Y}{C_Y} = -\frac{\dot{\theta}_1}{\theta_1}.$$

From (A.9):

$$\left[ \theta_1 \xi (1-\alpha) + \theta_2 (1-\xi) (1-\alpha)^2 \left( \frac{K}{L_Y} \right)^\alpha A^{1-\alpha} \right] = \left( \frac{1-\lambda}{1-L_Y} \right).$$

Combining the above with  $\theta_1 = \theta_2(1-\alpha)$ :

$$\begin{aligned}
\text{(A.13)} \quad & \theta_1 = \left( \frac{1-\lambda}{1-L_Y} \right) \left[ (1-\alpha) \left( \frac{K}{L_Y} \right)^\alpha A^{1-\alpha} \right]^{-1} \\
& \theta_2 = \left( \frac{1-\lambda}{1-L_Y} \right) \left[ (1-\alpha)^2 \left( \frac{K}{L_Y} \right)^\alpha A^{1-\alpha} \right]^{-1}.
\end{aligned}$$

Differentiating with respect to time:

$$\begin{aligned}
\frac{\dot{\theta}_1}{\theta_1} &= \frac{\dot{L}_Y}{1-L_Y} + \alpha \left( \frac{\dot{L}_Y}{L_Y} - \frac{\dot{K}}{K} \right) + \frac{\dot{\alpha}}{1-\alpha} - \dot{\alpha} \log\left(\frac{K}{AL_Y}\right) \\
\frac{\dot{\theta}_2}{\theta_2} &= \frac{\dot{L}_Y}{1-L_Y} + \alpha \left( \frac{\dot{L}_Y}{L_Y} - \frac{\dot{K}}{K} \right) + 2 \left( \frac{\dot{\alpha}}{1-\alpha} \right) - \dot{\alpha} \log\left(\frac{K}{AL_Y}\right)
\end{aligned}$$

Combining (A.13) with  $\frac{\lambda}{C_Y} = \theta_1 = \theta_2(1-\alpha)$ :

$$(A.14) \quad (1-L_Y) = \left( \frac{1-\lambda}{\lambda} \right) \frac{C_Y}{(1-\alpha) \left( \frac{K}{L_Y} \right)^\alpha A^{1-\alpha}}.$$

Differentiating with respect to time:

$$-\dot{L}_Y = \left[ (1-\alpha) \left( \frac{K}{L_Y} \right)^\alpha A^{1-\alpha} \right]^{-1} \left( \frac{1-\lambda}{\lambda} \right) \dot{C}_Y - \left( \frac{\alpha}{1-\alpha} \right) \left( \frac{1-\lambda}{\lambda} \right) C_Y K^{-\alpha-1} L_Y^\alpha A^{\alpha-1} \dot{K} +$$

$$\left( \frac{\alpha}{1-\alpha} \right) \left( \frac{1-\lambda}{\lambda} \right) C_Y K^{-\alpha} (L_Y A)^{\alpha-1} \dot{L}_Y + \left( \frac{1-\lambda}{\lambda} \right) C_Y \left[ \frac{\partial \left\{ (1-\alpha) \left( \frac{K}{L_Y} \right)^\alpha A^{1-\alpha} \right\}}{\partial \alpha} \right] \dot{\alpha},$$

where,

$$\frac{\partial \left\{ (1-\alpha) \left( \frac{K}{L_Y} \right)^\alpha A^{1-\alpha} \right\}}{\partial \alpha} = - \left[ \frac{(1-\alpha) \log \left( \frac{K}{AL_Y} \right) - 1}{(1-\alpha)^2 \left( \frac{K}{AL_Y} \right)^\alpha A^{1-\alpha}} \right].$$

Putting the above into growth rates and exploiting the fact that, by (A.14),

$$\left( \frac{\lambda}{1-\lambda} \right) \left( \frac{(1-\alpha) \left( \frac{K}{L_Y} \right)^\alpha A^{1-\alpha}}{C_Y} \right) = \frac{1}{1-L_Y},$$

we arrive at,

$$(A.15) \quad \frac{\dot{C}_Y}{C_Y} = \alpha \left[ \frac{\dot{K}}{K} - \frac{\dot{L}_Y}{L_Y} \right] + \left[ \log \left( \frac{K}{AL_Y} \right) - \left( \frac{1}{1-\alpha} \right) \right] \dot{\alpha} + \frac{\dot{L}_Y}{(1-L_Y)}.$$

Combing  $\theta_1 = \theta_2(1-\alpha)$  with (A.11) yields,

$$(A.16) \quad -\frac{\dot{\theta}_1}{\theta_1} = \alpha K^{\alpha-1} (AL_Y)^{1-\alpha} - \rho,$$

or,

$$(A.17) \quad \frac{\dot{C}_Y}{C_Y} = \alpha K^{\alpha-1} (AL_Y)^{1-\alpha} - \rho.$$

From (A.15) and the time-differentiation of (13) comes,

$$\frac{\dot{L}_Y}{1-L_Y} = \alpha \left( \frac{\dot{K}}{K} - \frac{\dot{L}_Y}{L_Y} \right) - \frac{\dot{\alpha}}{1-\alpha} + \dot{\alpha} \log \left( \frac{K}{AL_Y} \right) - [\alpha K^{\alpha-1} (AL_Y)^{1-\alpha} - \rho],$$

which, using (A.5) and (A.6), then becomes,

$$(A.18) \quad \frac{\dot{L}_Y}{1-L_Y} = \alpha \left( \frac{\xi I}{K} - \frac{\dot{L}_Y}{L_Y} \right) - (1-\xi)I + (1-\xi)(1-\alpha)I \log \left( \frac{K}{AL_Y} \right) - [\alpha K^{\alpha-1} (AL_Y)^{1-\alpha} - \rho]$$

Combing  $\theta_1 = \theta_2(1-\alpha)$  with (A.12) yields,

$$(A.19) \quad -\frac{\dot{\theta}_2}{\theta_2} = -\frac{\dot{\alpha}}{1-\alpha} + (1-\alpha)K^\alpha (AL_Y)^{1-\alpha} \log \left( \frac{K}{AL_Y} \right) - \rho,$$

implying, with (A.16), that,

$$(1-\alpha)K^\alpha (AL_Y)^{1-\alpha} \log \left( \frac{K}{AL_Y} \right) = \alpha K^{\alpha-1} (AL_Y)^{1-\alpha},$$

or,

$$(A.20) \quad \alpha = \frac{K \log \left( \frac{K}{AL_Y} \right)}{1 + K \log \left( \frac{K}{AL_Y} \right)}.$$

Rearranging (A.18) and using (A.20) yields,

$$(A.21) \quad \frac{\dot{L}_Y}{L_Y} = \left[ \frac{1-L_Y}{L_Y(1-\alpha)+\alpha} \right] \left[ \alpha \left( \frac{I}{K} \right) - (1-\xi)I - \left( \alpha K^{\alpha-1} (AL_Y)^{1-\alpha} - \rho \right) \right].$$

Differentiating (A.20) with respect to time:

$$(A.22) \quad \dot{\alpha} = K(1-\alpha)^2 \left\{ \left[ \log \left( \frac{K}{AL_Y} \right) + 1 \right] \left( \frac{\dot{K}}{K} \right) - \frac{\dot{L}_Y}{L_Y} \right\}.$$

Combining (A.22) with (A.6) and rearranging:

$$(A.23) \quad \frac{\dot{L}_Y}{L_Y} = \left( \frac{I}{K} \right) \left\{ \left[ 1 + \log \left( \frac{K}{AL_Y} \right) \right] \xi - \left( \frac{1-\xi}{1-\alpha} \right) \right\}.$$

Using (A.21) and (A.23) along with (A.5) and (A.6) results in, with good deal of rearranging,

$$(A.24) \quad \xi = \frac{(1-\alpha)(1-L_Y) \left[ \alpha - K - \left( \frac{K}{I} \right) \left( \alpha K^{\alpha-1} (AL_Y)^{1-\alpha} - \rho \right) \right] + L_Y(1-\alpha) + \alpha}{[L_Y(1-\alpha) + \alpha] \left[ \left( \frac{\alpha}{K} + 2 - \alpha \right) - \left( \frac{(1-\alpha)(1-L_Y)}{L_Y(1-\alpha) + \alpha} \right) K \right]}.$$

**Appendix B: Claim:  $\xi \geq \alpha$ .**

From (A.24) we can state that,

$$(B.1) \quad \xi = \frac{\left(\frac{1-L_Y}{L_Y(1-\alpha)+\alpha}\right)\left(\alpha - K - \frac{1}{I}\left(\alpha K^\alpha (AL_Y)^{1-\alpha} - \rho K\right)\right) + \frac{1}{1-\alpha}}{\left\{\left(\frac{\alpha}{1-\alpha} \frac{1}{K} + 1\right) + \frac{1}{(1-\alpha)} - \left(\frac{1-L_Y}{L_Y(1-\alpha)+\alpha}\right)K\right\}}.$$

So,

$$(B.2) \quad \frac{\alpha}{\xi} = \frac{\left(\frac{\alpha}{1-\alpha} \frac{\alpha}{K} + \alpha\right) + \frac{\alpha}{(1-\alpha)} - \left(\frac{(1-L)\alpha}{L_Y(1-\alpha)+\alpha}\right)K}{\left(\frac{1-L_Y}{L_Y(1-\alpha)+\alpha}\right)\left(\alpha - K - \frac{1}{I}\left(\alpha K^\alpha (AL_Y)^{1-\alpha} - \rho K\right)\right) + \frac{1}{1-\alpha}}.$$

Call A: numerator and B: denominator.

**Claim B1: If  $B > 0$  and  $A > 0$  then  $B > A$ .**

If  $B > 0$  then

$$(B.3) \quad \left(\frac{\alpha(1-L_Y)}{L_Y(1-\alpha)+\alpha}\right) + \frac{1}{1-\alpha} > \left(\frac{1-L_Y}{L_Y(1-\alpha)+\alpha}\right)\left(K + \frac{1}{I}\left(\alpha K^\alpha (AL_Y)^{1-\alpha} + \rho K\right)\right).$$

If  $A > 0$  then

$$(B.4) \quad \left(\frac{\alpha}{1-\alpha} \frac{\alpha}{K} + \alpha\right) + \frac{\alpha}{(1-\alpha)} > \left(\frac{(1-L_Y)\alpha}{L_Y(1-\alpha)+\alpha}\right)K.$$

From (B.3):

$$\left(\frac{\alpha(1-L_Y)}{L_Y(1-\alpha)+\alpha}\right) + \frac{1}{1-\alpha} > \left(\frac{1-L_Y}{L_Y(1-\alpha)+\alpha}\right)\left(K + \frac{1}{I}\left(\alpha K^\alpha (AL_Y)^{1-\alpha} - \rho K\right)\right).$$

Subtracting  $\left(\frac{\alpha^2}{1-\alpha} \frac{1}{K} + \alpha\right) + \frac{\alpha}{(1-\alpha)}$  from both sides yields.



$$(B.5) \quad \begin{aligned} & \left( \frac{\alpha(1-L_Y)}{L_Y(1-\alpha)+\alpha} \right) + \frac{1}{1-\alpha} - \alpha - \frac{\alpha}{1-\alpha} - \frac{\alpha^2}{1-\alpha} \frac{1}{K} > \\ & - \left[ \left( \frac{\alpha^2}{1-\alpha} \frac{1}{K} + \alpha \right) + \frac{\alpha}{(1-\alpha)} \right] + \\ & \left[ \left( \frac{\alpha}{1-\alpha} K + \frac{1}{\alpha} \right) + \frac{1}{(1-\alpha)} \right] \left( 1 + \frac{1}{I} \left( \alpha K^{\alpha-1} (AL_Y)^{1-\alpha} - \rho \right) \right) \end{aligned}$$

From (B.4),  $\left( \frac{\alpha^2}{1-\alpha} \frac{1}{K} + \alpha \right) + \frac{\alpha}{(1-\alpha)} > \left( \frac{(1-L_Y)\alpha}{L_Y(1-\alpha)+\alpha} \right) K$ , so,

$$\begin{aligned} & - \left[ \left( \frac{\alpha^2}{1-\alpha} \frac{1}{K} + \alpha \right) + \frac{\alpha}{(1-\alpha)} \right] + \\ & \left( \frac{(1-L_Y)}{L_Y(1-\alpha)+\alpha} \right) \left( K + \frac{K}{I} \left( \alpha K^{\alpha-1} (AL_Y)^{1-\alpha} - \rho \right) \right) \\ & < \left( \frac{(1-L_Y)}{L_Y(1-\alpha)+\alpha} \right) \left( K(1-\alpha) + \frac{K}{I} \left( \alpha K^{\alpha-1} (AL_Y)^{1-\alpha} - \rho \right) \right) \end{aligned}$$

Thus,

$$\begin{aligned} & \left( \frac{\alpha(1-L_Y)}{L_Y(1-\alpha)+\alpha} \right) + \frac{1}{1-\alpha} - \alpha - \frac{\alpha}{1-\alpha} - \frac{\alpha^2}{1-\alpha} \frac{1}{K} > \\ & \left( \frac{1-L_Y}{L_Y(1-\alpha)+\alpha} \right) K \left( (1-\alpha) + \frac{1}{I} \left( \alpha K^{\alpha-1} (AL_Y)^{1-\alpha} + \rho \right) \right) \end{aligned}$$

and,

$$\begin{aligned} & \left( \frac{\alpha(1-L_Y)}{L_Y(1-\alpha)+\alpha} \right) K - \frac{\alpha}{1-\alpha} - \alpha - \frac{\alpha^2}{1-\alpha} \frac{1}{K} > \\ & \left( \frac{1-L_Y}{L_Y(1-\alpha)+\alpha} \right) \left( K - \alpha + \frac{K}{I} \left( \alpha K^{\alpha-1} (AL_Y)^{1-\alpha} + \rho \right) \right) - \frac{1}{1-\alpha} \end{aligned}$$

Finally,

$$(B.6) \quad \frac{1}{1-\alpha} \frac{\alpha}{1-\alpha} + \frac{\alpha^2}{1-\alpha} \frac{1}{K} + \alpha - \left( \frac{\alpha(1-L_Y)}{L_Y(1-\alpha)+\alpha} \right) K < \left( \frac{1-L_Y}{L_Y(1-\alpha)+\alpha} \right) \left( \alpha - K - \frac{K}{I} \left( \alpha K^{\alpha-1} (AL_Y)^{1-\alpha} + \rho \right) \right) + \frac{1}{1-\alpha}$$

Therefore,

$$(B.7) \quad \frac{\alpha}{\xi} = \frac{\left( \frac{\alpha}{1-\alpha} \frac{\alpha}{K} + 1 \right) + \frac{\alpha}{(1-\alpha)} - \left( \frac{(1-L_Y)\alpha}{L_Y(1-\alpha)+\alpha} \right) K}{\left( \frac{1-L_Y}{L_Y(1-\alpha)+\alpha} \right) \left( \alpha - K - \frac{1}{I} \left( \alpha K^\alpha (AL_Y)^{1-\alpha} - \rho K \right) \right) + \frac{1}{1-\alpha}} < 1.$$

**Claim B2: If  $B < 0$  and  $A < 0$  then  $B < A$ .**

*Proof:*

If  $B < 0$  then,

$$(B.8) \quad \left( \frac{\alpha(1-L_Y)}{L_Y(1-\alpha)+\alpha} \right) + \frac{1}{1-\alpha} < \left( \frac{1-L_Y}{L_Y(1-\alpha)+\alpha} \right) \left( K + \frac{1}{I} \left( \alpha K^\alpha (AL_Y)^{1-\alpha} + \rho K \right) \right).$$

If  $A < 0$  then,

$$(B.9) \quad \left( \frac{\alpha}{1-\alpha} \frac{\alpha}{K} + \alpha \right) + \frac{\alpha}{(1-\alpha)} < \left( \frac{(1-L_Y)\alpha}{L_Y(1-\alpha)+\alpha} \right) K.$$

From (B.8):

$$\left( \frac{\alpha(1-L_Y)}{L_Y(1-\alpha)+\alpha} \right) + \frac{1}{1-\alpha} < \left( \frac{1-L_Y}{L_Y(1-\alpha)+\alpha} \right) \left( K + \frac{1}{I} \left( \alpha K^\alpha (AL_Y)^{1-\alpha} - \rho K \right) \right).$$

Subtracting  $\left( \frac{\alpha^2}{1-\alpha} \frac{1}{K} + \alpha \right) + \frac{\alpha}{(1-\alpha)}$  from both sides yields,

$$(B.10) \quad \begin{aligned} & \left( \frac{\alpha(1-L_Y)}{L_Y(1-\alpha)+\alpha} \right) + \frac{1}{1-\alpha} - \alpha - \frac{\alpha}{1-\alpha} - \frac{\alpha^2}{1-\alpha} \frac{1}{K} < \\ & - \left[ \left( \frac{\alpha^2}{1-\alpha} \frac{1}{K} + \alpha \right) + \frac{\alpha}{(1-\alpha)} \right] + \\ & \left[ \left( \frac{\alpha}{1-\alpha} K + \frac{1}{\alpha} \right) + \frac{1}{(1-\alpha)} \right] \left( 1 + \frac{1}{I} (\alpha K^{\alpha-1} (AL_Y)^{1-\alpha} - \rho) \right) \end{aligned}$$

From (B.9),  $\left( \frac{\alpha^2}{1-\alpha} \frac{1}{K} + \alpha \right) + \frac{\alpha}{(1-\alpha)} < \left( \frac{(1-L)\alpha}{L(1-\alpha)+\alpha} \right) K$ , so,

$$\begin{aligned} & - \left[ \left( \frac{\alpha^2}{1-\alpha} \frac{1}{K} + \alpha \right) + \frac{\alpha}{(1-\alpha)} \right] + \\ & \left( \frac{(1-L_Y)}{L_Y(1-\alpha)+\alpha} \right) \left( K + \frac{K}{I} (\alpha K^{\alpha-1} (AL_Y)^{1-\alpha} - \rho) \right) \\ & > \left( \frac{(1-L_Y)}{L_Y(1-\alpha)+\alpha} \right) \left( K(1-\alpha) + \frac{K}{I} (\alpha K^{\alpha-1} (AL_Y)^{1-\alpha} - \rho) \right) \end{aligned}$$

Thus,

$$\begin{aligned} & \left( \frac{\alpha(1-L_Y)}{L_Y(1-\alpha)+\alpha} \right) + \frac{1}{1-\alpha} - \alpha - \frac{\alpha}{1-\alpha} - \frac{\alpha^2}{1-\alpha} \frac{1}{K} < \\ & \left( \frac{1-L_Y}{L_Y(1-\alpha)+\alpha} \right) K \left( (1-\alpha) + \frac{1}{I} (\alpha K^{\alpha-1} (AL_Y)^{1-\alpha} + \rho) \right) \end{aligned}$$

and,

$$\begin{aligned} & \left( \frac{\alpha(1-L_Y)}{L_Y(1-\alpha)+\alpha} \right) K - \alpha - \frac{\alpha}{1-\alpha} - \frac{\alpha^2}{1-\alpha} \frac{1}{K} < \\ & \left( \frac{1-L_Y}{L_Y(1-\alpha)+\alpha} \right) \left( K - \alpha + \frac{K}{I} (\alpha K^{\alpha-1} (AL_Y)^{1-\alpha} + \rho) \right) - \frac{1}{1-\alpha} \end{aligned}$$

Finally,

$$(B.11) \quad \frac{\alpha^2}{1-\alpha} \frac{1}{K} + \alpha + \frac{\alpha}{1-\alpha} - \left( \frac{\alpha(1-L_Y)}{L_Y(1-\alpha)+\alpha} \right) K > \frac{1}{1-\alpha} - \left( \frac{1-L_Y}{L_Y(1-\alpha)+\alpha} \right) \left( K - \alpha + \frac{K}{I} \left( \alpha K^{\alpha-1} (AL_Y)^{1-\alpha} + \rho \right) \right)$$

Therefore,

$$(B.12) \quad \frac{\alpha}{\xi} = \frac{\left( \frac{\alpha}{1-\alpha} \frac{\alpha}{K} + \alpha \right) + \frac{\alpha}{(1-\alpha)} - \left( \frac{(1-L_Y)\alpha}{L_Y(1-\alpha)+\alpha} \right) K}{\left( \frac{1-L_Y}{L_Y(1-\alpha)+\alpha} \right) \left( \alpha - K - \frac{1}{I} \left( \alpha K^\alpha (AL_Y)^{1-\alpha} - \rho K \right) \right) + \frac{1}{1-\alpha}} < 1.$$

Having demonstrated **B1** and **B2**, the general claim that  $\xi \geq \alpha$  is established.

**Appendix C: Claim: Labor's share converges from above to a positive value.**

Labor's share is,

$$(C.1) \quad LSH = \frac{w_Y L_Y + w_X L_X}{P_Y Y + P_X X}.$$

Since the real wages (in terms of marginal utility units) are equated across sectors and total labor supply is unity we have,

$$(C.2) \quad LSH = \frac{\left(\frac{1-\lambda}{C_X}\right)B}{\frac{\lambda}{C_Y} K^\alpha (AL_Y)^{1-\alpha} + \frac{1-\lambda}{C_X} BL_X},$$

total and consumption output from the  $X$  sector are identical:

$$(C.3) \quad LSH = \frac{\left(\frac{1-\lambda}{1-L_Y}\right)}{\frac{\lambda}{C_Y} K^\alpha (AL_Y)^{1-\alpha} + (1-\lambda)} = \frac{\left(\frac{1}{1-L_Y}\right)}{1 + \frac{\lambda}{1-\lambda} \frac{K^\alpha (AL_Y)^{1-\alpha}}{C_Y}}.$$

Since  $\frac{1}{1-L_Y}$  unambiguously decreases during the transition, if we can demonstrate that

$\frac{K^\alpha (AL_Y)^{1-\alpha}}{C_Y}$  increases during the transition,  $LSH$  will unambiguously decrease towards

its steady-state value of  $LSH = \frac{1}{1 + \frac{\lambda}{1-\lambda} \frac{1}{\rho}}$ . The growth rate of the relevant ratio is,

$$(C.4) \quad \alpha \frac{\dot{K}}{K} + (1-\alpha) \frac{\dot{L}_Y}{L_Y} + \log\left(\frac{K}{AL_Y}\right) \dot{\alpha} - \frac{\dot{C}_Y}{C_Y}.$$

Furthermore, if we combine (C.4) with (A.15),

$$(A.15) \quad \frac{\dot{C}_Y}{C_Y} = \alpha \left[ \frac{\dot{K}}{K} - \frac{\dot{L}_Y}{L_Y} \right] + \left[ \log \left( \frac{K}{AL_Y} \right) - \left( \frac{1}{1-\alpha} \right) \right] \dot{\alpha} + \frac{\dot{L}_Y}{(1-L_Y)},$$

we arrive at,

$$(C.5) \quad \frac{\dot{L}_Y}{L_Y} + \frac{\dot{\alpha}}{1-\alpha} - \frac{\dot{L}_Y}{1-L_Y} = \frac{\dot{\alpha}}{1-\alpha} + \left\{ \frac{1}{L_Y} - \frac{1}{1-L_Y} \right\} \dot{L}_Y.$$

We know that  $\frac{\dot{\alpha}}{1-\alpha}$  is unambiguously non-negative and  $\dot{L}_Y$  is non-positive during

transition. So if  $\left\{ \frac{1}{L_Y} - \frac{1}{1-L_Y} \right\} \leq 0$ , then this is sufficient (though not necessary) for

*LSH* to be decreasing. Knowing that  $0 \leq L_Y \leq 1$ , we can numerical calculate that,

$$L_Y \geq 0.5 \quad \rightarrow \quad \text{then} \quad \left\{ \frac{1}{L_Y} - \frac{1}{1-L_Y} \right\} \leq 0.$$

So  $L_Y > 0.5$  is a sufficient condition: if the *Y* sector employs at least half of the labor supply and the transition path of *LSH* to its steady-state is monotonic, then *LSH* converges to its steady-state value from above.

Now we consider the case of  $L_Y < 0.5$ . From equation (C.5) it follows that

$$\frac{\dot{\alpha}}{1-\alpha} + \left\{ \frac{1}{L_Y} - \frac{1}{1-L_Y} \right\} \dot{L}_Y \geq 0$$

is a sufficient condition for *LISH* to be decreasing. From equation (4.2) in the main text we know that

$$\dot{\alpha} = K(1-\alpha)^2 \left\{ \left[ \log \left( \frac{K}{AL_Y} \right) + 1 \right] \left( \frac{\dot{K}}{K} - \frac{\dot{L}_Y}{L_Y} \right) \right\} \text{ so } \frac{\dot{\alpha}}{(1-\alpha)} = K(1-\alpha) \left\{ \left[ \log \left( \frac{K}{AL_Y} \right) + 1 \right] \left( \frac{\dot{K}}{K} - \frac{\dot{L}_Y}{L_Y} \right) \right\}$$

Therefore, the condition can be written as follows

$$(C.6) \quad K(1-\alpha) \left\{ \left[ \log\left(\frac{K}{AL_Y}\right) + 1 \right] \left( \frac{\dot{K}}{K} \right) \right\} + \left\{ \frac{1}{L_Y} - \frac{K(1-\alpha)}{L_Y} - \frac{1}{1-L_Y} \right\} \dot{L}_Y \geq 0$$

The first term of equation (C.6) is positive whenever savings are positive, so in order to prove that condition (C.6) holds, it suffices to prove that  $\left\{ \frac{1}{L_Y} - \frac{K(1-\alpha)}{L_Y} - \frac{1}{1-L_Y} < 0 \right\}$  or

$$1 - \frac{L_Y}{1-L_Y} < (1-\alpha)K.$$

If  $L_Y < 0.5$  it follows that  $1 - \frac{L_Y}{1-L_Y} < 1$ . So, given  $L_Y < 0.5$ ,  $K(1-\alpha) > 1$  is a sufficient

condition. Using equations (3.6) and (4.2) in the main text we get,

$$(C.7) \quad \frac{\dot{L}_Y}{L_Y} = \left( \frac{I}{K} \right) \left\{ \left[ 1 + \log\left(\frac{K}{AL_Y}\right) \right]^\xi - \left( \frac{1-\xi}{1-\alpha} \right) \right\},$$

and since  $\frac{\dot{L}_Y}{L_Y} \leq 0$  we know that  $(1-\alpha) \left[ 1 + \log\left(\frac{K}{AL_Y}\right) \right] < \left( \frac{1-\xi}{\xi} \right)$ .

Similarly from equation (3.5) we know that,

$$(C.8) \quad \frac{\dot{K}}{K} = \xi \left( \frac{I}{K} \right)$$

Using (C.7) and (C.8), equation (C.6) can be written as,

$$\xi \frac{I}{K} \left\{ \left[ K(1-\alpha) \left[ \log\left(\frac{K}{AL_Y}\right) + 1 \right] \right] + \left\{ 1 - K(1-\alpha) - \frac{L_Y}{1-L_Y} \right\} \left( \frac{1}{1-\alpha} \right) \left[ (1-\alpha) \left[ 1 + \log\left(\frac{K}{AL_Y}\right) \right] - \left( \frac{1-\xi}{\xi} \right) \right] \right\} \geq 0.$$

Rearranging,

(C.9)

$$\xi \frac{I}{K} \left[ \frac{1}{(1-\alpha)} \left\{ 1 - \frac{L_Y}{1-L_Y} \right\} (1-\alpha) \left[ 1 + \log \left( \frac{K}{AL_Y} \right) \right] - \left( \frac{1-\xi}{\xi} \right) \right] + K \left\{ \left( \frac{1-\xi}{\xi} \right) - \alpha \left[ 1 + \log \left( \frac{K}{AL_Y} \right) \right] \right\} \geq 0.$$

If  $L < 0.5$  then  $1 - \frac{L_Y}{1-L_Y} > 0$  and the following inequality is a sufficient condition:

$$K \left\{ \left( \frac{1-\xi}{\xi} \right) - \alpha \left[ 1 + \log \left( \frac{K}{AL_Y} \right) \right] \right\} > \frac{1}{(1-\alpha)} \left\{ 1 - \frac{L_Y}{1-L_Y} \right\} \left\{ \left( \frac{1-\xi}{\xi} \right) - (1-\alpha) \left[ 1 + \log \left( \frac{K}{AL_Y} \right) \right] \right\}.$$

Rearranging:

$$(1-\alpha) \left\{ K + \frac{\xi}{1-\xi} \left( \left\{ 1 - \frac{L_Y}{1-L_Y} - \alpha K \right\} \right) \left[ 1 + \log \left( \frac{K}{AL_Y} \right) \right] \right\} > \left\{ 1 - \frac{L_Y}{1-L_Y} \right\}.$$

We proved that  $\xi \geq \alpha \rightarrow 1 - \xi < 1 - \alpha$ , therefore if

$$(1-\alpha) \left\{ K + \frac{\xi}{1-\xi} \left( 1 - \frac{L_Y}{1-L_Y} - \alpha K \right) \left[ 1 + \log \left( \frac{K}{AL_Y} \right) \right] \right\} > \left\{ (1-\alpha)K + \xi \left( 1 - \frac{L_Y}{1-L_Y} - \alpha K \right) \right\} \left[ 1 + \log \left( \frac{K}{AL_Y} \right) \right]$$

then condition (C.9) holds.

So it suffices to prove that

$$(C.10) \quad \left\{ (1-\alpha)K + \xi \left( 1 - \frac{L_Y}{1-L_Y} - \alpha K \right) \right\} \left[ 1 + \log \left( \frac{K}{AL_Y} \right) \right] \geq \left\{ 1 - \frac{L_Y}{1-L_Y} \right\}.$$

First, if  $1 - \frac{L_Y}{1-L_Y} - \alpha K < 0$  then  $K > \frac{1}{\alpha} \left( 1 - \frac{L_Y}{1-L_Y} \right)$  and

$(1-\alpha)K > \frac{1-\alpha}{\alpha} \left( 1 - \frac{L_Y}{1-L_Y} \right)$ . Now, if  $(1-\alpha)K \geq \left( 1 - \frac{L_Y}{1-L_Y} \right)$ , then

$(1-\alpha)K \left[ 1 + \log \left( \frac{K}{AL_Y} \right) \right] \geq \left( 1 - \frac{L_Y}{1-L_Y} \right)$  because  $\left[ 1 + \log \left( \frac{K}{AL_Y} \right) \right] \geq 1$ . Therefore,

whenever  $\alpha \geq 0.5$  then  $K(1-\alpha) > 1$  holds.



Second, if  $1 - \frac{L_Y}{1 - L_Y} - \alpha K < 0$  and  $\alpha < 0.5$  then  $K > 2 \left[ 1 - \frac{L_Y}{1 - L_Y} \right]$  and

$K > \left[ 1 - \frac{L_Y}{1 - L_Y} \right]$  so  $K(1 - \alpha) > 1$  holds.

Third, if  $1 - \frac{L_Y}{1 - L_Y} - \alpha K \geq 0$  and  $\xi \geq \frac{1}{1 + \log\left(\frac{K}{AL_Y}\right)} \frac{1 - 2L_Y}{1 - 2L_Y - \alpha K(1 - L_Y)}$  then

condition (C.10) holds.

Fourth,  $1 - \frac{L_Y}{1 - L_Y} - \alpha K \geq 0$  and  $\xi < \frac{1}{1 + \log\left(\frac{K}{AL_Y}\right)} \frac{1 - 2L_Y}{1 - 2L_Y - \alpha K(1 - L_Y)}$  then

condition (C.10) holds whenever

$$(1 - \alpha)K \left[ 1 + \log\left(\frac{K}{AL_Y}\right) \right] \geq \left\{ 1 - \frac{L_Y}{1 - L_Y} \right\} \left( 1 - \xi \left[ 1 + \log\left(\frac{K}{AL_Y}\right) \right] \right) + \alpha K \xi \left[ 1 + \log\left(\frac{K}{AL_Y}\right) \right]$$

$1 - \frac{L}{1 - L} - \alpha K \geq 0$  so  $\frac{1}{K} \left( 1 - \frac{L}{1 - L} \right) \geq \alpha$  and it suffices to prove that

$$(C.11) \quad (1 - \alpha)K \left[ 1 + \log\left(\frac{K}{AL_Y}\right) \right] \geq \left\{ 1 - \frac{L_Y}{1 - L_Y} \right\}.$$

Since  $1 - \alpha = \frac{1}{1 + K \log\left(\frac{K}{AL_Y}\right)}$  we can rewrite (C.11) as follows:

$$\frac{K + \log\left(\frac{K}{AL_Y}\right)}{1 + K \log\left(\frac{K}{AL_Y}\right)} \geq \left\{ 1 - \frac{L_Y}{1 - L_Y} \right\}.$$

So a sufficient condition is  $K \geq 1$ .

**Appendix D: Claim:** *There exists a  $\tilde{K}$  such that  $\forall K > \tilde{K}$ ,  $\left(\frac{K}{AL_{yt}}\right) < \left(\frac{1}{\rho}\right)^{\frac{1}{1-\alpha}}$ .*

Define  $\tilde{K} = \frac{1}{\log\left(\frac{1}{\rho}\right)}$ . Since  $\rho > 0$  then  $\log\left(\frac{1}{\rho}\right)$  is a finite number and so is  $\tilde{K}$ .

Note that  $\tilde{K} \geq \frac{\alpha}{\log\left(\frac{1}{\rho}\right)}$  because  $\alpha \leq 1$ . Therefore,  $\log\left(\frac{1}{\rho}\right) \geq \frac{\alpha}{\tilde{K}}$  and,

$$(D.1) \quad \frac{1}{1-\alpha} \log\left(\frac{1}{\rho}\right) \geq \frac{1}{1-\alpha} \frac{\alpha}{\tilde{K}}$$

Combining equation (4.1) from the main text and (D.1) yields,

$$\frac{1}{1-\alpha} \log\left(\frac{1}{\rho}\right) \geq K \log\left(\frac{K}{AL}\right) \frac{1}{\tilde{K}}$$

Rearranging:

$$(D.2) \quad \log\left(\frac{1}{\rho}\right) \frac{\tilde{K}}{K} \geq (1-\alpha) \log\left(\frac{K}{AL}\right)$$

Since we are considering  $K > \tilde{K}$ ,  $\log\left(\frac{1}{\rho}\right) > (1-\alpha) \log\left(\frac{K}{AL}\right)$  and  $\left(\frac{1}{\rho}\right) > \left(\frac{K}{AL}\right)^{(1-\alpha)}$ .

Therefore  $\left(\frac{1}{\rho}\right) > \left(\frac{K}{AL}\right)^{(1-\alpha)}$  holds for any  $K > \tilde{K}$ .

**Appendix E: Claim:**  $\left(\frac{K}{AL_Y}\right)^{\alpha-1}$  *increases as K increases.*

Consider a function,

$$(E.1) \quad G(K, L) = (\alpha - 1) \log \left( \frac{K}{AL_Y} \right).$$

Differentiating with respect to time:

$$\frac{\partial G(K, L)}{\partial t} = \dot{\alpha} \log \left( \frac{K}{AL_Y} \right) - (1 - \alpha) \left( \frac{\dot{K}}{K} - \frac{\dot{L}}{L} \right).$$

The above can be combined with (4.2) from the main text to yield,

(E.3)

$$\frac{\partial G(K, L)}{\partial t} = (1 - \alpha) \left( K (1 - \alpha) \left[ \log \left( \frac{K}{AL_Y} \right) \frac{\dot{K}}{K} \right] \log \left( \frac{K}{AL_Y} \right) - \left( \frac{\dot{K}}{K} - \frac{\dot{L}}{L} \right) \left( 1 - K (1 - \alpha) \log \left( \frac{K}{AL_Y} \right) \right) \right).$$

Using equation (4.1) and rearranging:

$$(E.4) \quad \frac{\partial G(K, L)}{\partial t} = (1 - \alpha)^2 \left( \left( \log \left( \frac{K}{AL_Y} \right) \right)^2 \dot{K} - \left( \frac{\dot{K}}{K} - \frac{\dot{L}}{L} \right) \right).$$

Therefore if  $\dot{K} \left( \left( \log \left( \frac{K}{AL_Y} \right) \right)^2 - \frac{1}{K} \right) > - \frac{\dot{L}}{L}$  then  $\frac{\partial G(K, L)}{\partial t} > 0$ .

Using (A.23) and (A.5) we can rewrite the above condition as,

$$(E.5) \quad \xi \left( \left( \log \left( \frac{K}{AL_Y} \right) \right)^2 - \frac{1}{K} \right) > - \left\{ \left( \left( \frac{\alpha}{1 - \alpha} \right) \frac{1}{K} + 1 \right) \xi - \frac{(1 - \xi)}{(1 - \alpha)} \right\}.$$

Rearranging,

$$(E.6) \quad \frac{(1 - \alpha)}{\alpha} \left( \alpha \left( \log \left( \frac{K}{AL_Y} \right) \right)^2 + \alpha \log \left( \frac{K}{AL_Y} \right) + \alpha - \frac{\alpha}{K} \right) > \frac{(1 - \xi)}{\xi}.$$

We already know that  $\alpha \leq \xi$  so it remains to prove that,

$$(E.7) \quad \alpha \left( \log \left( \frac{K}{AL_Y} \right) \right)^2 + \alpha \log \left( \frac{K}{AL_Y} \right) + \alpha - \frac{\alpha}{K} > 1.$$

The function  $\alpha \log \left( \frac{K}{AL_Y} \right)$  is strictly increasing in  $K$  and  $\lim_{K \rightarrow \infty} \alpha \log \left( \frac{K}{AL_Y} \right) = \infty$ .

Therefore, there exists a capital stock  $\bar{K}$  such that  $\alpha \log \left( \frac{K}{AL_Y} \right) > 1$  for any  $K > \bar{K}$ .

Consequently, if  $K > \bar{K}$  then  $\log \left( \frac{K}{AL_Y} \right) \left( \alpha \log \left( \frac{K}{AL_Y} \right) - 1 \right) > 1 - \alpha$ . rearranging it yields,

$$(E.8) \quad \alpha \left( \log \left( \frac{K}{AL_Y} \right) \right)^2 + \alpha \log \left( \frac{K}{AL_Y} \right) + \alpha - \frac{\alpha}{K} > 1.$$

**Appendix F: Claim:** *There exists a  $K^{**}$  such that for any initial  $K_0 > K^{**}$ ,  $\frac{\dot{C}}{C} > 0$ .*

Define  $K^{**} = \max(\tilde{K}, \bar{K})$ . From the previous two propositions it follows that for any initial capital stock  $K_0 > K^{**}$  the optimal growth rate of consumption is positive.

**Appendix G: Claim:** *(i) For any  $\lambda > 0$  there exists a stock of capital  $\tilde{K}$  such that if  $K > \tilde{K}$  then  $\frac{\dot{C}}{C} \leq \lambda(A - \rho)$ ; and (ii) if  $\lambda \geq \frac{1}{2}$  then  $\frac{\dot{C}}{C} \leq \lambda(A - \rho)$ .*

We have to prove that  $-\frac{\dot{L}_Y}{1-L_Y} \leq \lambda(A - \rho)$

$$\frac{\dot{L}_Y}{L_Y} = \frac{I(1-L_Y)}{K(1-\alpha)} \left\{ \frac{\left( \frac{\alpha}{K} + 2 - \alpha \right) \left( \alpha - K - \frac{K}{I} (\alpha AK^{\alpha-1} L_Y^{1-\alpha} - \rho) \right) + K}{\left\{ \left( \frac{\alpha}{K} + 2 - \alpha \right) \left( L_Y + \frac{\alpha}{1-\alpha} \right) - (1-L_Y)K \right\}} \right\}$$

$$-\frac{\dot{L}_Y}{1-L_Y} = \left\{ \frac{\left( \frac{\alpha}{K} + 2 - \alpha \right) \left( IL_Y - (\alpha AK^{\alpha-1} L_Y^{1-\alpha} - \rho)L_Y - \alpha \frac{I}{K} L_Y \right) - IL_Y}{\left\{ \left( \frac{\alpha}{K} + 2 - \alpha \right) ((1-\alpha)L_Y + \alpha) - (1-\alpha)(1-L_Y)K \right\}} \right\}$$

So we have to prove that

$$\lambda(A - \rho) > \frac{\left( \frac{\alpha}{K} + 2 - \alpha \right) \left( IL_Y \left( 1 - \frac{\alpha}{K} \right) + (\alpha AK^{\alpha-1} L_Y^{1-\alpha} - \rho)L_Y \right) - IL_Y}{\left( \frac{\alpha}{K} + 2 - \alpha \right) ((1-\alpha)L_Y + \alpha) - (1-\alpha)(1-L_Y)K}$$

Rearranging,

$$\lambda(A - \rho) > \frac{\left( \alpha AK^{\alpha-1} L_Y^{1-\alpha} - \rho \right) L_Y - IL_Y \left( \frac{\alpha}{K} + \frac{\alpha}{\left( \frac{\alpha}{K} + 2 - \alpha \right)} + (1-\alpha) \right)}{\left( (1-\alpha)L_Y + \alpha \right) - \frac{(1-\alpha)(1-L_Y)K}{\left( \frac{\alpha}{K} + 2 - \alpha \right)}}$$

$$\lambda(A - \rho) \left( ((1 - \alpha)L_Y + \alpha) - \frac{(1 - \alpha)(1 - L_Y)K}{\left(\frac{\alpha}{K} + (2 - \alpha)\right)} - \frac{L_Y}{\lambda} \right) > -L_Y \left( \frac{\alpha}{K} + \frac{\frac{\alpha}{K} + (1 - \alpha)}{\left(\frac{\alpha}{K} + (2 - \alpha)\right)} \right)$$

Therefore it suffices to prove that,

$$\left( ((1 - \alpha)L_Y + \alpha) - \frac{(1 - \alpha)(1 - L_Y)K}{\left(\frac{\alpha}{K} + (2 - \alpha)\right)} - \frac{L_Y}{\lambda} \right) > 0 \text{ or}$$

$$\left( \alpha - \frac{(1 - \alpha)K}{\left(\frac{\alpha}{K} + (2 - \alpha)\right)} - L_Y \left( \frac{1 - \lambda(1 - \alpha)}{\lambda} \right) \right) > 0$$

First, note that

$$\alpha - \frac{(1 - \alpha)K}{\left(\frac{\alpha}{K} + (2 - \alpha)\right)} = \alpha \left( 1 + \frac{K}{\left(\frac{\alpha}{K} + (2 - \alpha)\right)} \right) - \frac{K}{\left(\frac{\alpha}{K} + (2 - \alpha)\right)}$$

and

$$\left( \alpha \left( \frac{\frac{\alpha}{K} + (2 - \alpha) + K}{\left(\frac{\alpha}{K} + (2 - \alpha)\right)} \right) - \frac{K}{\left(\frac{\alpha}{K} + (2 - \alpha)\right)} \right) = \left( \alpha \left( \frac{\frac{\alpha}{K} + (2 - \alpha) + K - \frac{\alpha}{K}}{\left(\frac{\alpha}{K} + (2 - \alpha)\right)} \right) \right)$$

Therefore,

$$\left( \frac{\frac{\alpha}{K} + (2 - \alpha) + K - \frac{\alpha}{K}}{\left(\frac{\alpha}{K} + (2 - \alpha)\right)} \right) = \frac{(2 - \alpha) + K}{\left(\frac{\alpha}{K} + (2 - \alpha)\right)} > 1 + K$$

So it suffices to prove that,

$$1 + K > L_Y \left( \frac{1 - \lambda(1 - \alpha)}{\lambda} \right) \text{ or } \frac{1}{L_Y} + k_Y > \left( \frac{1 - \lambda(1 - \alpha)}{\lambda} \right)$$

$$\text{So if } \frac{1}{L_Y} + k_Y > \left( \frac{1 - \lambda(1 - \alpha)}{\lambda} \right) \text{ then } \frac{\dot{C}}{C} \leq \lambda(A - \rho)$$

Therefore,

- (i) For any  $\lambda > 0$  there exists a stock of capital  $\tilde{K}$  such that if  $K > \tilde{K}$  then
- $$\frac{\dot{C}}{C} \leq \lambda(A - \rho)$$
- (ii) If  $\lambda \geq \frac{1}{2}$  then  $\frac{\dot{C}}{C} \leq \lambda(A - \rho)$ .