Robust Rational Turnout

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Abstract

We establish that, except for a finite set of common costs of participation, all equilibria of a class of complete information voting games (as in Palfrey and Rosenthal (1983)) are regular. Thus, all the equilibria of these games (including those exhibiting high turnout rates) are robust to small but arbitrary payoff perturbations, and survive in nearby games with incomplete information about voting costs and/or about the fraction of supporters of the two candidates. We also show that all the equilibria of these complete information games exhibit minimal heterogeneity of behavior, so that the strategies of indifferent players are characterized by at most two probabilities.

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1 Introduction

Fully strategic theories of voter participation in which voters' calculus is solely based on a comparison of their cost of voting with the probability of being pivotal in elections (*e.g.,* Ledyard (1984), Palfrey and Rosenthal (1983, 1985)) are considered to be inconsistent with the high turnout rates typically observed in elections. This perception is widespread, to the degree that scholars have even conceded that turnout is "the paradox that ate rational choice theory" (Fiorina (1990), page 334). Nevertheless, the possibility of Nash equilibrium with high turnout rates in large electorates has been illustrated in games of complete information by Palfrey and Rosenthal (1983). At least part of the scepticism, then, stems from the belief that such equilibria with high turnout rates are not robust. The question of robustness of these high turnout equilibria is raised, for example, by Feddersen and Sandroni (2006, p. 1272). Consistent with this intuition, Palfrey and Rosenthal (1985) analyze rational turnout in games of incomplete information. They consider a Bayesian game in which individual voting costs are private information drawn from some common probability distribution. Holding the distribution of voting costs fixed, Palfrey and Rosenthal study a sequence of Bayesian Nash equilibria as the electorate gets large. They show that equilibrium turnout rate goes to zero as the electorate goes to infinity.

A prevalent but imprecise rendition of this result in the literature is that while high turnout rate in large electorates is possible in games of complete information, it is not possible in games of incomplete information. This sharp divide between the two types of games, though, is not valid for *regular* games in normal form. Harsanyi's purification theorem (Harsanyi (1973a)) guarantees that every regular equilibrium of a complete information game can be approximated by a Bayesian Nash equilibrium of a nearby game of incomplete information. As a result, *if* the equilibria with high turnout in games of complete information are regular, then high turnout equilibria also exist in nearby games of incomplete information. Harsanyi (1973a,b) has shown that almost all games in normal form are regular, so that all of their equilibria are regular equilibria. Unfortunately, this result does not apply to turnout games. The anonymity built into these games, as well as the fact that the margin of victory does not affect players' payoffs, imply a much lower dimensional payoff space than that required in Harsanyi's theorem.

When heterogeneity in voting costs is allowed, generic regularity of participation games has been shown by De Sinopoli and Iannantuoni (2005), who consider general n-candidate games.[1] Yet, unlike games with equal voting costs, as in Palfrey and Rosenthal (1983), the heterogeneity of players' voting costs makes it hard to establish in any generality that these games admit high turnout equilibria with large electorates. Although examples are easy to construct, such equilibria may hinge on properties of the distribution of the heterogeneous individual voting costs. It may be the case that for a class of distributions of individual voting costs (in complete information games) the only equilibria with large electorates involve low turnout rates. On the other hand, the games with equal voting costs analyzed by Palfrey and Rosenthal admit high turnout equilibria but are not covered by the results of De Sinopoli and Iannantuoni (2005), or Kalandrakis (2006), so that these games may not be regular. In fact, we would expect these games not to be regular, given their extremely low-dimensional payoff space, which makes it likely that singularities are present in the equations characterizing their Nash equilibria. Thus, the state of our knowledge on these questions can be summarized in table 1.

[1] Kalandrakis (2006), focusing on two-candidate games, provides an alternative proof of regularity, characterizes equilibria as stationary points of the minmax program of an artificial two-player zero-sum game, and establishes the existence of an intuitive refined set of equilibria.

	Common Cost	Heterogeneous Costs
High Turnout	Yes	?
Regularity	?	Yes

Table 1: Complete Information Games: Regularity and High Turnout Equilibria.

In this paper we attempt to resolve both open questions we outlined in the above discussion by studying the regularity of equilibria in games with equal voting costs. Contrary to our own expectation, we establish that all the equilibria of these games are regular, outside a null set consisting of a finite number of possible common costs of voting. As a result, for arbitrarily large electorates there exist equilibria with high turnout even in games of incomplete information. These incomplete information games can satisfy the full support requirement on the distribution of costs imposed by Palfrey and Rosenthal (1985). The equilibria are also robust to the presence of uncertainty on the fraction of supporters for each candidate, as is assumed in the games of Ledyard (1984), and more recently that by Borgers (2004).

Furthermore, by standard continuity implications of equilibrium which follow from regularity, games with heterogeneous voting costs, or more elaborate payoff motivations, that are close to the assumed payoffs with a common voting cost, also posses equilibria with high turnout rates. These continuity properties strengthen the relevance of asymmetric equilibria in these games. In particular, asymmetric behavior can be attributed to differences in characteristics across groups or voters in the electorate that are assumed away by the analyst via the blanket assumption of common voting costs. Thus, one implication of our findings is that the actual heterogeneity that supports asymmetric equilibria can be restored (in the model) via appropriate payoff perturbations of the simpler game with common costs. Experimental evidence (*e.g.*, Levine and Palfrey (2006)) strongly suggests the presence of individual heterogeneity, even when participants in experiments receive identical treatments.

Because we show regularity of games with a single payoff parameter and a large number of players, our proof cannot rely on the typical line of argument that invokes some version of Sard's theorem. Rather, we exploit the simple structure of these games in order to transform the problem of establishing generic regularity into a more tractable mathematical question. In this process, we obtain a characterization of equilibrium that dramatically narrows down the extent of heterogeneity of behavior we may observe in any equilibrium of these games. Our characterization allows us to pose the question of generic regularity of equilibria as a question on the finite solvability of a system of polynomial equations. This polynomial system has a simple triangular structure, so that finite solvability (and regularity) obtains simply from the Fundamental Theorem of Algebra. The analysis is general in that we cover elections with arbitrary (biased) threshold rules, and consider both a coin-toss rule and a status-quo rule for the resolution of ties. As a result, the analysis also covers, as a special case, the public good participation games without refunds analyzed by Palfrey and Rosenthal (1984).

When it comes to the applicability of our results to Bayesian games, we emphasize that the fact that they imply feasibility of Nash equilibrium with high turnout rates in such Bayesian games is not inconsistent with the results of Palfrey and Rosenthal (1985). The apparent inconsistency stems from the fact that the answer to the question on the possibility of high turnout in incomplete information games with large electorates hinges on the "order of limits" with which we attempt to resolve the question under their assumptions. In particular, in Palfrey and Rosenthal (1985) the limit is taken as the electorate gets large holding the distribution of voting costs fixed. The present analysis, on the other hand, ensures that, if we hold an arbitrarily large electorate fixed, there exist distributions of voting costs that are sufficiently 'close' to the game of complete information, so that the regular equilibria of the complete information game survive in the corresponding incomplete information version with those distributions. We emphasize that it does not follow from the above discussion that the variance of the aggregate distribution of voting costs must be vanishingly small in large electorates in order to get Bayesian Nash equilibria with high turnout. At the end of section 6, we discuss how we can combine incomplete information and subgroup heterogeneity within the group of supporters of each of the candidates in order to represent realistically dispersed aggregate distributions of voting costs in the electorate, while at the same time obtaining high equilibrium turnout rates.

Work on this paper coincided with a period of renewed interest on theories of turnout with costly voting. Coate and Conlin (2005), and Feddersen and Sandroni (2006) have proposed a theory of turnout under the assumption that voters are rule utilitarians so that their calculus is not (solely) based on the probability that they are pivotal in elections. Coate and Conlin (2005) provide empirical evidence for this theory using data from Texas liquor referandums. As is the case for the present study, these studies draw on the thinking of Harsanyi, in their case on Harsanyi (1980). Roemer (2006) analyzes alternative but related motivations for voters in an application of his notion of Kantian equilibrium, but reaches different conclusions on the possibility of high turnout rates. Herrera and Martinelli (forthcoming) have developed a theory of participation that is based on the endogenous formation of subgroups in the population which follow the participation mandates of their leaders. Levine and Palfrey (2006) provide evidence in support of the more conventional strategic turnout model with incomplete information using experimental data. Taylor and Yildirim (2006) derive a sufficient condition for uniqueness of symmetric equilibria in an incomplete information model, and characterize the distribution of voter participation as the electorate goes to infinity. Unlike the model of Palfrey and Rosenthal (1985), Taylor and Yildirim (2006) allow for uncertainty on the size of support for the two candidates as do the models of Borgers (2004) and Ledyard (1984), who also derives a uniqueness condition for symmetric equilibria in the context of his model. Comprehensive review of earlier literature can be found in the above studies and in Aldrich (1993).

The analysis proceeds as follows. We introduce the model and some preliminary results in section 2. In section 3 we offer a characterization of equilibrium. In section 4 we develop a simple condition in order for an equilibrium to be regular. We establish generic regularity in section 5. In section 6 we discuss implications.

2 Model & Preliminaries

The set of players consists of two groups of voters who may participate in an election with two candidates. We index a group by $t = 1, 2$. Players that support candidate 1 are given by a set $N^1 = \{1, ..., n^1\}$, while the supporters of candidate 2 are given by $N^2 = \{n^1 + 1, ..., n^1 + n^2\}$. We assume $n^1 + n^2 \geq 2$. We normalize the benefit from victory of each player's favorite candidate to one, while players whose favorite candidate loses the election receive zero. Players can cast a vote for their favorite candidate at a cost $c_i^t \in \mathbf{R}_{++}$, $i \in N^t$, $t =$ 1,2, or abstain. Although our notation allows for heterogeneous costs, our goal is to study the version of this participation game with equal voting costs. The strategy of player i in group t, is represented by $\sigma_i^t \in [0, 1]$, where σ_i^t

is the probability this player votes. We let $\sigma \in [0,1]^{n^1+n^2}$ represent a vector consisting of all players' voting strategies. Candidate 1 wins the election if she receives at least $\kappa+1$, $\kappa \in \mathbf{Z}$, more votes than candidate 2, while candidate 2 wins if candidate 1 receives at most $\kappa-1$ more votes than candidate 2. In the case of a tie, when the number of votes for candidate 1 minus those received by candidate 2 is exactly equal to κ, candidate 1 wins with probability $\pi \in \{1, \frac{1}{2}\}$, and candidate 2 wins with probability $1-\pi$. The case $\pi = \frac{1}{2}$ represents the familiar coin-toss rule, while the case $\pi = 1$ corresponds to a status-quo rule such that ties are resolved in favor of candidate 1. We need not consider the rule $(\kappa, \pi = 0)$, as it is equivalent to the rule $(\kappa+1, \pi = 1)$. For clarity, we follow the convention of using superscripts to indicate a group, and reserve subscripts to indicate players. When referring to group $t = 1, 2$ in the abstract, we shall use the notation $-t$ to indicate the group competing with group t. We also use the standard notation σ_{-i} to indicate the strategies of all players except $i \in N^1 \cup N^2$. Most of the analysis concerns the case costs of voting are equal, so that $c_i^t = c \in \mathbf{R}_{++}$, for all players $i \in N^t$, $t = 1, 2$. We denote the game with common cost of voting, c, by $\Gamma(c, \pi, \kappa, n^1, n^2)$.

For voting strategies σ, denote the actual total votes cast by group 1 by m^1 those by group two by m^2. Also denote the (adjusted) difference in actual votes cast between the two groups by $m = m^1 - m^2 - \kappa$. The voting strategies, σ, induce a probability mass function over $m \in \mathbf{Z}$. We can calculate this probability mass function separately for each set of players $(N^1 \cup N^2)\backslash C$, *i.e.*, excluding players in set $C \subset N^1 \cup N^2$. Denote the associated probability mass function by $f_\sigma(m; C)$. Now, $f_\sigma(m; C)$ represents the probability that the adjusted difference between the actual votes cast by all players except those in $C \subset N^1 \cup N^2$ is m, given strategy choices σ. Of course, $f_\sigma(m; C)$ does not depend on the coordinates of σ that correspond to the strategies of players in the excluded set C. Also, we must have $f_\sigma(m; \emptyset) = 0$ for all $m \notin \{-n^2 - \kappa, ..., n^1 - \kappa\}$ and all σ.

Under our assumptions about the voting rule and the resolution of ties (*i.e.*, parameters κ, π), the probability of victory of group 1 given strategy choices, σ, is a function

$$F^1(\sigma) = \sum_{m=1}^{+\infty} f_\sigma(m; \emptyset) + \pi f_\sigma(0; \emptyset).$$

The corresponding probability of victory of group 2 is obtained naturally as a function

$$F^2(\sigma) = 1 - F^1(\sigma) = \sum_{m=-1}^{-\infty} f_\sigma(m; \emptyset) + (1-\pi) f_\sigma(0; \emptyset).$$

Note (see Kalandrakis, 2006) that by making use of the fact that

$$\begin{aligned} f_\sigma(m; C) &= \sigma_i^1 f_\sigma(m-1; C \cup \{i\}) + (1-\sigma_i^1) f_\sigma(m; C \cup \{i\}) \\ &= \sigma_j^2 f_\sigma(m+1; C \cup \{j\}) + (1-\sigma_j^2) f_\sigma(m; C \cup \{j\}), \end{aligned} \tag{1}$$

we can obtain the derivative of $F^1(\sigma)$ with respect to σ_i^1, which we denote by $F_i^1(\sigma)$. This derivative represents the "marginal" benefit from player i's participation which is a function of the probability of events such that i is *pivotal* and the probability with which ties are resolved:

$$\begin{aligned} F_i^1(\sigma) &= \sum_{m=1}^{+\infty} (f_\sigma(m-1; \{i\}) - f_\sigma(m; \{i\})) + \pi(f_\sigma(-1; \{i\}) - f_\sigma(0, \sigma, \{i\})) \\ &= \pi f_\sigma(-1; \{i\}) + (1-\pi) f_\sigma(0; \{i\}). \end{aligned} \tag{2}$$

The analogous derivative for player $i \in N^2$ is obtained as

$$F_i^2(\sigma) = (1-\pi) f_\sigma(1; \{i\}) + \pi f_\sigma(0; \{i\}). \tag{3}$$

Another application of (1) is that it can be used to derive an inductive proof of the fact that the distribution $f_\sigma(m;C)$ is strictly unimodal or strictly log-concave (see Keilson and Gerber, 1971). Strict log-concavity implies

$$f_\sigma(m;C)^2 \geq f_\sigma(m+1;C) f_\sigma(m-1;C), \text{ for all } m \in \mathbf{Z}, \tag{4}$$

with the inequality strict for every m such that $f_\sigma(m;C) > 0$. This strict log-concavity property plays a key role in the sequel. In the following section, we shall make use of the inequalities in (4) in order to show, in lemma 2, that certain pairs of cross-partial derivatives of probability of victory functions cannot both be equal to zero. In turn, that lemma will be used to establish the main results of sections 3 and 4.

We will now define three increasingly stronger notions of equilibrium for the turnout game we consider. First, we define a *Nash* equilibrium as follows:

Definition 1 *σ is a Nash equilibrium if*

$$\left.\begin{array}{ccc} \sigma_h^t = 1 & \textit{if} & F_h^t(\sigma) > c_h^t \\ \sigma_h^t \in [0,1] & \textit{if} & F_h^t(\sigma) = c_h^t \\ \sigma_h^t = 0 & \textit{if} & F_h^t(\sigma) < c_h^t \end{array}\right\}, \tag{5}$$

for all players $h \in N^t, t = 1{,}2$.

A Nash equilibrium is *quasi-strict* if all players using pure strategies play a strict best response.

Definition 2 *A Nash equilibrium, σ, is quasi-strict if*

$$\begin{array}{c} \sigma_h^t = 1 \Longrightarrow F_h^t(\sigma) > c_h^t, \textit{ and} \\ \sigma_h^t = 0 \Longrightarrow F_h^t(\sigma) < c_h^t, \end{array}$$

for all players $h \in N^t, t = 1{,}2$.

Lastly, the strongest refinement we consider and which shall be the focus of the analysis is that of a *regular* equilibrium. Definitions of regularity can be found in Harsanyi, 1973, or Van Damme, 1987, for general normal form games. In the two action game we consider these definitions boil down to the following:

Definition 3 *A Nash equilibrium, σ, is regular if it is quasi-strict and the system of equations*

$$F_h^t(\sigma) = c_h^t, \; h : \sigma_h^t \in (0,1),$$

has a non-singular Jacobian.

Note that the Jacobian that must not be singular is obtained as

$$J(\sigma) = \begin{bmatrix} 0 & F^1_{hi}(\sigma) & \cdots & F^1_{hj}(\sigma) & F^1_{hg}(\sigma) \\ F^1_{ih}(\sigma) & 0 & \cdots & F^1_{ij}(\sigma) & F^1_{ig}(\sigma) \\ \vdots & \vdots & & \vdots & \vdots \\ F^2_{jh}(\sigma) & F^2_{ji}(\sigma) & \cdots & 0 & F^2_{jg}(\sigma) \\ F^2_{gh}(\sigma) & F^2_{gi}(\sigma) & \cdots & F^2_{gj}(\sigma) & 0 \end{bmatrix}, \tag{6}$$

where players i, $h \in N^1$, and players j, $g \in N^2$. Our proof of regularity will hinge on our ability to demonstrate that this matrix is not singular. Thus, while Nash equilibrium amounts to a set of conditions involving first derivatives of the probability of winning function, $F^t(\sigma)$, $t = 1, 2$, regularity requires a condition involving second derivatives of this function. It is straightforward to obtain expressions for these second derivatives using (1), and we do so in lemma 9 in the appendix. Anticipating the arguments in the next two sections, in the next lemma we show that voting strategies and second derivatives are related in a direct manner with individual costs of voting. These relations restrict certain entries of the Jacobian, $J(\sigma)$, to zero under the assumption that voting costs are equal.

Lemma 1 *For every game with parameters π, κ, n^1, n^2, all voting strategies σ, and any players i, $h \in N^t$, $j \in N^{-t}$, $t = 1, 2$:*

(a) If the indifference conditions $F^t_i(\sigma) = c^t_i$, $F^t_h(\sigma) = c^t_h$ hold, then

$$(\sigma^t_h - \sigma^t_i)F^t_{ih}(\sigma) = c^t_i - c^t_h, \text{ and} \tag{7}$$

(b) if the indifference conditions $F^t_i(\sigma) = c^t_i$, $F^t_j(\sigma) = c^{-t}_j$ hold, then

$$(\sigma^{-t}_j + \sigma^t_i - 1)F^t_{ij}(\sigma) = c^t_i - c^{-t}_j. \tag{8}$$

Proof. It suffices to consider the case $t = 1$.[2] For part (a), we start by expanding (2) using (1) as in the proof of lemma 9 in the appendix and substituting from (21) to obtain

$$\begin{aligned} F^1_i(\sigma) &= \pi f_\sigma(-1; \{i\}) + (1-\pi) f_\sigma(0; \{i\}) \\ &= \sigma^1_h F^1_{ih}(\sigma) + \pi f_\sigma(-1; \{i, h\}) + (1-\pi) f_\sigma(0; \{i, h\}). \end{aligned}$$

Analogously expanding $F^1_h(\sigma)$ yields

$$F^1_h(\sigma) = \sigma^1_i F^1_{ih}(\sigma) + \pi f_\sigma(-1; \{i, h\}) + (1-\pi) f_\sigma(0; \{i, h\}).$$

We now substitute from above in the indifference equations $F^1_i(\sigma) = c^1_i$ and $F^1_h(\sigma) = c^1_h$, and take differences to obtain

$$(\sigma^1_h - \sigma^1_i)F^1_{ih}(\sigma) = c^1_i - c^1_h.$$

This completes the proof of part (a). We proceed similarly for part (b). With a different application

[2]Here and in analogous arguments in the sequel, if the conclusion holds for $t = 1$ for all games with parameters $\pi, \kappa = \kappa', n^1 = n^{1\prime}, n^2 = n^{2\prime}$, then it also holds for $t = 2$ and every game with $\kappa = -\kappa', n^1 = n^{2\prime}, n^2 = n^{1\prime}$, if $\pi = \frac{1}{2}$, or for $t = 2$ and every game with $\kappa = 1 - \kappa', n^1 = n^{2\prime}, n^2 = n^{1\prime}$, if $\pi = 1$.

of (1), we expand (2) and use (23) of lemma 9 in the appendix to get:

$$F_i^1(\sigma) = \sigma_j^2 F_{ij}^1(\sigma) + \pi f_\sigma(-1; \{i,j\}) + (1-\pi) f_\sigma(0; \{i,j\}).$$

Analogously expanding (3) and using (23) we write

$$F_j^2(\sigma) = -\sigma_i^1 F_{ij}^1 + \pi f_\sigma(0; \{i,j\}) + (1-\pi) f_\sigma(1; \{i,j\}).$$

We now start with the indifference equations $F_i^1(\sigma) = c_i^1$ and $F_j^2(\sigma) = c_j^2$, substitute from the above and take differences to obtain

$$(\sigma_j^2 + \sigma_i^1) F_{ij}^1(\sigma) - ((1-\pi) f_\sigma(1; \{i,j\}) - (1-2\pi) f_\sigma(0; \{i,j\}) - \pi f_\sigma(-1; \{i,j\})) = c_i^1 - c_j^2.$$

Substituting one last time from (23) we complete the proof with the desired conclusion

$$(\sigma_j^2 + \sigma_i^1 - 1) F_{ij}^1(\sigma) = c_i^1 - c_j^2.$$

■

With these preliminaries established, we now move to the analysis of games with equal voting costs.

3 Equilibrium Characterization

Throughout the following three sections, we assume equal voting costs $c_i^t = c$, for all players i and both groups $t = 1, 2$. We shall show that with equal voting costs, Nash equilibrium imposes considerable structure on players' participation behavior in game $\Gamma(c, \pi, \kappa, n^1, n^2)$. In order to make this more precise, for every strategy σ, we define the set $\widehat{N}(\sigma) \subseteq N^1 \cup N^2$ that consists of the players that are indifferent between voting and abstaining, *i.e.*,

$$\widehat{N}(\sigma) = \{h \in N^t : F_h^t(\sigma) = c, t = 1, 2\}.$$

For every probability $p \in [0,1]$, we also define a subset of $\widehat{N}(\sigma)$, which we denote by $\widehat{N}_p(\sigma) \subseteq \widehat{N}(\sigma)$, and is given by

$$\widehat{N}_p(\sigma) = \{h \in \widehat{N}(\sigma) : \sigma_h^1 = p \text{ or } \sigma_h^2 = 1-p\}.$$

Thus, $\widehat{N}_p(\sigma)$ consists of those among the players that are indifferent, given strategies σ, *and* participate with probability p, if they support candidate 1, or with probability $1-p$, if they support candidate 2.

Before we establish the main result of this section, we show a lemma concerning second derivatives of probabilities of victory. The lemma is a direct consequence of the strict log-concavity property of the probability mass function, $f_\sigma(m; C)$, over the margin of victory induced by the players' strategies, and is obtained by application of (1) and (4).

Lemma 2 *For game $\Gamma(c, \pi, \kappa, n^1, n^2)$, consider strategies σ, and probabilities $p, q \in [0,1]$ with $p \neq q$. If players $i \in \widehat{N}(\sigma)$, $h \in \widehat{N}_p(\sigma)$, $j \in \widehat{N}_q(\sigma)$, then the system*

$$\left.\begin{array}{l} F_{ih}^t(\sigma) = 0 \\ F_{ij}^t(\sigma) = 0 \end{array}\right\}, \ t = 1,\ 2, \tag{9}$$

is inconsistent.

Proof. It suffices to consider the case players i, $h \in N^1$. We write (9) as

$$\left.\begin{array}{c} F^1_{ih}(\sigma) = -F^2_{ih}(\sigma) = 0 \\ F^1_{ij}(\sigma) = -F^2_{ij}(\sigma^*) = 0 \end{array}\right\}, \tag{10}$$

where we make use of the fact that $F^1(\sigma) = 1 - F^2(\sigma)$. We now have two possibilities: $j \in N^1$ or $j \in N^2$. If $j \in N^1$, then after substituting from (21) of lemma 9 in the appendix, (10) becomes

$$\left.\begin{array}{c} \pi f_\sigma(-2;\{i,h\}) + (1-2\pi) f_\sigma(-1;\{i,h\}) - (1-\pi) f_\sigma(0;\{i,h\}) = 0 \\ \pi f_\sigma(-2;\{i,j\}) + (1-2\pi) f_\sigma(-1;\{i,j\}) - (1-\pi) f_\sigma(0;\{i,j\}) = 0 \end{array}\right\}.$$

Using $\sigma^{1*}_j = q$, $\sigma^{1*}_h = p$, and applying (1) we expand the left hand side of the two equalities to get

$$\left.\begin{array}{c} \pi f_\sigma(-2;C) + (1-2\pi) f_\sigma(-1;C) - (1-\pi) f_\sigma(0;C) \\ +q(\pi f_\sigma(-3;C) + (1-3\pi) f_\sigma(-2;C) - (2-3\pi) f_\sigma(-1;C) + (1-\pi) f_\sigma(0;C)) = 0 \\ \pi f_\sigma(-2;C) + (1-2\pi) f_\sigma(-1;C) - (1-\pi) f_\sigma(0;C) \\ +p(\pi f_\sigma(-3;C) + (1-3\pi) f_\sigma(-2;C) - (2-3\pi) f_\sigma(-1;C) + (1-\pi) f_\sigma(0;C)) = 0 \end{array}\right\},$$

where $C = \{i, h, j\}$. Subtracting the second from the first left hand side, we equivalently get

$$\left.\begin{array}{c} (q-p)(\pi f_\sigma(-3;C) + (1-3\pi) f_\sigma(-2;C) - (2-3\pi) f_\sigma(-1;C) + (1-\pi) f_\sigma(0;C)) = 0 \\ \pi f_\sigma(-2;C) + (1-2\pi) f_\sigma(-1;C) - (1-\pi) f_\sigma(0;C) \\ +p(\pi f_\sigma(-3;C) + (1-3\pi) f_\sigma(-2;C) - (2-3\pi) f_\sigma(-1;C) + (1-\pi) f_\sigma(0;C)) = 0 \end{array}\right\}.$$

Since $(q-p) \neq 0$, we deduce

$$\left.\begin{array}{c} \pi f_\sigma(-3;C) + (1-3\pi) f_\sigma(-2;C) - (2-3\pi) f_\sigma(-1;C) + (1-\pi) f_\sigma(0;C) = 0 \\ \pi f_\sigma(-2;C) + (1-2\pi) f_\sigma(-1;C) - (1-\pi) f_\sigma(0;C) = 0 \end{array}\right\}.$$

Adding the left hand side of the second equation to the first we get

$$\left.\begin{array}{c} \pi f_\sigma(-3;C) + (1-2\pi) f_\sigma(-2;C) - (1-\pi) f_\sigma(-1;C) = 0 \\ \pi f_\sigma(-2;C) + (1-2\pi) f_\sigma(-1;C) - (1-\pi) f_\sigma(0;C) = 0 \end{array}\right\}. \tag{11}$$

We now consider two cases. Case 1, $\pi = 1$: from (2) we have $F^1_i(\sigma) = f_\sigma(-1;\{i\}) = c > 0$ so that we must have $\sum_{m=-3}^{-1} f_\sigma(m;C) > 0$. Hence, from (11) we obtain

$$f_\sigma(-3;C) = f_\sigma(-2;C) = f_\sigma(-1;C) > 0.$$

But, strict log-concavity of $f_\sigma(m;C)$ requires

$$f_\sigma(-2;C)^2 > f_\sigma(-3;C) f_\sigma(-1;C) = f_\sigma(-2;C)^2,$$

a contradiction. Case 2, $\pi = \frac{1}{2}$: in this case, the indifference condition $F^1_i(\sigma) = c > 0$ requires $\sum_{m=-3}^{0} f_\sigma(m;C) > 0$. Furthermore, (11) yields $f_\sigma(-3;C) = f_\sigma(-1;C)$, and $f_\sigma(-2;C) = f_\sigma(0;C)$. From (4) we now have

$$\left.\begin{array}{c} f_\sigma(-1;C)^2 \geq f_\sigma(0;C) f_\sigma(-2;C) = f_\sigma(-2;C)^2 \\ f_\sigma(-2;C)^2 \geq f_\sigma(-3;C) f_\sigma(-1;C) = f_\sigma(-1;C)^2 \end{array}\right\},$$

so that we have concluded

$$f_\sigma(-3;C) = f_\sigma(-2;C) = f_\sigma(-1;C) = f_\sigma(0;C) > 0.$$

But, once more, strict log-concavity of $f_\sigma(m;C)$ requires

$$f_\sigma(-2;C)^2 > f_\sigma(-3;C)f_\sigma(-1;C) = f_\sigma(-2;C)^2,$$

again a contradiction. Thus, system (9) is inconsistent.

In the case $j \in N^2$, the identical expansion of $F^1_{ih}(\sigma^*)$, $F^1_{ij}(\sigma)$ from (21) and (23) of lemma 9 using $\sigma^{2*}_j = 1 - q$ and (1) yields that (9) is equivalent to

$$\left.\begin{array}{c} \pi f_\sigma(-1;C) + (1-2\pi)f_\sigma(0;C) - (1-\pi)f_\sigma(1;C) \\ +q(\pi f_\sigma(-2;C) + (1-3\pi)f_\sigma(-1;C) - (2-3\pi)f_\sigma(0;C) + (1-\pi)f_\sigma(1;C)) = 0 \\ -\pi f_\sigma(-1;C) - (1-2\pi)f_\sigma(0;C) + (1-\pi)f_\sigma(1;C) \\ -p(\pi f_\sigma(-2;C) + (1-3\pi)f_\sigma(-1;C) - (2-3\pi)f_\sigma(0;C) + (1-\pi)f_\sigma(1;C)) = 0 \end{array}\right\},$$

Since $(q-p) \neq 0$, we can execute analogous steps to get

$$\left.\begin{array}{c} \pi f_\sigma(-2;C) + (1-2\pi)f_\sigma(-1;C) - (1-\pi)f_\sigma(0;C) = 0 \\ \pi f_\sigma(-1;C) + (1-2\pi)f_\sigma(0;C) - (1-\pi)f_\sigma(1;C) = 0 \end{array}\right\}. \tag{12}$$

Now, the fact that $F^1_i(\sigma) = c > 0$, analogously implies a contradiction of strict log-concavity of $f_\sigma(m;C)$. ■

Note that since we are considering a finite game, for any Nash equilibrium, σ^*, the set of indifferent players $\widehat{N}_p(\sigma^*)$ must be empty for almost all $p \in [0,1]$: the number of distinct mixing probabilities cannot exceed the number of players in the game. Remarkably, lemmas 1 and 2 jointly imply that in every Nash equilibrium the behavior of indifferent players is characterized by at most *two* distinct probabilities.

Theorem 1 *If σ^* is a Nash equilibrium of game $\Gamma(c,\pi,\kappa,n^1,n^2)$, then there exist probabilities $p,q \in [0,1]$ with $p \neq q$ such that the set of indifferent players, $\widehat{N}(\sigma^*)$, is partitioned in two sets $\widehat{N}_p(\sigma^*)$ and $\widehat{N}_q(\sigma^*)$, i.e.,*

$$\widehat{N}(\sigma^*) = \widehat{N}_p(\sigma^*) \cup \widehat{N}_q(\sigma^*).$$

Proof. We shall derive a proof by contradiction. Assume, contrary to the theorem, that there exist probabilities $p,q,r \in [0,1]$ with $p \neq q$, $p \neq r$, and $q \neq r$, such that all three sets $\widehat{N}_p(\sigma^*)$, $\widehat{N}_q(\sigma^*)$, and $\widehat{N}_r(\sigma^*)$ are non-empty. Consider players $i \in \widehat{N}_r(\sigma^*)$, $j \in \widehat{N}_q(\sigma^*)$, and $h \in \widehat{N}_p(\sigma^*)$. Let $i \in N^t$. Then, since $c^t_g = c$, for all $g \in N^t$ and all $t = 1,2$, parts (a) and (b) of lemma 1 imply

$$\left.\begin{array}{c} F^t_{ih}(\sigma^*) = 0 \\ F^t_{ij}(\sigma^*) = 0 \end{array}\right\}.$$

But this is impossible by lemma 2. ■

Of course, the theorem does no assert that there exist two distinct sets $\widehat{N}_p(\sigma^*)$ and $\widehat{N}_q(\sigma^*)$ that are non-empty for every equilibrium σ^*. Two is the maximum number of probabilities necessary to describe the behavior of indifferent players, but there certainly exist instances of game

$\Gamma(c,\pi,\kappa,n^1,n^2)$ with equilibria in which just one probability suffices. We will now provide three examples that illustrate the possibilities for Nash equilibrium permitted by theorem 1. In the first equilibrium all players' mixed strategies are characterized by a single probability.

Example 1 *Assume $n^1 = 3$ and $n^2 = 2$, $\kappa = 0$, $\pi = \frac{1}{2}$, and $c = \frac{4}{9}$. There exists a totally mixed Nash equilibrium, σ^*, with $\widehat{N}(\sigma^*) = \widehat{N}_p(\sigma^*) = N^1 \cup N^2$ and $p = \frac{1}{3}$. Indeed, in this case we calculate*

$$F_i^1(\sigma^*) = F_j^2(\sigma^*) = \tfrac{1}{2}\tbinom{4}{2}p^2(1-p)^2 + \tfrac{1}{2}\tbinom{4}{2}(1-p)^3 p,$$

which is equal to $c = \frac{4}{9}$ when $p = \frac{1}{3}$.

In the second example, we present a totally mixed equilibrium characterized by two distinct probabilities.

Example 2 *Assume $n^1 = 3$ and $n^2 = 2$, $\kappa = 0$, $\pi = \frac{1}{2}$, and $c = \frac{81}{224}$. There exists a totally mixed Nash equilibrium, σ^*, with $\widehat{N}_q(\sigma^*) = \{i,h\} \subset N^1$, and $\widehat{N}_p(\sigma^*)$ consisting of the remaining three players from the two groups. In this case $q = \frac{4}{7}$ and $p = \frac{1}{4}$. Indeed, after some simplifications we have*

$$F_i^1(\sigma^*) = F_h^1(\sigma^*) = \tfrac{1}{2}((1-3p+2p^3)q + 3p(1-p)),$$

and

$$F_j^t(\sigma^*) = \tfrac{1}{2}((2p^2-1)q^2 + 2(1-2p)q + (2-p)p),\ j \neq i,h,\ j \in N^t,\ t = 1,2,$$

and we can verify that $q = \frac{4}{7}$ and $p = \frac{1}{4}$ solve $F_g^t(\sigma^*) = c$ for all $g \in N^t$.

Note that the equilibrium in the above example exhibits heterogeneity of behavior within group 1. Both of the above examples involved totally mixed equilibria. The third example we present involves a group of players using pure strategies. This example is taken from an exercise in Osborne, 2003, page 108.

Example 3 *Assume $n^1 > n^2 > 1$, $\kappa = 0$, $\pi = \frac{1}{2}$, and $c \in (0,\frac{1}{2})$. Consider an equilibrium, σ^*, in which n^2 members of group 1 vote with probability one, and $n^1 - n^2$ with probability zero, and all members of group 2 vote with probability $1-p$, i.e., $\widehat{N}(\sigma^*) = \widehat{N}_p(\sigma^*) = N^2$. For $i \in N^2$ we have*

$$F_i^1(\sigma) = \tfrac{1}{2}(1-p)^{n^2-1} = c,$$

which yields $p = 1 - \sqrt[n^2-1]{2c}$. We also verify substituting from the above equation that for players $i \in N^1$ with $\sigma_i^1 = 1$ we have

$$F_i^1(\sigma) = \tfrac{1}{2}\tbinom{n^2}{n^2-1}(1-p)^{n^2-1}p + \tfrac{1}{2}(1-p)^{n^2} = (\tbinom{n^2}{n^2-1}p + (1-p))c > c.$$

Finally, for players $i \in N^1$ with $\sigma_i^1 = 0$ we have

$$F_i^1(\sigma) = \tfrac{1}{2}(1-p)^{n^2} = (1-p)c < c.$$

Observe that the turnout rate in the above example approaches $\frac{2n^2}{n^1+n^2}$ as n^2 gets large. Furthermore, the equilibrium in the example is quasi-strict, since the players that use pure strategies from group 1 have a strict preference for these strategy choices. If, in addition to being quasi-strict, this equilibrium is regular, then the high equilibrium turnout rate also survives in nearby games of incomplete information or games with payoff heterogeneity. In the next section we derive a straightforward test of regularity for such a quasi-strict equilibrium.

4 A Condition for Regularity

Our task in this section is to derive a simple condition in order for a quasi-strict Nash equilibrium of game $\Gamma(c, \pi, \kappa, n^1, n^2)$ to be regular. As we discussed in section 2, verifying regularity of such equilibria amounts to ascertaining the non-singularity of the Jacobian of the indifferent players' equilibrium conditions, which is specified in (6). We start by showing that a group of entries in this Jacobian are identical in absolute value.

Lemma 3 *Consider game $\Gamma(c, \pi, \kappa, n^1, n^2)$, and strategies, σ, such that $\sigma_i^t = \sigma_h^t = p$, and $\sigma_j^{-t} = 1 - p$, for $t = 1, 2$, $p \in [0, 1]$, and distinct players i, h, j. Then*

$$F_{ih}^t(\sigma) = -F_{ij}^t(\sigma) = F_{ij}^{-t}(\sigma). \tag{13}$$

Proof. Assume (without loss of generality) that players $i, h \in N^1$, that player $j \in N^2$, and set $C = \{i, h, j\}$. Applying (1) to (21) and (23) from lemma 9 we rewrite

$$\begin{aligned} F_{ih}^1(\sigma) &= \pi f_\sigma(-2; \{i, h\}) + (1 - 2\pi) f_\sigma(-1; \{i, h\}) - (1 - \pi) f_\sigma(0; \{i, h\}) \\ &= \pi f_\sigma(-1; C) + (1 - 2\pi) f_\sigma(0; C) - (1 - \pi) f_\sigma(1; C) \\ &\quad + p(\pi f_\sigma(-3; C) + (1 - 3\pi) f_\sigma(-2; C) - (2 - 3\pi) f_\sigma(-1; C) + (1 - \pi) f_\sigma(0; C)) \end{aligned}$$

and

$$\begin{aligned} F_{ij}^1(\sigma) &= -\pi f_\sigma(-1; \{i, j\}) - (1 - 2\pi) f_\sigma(0; \{i, j\}) + (1 - \pi) f_\sigma(-1; \{i, j\}) \\ &= -\pi f_\sigma(-1; C) - (1 - 2\pi) f_\sigma(0; C) + (1 - \pi) f_\sigma(1; C) \\ &\quad - p(\pi f_\sigma(-3; C) + (1 - 3\pi) f_\sigma(-2; C) - (2 - 3\pi) f_\sigma(-1; C) + (1 - \pi) f_\sigma(0; C)). \end{aligned}$$

Now the above along with the fact that $F^1(\sigma) = 1 - F^2(\sigma)$ imply $F_{ih}^1(\sigma) = -F_{ij}^1(\sigma) = F_{ij}^2(\sigma)$, as desired. ■

Now, our characterization in theorem 1, along with lemmas 1 and 3 imply that the Jacobian $J(\sigma^*)$ assumes a very simple, block diagonal structure at a Nash equilibrium σ^*. It is thus possible to directly calculate the determinant of this Jacobian so that we can obtain a straightforward condition for a quasi-strict equilibrium to be regular. We state this condition, which will be put to extensive use in the next section, in the following lemma.

Lemma 4 *Consider a quasi-strict Nash equilibrium, σ^*, of game $\Gamma(c, \pi, \kappa, n^1, n^2)$. If σ^* is not a regular equilibrium then either $|\widehat{N}_p(\sigma^*)| = 1$ for some $p \in (0, 1)$, or $F_{ij}^t(\sigma^*) = 0$ for some players $i, j \in \widehat{N}_p(\sigma^*)$, $i \neq j$, $p \in (0, 1)$ (or both).*

Proof. Since σ^* is quasi-strict but not regular, the Jacobian, $J(\sigma^*)$, of the indifference conditions

$$F_i^t(\sigma^*) - c = 0,\ i \in \widehat{N}(\sigma^*),$$

is singular. By theorem 1 and the fact that σ^* is quasi-strict, we have $\widehat{N}(\sigma^*) = \widehat{N}_p(\sigma^*) \cup \widehat{N}_q(\sigma^*)$ for some $p, q \in (0,1)$, $p \neq q$. Let $\widehat{n}_p = |\widehat{N}_p(\sigma^*)|$ and $\widehat{n}_q = |\widehat{N}_q(\sigma^*)|$. By appropriately arranging rows and columns, lemma 1 allows us to write $J(\sigma^*)$ in the form

$$J(\sigma^*) = \begin{bmatrix} A_p(\sigma^*) & \mathbf{0} \\ \mathbf{0} & A_q(\sigma^*) \end{bmatrix},$$

where $A_p(\sigma^*)$ is a $\widehat{n}_p$ by $\widehat{n}_p$ matrix corresponding to the cross-partial derivatives $F^t_{ih}(\sigma^*)$ of players $i, h \in \widehat{N}_p(\sigma^*)$, while $A_q(\sigma^*)$ is a $\widehat{n}_q$ by $\widehat{n}_q$ matrix corresponding to the cross-partial derivatives $F^t_{jg}(\sigma^*)$ of players $j, g \in \widehat{N}_q(\sigma^*)$. Note that all cross-partials of the form $F^t_{ij}(\sigma^*)$, for players $i \in \widehat{N}_p(\sigma^*)$, $j \in \widehat{N}_q(\sigma^*)$ equal zero from lemma 1. Furthermore, if we set $\alpha = F^t_{ih}(\sigma^*)$, for some $i, h \in \widehat{N}_p(\sigma^*)$, $i \neq h$, $i \in N^t$, then lemma 3 implies that the determinant of $A_p(\sigma^*)$ is obtained as

$$Det[A_p(\sigma^*)] = (-1)^k Det \begin{bmatrix} 0 & \alpha & \cdots & \alpha \\ \alpha & 0 & \cdots & \alpha \\ \vdots & \vdots & & \vdots \\ \alpha & \alpha & \cdots & 0 \end{bmatrix},$$

where $k = |\widehat{N}_p(\sigma^*) \cap N^{-t}|$ if $h \in N^t$, or $k = |\widehat{N}_p(\sigma^*) \cap N^t|$ otherwise. In particular, k is the number of columns of $A_p(\sigma^*)$ with non-diagonal entry equal to $-\alpha$.

Now using induction we calculate

$$Det \begin{bmatrix} 0 & \alpha & \cdots & \alpha \\ \alpha & 0 & \cdots & \alpha \\ \vdots & \vdots & & \vdots \\ \alpha & \alpha & \cdots & 0 \end{bmatrix} = (-1)^{\widehat{n}_p - 1}(\widehat{n}_p - 1)\alpha^{\widehat{n}_p}.$$

As a consequence, we have that

$$Det[J(\sigma^*)] = \pm(\widehat{n}_p - 1)(\widehat{n}_q - 1)\alpha^{\widehat{n}_p}\beta^{\widehat{n}_q}, \tag{14}$$

where $\beta = F^t_{jg}(\sigma^*)$, for some distinct $j, g \in \widehat{N}_q(\sigma^*)$. Since the quasi-strict equilibrium σ^* is regular if and only if $Det[J(\sigma^*)] \neq 0$, the lemma follows from (14). ■

Lemma 4 identifies two classes of equilibria that are not regular. In one of the two classes, a non-regular equilibrium, say σ^*, is such that there exists a unique player, i, that both, is indifferent *and* uses a particular voting strategy $\sigma_i^{1*} = p$ or $\sigma_i^{2*} = 1 - p$, so that $|\widehat{N}_p(\sigma^*)| = 1$. In the next lemma we shall show that if such an equilibrium exists for some common cost of voting, c, then there exists a continuum of equilibria which are not regular.

Lemma 5 (Odd One Out) *Consider a Nash equilibrium, σ^*, of game $\Gamma(c, \pi, \kappa, n^1, n^2)$. If $\widehat{N}_p(\sigma^*) = \{i\}$ for some $p \in [0,1]$, then $(\sigma^*_{-i}, \sigma^t_i)$ is also a Nash equilibrium that is not regular for all $\sigma^t_i \in [0,1]$.*

Proof. Without loss of generality, assume $i \in N^1$ that satisfies the premises of the lemma. We shall show that $(\sigma^*_{-i}, \sigma^1_i)$ is a Nash equilibrium. First, we note that by assumption any strategy $\sigma^1_i \in [0,1]$ is a best response for i given σ^*_{-i}. We shall now verify that all remaining players also play best responses at $(\sigma^*_{-i}, \sigma^1_i)$ for all $\sigma^1_i \in [0,1]$. For $h \neq i$, we distinguish five cases:

1. $h \in N^t$, and $F_h^t(\sigma^*) = c$, $t = 1, 2$: Since $h \in \widehat{N}_q(\sigma^*)$, $q \neq p$, we have $F_{hi}^t(\sigma^*) = 0$ from lemma 1. If $h \in N^1$, then expanding $F_h^1(\sigma^*)$ using (2) and (1) we obtain

$$F_h^1(\sigma^*) = \pi f_{\sigma^*}(-1; \{i,h\}) + (1-\pi) f_{\sigma^*}(0; \{i,h\}) + \sigma_i^{1*} F_{hi}^1(\sigma^*).$$

Since $f_\sigma(m; C)$ does not depend on coordinates σ_g^t, $g \in C$, we have $f_{\sigma^*}(m; C) = f_{(\sigma_{-g}^*, \sigma_g^t)}(m; C)$ for all m, all $\sigma_g^t \in [0,1]$, and all $g \in C$. Thus, we have $F_{hi}^t(\sigma^*) = F_{hi}^t(\sigma_{-i}^*, \sigma_i^1) = 0$. We then obtain

$$F_h^1(\sigma_{-i}^*, \sigma_i^1) = \pi f_{(\sigma_{-i}^*, \sigma_i^1)}(-1; \{i,h\}) + (1-\pi) f_{(\sigma_{-i}^*, \sigma_i^1)}(0; \{i,h\}) = F_h^1(\sigma^*) = c,$$

as we wished to show. If $h \in N^2$, we analogously obtain

$$F_h^2(\sigma_{-i}^*, \sigma_i^1) = \pi f_{(\sigma_{-i}^*, \sigma_i^1)}(0; \{i,h\}) + (1-\pi) f_{(\sigma_{-i}^*, \sigma_i^1)}(1; \{i,h\}) = F_h^2(\sigma^*) = c.$$

2. $h \in N^1$, and $F_h^1(\sigma^*) > c$: Then we have $\sigma_i^{1*} < \sigma_h^{1*} = 1$. From $F_i^1(\sigma^*) = c$ we obtain, after substitution from (2), using (1), and $\sigma_h^{1*} = 1$, that

$$F_i^1(\sigma^*) = \pi f_{\sigma^*}(-2; \{i,h\}) + (1-\pi) f_{\sigma^*}(-1; \{i,h\}) = c. \tag{15}$$

We wish to show that $F_h^1(\sigma_{-i}^*, \sigma_i^1) \geq c$, which using (2) and (1) is equivalent to

$$\begin{aligned}(1-\sigma_i^1)(\pi f_{(\sigma_{-i}^*, \sigma_i^1)}(-1; \{i,h\}) + (1-\pi) f_{(\sigma_{-i}^*, \sigma_i^1)}(0; \{i,h\}))\\ +\sigma_i^1(\pi f_{(\sigma_{-i}^*, \sigma_i^1)}(-2; \{i,h\}) + (1-\pi) f_{(\sigma_{-i}^*, \sigma_i^1)}(-1; \{i,h\})) \geq c,\end{aligned}$$

which using (15) is also equivalent to

$$(1-\sigma_i^1)(\pi f_{(\sigma_{-i}^*, \sigma_i^1)}(-1; \{i,h\}) + (1-\pi) f_{(\sigma_{-i}^*, \sigma_i^1)}(0; \{i,h\})) \geq (1-\sigma_i^1)c.$$

But the strict version of the last inequality is true for $\sigma_i^1 = \sigma_i^{1*} < 1$, so $F_h^1(\sigma_{-i}^*, \sigma_i^1) \geq c$ holds for all $\sigma_i^1 \in [0,1]$.

3. $h \in N^1$, and $F_h^1(\sigma^*) < c$: Then we have $\sigma_i^{1*} > \sigma_h^{1*} = 0$. From $F_i^1(\sigma^*) = c$ we obtain, after substitution from (2), and using (1) and $\sigma_h^{1*} = 0$,

$$F_i^1(\sigma^*) = \pi f_{\sigma^*}(-1; \{i,h\}) + (1-\pi) f_{\sigma^*}(0; \{i,h\}) = c. \tag{16}$$

We now start with $F_h^1(\sigma_{-i}^*, \sigma_i^1) \leq c$ which, using (1) and (2), is equivalent to

$$\begin{aligned}(1-\sigma_i^1)(\pi f_{(\sigma_{-i}^*, \sigma_i^1)}(-1; \{i,h\}) + (1-\pi) f_{(\sigma_{-i}^*, \sigma_i^1)}(0; \{i,h\}))\\ +\sigma_i^1(\pi f_{(\sigma_{-i}^*, \sigma_i^1)}(-2; \{i,h\}) + (1-\pi) f_{(\sigma_{-i}^*, \sigma_i^1)}(-1; \{i,h\})) \leq c,\end{aligned}$$

which using (16) becomes

$$\sigma_i^1(\pi f_{(\sigma_{-i}^*, \sigma_i^1)}(-2; \{i,h\}) + (1-\pi) f_{(\sigma_{-i}^*, \sigma_i^1)}(-1; \{i,h\})) \leq \sigma_i^1 c,$$

which is strictly true for $\sigma_i^1 = \sigma_i^{1*} > 0$, hence $F_h^1(\sigma_{-i}^*, \sigma_i^1) \leq c$ is true with equality for $\sigma_i^1 = 0$ and with strict inequality for all $\sigma_i^1 > 0$.

4. $h \in N^2$, and $F_h^2(\sigma^*) > c$: Then we have $\sigma_h^{2*} = 1$ and $\sigma_i^{1*} > 0$. From $F_i^1(\sigma^*) = c$ we obtain

after substitution from (2) and using (1) and $\sigma_h^{2*} = 1$

$$F_i^1(\sigma^*) = \pi f_{\sigma^*}(0; \{i,h\}) + (1-\pi) f_{\sigma^*}(1; \{i,h\}) = c. \tag{17}$$

Now $F_h^2(\sigma_{-i}^*, \sigma_i^1) \geq c$ is, by (1) and (3), equivalent to

$$\begin{aligned}&(1-\sigma_i^1)((1-\pi) f_{(\sigma_{-i}^*,\sigma_i^1)}(1; \{i,h\}) + \pi f_{(\sigma_{-i}^*,\sigma_i^1)}(0; \{i,h\})) \\ &+\sigma_i^1((1-\pi) f_{(\sigma_{-i}^*,\sigma_i^1)}(0; \{i,h\}) + \pi f_{(\sigma_{-i}^*,\sigma_i^1)}(-1; \{i,h\})) \geq c,\end{aligned}$$

which using (17) becomes

$$\sigma_i^1((1-\pi) f_{(\sigma_{-i}^*,\sigma_i^1)}(0; \{i,h\}) + \pi f_{(\sigma_{-i}^*,\sigma_i^1)}(-1; \{i,h\})) \geq \sigma_i^1 c.$$

The last inequality is strictly true for $\sigma_i^1 = \sigma_i^{1*} > 0$, hence $F_h^2(\sigma_{-i}^*, \sigma_i^1) \geq c$ for all $\sigma_i^1 \in [0,1]$.

5. Lastly, $h \in N^2$, and $F_h^2(\sigma^*) < c$: Then we have $\sigma_h^{2*} = 0$ and $\sigma_i^{1*} < 1$. From $F_i^1(\sigma^*) = c$ we obtain after substitution from (2), using (1) and $\sigma_h^{2*} = 0$

$$F_i^1(\sigma^*) = \pi f_{\sigma^*}(-1; \{i,h\}) + (1-\pi) f_{\sigma^*}(0; \{i,h\}) = c. \tag{18}$$

Using (1) and (3), $F_h^2(\sigma_{-i}^*, \sigma_i^1) \leq c$ is equivalent to

$$\begin{aligned}&(1-\sigma_i^1)((1-\pi) f_{(\sigma_{-i}^*,\sigma_i^1)}(1; \{i,h\}) + \pi f_{(\sigma_{-i}^*,\sigma_i^1)}(0; \{i,h\})) \\ &+\sigma_i^1((1-\pi) f_{(\sigma_{-i}^*,\sigma_i^1)}(0; \{i,h\}) + \pi f_{(\sigma_{-i}^*,\sigma_i^1)}(-1; \{i,h\})) \geq c,\end{aligned}$$

which using (18) is also equivalent to

$$(1-\sigma_i^1)((1-\pi) f_{(\sigma_{-i}^*,\sigma_i^1)}(1; \{i,h\}) + \pi f_{(\sigma_{-i}^*,\sigma_i^1)}(0; \{i,h\})) < (1-\sigma_i^1)c.$$

Since the above is true for $\sigma_i^1 = \sigma_i^{1*} < 1$, $F_h^2(\sigma_{-i}^*, \sigma_i^1) \leq c$ is true for all $\sigma_i^1 \in [0,1]$.

Thus, we have shown that all players $h \neq i$ play best responses at $(\sigma_{-i}^*, \sigma_i^1)$, for all $\sigma_i^1 \in [0,1]$, so that $(\sigma_{-i}^*, \sigma_i^1)$ is indeed a Nash equilibrium. That this equilibrium is not regular follows by the fact that the equilibrium is not quasi strict for $\sigma_i^1 \in \{0,1\}$. Non-regularity of equilibrium $(\sigma_{-i}^*, \sigma_i^1)$ with $\sigma_i^1 \in (0,1)$ follows by lemma 4 for the case $\widehat{N}_{\sigma_i^1}(\sigma_{-i}^*, \sigma_i^1) = \{i\}$, and by the continuity of the determinant of the Jacobian $J(\sigma_{-i}^*, \sigma_i^1)$, otherwise. Hence, the proof is complete. ■

The main use of lemma 5 in the sequel is that it allows us to focus our genericity analysis on equilibria, σ^*, such that $|\widehat{N}_p(\sigma^*)| \neq 1$ for all $p \in [0,1]$. If we can rule out irregular equilibria with $|\widehat{N}_p(\sigma^*)| \neq 1$ for certain common voting costs, c, then there cannot exist equilibria with $|\widehat{N}_p(\sigma^*)| = 1$ for these voting costs. To summarize, in this section we sought a simple necessary and sufficient condition in order for a quasi-strict Nash equilibrium to be regular. Our investigation has yielded simplicity indeed. In effect, barring the pathological situation dealt with in lemma 5, we have reduced the onerous task of ascertaining the non-singularity of the Jacobian in (6) to a test involving at most two entries of that matrix. We now proceed to show that for almost all common voting costs, $c \in \mathbf{R}_{++}$, all equilibria of game $\Gamma(c, \pi, \kappa, n^1, n^2)$ are regular.

5 Regular Games

A game is *regular*, if all of its Nash equilibria are regular equilibria. In this section, we use results from sections 3 and 4 in order to show that almost all games with common voting costs are regular. In particular, the main result of this section is:

Theorem 2 *Outside a null set $\widetilde{C} = \{c_1, ..., c_{\widetilde{s}}\}$ consisting of a finite number of common voting costs $c \in \mathbf{R}_{++}$, every Nash equilibrium of game $\Gamma(c, \pi, \kappa, n^1, n^2)$ is regular.*

Theorem 2 is surprising. We are studying arbitrarily large games in terms of the number of players. These games involve extensive payoff restrictions: all players' payoffs take at most four distinct values in these games. This payoff structure would lead one to conjecture that the equations characterizing Nash equilibrium are fraught with singularities. Instead, we show that in the space of a single payoff parameter, the common voting cost, c, all equilibria of the game $\Gamma(c, \pi, \kappa, n^1, n^2)$ are generically regular. In the next section, we discuss in some detail the implications of this theorem for the theory of turnout and participation. The rest of this section is devoted to proving theorem 2.

We start by observing, using lemma 5, that if a game $\Gamma(c, \pi, \kappa, n^1, n^2)$ is not regular, then it has a non-regular Nash equilibrium, σ^*, such that $|\widehat{N}_p(\sigma^*)| \neq 1$ for all $p \in (0, 1)$. By theorem 1, such an equilibrium σ^* can logically take only one of three forms:

1. equilibrium σ^* is such that $|\widehat{N}_p(\sigma^*)| \geq 2$ and $|\widehat{N}_q(\sigma^*)| \geq 2$, for some $p, q \in (0, 1)$, $p \neq q$, or,
2. equilibrium σ^* is such that $|\widehat{N}_p(\sigma^*)| = |\widehat{N}(\sigma^*)| \neq 1$, for some $p \in (0, 1)$, or,
3. equilibrium σ^* is not quasi-strict so that $\widehat{N}_p(\sigma^*) \neq \emptyset$, for some $p \in \{0, 1\}$.

In lemmas 6, 7, and 8, respectively, we consider these three types of equilibria in turn. We show that equilibria of the first type are regular for all common voting costs, while the second type of equilibrium are regular for almost all common voting costs. Lastly, we show that non-quasi strict equilibria of the third type can exist only for a finite number of possible voting costs. These three lemmas essentially constitute the proof of theorem 2.

We now proceed to execute the plan of analysis we just outlined. First we show:

Lemma 6 *For all costs of voting $c \in \mathbf{R}_{++}$, every Nash equilibrium, σ^*, of game $\Gamma(c, \pi, \kappa, n^1, n^2)$ satisfying $|\widehat{N}_p(\sigma^*)| \geq 2$ and $|\widehat{N}_q(\sigma^*)| \geq 2$, $p, q \in (0, 1)$, $p \neq q$, is regular.*

Proof. By theorem 1, σ^* is a quasi strict equilibrium since we must have $\widehat{N}_0(\sigma^*) = \widehat{N}_1(\sigma^*) = \emptyset$. Let distinct players $i, h \in \widehat{N}_p(\sigma^*)$ and $j, g \in \widehat{N}_q(\sigma^*)$. By lemma 1, we have $F^t_{ij}(\sigma^*) = 0$. By lemma 4, in order for σ^* not to be regular, it must be that either $F^t_{ih}(\sigma^*) = 0$ or $F^t_{jg}(\sigma^*) = 0$, or both. In the first case we obtain the system

$$\left.\begin{array}{l} F^t_{ij}(\sigma^*) = 0 \\ F^t_{ih}(\sigma^*) = 0 \end{array}\right\},$$

while in the latter case we obtain

$$\left.\begin{array}{l} F^t_{ij}(\sigma^*) = 0 \\ F^t_{jg}(\sigma^*) = 0 \end{array}\right\}.$$

But both of these systems are inconsistent by lemma 2. Hence σ^* is regular. ■

Observe that the regularity of equilibria with $|\widehat{N}_p(\sigma^*)| \geq 2$ and $|\widehat{N}_q(\sigma^*)| \geq 2$, $p, q \in (0,1)$, follows *for all* voting costs. Equilibria of this type are guaranteed to be both quasi-strict and regular. In the next lemma we consider the more pedestrian type of quasi-strict equilibrium that satisfies $|\widehat{N}_p(\sigma^*)| = |\widehat{N}(\sigma^*)| \neq 1$, $p \in (0,1)$. Unlike the more nuanced form of equilibrium considered in lemma 6, this second type of equilibrium may not be regular, but only for a finite number of voting costs. Thus:

Lemma 7 *Outside a null set $\widehat{C} = \{c_1, ..., c_{\widehat{s}}\}$ consisting of a finite number of voting costs $c \in \mathbf{R}_{++}$, every equilibrium, σ^*, of game $\Gamma(c, \pi, \kappa, n^1, n^2)$ satisfying $|\widehat{N}(\sigma^*)| = |\widehat{N}_p(\sigma^*)| \neq 1$, $p \in (0,1)$, is regular.*

Proof. First, if $\widehat{N}(\sigma^*) = \emptyset$, we have a quasi-strict equilibrium in pure strategies, which is regular. Thus, we focus on mixed strategy equilibria with $|\widehat{N}(\sigma^*)| = |\widehat{N}_p(\sigma^*)| > 1$. We shall derive necessary conditions in order to obtain such a non-regular equilibrium, σ^*, and then show that these necessary conditions are met only for a finite set of common voting costs. First, observe that equilibrium σ^*, is characterized by three objects: the mixing probability parameter p, the set $V \subset N^1 \cup N^2$ containing players that vote with probability 1, and the set $M \subseteq N^1 \cup N^2$, $|M| > 1$, consisting of players that use non-degenerate mixed strategies. Fix arbitrary sets $M, V, V \cap M = \emptyset$, $|M| > 1$. We represent a class of strategies, $\tilde{\sigma}(p)$, as a function of $p \in (0,1)$, as follows: we set $\tilde{\sigma}_i^t(p) = 1$, $i \in V$; $\tilde{\sigma}_i^t(p) = 0$, $i \in (N^1 \cup N^2) \backslash (V \cup M)$; and, $\tilde{\sigma}_i^1(p) = p$, $\tilde{\sigma}_j^2(p) = 1 - p$, for all $i \in M \cap N^1$, $j \in M \cap N^2$. In order for strategies $\tilde{\sigma}(p)$ to constitute an equilibrium, we must have $F_i^t(\tilde{\sigma}(p)) = c$, $i \in M \cap N^t$. Furthermore, by lemma 4, in order for such an equilibrium to be quasi strict but not regular, it must be that $F_{ih}^t(\tilde{\sigma}(p)) = 0$ for players $i, h \in M$, $i \neq h$. We conclude that if strategies $\tilde{\sigma}(p)$ with indifferent players in set $M = \widehat{N}(\tilde{\sigma}(p))$ constitute an equilibrium that is not regular, there must exist a pair of $p \in (0,1)$, and $c > 0$ that solve the system of equations:

$$\left.\begin{array}{r} F_i^t(\tilde{\sigma}(p)) = c,\ i \in M \cap N^t \\ F_{ih}^t(\tilde{\sigma}(p)) = 0,\ i, h \in M, i \neq h \end{array}\right\}. \tag{19}$$

Then, the proof is complete if we can show that there can exist at most a finite number of costs of voting, $\widehat{C}(M, V) = \{c_1, ..., c_s\}$, for which the system (19) is consistent, for *some* $i, h \in M$. If the above were true, and since there are only a finite number of possible pairs of sets, M, V, the union of all possible $\widehat{C}(M, V)$ is also a finite set, say $\widehat{C} = \{c_1, ..., c_{\widehat{s}}\}$. Then, for all $c \notin \widehat{C}$, every equilibrium σ^* of game $\Gamma(c, \pi, \kappa, n^1, n^2)$ with $|\widehat{N}(\sigma^*)| = |\widehat{N}_p(\sigma^*)| \neq 1$, $p \in (0,1)$, is regular. Thus, we will devote the rest of the proof to show that system (19) is consistent only for a finite number of costs of voting, $\widehat{C}(M, V) = \{c_1, ..., c_s\}$.

We will only consider the case $i \in N^1$, as the argument is identical in the case $i \in N^2$. From lemma 9 in the appendix, the second equation in (19) is equivalent to:

$$f_{\tilde{\sigma}(p)}(m; \{i, h\}) - f_{\tilde{\sigma}(p)}(m + k; \{i, h\}) = 0,$$

where $m = -2$, $k = 1$, if $\pi = 1$, $h \in N^1$; $m = -2$, $k = 2$, if $\pi = \frac{1}{2}$, $h \in N^1$; $m = -1$, $k = 1$, if $\pi = 1$, $h \in N^2$; and $m = -1$, $k = 2$, if $\pi = \frac{1}{2}$, $h \in N^2$. We now distinguish two cases.
Case 1 ($f_{\tilde{\sigma}(p)}(m; \{i, h\}) + f_{\tilde{\sigma}(p)}(m + k; \{i, h\}) > 0$, for some $p \in (0,1)$): Then, by lemma 11 in the appendix, the second equation of (19) is satisfied only for a finite set, $P(M, V) = \{p_1, ..., p_s\}$, of

probabilities $p \in (0,1)$. Thus, (19) is consistent for at most s possible values of cost $c \in \widehat{C}(M,V) = \{F_i^1(\tilde{\sigma}(p_1)), ..., F_i^1(\tilde{\sigma}(p_s))\}$, which is what we wished to show.
Case 2 ($f_{\tilde{\sigma}(p)}(m;\{i,h\}) + f_{\tilde{\sigma}(p)}(m+k;\{i,h\}) = 0$, for all $p \in (0,1)$): In this case, the second equation of (19) is satisfied for all $p \in (0,1)$, but we shall show that either the first equation is never satisfied (when $\pi = 1$) or that it is satisfied only for a single possible value of c (when $\pi = \frac{1}{2}$). From (2), the first equation of (19) can be written as

$$\pi f_{\tilde{\sigma}(p)}(-1;\{i\}) + (1-\pi) f_{\tilde{\sigma}(p)}(0;\{i\}) = c.$$

If $h \in N^1$, then expanding the above using (1) we get

$$\pi(p f_{\tilde{\sigma}(p)}(-2;\{i,h\}) + (1-p) f_{\tilde{\sigma}(p)}(-1;\{i,h\})) + (1-\pi)(p f_{\tilde{\sigma}(p)}(-1;\{i,h\}) + (1-p) f_{\tilde{\sigma}(p)}(0;\{i,h\})) = c.$$

If $h \in N^2$, the same expansion using (1) yields

$$\pi((1-p) f_{\tilde{\sigma}(p)}(-1;\{i,h\}) + p f_{\tilde{\sigma}(p)}(0;\{i,h\})) + (1-\pi)((1-p) f_{\tilde{\sigma}(p)}(0;\{i,h\}) + p f_{\tilde{\sigma}(p)}(1;\{i,h\})) = c.$$

If $\pi = 1$, $k = 1$ then both equations reduce to $0 = c$ (because $f_{\tilde{\sigma}(p)}(m;\{i,h\}) = f_{\tilde{\sigma}(p)}(m+1;\{i,h\}) = 0$), so that (19) is inconsistent in this case. If on the other hand $\pi = \frac{1}{2}$, $k = 2$, then we get $\frac{1}{2} f_{\tilde{\sigma}(p)}(m+1;\{i,h\}) = c$. But since $f_{\tilde{\sigma}(p)}(m;\{i,h\}) = f_{\tilde{\sigma}(p)}(m+2;\{i,h\}) = 0$, we conclude that the only possibility for this equation to hold is if $M = \{i,h\}$ and $f_{\tilde{\sigma}(p)}(m+1;\{i,h\}) = 1$. Then (19) is consistent only for $c = \frac{1}{2}$. This concludes the proof. ■

Because of theorem 1 and lemma 5, with lemmas 6, and 7 we have effectively shown that for almost all voting costs all quasi-strict equilibria are regular. In the final lemma of this section we show that for almost all costs of voting, all equilibria are quasi-strict.

Lemma 8 *Outside a null set $C^* = \{c_1, ..., c_{s^*}\}$ consisting of a finite number of voting costs $c \in \mathbf{R}_{++}$, every equilibrium of game $\Gamma(c, \pi, \kappa, n^1, n^2)$ is quasi-strict.*

Proof. If an equilibrium σ^* is not quasi-strict, then either $\widehat{N}_0(\sigma^*) \neq \emptyset$ or $\widehat{N}_1(\sigma^*) \neq \emptyset$, or both. We distinguish two cases for an equilibrium σ^* that is not quasi-strict:

Case 1, σ^* is a pure strategy equilibrium. The only possibility for players to be indifferent in a pure strategy equilibrium is if $\pi = c = \frac{1}{2}$.

Case 2, $\widehat{N}_p(\sigma^*) \neq \emptyset$ for some $p \in (0,1)$. Then, by theorem 1 and the fact that σ^* is not quasi-strict, we decompose the set of all indifferent players to $\widehat{N}(\sigma^*) = \widehat{N}_q(\sigma^*) \cup \widehat{N}_p(\sigma^*)$, with $q \in \{0,1\}$. If $\widehat{N}_p(\sigma^*) = \{i\}$ then by lemma 5 there exists a pure strategy equilibrium (σ^*_{-i}, q) or $(\sigma^*_{-i}, 1-q)$ that is not quasi-strict, hence we revert to the conclusion reached in case 1. Hence, assume $|\widehat{N}_p(\sigma^*)| > 1$. We now borrow notation from lemma 7 and let $V, M \subseteq N^1 \cup N^2$, $M \cap V = \emptyset$, $|M| > 1$, denote the set of players voting and mixing in such an equilibrium, and use $\tilde{\sigma}(p)$ to denote the corresponding class of strategies as a function of p. By lemma 1 and by (13), if an equilibrium with strategies $\tilde{\sigma}(p)$ is not quasi-strict, then

$$\left.\begin{array}{c} F_i^t(\tilde{\sigma}(p)) = c, i \in M \cap N^t \\ F_{ij}^t(\tilde{\sigma}(p)) = 0, j \in (N^1 \cup N^2) \backslash M \end{array}\right\}. \tag{20}$$

Our goal, then, is to show that (20) is true only for a finite number of voting costs $C^*(M,V) = \{c_1, ..., c_s\}$ for some $i \in M \cap N^t$ and $j \in (N^1 \cup N^2) \backslash M$. For costs $c \notin C^*(M,V)$, any equilibrium

$\tilde{\sigma}(p)$ must be quasi strict.

As in the proof of lemma 7, we will only consider the case $i \in N^1$. From lemma 9 in the appendix, the second equation in (20) is equivalent to:

$$f_{\tilde{\sigma}(p)}(m;\{i,j\}) - f_{\tilde{\sigma}(p)}(m+k;\{i,j\}) = 0,$$

where $m = -2$, $k = 1$, if $\pi = 1$, $j \in N^1$; $m = -2$, $k = 2$, if $\pi = \frac{1}{2}$, $j \in N^1$; $m = -1$, $k = 1$, if $\pi = 1$, $j \in N^2$; and $m = -1$, $k = 2$, if $\pi = \frac{1}{2}$, $j \in N^2$. We now distinguish two subcases:
Subcase 2a ($f_{\tilde{\sigma}(p)}(m;\{i,j\}) + f_{\tilde{\sigma}(p)}(m+k;\{i,j\}) > 0$, for some $p \in (0,1)$): Then, by lemma 11 in the appendix, the second equation of (20) is satisfied only for a finite set, $P(M,V) = \{p_1, ..., p_s\}$, of probabilities $p \in (0,1)$. Thus, (20) is consistent for at most s possible values of cost $c \in \widehat{C}(M,V) = \{F_i^1(\tilde{\sigma}(p_1)), ..., F_i^1(\tilde{\sigma}(p_s))\}$, which is what we wished to show.
Subcase 2b ($f_{\tilde{\sigma}(p)}(m;\{i,j\}) + f_{\tilde{\sigma}(p)}(m+k;\{i,j\}) = 0$, for all $p \in (0,1)$): In this subcase, the second equation of (20) is satisfied for all $p \in (0,1)$, but we shall show that the first equation is never satisfied. From (2), the first equation of (20) can be written as

$$\pi f_{\tilde{\sigma}(p)}(-1;\{i\}) + (1-\pi) f_{\tilde{\sigma}(p)}(0;\{i\}) = c.$$

If $j \in N^1$, then expanding the above using (1) we get

$$\pi(q f_{\tilde{\sigma}(p)}(-2;\{i,j\}) + (1-q) f_{\tilde{\sigma}(p)}(-1;\{i,j\})) + (1-\pi)(q f_{\tilde{\sigma}(p)}(-1;\{i,j\}) + (1-q) f_{\tilde{\sigma}(p)}(0;\{i,j\})) = c.$$

If $j \in N^2$, the same expansion using (1) yields

$$\pi((1-q) f_{\tilde{\sigma}(p)}(-1;\{i,j\}) + q f_{\tilde{\sigma}(p)}(0;\{i,j\})) + (1-\pi)((1-q) f_{\tilde{\sigma}(p)}(0;\{i,j\}) + q f_{\tilde{\sigma}(p)}(1;\{i,j\})) = c.$$

If $\pi = 1$, $k = 1$ then both equations reduce to $0 = c$, whether $q = 0$ or $q = 1$, because $f_{\tilde{\sigma}(p)}(m;\{i,j\}) = f_{\tilde{\sigma}(p)}(m+k;\{i,j\}) = 0$. Thus, (20) is inconsistent in this case. If on the other hand $\pi = \frac{1}{2}$, $k = 2$, then we have $\frac{1}{2} f_{\tilde{\sigma}(p)}(m+1;\{i,j\}) = c$. But since $f_{\tilde{\sigma}(p)}(m;\{i,h\}) = f_{\tilde{\sigma}(p)}(m+2;\{i,h\}) = 0$, we conclude that the only possibility for this equation to hold is if $0 < f_{\tilde{\sigma}(p)}(m+1;\{i,j\}) = 1$, which can only happen if $M = \{i\}$, a contradiction. Thus (20) is inconsistent.

Combining cases 1 and 2, an equilibrium $\tilde{\sigma}(p)$ characterized by the sets M, V, $|M| \geq 0$, can fail to be quasi-strict only for a finite set of common voting costs $C^*(M,V) = \{c_1, ..., c_s\}$. Taking the finite union of these costs over all possible sets M, V we get a finite set

$$C^* = \{c_1, ..., c_{s^*}\},$$

outside which all equilibria are quasi-strict. ■

We can now combine the above results to prove theorem 2.

Proof of Theorem 2. We define the finite set $\widetilde{C} = C^* \cup \widehat{C}$, where $\widehat{C}$ and C^* are as specified in lemma 7 and 8, respectively. By lemmas 6, 7, and 8, for all common voting costs $c \notin \widetilde{C}$, all Nash equilibria σ^* of game $\Gamma(c, \pi, \kappa, n^1, n^2)$ with $|\widehat{N}_p(\sigma^*)| \neq 1$ for all $p \in [0,1]$, are regular. Then, by lemma 5, for all common voting costs $c \notin \widetilde{C}$, all games $\Gamma(c, \pi, \kappa, n^1, n^2)$ are regular, as we set out to prove. ■

In addition to establishing the determinacy of non-regular games, $\Gamma(c, \pi, \kappa, n^1, n^2)$, we have obtained rather precise conditions that must be satisfied by such non-regular games. Because of the particular structure of these polynomial conditions, we can in fact easily compute and enumerate the set of irregular games $\Gamma(c, \pi, \kappa, n^1, n^2)$, $c \in \widetilde{C}$, by essentially solving either a linear or a quadratic equation (see proof of lemma 11). The following example illustrates this possibility.

Example 4 *Let $n^1 = 4, n^2 = 3$, $\kappa = 0$, and $\pi = \frac{1}{2}$. We seek a totally mixed non-regular equilibrium, σ^*. By lemmas 4 and 6, we conclude that we must have $\widehat{N}(\sigma^*) = \widehat{N}_p(\sigma^*)$, $p \in (0,1)$ and*

$$F^1_{ih}(\sigma^*) = \tfrac{1}{2}(\tbinom{5}{4}(1-p)^6 p - \tbinom{5}{2}p^3(1-p)^4) = 0.$$

The latter can be factored as follows

$$F^1_{ih}(\sigma^*) = \tfrac{1}{2}p(1-p)^4(\tbinom{5}{4}(1-p)^2 - \tbinom{5}{2}p^2) = 0,$$

so that p must solve the quadratic $\binom{5}{4}(1-p)^2 - \binom{5}{2}p^2$. We calculate $p = \sqrt{2} - 1$ as a solution. In order for this solution to constitute an equilibrium, we must have

$$F^1_i(\sigma^*) = F^2_j(\sigma^*) = \tfrac{1}{2}(\tbinom{6}{4}(1-p)^4 p - \tbinom{6}{3}p^3(1-p)^2) = c,$$

Substituting $p = \sqrt{2} - 1$ above we obtain that σ^* is indeed an irregular Nash equilibrium for game $\Gamma(c, \pi, \kappa, n^1, n^2)$ with $c = 290\sqrt{2} - 410$.

In the following section we give some of the reasons why ensuring regularity is warranted in the context of turnout games. We note that although we focus our discussion on the case when the model represents an election and the players' participation decision is whether to vote or not, our results also cover the public good participation games without refunds analyzed by Palfrey and Rosenthal (1984).

6 Implications

In this section we discuss implications of our analysis for the theory of rational turnout. We have organized this discussion in two parts. We first consider continuity properties of the equilibrium set of game $\Gamma(c, \pi, \kappa, n^1, n^2)$ with respect to perturbations of the game that preserve the complete information character of the game. Then, we consider the robustness of equilibria to the introduction of various forms of incomplete information. Both sets of results constitute well known implications of regularity.[3]

6.1 Payoff Heterogeneity and Voter Motivations

We first discuss perturbations of game $\Gamma(c, \pi, \kappa, n^1, n^2)$ that maintain the complete information nature of the game. Note that, in principle, game $\Gamma(c, \pi, \kappa, n^1, n^2)$ has a total of $k^* = 2^{n^1+n^2}$ outcomes. With $n^1 + n^2$ players, the payoff space of the game can be considered a subset of $\mathbf{R}^{(n^1+n^2)k^*}$. Let $p(c) \in \mathbf{R}^{(n^1+n^2)k^*}$ be the payoffs we have assumed in the definition of game $\Gamma(c, \pi, \kappa, n^1, n^2)$, and let $U \subset \mathbf{R}^{(n^1+n^2)k^*}$ be an open set around these payoffs. Now U contains

[3] In fact, regular equilibria were defined by Harsanyi (1973a,b) exactly in order to obtain results of the type we present in the second part of this section.

very general perturbations of game $\Gamma(c, \pi, \kappa, n^1, n^2)$, allowing for, *e.g.*, heterogeneity in individual voting costs, or for players that care about the margin of victory in the election, conformity with the behavior of their peers, etc. An application of the implicit function theorem, ensures that every regular equilibrium σ^* of $\Gamma(c, \pi, \kappa, n^1, n^2)$ is robust to such payoff perturbations.

Theorem 3 *For all sizes of the electorate n^1, n^2, for almost all c, and any Nash equilibrium, σ^*, of game $\Gamma(c, \pi, \kappa, n^1, n^2)$, there exist neighborhoods $U \subset \mathbf{R}^{k^*}$ of $p(c)$ and $V \subset \mathbf{R}^{n^1+n^2}$ of σ^*, and a function $\widetilde{\sigma} : U \to V$, such that:*

1. *the function $\widetilde{\sigma} : U \to V$, is differentiable and satisfies $\sigma(p(c)) = \sigma^*$, and*
2. *the set of equilibria $E(p)$, satisfies $\widetilde{\sigma}(p) \in E(p)$ and $|E(p) \cap V| = 1$, for every $p \in U$.*

Proof. Follows from theorem 2 and the arguments in van Damme (1987), page 40. ■

Theorem 3 strengthens the relevance of the asymmetric equilibria of game $\Gamma(c, \pi, \kappa, n^1, n^2)$. In particular, asymmetric behavior can be attributed to differences in characteristics across subgroups in the electorate that are assumed away by the analyst via the assumption of common voting costs. These differences can be restored (in the model) via appropriate payoff perturbations of game $\Gamma(c, \pi, \kappa, n^1, n^2)$. Obviously, a game with payoff heterogeneity is a more realistic one in the class of complete information games. Individual heterogeneity is persistent and manifest in laboratory experiments even when experimental treatments posit homogeneous players in these settings. Indeed, the main reason these games are not considered in the literature is the fact that they are analytically intractable. Theorem 3 clearly allows for individual heterogeneity and players with different individual voting costs. But, even if we maintain homogeneity of players within groups, theorem 3 also allows for additional interesting payoff perturbations. For instance, players's payoff may depend (monotonically or not) on the margin of victory (or defeat) of their favorite candidate. We can also model ethical voters of some form, such as those that would be concerned about the solidarity of behavior of peers within their group, etc.

It is significant to note that the theorem ensures robustness of all equilibria of the (regular) complete information game. As example 3 illustrates, it is easy to construct equilibria of game $\Gamma(c, \pi, \kappa, n^1, n^2)$ that maintain high turnout rates as the electorate gets arbitrarily large. Quasi-symmetric equilibria with this property are discussed in Palfrey and Rosenthal, 1983. Thus, we conclude that large turnout under complete information can occur in games in which players have heterogeneous voting costs. The fact that theorem 3 deals with small heterogeneity, resolves the question on the possibility of high turnout in these games under the hardest possible scenario. To substantiate the above claim, consider the case heterogeneous voting costs are allowed to differ significantly from some benchmark cost, c, instead. Then large turnout equilibria are easier to obtain. We can start with a Nash equilibrium of a smaller game played among the highest cost voters, and then add low cost voters in equal numbers in each group, $t = 1, 2$. If the assigned costs of the added low cost voters are sufficiently smaller, all these added players (which can be in the millions) have a strict incentive to participate. These low cost voters do not affect the incentives of high cost voters since they are added in equal number to the two teams. Hence, when voting costs can differ significantly among players in the same group, (regular) high turnout Nash equilibria are possible. Theorem 3 ensures that we can also get high turnout equilibria even when voting costs have to be arbitrarily close to each other for all players.

6.2 Incomplete Information and/or Group Size Uncertainty

In the previous subsection we discussed continuity properties of the equilibrium set to payoff perturbations. The next question we turn to, concerns the robustness of the equilibrium set to the introduction of incomplete information. One form of incomplete information we consider (*e.g.*, Ledyard (1984), Palfrey and Rosenthal (1985)), is with regard to individual costs of voting. The second concerns the size of the two groups of voters (*e.g.*, Ledyard (1984), Borgers (2004)). Specifically concerning this second type of incomplete information, note that throughout the analysis we have assumed that players have a binary choice: to vote for their favorite candidate or abstain. Of course, in reality voters have an option to vote for their non-favorite candidate. This third strategy is strictly dominated in the complete information game $\Gamma(c, \pi, \kappa, n^1, n^2)$. As a consequence, adding this strategy does not affect our analysis up to this section. The availability of this third strategy, though, is relevant in an incomplete information game in which the number of supporters of candidates 1 and 2 are not known. In such a game, players may use any one of the three strategies, depending on the realization of their type. Hence, let $\Gamma^*(c, \kappa, \pi, n^1, n^2)$ denote the complete information game in which this third strategy is added to each player's strategy set, but is otherwise identical to the game we have analyzed. Theorem 2 implies the following:

Theorem 4 *For arbitrary parameters π, κ, n^1, n^2, construct a sequence of Bayesian games Γ_n^* such that for $\epsilon > 0$:*

(i) the vector of voting costs $(c_i^t)_{i \in N^t, t=1,2}$, is private information drawn from joint density g_n, with support that contains an open subset (possibly different across n) of $[-\epsilon, +\infty]^{n^1+n^2}$, or

(ii) the vector of benefits $(b_i^t)_{i \in N^t, t=1,2}$ from the victory of candidate $t = 1, 2$ is private information drawn from joint density g_n, with support that contains an open subset (possibly different across n) of $[-1, 1]^{n^1+n^2}$, (while the benefit from the victory of candidate, $-t$, is fixed at 0 for each $i \in N^t$), or

(iii) the vector of pairs of the above cost and benefit $(c_i^t, b_i^t)_{i \in N^t, t=1,2}$ are private information drawn from joint density g_n, with support that contains an open subset (possibly different across n) of $([-\epsilon, +\infty] \times [-1, 1])^{n^1+n^2}$.

Assume g_n converges to $(c, ..., c)$ in case (i), to $(1, ..., 1)$ in case (ii), or $((c, 1)..., (c, 1))$ in case (iii). For almost all c and every Nash equilibrium, σ^*, of $\Gamma^*(c, \pi, \kappa, n^1, n^2)$, and for all $\varepsilon > 0$, there exists large enough n', such that the Bayesian game Γ_n^*, $n > n'$, has an essentially strict Bayesian Nash equilibrium that induces a distribution of voter participation that is arbitrarily close (within ε) to that induced by the complete information game equilibrium, σ^*.

Proof. Theorem 2 ensures that game $\Gamma^*(c, \pi, \kappa, n^1, n^2)$ is regular for almost all c. Now, for all three possibilities (i's type is c_i^t, or b_i^t, or (c_i^t, b_i^t)) in the perturbed Bayesian game, the distribution of individual players' types, g_n, satisfies the conditions of, *e.g.*, Govindan, Reny, and Robson, 2003, rendering a particular player i indifferent between any pair of actions only for a set of measure zero in the support of i's type. ■

In the Bayesian Nash Equilibrium, the implied mixing probability ratio for the players that use mixed strategies approaches the ratios that prevail in σ^*. Since the players that use pure strategies play strict best responses in the complete information version, then the strict inequalities characterizing their equilibrium behavior are also satisfied for the bulk of the mass of types in the support of the density g_n. Thus, the participation/voting rates of these players converge to their strategies in the equilibrium, σ^*, of the complete information game. Without exhausting the level of generality that can be achieved in this context, theorem 4 allows for very general forms of private information, including correlation of types. Among the possibilities, is that covered in cases (ii) and (iii) of the theorem, where player $i \in N^t$ may prefer a victory by candidate $-t$ (for a benefit of 0 when $b_i^t < 0$). Thus, the fraction of supporters for the two candidates is not known with certainty in such a Bayesian game. As a consequence, the equilibria of game $\Gamma(c, \pi, \kappa, n^1, n^2)$ are robust to the introduction of uncertainty either on the size of support for the two candidates, or the cost of voting, or both.

As we discussed in the introduction, theorem 4 does not contradict the results of Palfrey and Rosenthal, 1985. They conclude that high turnout equilibria are impossible in games of incomplete information by performing a different limit calculation than that performed in theorem 4. Instead of holding the size of the electorate fixed, as we do, they hold the distribution of costs fixed and take a limit as $n^1 + n^2$ goes to infinity.[4] Myerson, 1998, discusses an application of poisson games of population uncertainty to turnout games similar to those analyzed by Palfrey and Rosenthal (1983, 1985). Via an example, he illustrates how a unique low turnout equilibrium obtains in the context of these games when the electorate is large. In our setup, we model "population uncertainty" indirectly by allowing (as we do in theorem 4) players' set of possible types to include costs of voting that exceed the maximum possible benefit from winning the election, and interpreting such high cost types as players that are *de facto* not playing the game. Furthermore, unlike population uncertainty games as in Myerson (1998), we allow the same types of different players to behave differently. On the contrary, in Myerson (1998) the number of players of a specific type is a poisson random variable and players of the same type that end up playing the game exhibit homogeneous behavior.[5] It is conceivable that population uncertainty games in which types are not necessarily poisson distributed and satisfy the type-homogeneity desideratum imposed by Myerson can yield high turnout equilibria as these games get close to models with population certainty. In particular, if population uncertainty is small, these games should approximate the type homogeneous equilibria of the game with population certainty.

We conclude with a discussion that combines the insights from theorem 3, in particular the possibility of high turnout with individual heterogeneity, and theorem 4. Note that in theorem 4, Bayesian Nash equilibria that are close to the equilibria of the complete information game, $\Gamma(c, \pi, \kappa, n^1, n^2)$, arise when the distribution of voting costs gets more tightly concentrated around cost level, c, as the number of players increases. Such distribution of voting costs may be implausible in a large population. We illustrate this point in figure 1. For instance, a realistically dispersed distribution of costs in the population may be that in figure 1.(a), rather than that in figure 1.(b). But if realism (however ill-defined this may be in the context of our models) is the goal, then we wouldn't expect all individuals in the population to have voting costs drawn from a common distribution such as that in figure 1.(a), either. Voter participation differs significantly and systematically conditional on a person's income, education, age, etc. Thus, the aggregate distribution in figure 1.(a) may arise as a 'mixture' of distributions, one for each demographic group. The distribution of costs within each group can have full support but be concentrated in distinct consecutive

[4] Palfery and Rosenthal, 1985, also focus on equilibria in which players within each group behave symmetrically, using a common cutoff-point participation rule.

[5] If we can split types in a payoff irrelevant manner, this homogeneity assumption is a restriction in general games of population uncertainty, but (essentially) constitutes an equilibrium property in the case of poisson games (Myerson (1998), theorem 4, p. 386).

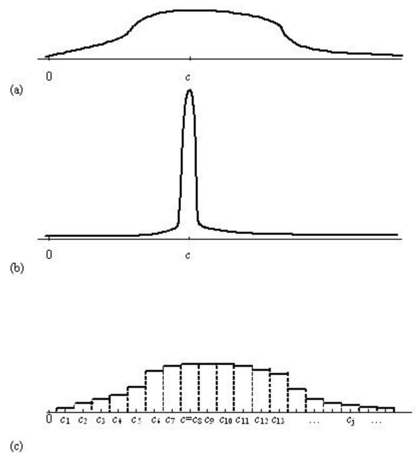

Figure 1: Heterogeneity & Incomplete Information.

intervals across groups. Such a distribution (except for the full support requirement) that emulates that of figure 1.(a) is illustrated with consecutive uniform distributions in figure 1.(c). Now, while the distribution of individual voting costs in figure 1.(a) may be inconsistent with high turnout rate equilibria in a large electorate, such high turnout may be possible for the same population size and the distribution in figure 1.(c). In particular, the heterogeneity of sub-populations within each group, allows us to build equilibria in which subgroups with distributions centered at lower cost levels participate with probability that is (almost) equal to one, while 'mixing' (in the sense of the *ex ante* probability of voting) occurs only for subgroups of high or intermediate cost level.

7 Conclusion

We have shown that the starkest possible rational participation model is remarkably well behaved. By that we mean that outside a negligible set of common voting costs, all the equilibria of this model are regular. Regularity is a very strong refinement of Nash equilibrium, stronger than the typical menu of refinements encountered in applications (*e.g.*, van Damme, 1987). Our analysis implies that, generically, all the equilibria of these games are robust to the introduction of incomplete information or to the introduction of general payoff perturbations. One implication of this property is that the Nash equilibria of these games are reasonable approximations of equilibria in the more realistic nearby games that admit individual heterogeneity. A second implication is that the Nash equilibria of the complete information model that exhibit high turnout rates persist even when incomplete information, or heterogeneity, or both are introduced in the model.

Thus, criticism of rational models of turnout cannot be based on the unqualified assertion that they are inconsistent with high participation rates in large electorates. This is not to say that richer models of participation behavior are not necessary in order to account for the observed participation behavior. Rather, it is to emphasize that rejection of these models on empirical grounds should be based on aspects of the data other than the observed aggregate turnout rates. Whether the 'correct' model(s) that ideally account for observed participation behavior are rational or not leads us well beyond the scope of this study. Instead of attempting to adjudicate the controversies sparked by various theories of turnout, our objective was to deepen our understanding of a specific class of rational models of individual participation behavior. We pursued this objective because we believe these models are foundational to our understanding of individual participation behavior. Aside from the fact that many deductions from these models receive considerable support in the laboratory (Levine and Palfrey, 2006), reasoning from these models survives in one way or another in any non-trivial theory of participation, short of assuming that individuals vote because they prefer it independent of the incentives generated by the behavior of their peers.

APPENDIX

In this appendix, we prove three lemmas. In the first lemma we express certain second derivatives of probability of winning functions in terms of probabilities of the form $f_\sigma(m; C)$.

Lemma 9 *For players* $i,\ h \in N^1$ *and* $j, g \in N^2$ *we have:*

$$F^1_{ih}(\sigma) = \pi f_\sigma(-2; \{i, h\}) + (1 - 2\pi) f_\sigma(-1; \{i, h\}) - (1 - \pi) f_\sigma(0; \{i, h\}), \tag{21}$$

$$F^2_{jg}(\sigma) = (1 - \pi) f_\sigma(2; \{j, g\}) - (1 - 2\pi) f_\sigma(1; \{j, g\}) - \pi f_\sigma(0; \{j, g\}), \text{ and} \tag{22}$$

$$F^1_{ij}(\sigma) = -F^2_{ij}(\sigma) = (1 - \pi) f_\sigma(1; \{i, j\}) - (1 - 2\pi) f_\sigma(0; \{i, j\}) - \pi f_\sigma(-1; \{i, j\}). \tag{23}$$

Proof. Expanding (2) using (1) we get

$$\begin{aligned} F^1_i(\sigma) &= \pi f_\sigma(-1; \{i\}) + (1 - \pi) f_\sigma(0; \{i\}) \\ &= \pi(\sigma^1_h f_\sigma(-2; \{i, h\}) + (1 - \sigma^1_h) f_\sigma(-1; \{i, h\}) \\ &+ (1 - \pi)(\sigma^1_h f_\sigma(-1; \{i, h\}) + (1 - \sigma^1_h) f_\sigma(0; \{i, h\}), \end{aligned}$$

which yields (21) after differentiation with respect to σ^1_h. Applying identical arguments to (3) for players $j,\ g \in N^2$ we also get

$$\begin{aligned} F^2_j(\sigma) &= (1 - \pi) f_\sigma(1; \{j\}) + \pi f_\sigma(0; \{j\}) \\ &= (1 - \pi)(\sigma^2_g f_\sigma(2; \{j, g\}) + (1 - \sigma^2_g) f_\sigma(1; \{j, g\})) \\ &+ \pi(\sigma^2_g f_\sigma(1; \{j, g\}) + (1 - \sigma^2_g) f_\sigma(0; \{j, g\})), \end{aligned}$$

which yields (22) after differentiation with respect to σ^2_g. Finally, for players $i \in N^1$, and $j \in N^2$, we rewrite (2)

$$\begin{aligned} F^1_i(\sigma) &= \pi f_\sigma(-1; \{i\}) + (1 - \pi) f_\sigma(0; \{i\}) \\ &= \pi(\sigma^2_j f_\sigma(0; \{i, j\}) + (1 - \sigma^2_j) f_\sigma(-1; \{i, j\}) \\ &+ (1 - \pi)(\sigma^2_j f_\sigma(1; \{i, j\}) + (1 - \sigma^2_j) f_\sigma(0; \{i, j\}), \end{aligned}$$

and get (23) by differentiation with respect to σ^2_j and using the fact that $F^1(\sigma) = 1 - F^2(\sigma)$. ■

Next, we evaluate the probability $f_\sigma(m; C)$ when players' non-degenerate mixed strategies are characterized by some probability p.

Lemma 10 *For game* $\Gamma(c,\pi,\kappa,n^1,n^2)$*, fix* $M, V \subseteq N^1 \cup N^2$*, such that* $V \cap M \neq \emptyset$*. For* $p \in (0,1)$*, let* $\tilde{\sigma}(p)$ *denote players' strategies such that* $\tilde{\sigma}_i^t(p) = 1$ *for all* $i \in V$*,* $\tilde{\sigma}_i^t(p) = 0$ *for all* $i \in (N^1 \cup N^2)\backslash(V \cup M)$*,* $t = 1,2$*, while* $\tilde{\sigma}_i^1(p) = p$*, for all* $i \in M \cap N^1$*, and* $\tilde{\sigma}_j^2(p) = 1-p$ *for all* $j \in M \cap N^2$*. Then,*

$$f_{\tilde{\sigma}(p)}(m;C) = \binom{\mu^1+\mu^2}{\mu^1-d}(1-p)^{\mu^1-d}p^{\mu^2+d},$$

where $\mu^t = |(M \cap N^t)\backslash C|$*, and* $d = \kappa + m + |(V \cap N^2)\backslash C| - |(V \cap N^1)\backslash C|$*.*

Proof. We compute the probability $f_{\tilde{\sigma}(p)}(m;C)$ as follows:

$$\begin{aligned} f_{\tilde{\sigma}(p)}(m;C) &= \sum_{k=\max\{0,d\}}^{\min\{\mu^1,\mu^2+d\}} \binom{\mu^1}{k} p^k(1-p)^{\mu^1-k}\binom{\mu^2}{k-d}(1-p)^{k-d}p^{\mu^2+d-k} \\ &= \sum_{k=\max\{0,d\}}^{\min\{\mu^1,\mu^2+d\}} \binom{\mu^1}{k}\binom{\mu^2}{k-d}(1-p)^{\mu^1-d}p^{\mu^2+d} \\ &= \binom{\mu^1+\mu^2}{\mu^1-d}(1-p)^{\mu^1-d}p^{\mu^2+d}, \end{aligned}$$

where the last step is obtained by application of the Vandermonde identity. ■

The last lemma provides sufficient conditions for the finite solvability of a certain class of equations. This result is used in the proof of lemmas 7 and 8.

Lemma 11 *For game* $\Gamma(c,\pi,\kappa,n^1,n^2)$*, fix* $M, V \subseteq N^1 \cup N^2$*, such that* $V \cap M \neq \emptyset$*. For* $p \in (0,1)$*, let* $\tilde{\sigma}(p)$ *denote players' strategies such that* $\tilde{\sigma}_i^t(p) = 1$ *for all* $i \in V$*,* $\tilde{\sigma}_i^t(p) = 0$ *for all* $i \in (N^1 \cup N^2)\backslash(V \cup M)$*,* $t = 1,2$*, while* $\tilde{\sigma}_i^1(p) = p$*, for all* $i \in M \cap N^1$*, and* $\tilde{\sigma}_j^2(p) = 1-p$ *for all* $j \in M \cap N^2$*. If* $f_{\tilde{\sigma}(p)}(m;C) + f_{\tilde{\sigma}(p)}(m+k;C) > 0, k = 1,2$*, for some* $p \in (0,1)$*, then* $f_{\tilde{\sigma}(p)}(m;C) - f_{\tilde{\sigma}(p)}(m+k;C) \neq 0$*, except for a finite set* $P(M,V) = \{p_1, ..., p_s\}$ *of* $p \in (0,1)$*.*

Proof. If either $f_{\tilde{\sigma}(p)}(m;C)$, or $f_{\tilde{\sigma}(p)}(m+k;C)$, is a constant in p, then the conclusion holds immediately from lemma 10, since the difference of the two is a non-degenerate polynomial in p of finite degree. If both quantities are a constant in p, then one of the two must equal 0 and the other 1, and the conclusion follows. Thus, we only need consider the case in which both $f_{\tilde{\sigma}(p)}(m;C)$ and $f_{\tilde{\sigma}(p)}(m+k;C)$ are non-degenerate polynomials of p, given by the expression in lemma 10. Then, the difference $f_{\tilde{\sigma}(p)}(m;C) - f_{\tilde{\sigma}(p)}(m+k;C)$ is equal to

$$\binom{\mu^1+\mu^2}{\mu^1-d}(1-p)^{\mu^1-d}p^{\mu^2+d} - \binom{\mu^1+\mu^2}{\mu^1-d-k}(1-p)^{\mu^1-d-k}p^{\mu^2+d+k},$$

where, as in lemma 10, $\mu^t = |(M \cap N^t)\backslash C|$, and $d = \kappa + m + |(V \cap N^2)\backslash C| - |(V \cap N^1)\backslash C|$. We can then factor the above into

$$\left(\binom{\mu^1+\mu^2}{\mu^1-d}(1-p)^k - \binom{\mu^1+\mu^2}{\mu^1-d-k}p^k\right)(1-p)^{\mu^1-d-k}p^{\mu^2+d}.$$

In order for the above to equal 0 when $p \in (0,1)$, we must have either

$$\binom{\mu^1+\mu^2}{\mu^1-d} - \left(\binom{\mu^1+\mu^2}{\mu^1-d} + \binom{\mu^1+\mu^2}{\mu^1-d-1}\right) p = 0, \text{ when } k=1, \textit{or}$$

$$\binom{\mu^1+\mu^2}{\mu^1-d}(1-2p) + \left(\binom{\mu^1+\mu^2}{\mu^1-d} - \binom{\mu^1+\mu^2}{\mu^1-d-k}\right) p^2 = 0, \text{ when } k=2.$$

Since both $\binom{\mu^1+\mu^2}{\mu^1-d} > 0$ and $\binom{\mu^1+\mu^2}{\mu^1-d} > 0$, there are at most two values of $p \in (0,1)$ that satisfy the above equations. This completes the proof. ■

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