Regularity of Pure Strategy Equilibrium Points in a Class of Bargaining Games

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by

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Abstract: For a class of n-player ($n \geq 2$) sequential bargaining games with probabilistic recognition and general agreement rules, we characterize pure strategy Stationary Subgame Perfect (PSSP) equilibria via a finite number of equalities and inequalities. We use this characterization and the degree theory of Shannon, 1994, to show that when utility over agreements has negative definite second (contingent) derivative, there is a finite number of PSSP equilibrium points for almost all discount factors. If in addition the space of agreements is one-dimensional, the theorem applies for all SSP equilibria. And for oligarchic voting rules (which include unanimity) with agreement spaces of arbitrary finite dimension, the number of SSP equilibria is odd and the equilibrium correspondence is lower-hemicontinuous for almost all discount factors. Finally, we provide a sufficient condition for uniqueness of SSP equilibrium in oligarchic games.

Keywords: Local Uniqueness of Equilibrium, Regularity, Sequential Bargaining.

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1. INTRODUCTION

Sequential bargaining models of complete information starting with Rubinstein, 1982, have provided a fruitful environment for the study of the resolution of disagreements among agents. Unlike cooperative formulations which are silent about the underlying actions of bargaining parties, these games juxtapose equilibrium conditions based on a scrutiny of the optimality of individual choices via which proposals emerge and agreements are crafted. In typical situations, (refined) equilibria of these games exist allowing applications to numerous areas of social interaction. In legislative or other political environments where the non-existence of equilibrium of the cooperative genre is pervasive, such models have been welcomed in celebratory spirits.

Our goal in this paper is to study the structure of the set of equilibria for an important class of these bargaining games and analyze their stability to perturbations of the model. By stability in what follows we mean the property that the number of equilibria of these games is finite and each equilibrium is locally expressible as a continuous function of model parameters. Then, a slight change in the bargaining environment results in small changes in equilibrium behavior. Importantly, if we calculate equilibria using parameter values that do not exactly coincide with their true values, the equilibria we obtain are still close to the true equilibria.

Besides its obvious epistemological significance, such a property seems essential in order to build richer models of political interaction. In parliamentary systems, for example, government formation following elections requires us to append a bargaining model at the end of a game preceded by an electoral stage. In these situations, it is important to ensure conditions such that changes in the bargaining environment induced at the electoral stage, induce continuous changes in the distribution of subsequent agreements and policy outcomes.

We focus our investigation on Stationary Subgame Perfect equilibria in pure strategies (PSSP). We allow general agreement rules, although our results are stronger for a subclass of these. We assume players are recognized to make proposals with some probability fixed across periods. In the main paper we focus on the case of bargaining with discounting, although in the appendix we show our analysis also applies in the case of fixed delay costs. With varying degrees of generality, such models have been analyzed by, for example, Binmore, 1987, Baron and Ferejohn, 1989, Harrington, 1990, Baron, 1991, Merlo and Wilson, 1995, Banks and Duggan, 2000, Jackson and Moselle, 2002, Eraslan, 2002, Eraslan and Merlo, 2002, etc.

Except in special cases, the behavior of SSP equilibria for these games is not fully understood. Banks and Duggan, 2000, studied discounted games such as those we analyze and showed upper-hemicontinuity of SSP equilibria with respect to parameters. But they also provided an example of a majority rule game that has a continuum of PSSP and an equilibrium correspondence that fails lower-hemicontinuity.

Stronger results arise in certain types of unanimity games. For such n-player games with discounting, Merlo and Wilson, 1995, have shown that stationary equilibria are unique when bargaining emulates the division of a possibly stochastic cake and a contraction condition is met. Thus, for the subset of games in Merlo and Wilson for which the upper-hemicontinuity result of Banks and Duggan holds, the equilibrium correspondence simply becomes a continuous function of the parameters.

But collective bargaining will often not emulate the division of a cake, due to the public goods aspect of agreements or the ideological nature of disagreements. For such an ideological space, in section 3 of this paper we provide an example of a four-player discounted unanimity game that admits a continuum of PSSP equilibria.[2]

In view of the above, at most we can hope to show that this type of pathological behavior of the equilibrium set is not generic. Binmore, 1987, showed this is so in Rubinstein's two player game with delay costs, but we know of no general arguments to that effect. Our theorem specializes in three versions of decreasing strength which guarantee that, *for almost all discount factors*: (a) when the agreement rule is oligarchic, a class that includes unanimity rule, the number of SSP equilibria is odd and equilibrium correspondence is lower-hemicontinuous. (b) If the space of agreements is one-dimensional, then all SSP equilibria are locally unique and finite in number for general agreement rules. Finally, (c) for non-oligarchic rules in multidimensional agreement spaces there is a finite number (possibly zero) of PSSP equilibria.

Our results are weaker in the last case since PSSP equilibria may not exist in these games. Still, this theorem has been used in a majority rule application in Kalandrakis, 2003, to show that minority governments in parliamentary government formation bargaining almost always occur

[2] The situation is less encouraging in the case of fixed delay costs. From Rubinstein's original paper a continuum of PSSP can emerge in such games even in the two-player unanimity case, even though SSP (and mere Subgame Perfect) equilibrium is unique in the discounted version. Rubinstein's game involves alternating offers, but the same holds for the probabilistic recognition rules we consider.

with positive probability when utility from cabinet positions is small relative to the ideological disagreements of political parties.

Before we move to the formal arguments, we tie our analysis to two strands of related literature offering a guide to the arguments that permitted our results. The first literature sprung from the study of general equilibrium economies and was pioneered by, Debreu, 1970. There is also a related game-theoretic literature starting with, Harsanyi, 1973, who provided an alternative proof of the fact that almost all finite games in normal form have an odd number of Nash equilibria. Similarly, Kreps and Wilson, 1982, showed that the equilibrium outcome distributions of finite games in the extensive form are finite in number. The same was shown true for a class of cheap talk games by In-Uck Park, 1997. Haller and Lagunoff, 2000, showed genericity of behavior in Markovian equilibria of dynamic games with finite action and state spaces.

When it comes to games with continuous action spaces, Dubey, 1986, offered a general result for simultaneous move games. The bargaining games we analyze involve both a continuous action space for the proposer as well as multi-period dynamic interaction. Thus, these games are not covered by any existing studies. Yet, due to their particular structure, these games are amenable to similar techniques. A key insight is that the proposer in each period chooses among a finite number only of winning coalitions, even though she has to propose from a continuum of agreements. Exploiting this fact, our proof strategy proceeds as follows.

First, we introduce the notion of an *agenda setting plan* corresponding to a player/proposer and a winning coalition: it is a mapping from the possible reservation values of players to optimal proposals by the proposer that are acceptable by members of the coalition. The role of agenda setting plans in our analysis is very akin to that of demand functions in the study of economic equilibrium. Much like demand functions shift the focus from the consumers' optimization problem to reduce economic equilibrium to a set of equations that ensure that markets clear, agenda setting plans sidestep the proposer's optimization problem and reduce equilibrium to a set of equations that ensure that players' coalition choices produce reservation values that are consistent with these choices.[3]

PSSP equilibria emerge when each proposer chooses a unique coalition. Since there is a finite

[3]Kalandrakis, n.d., has used similar arguments to provide a proof of existence of SSP equilibria via Brouwer's theorem.

number of possible combinations of coalition choices by the players, we ensure that every PSSP can be expressed as the solution to one among a finite number of systems of equations. Thence, our result only requires that each of these systems of equations has finitely many solutions.

Because they involve solutions to optimization problems (the agenda setting plans) our equilibrium equations are not sufficiently smooth to allow us to apply Sard's theorem (or the transversality theorem). In the theory of general economic equilibrium this problem has been confronted early on by Rader, 1973, who was able to extend Debreu's 1970 results to cases when the demand functions are not differentiable but satisfy certain stability properties. More recently, Shannon, 1994, developed a degree theory for non-smooth equations. As an application, she strengthened Rader's work to a conclusion similar to Dierker's, 1972. Our theorem is derived by applying homotopy arguments based on the degree theory of Shannon, 1994, and using a theorem of Rader, 1973.

We have organized our analysis in the remainder as follows. In section 2 we present the bargaining model analyzed. In section 3, we provide an example of a unanimity game that admits a continuum of PSSP equilibria. Our analysis culminates in section 4 where we show that the example in section 3 is not generic in the space of discount factors. We conclude in section 5.

2. MODEL

Consider a set of $n \geq 2$ players $N = \{1, ..., n\}$. They convene in periods $t = 1, 2, ...$ to reach an agreement $\mathbf{x}$ drawn from a set X. We assume X a convex, compact subset of $\mathbb{R}^d$, $d \geq 1$. Much of related literature sidesteps the underlying space of agreements X to work with the space of payoffs generated from X. We will eventually resort to similar arguments, but we find it enlightening to build from the primitives of the model.

An agreement requires the approval of a winning coalition, $C \subseteq N$. The set of winning coalitions is determined by the underlying voting rule and is denoted by $\mathcal{D} \subset 2^N \backslash \emptyset$, $\mathcal{D} \neq \emptyset$. For example, if all players have one vote and the voting rule is simple majority, $\mathcal{D}$ consists of all coalitions with more than $\frac{n}{2}$ members.

Our strongest results concern the class of *oligarchic* rules. Voting rule $\mathcal{D}$ is *oligarchic* if there exists coalition $C_o \in \mathcal{D}$ such that $C_o = \bigcap_{C \in \mathcal{D}} C$. One important member of the class of oligarchic rules is *unanimity*, when $\mathcal{D} = \{N\}$ and $C_o = N$. Since some of our basic arguments afford greater generality, we admit a much wider class of agreement rules. Hence, as in Banks and Duggan, 2000, the only restriction on $\mathcal{D}$ is *monotonicity*: for any two coalitions A, B with $A \subseteq B \subseteq N$, we have $A \in \mathcal{D} \Longrightarrow B \in \mathcal{D}$.

Players bargain as follows. In each period $t = 1, 2, ...$, one of the players is chosen by nature (or 'recognized' in legislative language) to make a proposal $\mathbf{z} \in X$. Having observed the proposal, players vote *yes* or *no*. If a winning coalition vote *yes*, then the game ends with $\mathbf{z}$ being implemented. Otherwise, the game moves to the next period, a player is recognized anew to make a proposal, and so on until an agreement is reached. The probability that player $i \in N$ is recognized to make a proposal is constant across periods and equal to $\pi_i \geq 0$. Obviously $\sum_{i \in N} \pi_i = 1$.

Legislator $i \in N$ derives von Neuman-Morgenstern stage utility $u_i : X \longrightarrow \mathbb{R}$ from the agreement $\mathbf{x}$. We assume throughout that u_i is continuously differentiable and concave, and that the agreements are desirable so that $u_i(\mathbf{x}) > 0$, for all $\mathbf{x} \in X$, all $i \in N$. Our analysis will require additional assumptions on u_i, $i \in N$, which we state in the sequel as necessary.

Players discount the future by a factor $\delta_i \in (0, 1)$, $i \in N$. Thus, the payoff of player i from a decision $\mathbf{x} \in X$ reached in period $t \geq 1$ is given by $\delta_i^{t-1} u_i(\mathbf{x})$, and it is zero in the case of perpetual disagreement. We denote the vector of discount factors for players $i = 1, ..., n$ by $\boldsymbol{\delta} = (\delta_1, ..., \delta_n) \in D$, where $D \equiv (0, 1)^n$. Highlighting the fact that our genericity results are cast in terms of discounting parameters $\boldsymbol{\delta}$, we denote specific games by $\Gamma_{\boldsymbol{\delta}}$. In the appendix we discuss how our arguments extend naturally to the case players incur a delay cost $c_i \in \mathbb{R}_{++}$, $i \in N$ instead of discounting the future.

We shall focus our attention to pure strategy stationary subgame perfect (PSSP) equilibria. A pure stationary proposal strategy for player $i \in N$ is an agreement $\mathbf{z}_i \in X$ proposed when i is recognized. A stationary voting strategy for player i is specified by an acceptance set $A_i \subseteq X$. A_i is the set of proposals on which player i votes *yes*. A stationary strategy for player i is a pair $\sigma_i = (\mathbf{z}_i, A_i)$ that consists of a proposal strategy and an acceptance set.

Given stationary strategies $\sigma_i = (\mathbf{z}_i, A_i)$, we calculate players' continuation value, v_i, $i \in N$, which is defined as the expected utility if the game moves in the next period. Using the continuation value, we define the *reservation value* of player i as:

Definition 1 *The* reservation value *of player i, $i \in N$, for continuation value v_i is given by $r_i \equiv \delta_i v_i$.*

In general, it is necessary to restrict voting strategies in order to rule out implausible equilibria in which winning coalitions approve undesirable proposals or reject desirable proposals solely because each of the players is not pivotal and hence is indifferent between her voting actions. In effect, we require that for reservation values $\mathbf{r} \in \mathbb{R}^n$, voting strategies satisfy:

$$\mathbf{x} \in A_i \Longleftrightarrow u_i(\mathbf{x}) \geq r_i,\ i \in N \tag{1}$$

i.e. players approve proposals *if and only if* they weakly prefer them over their reservation value. Following Baron and Kalai, 1993, we call such voting strategies *stage-undominated*.

A PSSP equilibrium in stage-undominated voting strategies is an n-tuple of stationary strategies $\sigma_i, i \in N$ such that A_i are stage undominated given the corresponding reservation values, and players are sequentially rational. It can be shown (see Banks and Duggan, 2000) that all SSP equilibria in the class of games we consider involve no delay, i.e. all equilibrium proposals are approved given equilibrium voting strategies. In what follows we shall omit reference to the refinement on voting strategies and the no-delay property of equilibria for compactness. Thus, a PSSP equilibrium will be taken to imply the use of stage-undominated voting strategies and involve no delay.

3. AN EXAMPLE WITH A CONTINUUM OF PSSP

In this section we shall provide an example of a game satisfying our assumptions that has a continuum of PSSP equilibria. It involves four players bargaining under unanimity rule over agreements drawn from a one-dimensional policy space. This example illustrates the gap between the conditions that ensure uniqueness of equilibrium in the unanimity games of Merlo and Wilson, 1995, and agreement spaces typical in political environments.

[insert figure 1 about here]

Example 1 *(See Figure 1) Assume* $X = \left[-\sqrt{\frac{7}{6}}, \sqrt{\frac{7}{6}}\right]$, $N = \{1,2,3,4\}$, $\mathcal{D} = \{N\}$, $\pi_i = \frac{1}{4}$, $i \in N$, $\delta_i = \frac{1}{5}$, $i = 1,4$, $\delta_i = \frac{4}{5}$, $i = 2,3$, *and the following utility functions over* $x \in X$: $u_1(x) = -x^2 - 3x + 10$, *and*

$$u_2(x) = \begin{cases} -x^2 + \frac{7}{2} & \text{if} \quad x \in \left[-\sqrt{\frac{7}{6}}, 0\right] \\ -\frac{1}{4}x^{30} - \frac{1}{4}x^2 + \frac{7}{2} & \text{if} \quad x \in [0,1] \\ -4x^2 + 7 & \text{if} \quad x \in \left[1, \sqrt{\frac{7}{6}}\right] \end{cases}$$

Also $u_4(x) = u_1(-x)$ *and* $u_3(x) = u_2(-x)$*. Notice that* $u_i, i = 1,2,3,4$ *are strictly concave and in fact* C^1 *and that* $u_i > 0$*, for all* $x \in X$*.*

There is a continuum of PSSP equilibria where players 2 and 3 propose 0, player 1 proposes $\alpha \in \left[-\sqrt{\frac{7}{6}}, -1\right]$*, while player 4 proposes* $-\alpha$*. Indeed, with these proposals, players' reservation values are as follows:*

$$\begin{aligned} r_1(\alpha) &= \tfrac{1}{5}\left(\tfrac{1}{2}u_1(0) + \tfrac{1}{4}u_1(\alpha) + \tfrac{1}{4}u_1(-\alpha)\right) = 2 - \tfrac{1}{10}\alpha^2 \\ r_2(\alpha) &= \tfrac{4}{5}\left(\tfrac{1}{2}u_2(0) + \tfrac{1}{4}u_2(-\alpha) + \tfrac{1}{4}u_2(\alpha)\right) = -\alpha^2 + \tfrac{7}{2} = u_2(\alpha) \end{aligned}$$

It is straightforward to verify that $u_1(z) > r_1(\alpha)$*, for* $z \in \{\alpha, 0, -\alpha\}$ *and that* $u_2(-\alpha) > r_2(\alpha)$ *for* $\alpha \in \left(-\sqrt{\frac{7}{6}}, -1\right]$*, while* $u_2\left(\sqrt{\frac{7}{6}}\right) = r_2\left(-\sqrt{\frac{7}{6}}\right)$*. Thus, the unanimously acceptable proposals are* $[\alpha, -\alpha] \subseteq X$*, when* $\alpha \in \left(-\sqrt{\frac{7}{6}}, -1\right]$*. For these acceptable agreements, proposal strategies are optimal. Hence these strategies constitute a PSSP for each* $\alpha \in \left[-\sqrt{\frac{7}{6}}, -1\right]$*.*

Note that we give an example with an oligarchic voting rule, but this example can be modified to obtain a continuum of equilibria with non-oligarchic rules, such as majority rule. Also the example admits a number of other modifications, such as a fifth player with a 'bliss' point at zero (the median), additional policy dimensions, etc. The key to the result is that the balanced contraction of the two extreme equilibrium proposals (α and $-\alpha$) produces a net change in the reservation values of players 2 and 3 that is exactly equal to the change in utility from the one of these two proposals that renders each player exactly indifferent.

For this to occur under unanimity in the presence of discounting, utility must change at a much faster rate at the agreement which these players strictly approve as shown in Figure 1. Clearly, if players 2 and 3 are strictly indifferent between both proposals α and $-\alpha$, such differential rate of change in utilities is not possible without destroying the feasibility of one of the two proposals. Thus, this type of multiplicity of equilibrium points cannot emerge under the conditions that ensure uniqueness of equilibrium in Merlo and Wilson, 1995. In their analysis, the assumption that bargaining amounts to the division of a (stochastic) cake and a contraction condition ensure that players other than the proposer receive exactly their reservation value under all equilibrium agreements.

In the next section we shall show that the pathological behavior of the equilibrium set in the above example can occur for at most a set of Lebesgue measure zero in the space of discount factors.

4. GENERIC REGULARITY

We have divided this section into four subsections. We start with subsection (i), in which we offer a characterization of PSSP equilibria. The advantage of this characterization is that it reduces the expression of equilibrium to a finite number of equalities and inequalities. In subsection (ii), we use this characterization in order to apply the degree theory for non-smooth equations developed by Shannon, 1994, on a weaker notion of equilibrium that ignores the inequalities among equilibrium conditions. We define regular games Γ_δ, and develop an index theory for the PSSP equilibria of regular games. In subsection (iii), we establish sufficient conditions on players' utilities that ensure that almost all such games are regular. We conclude this section in subsection (iv), where we provide a sufficient condition for uniqueness of PSSP equilibrium in oligarchic games.

i. Agenda Setting Plans and PSSP

In order to state our characterization of PSSP equilibria, we introduce some necessary concepts and notation. First, we construct a space of possible continuation values by considering all possible lotteries over proposals that may prevail in period $t+1$ if delay occurs in period t. Let $\mathcal{P}[X]$ be the set of Borel probability measures over X. We obtain the space of possible continuation values, V, as the image $\mathbf{v}(\mathcal{P}[X])$ of the mapping $\mathbf{v} : \mathcal{P}[X] \longrightarrow \mathbf{R}^n$, where $v_i(\mu) \equiv \int_X u_i(\mathbf{x})\, \mu(d\mathbf{x})$, $\mu \in \mathcal{P}[X]$.[4]

Next we define the set of reservation values for game $\Gamma_{\boldsymbol{\delta}}$. We denote this set by $R_{\boldsymbol{\delta}}$, since it depends on discount factors $\boldsymbol{\delta} \in D$ by the definition of the reservation values $r_i \equiv \delta_i v_i$. Thus we have:

$$R_{\boldsymbol{\delta}} \equiv \{\mathbf{x} \in \mathbb{R}^n : x_i = \delta_i v_i, \text{ all } i \in N, \text{ all } \mathbf{v} \in V\}$$

We also define a set $R \subset \mathbb{R}^n$ that contains all possible reservation values for all possible discount factors $\boldsymbol{\delta} \in D$. Specifically,

$$R \equiv \bigcup_{\delta \in D} R_{\boldsymbol{\delta}}.$$

[4] Even though we focus on pure proposal strategies, the need to consider lotteries over proposals arises from the fact that proposers are chosen probabilistically.

Next consider the subset of winning coalitions that include player i and are *minimum winning* in the sense that if any player $j \neq i$ is removed from the coalition, the coalition ceases to be a winning coalition. Denote this set of coalitions by $\Xi_i \subset \mathcal{D}$, more formally defined as[5]

$$\Xi_i \equiv \{C \in \mathcal{D} : i \in C \text{ and } C \setminus \{j\} \notin \mathcal{D}, \forall j \in C, j \neq i\}$$

Let the number of coalitions in Ξ_i be $\xi_i \equiv |\Xi_i|$. By non-emptiness and monotonicity of the agreement rule, $\mathcal{D}$, we are guaranteed that $\xi_i \geq 1$, for all $\in N$. In what follows we index the elements of Ξ_i, $C_\omega \in \Xi_i$, by $\omega = 1, ..., \xi_i$.

We now introduce the concept of an *agenda setting plan.* Roughly speaking, an agenda setting plan specifies optimal agreements for the proposer that meet the approval of a particular winning coalition for all possible reservation values that members in that coalition may have. We denote the agenda setting plan of proposer i and coalition $C_\omega \in \Xi_i$ by f^i_ω. Formally it is a correspondence $f^i_\omega : R \rightrightarrows X$, given by

$$f^i_\omega (\mathbf{r}) \equiv \arg\max_{\mathbf{x}} \{u_i (\mathbf{x}) : u_j (\mathbf{x}) \geq r_j, j \in C_\omega\}.$$

The significance of agenda setting plans becomes obvious from the following lemma:

Lemma 1 *Consider any reservation values $\mathbf{r} \in R_\delta$ and determine voting strategies A_i according to (1). Then,*

$$\arg\max \left\{u_i (\mathbf{x}) : \mathbf{x} \in \bigcup_{C\in\mathcal{D}} \left[\bigcap_{h\in C} A_h\right]\right\} \subseteq \bigcup_{\omega=1}^{\xi_i} f^i_\omega (\mathbf{r}) \tag{2}$$

Proof. It suffices to show that optimal agreements $\mathbf{y} \in \arg\max \left\{u_i (\mathbf{x}) : \mathbf{x} \in \bigcup_{C\in\mathcal{D}} \left[\bigcap_{h\in C} A_h\right]\right\}$ that are acceptable by any coalition $C \in \mathcal{D}$ such that $C \notin \Xi_i$ are also acceptable by some coalition $C_\omega \in \Xi_i$, so that $\mathbf{y} \in f^i_\omega (\mathbf{r})$. Since $\mathbf{y}$ is optimal, $u_i (\mathbf{y}) \geq u_i (\mathbf{x}) \geq r_i$, for all $\mathbf{x} \in \bigcup_{\omega=1}^{\xi_i} f^i_\omega (\mathbf{r})$. By the monotonicity of the agreement rule $C \bigcup \{i\} \in \mathcal{D}$ and there exists $C_\omega \subseteq C \bigcup \{i\}$ such that $C_\omega \in \Xi_i$, by the definition of Ξ_i. Thus, $\mathbf{y} \in \bigcap_{h\in C_\omega} A_h$ hence $\mathbf{y} \in f^i_\omega (\mathbf{r})$, since $\mathbf{y} \in \arg\max \left\{u_i (\mathbf{x}) : \mathbf{x} \in \bigcup_{C\in\mathcal{D}} \left[\bigcap_{h\in C} A_h\right]\right\}$. ■

In general, an agenda setting plan f^i_ω is a multi-valued correspondence. But in certain modal cases, as we establish in lemma 3 at the end of subsection (iii), agenda setting plans f^i_ω are functions. In those cases, we can characterize PSSP equilibria by representing the proposal strategy of player

[5] Note that coalitions in Ξ_i need not be minimum winning in the traditional sense since for coalition $C \in \Xi_i$ it may be that $C \setminus \{i\} \in \mathcal{D}$, *i.e.* the coalition is still winning when the proposer is removed.

$i \in N$ as a choice of coalition from Ξ_i for given reservation values, instead of a choice of agreement from X. Let $\Omega \equiv \{1, ..., \xi_i\} \times_{i=1}^{n}$, so that $\boldsymbol{\omega} \in \Omega$ represents a choice of coalition from Ξ_i by each of the n players $i \in N$. We have:

Theorem 1 *Consider game $\Gamma_{\boldsymbol{\delta}}$ for which f_{ω}^{i} is a function for all $i \in N$, $\omega = 1, ..., \xi_i$. Strategies $\{\mathbf{z}_i^*, A_i^*\}_{i=1}^{n}$ are a PSSP equilibrium of game $\Gamma_{\boldsymbol{\delta}}$ if and only if there exists a vector of coalition choices $\boldsymbol{\omega} \in \Omega$ and a vector of reservation values $\mathbf{r}^* \in R_{\boldsymbol{\delta}}$, such that $\mathbf{z}_i^* = f_{\omega_i}^{i}(\mathbf{r}^*)$, $i \in N$, and:*

$$r_i^* = \delta_i \sum_{h=1}^{n} \pi_h u_i \left(f_{\omega_h}^{h}(\mathbf{r}^*) \right), \textit{ for all } i \in N, \tag{3}$$

$$u_i \left(f_{\omega_i}^{i}(\mathbf{r}^*) \right) \geq u_i \left(f_{\omega'}^{i}(\mathbf{r}^*) \right), \textit{ for all } \omega' = 1, ..., \xi_i,\ i \in N \tag{4}$$

Proof. ($\Longrightarrow$) Suppose strategies $\{\mathbf{z}_i^*, A_i^*\}_{i=1}^{n}$ form a PSSP equilibrium. We shall show there exist $\boldsymbol{\omega} \in \Omega$, $\mathbf{r}^* \in R_{\boldsymbol{\delta}}$ such that $\mathbf{z}_i^* = f_{\omega_i}^{i}(\mathbf{r}^*)$, $i \in N$, and (3) and (4) hold. Construct the reservation values $\mathbf{r}^* \in R_{\boldsymbol{\delta}}$ according to

$$r_i^* = \delta_i \sum_{h=1}^{n} \pi_h u_i(\mathbf{z}_i^*), \text{ for all } i \in N.$$

For these reservation values $\mathbf{r}^*$, we now obtain voting strategies A_i' according to (1). Since $\{\mathbf{z}_i^*, A_i^*\}_{i=1}^{n}$ is a no-delay equilibrium (Banks and Duggan, 2000, theorem 1(ii)), we must have $A_h' = A_h^*$ and

$$\mathbf{z}_i^* \in \arg\max \left\{ u_i(\mathbf{x}) : \mathbf{x} \in \bigcup\nolimits_{C \in \mathcal{D}} \left[\bigcap\nolimits_{h \in C} A_h' \right] \right\}.$$

Thus, we must have $\mathbf{z}_i^* \in \left\{ f_1^i(\mathbf{r}^*), ..., f_{\xi_i}^i(\mathbf{r}^*) \right\}$, by lemma 1. Thence, there exists $\omega_i^* \in \{1, ..., \xi_i\}$ such that $u_i\left(f_{\omega_{i^*}}^{i}(\mathbf{r}^*)\right) \geq u_i\left(f_{\omega}^{i}(\mathbf{r}^*)\right)$, for all $\omega = 1, ..., \xi_i$, all $i \in N$. Set $\boldsymbol{\omega} = (\omega_1^*, ..., \omega_n^*)$, and this part of the proof is complete.

($\Longleftarrow$) Consider $\boldsymbol{\omega} \in \Omega$, $\mathbf{r}^* \in R_{\boldsymbol{\delta}}$ that satisfy (3) and (4). Construct stationary strategies $\{\mathbf{z}_i', A_i'\}_{i=1}^{n}$ by setting $\mathbf{z}_i' = f_{\omega_i}^{i}(\mathbf{r}^*)$ and $A_i' = \{\mathbf{x} \in X : u_i(\mathbf{x}) \geq r_i^*\}$. We wish to show that $\{\mathbf{z}_i', A_i'\}_{i=1}^{n}$ form a PSSP equilibrium. Condition (4) and lemma 1 ensure that

$$\mathbf{z}_i' \in \arg\max \left\{ u_i(\mathbf{x}) : \mathbf{x} \in \bigcup\nolimits_{C \in \mathcal{D}} \left[\bigcap\nolimits_{h \in C} A_h' \right] \right\}$$

and A_h' are stage-undominated. As a result, strategies $\{\mathbf{z}_i', A_i'\}_{i=1}^{n}$ do not admit profitable one period deviations and constitute a PSSP equilibrium by the one-stage deviation principle. ■

The characterization in theorem 1 allows us to study the PSSP equilibrium set by applying homotopy arguments on the n equations determined by (3). For that purpose we define a weaker notion of equilibrium:

Definition 2 *Consider game $\Gamma_{\boldsymbol{\delta}}$ for which f^i_ω is a function for all $i \in N$, $\omega = 1, ..., \xi_i$. A pseudo-PSSP equilibrium is a vector of reservation values $\mathbf{r}^* \in R$ and a vector of coalition choices $\boldsymbol{\omega} \in \Omega$ such that,*

$$r^*_i = \delta_i \sum_{h=1}^{n} \pi_h u_i \left(f^h_{\omega_h} (\mathbf{r}^*) \right), \textit{ for all } i \in N. \tag{5}$$

Clearly, from theorem 1 and definition 2, genuine no-delay equilibria differ from pseudo-PSSP in that the latter do not necessarily satisfy the inequalities in (4). Thus, every PSSP equilibrium is a pseudo-PSSP equilibrium but not *vice-versa*. In the special case of oligarchic games, we have $\xi_i = 1, i \in N$, hence Ω is a singleton and pseudo-PSSP and PSSP equilibria coincide.

ii. Regular Games

Our analysis has permitted the expression of (pseudo)equilibria as solutions to systems of equations. In this subsection we shall use this formulation to study the behavior of (pseudo)PSSP equilibria of game $\Gamma_{\boldsymbol{\delta}}$ and derive an index theory for the equilibrium set.

Define the set of solutions to equations (5) for given $\boldsymbol{\omega} \in \Omega$ by $E^{\boldsymbol{\omega}}_{\boldsymbol{\delta}} \equiv \{\mathbf{r} \in R_{\boldsymbol{\delta}}$: (5) holds$\}$. Next, assuming f^i_ω is a function for all $i \in N$, $\omega = 1, ..., \xi_i$, define for each $\boldsymbol{\omega} \in \Omega$ the mapping $G^{\boldsymbol{\omega}} : R \longrightarrow \mathbb{R}^n$ as:[6]

$$G^{\boldsymbol{\omega}}_i (\mathbf{r}) \equiv \frac{r_i}{\sum_{h=1}^{n} \pi_h u_i \left(f^h_{\omega_h} (\mathbf{r}) \right)}, \; i = 1, .., n$$

Clearly $\mathbf{r} \in E^{\boldsymbol{\omega}}_{\boldsymbol{\delta}}$ if and only if $G^{\boldsymbol{\omega}} (\mathbf{r}) = \boldsymbol{\delta}$.

Following Rader, 1973, and Shannon, 1994, we say that $y \in Y$ is a *regular value* of the function $g : X \longrightarrow Y$ if both $Dg(\mathbf{x})$ *exists and is non-singular* for all $x \in X$ such that $g(x) = y$. We say that $y \in Y$ is a *critical value* if it is not a regular value. We can show the following:

Lemma 2 *Consider game $\Gamma_{\boldsymbol{\delta}}$ and assume f^i_ω a continuous function for all $i \in N$, $\omega = 1, ..., \xi_i$. If $\boldsymbol{\delta} \in D$ is a regular value of $G^{\boldsymbol{\omega}}$, for some $\boldsymbol{\omega} \in \Omega$, then*

(i) $|E^{\boldsymbol{\omega}}_{\boldsymbol{\delta}}| < +\infty$, is an odd number, and

[6] Recall that $u_i(\mathbf{x}) > 0$ for all $\mathbf{x} \in X$, $i \in N$, so that $\sum_{h=1}^{n} \pi_h u_i \left(f^h_{\omega_h} (\mathbf{r}) \right) > 0$.

(ii) $E^{\boldsymbol{\omega}}_{\boldsymbol{\delta}}$ *is lower-hemicontinuous at* $\boldsymbol{\delta}$*.*

Proof. Let ∂R denote the boundary of R.[7] We claim first that:

Claim: *For each* $\delta \in D$, $R_{\boldsymbol{\delta}} \bigcap \partial R = \varnothing$*.* Since $R_{\boldsymbol{\delta}} \subseteq R$, it suffices to show that R is an open set. To see that R is open, notice that R can also be expressed as the union $\bigcup_{\mathbf{v}\in V} R_{\mathbf{v}}$ of the product sets $R_{\mathbf{v}} \equiv D\{\mathbf{v}\}$, $\mathbf{v} \in V$. Clearly, $R_{\mathbf{v}} = (0, v_i) \times_{i=1}^{n}$, and since $v_i > 0$ for all $\mathbf{v} \in V$, $i \in N$, $R_{\mathbf{v}}$ is open. Hence, R is open as the union of open sets.

Now define $G^1 : R \longrightarrow \mathbb{R}^n$ as $G^1(\mathbf{r}) \equiv G^{\boldsymbol{\omega}}(\mathbf{r}) - \boldsymbol{\delta}$. Since X is compact and u_i continuous, $\underline{u}_i \equiv \min_{\mathbf{x}\in X} u_i(\mathbf{x})$ is well defined and $\underline{u}_i > 0$. Let $G^0 : R \longrightarrow \mathbb{R}^n$ be given by $G^0_i(\mathbf{r}) \equiv \dfrac{r_i}{\underline{u}_i} - \delta_i$, $i = 1, ..., n$. Consider the continuous function $H_t = (1-t)\, G^0 + tG^1$. Let $H_t^{-1}(\mathbf{0}) \equiv \{\mathbf{r} \in R : H_t(\mathbf{r}) = \mathbf{0}\}$. In order to apply homotopy arguments, we wish to show that $H_t^{-1}(\mathbf{0}) \bigcap \partial R = \varnothing$ for all $t \in [0,1]$.

For $\mathbf{r} \in H_t^{-1}(\mathbf{0})$ we have $(1-t)\, G^0 + tG^1 = \mathbf{0} \Longrightarrow r_i = \dfrac{\delta_i \underline{u}_i \sum_{h=1}^{n} \pi_h u_i \left(f^h_{\omega_h}(\mathbf{r})\right)}{(1-t)\sum_{h=1}^{n} \pi_h u_i \left(f^h_{\omega_h}(\mathbf{r})\right) + t\underline{u}_i}$. Set $\delta'_i \equiv \dfrac{\delta_i \underline{u}_i}{(1-t)\sum_{h=1}^{n} \pi_h u_i \left(f^h_{\omega_h}(\mathbf{r})\right) + t\underline{u}_i}$, $i = 1, ..., n$, so that we have $r_i = \delta'_i \sum_{h=1}^{n} \pi_h u_i \left(f^h_{\omega_h}(\mathbf{r})\right)$. We must have $\sum_{h=1}^{n} \pi_h u_i \left(f^h_{\omega_h}(\mathbf{r})\right) \geq \underline{u}_i$ for all $i \in N$, hence we have $\delta'_i \in (0, \delta_i]$ for all $i \in N$ and all $t \in [0,1]$. Then, $\boldsymbol{\delta}' \in D$ and we have $\mathbf{r} \in R_{\boldsymbol{\delta}'}$. But $R_{\boldsymbol{\delta}'} \bigcap \partial R = \varnothing$, by the claim. Hence, for all $t \in [0,1]$ and all $\mathbf{r} \in H_t^{-1}(\mathbf{0})$, we have $\mathbf{r} \notin \partial R$.

Note that, since $\boldsymbol{\delta}$ is a regular value of $G^{\boldsymbol{\omega}}$ and by theorem 10, page 159, in Shannon, 1994, the *degree* of $G^{\boldsymbol{\omega}}$ is given by $d(G^{\boldsymbol{\omega}}, R, \boldsymbol{\delta}) \equiv \sum_{\mathbf{r}\in E^{\boldsymbol{\omega}}_{\boldsymbol{\delta}}} \mathit{sign} \det DG^{\boldsymbol{\omega}}(\mathbf{r})$. But, by homotopy invariance, $d\left(G^0, R, \mathbf{0}\right) = d\left(G^1, R, \mathbf{0}\right)$, and $d\left(G^0, R, \mathbf{0}\right) = 1$. Thence, we have $d\left(G^1, R, \mathbf{0}\right) = d(G^{\boldsymbol{\omega}}, R, \boldsymbol{\delta}) = 1$. Since R is a bounded set, we have proven (i): $G^{\boldsymbol{\omega}}$ has an odd, finite number (at least one) of solutions.

To show (ii) notice that, since $\boldsymbol{\delta}$ is a regular value, for every $\mathbf{r}_l \in E^{\boldsymbol{\omega}}_{\boldsymbol{\delta}}$, $l = 1, ..., |E^{\boldsymbol{\omega}}_{\boldsymbol{\delta}}|$ there exists a neighborhood of $\mathbf{r}_l$, $N_l \subset R$, such that $E^{\boldsymbol{\omega}}_{\boldsymbol{\delta}} \bigcap \overline{N_l} = \{\mathbf{r}_l\}$ (e.g. Shannon, 1994, theorem 1, page 150). Then, since $\boldsymbol{\delta} \notin G^{\boldsymbol{\omega}}(\partial N_l)$, there exists $\varepsilon > 0$ such that $|\boldsymbol{\delta} - \mathbf{d}| > \varepsilon$ for all $\mathbf{d} \in G^{\boldsymbol{\omega}}(\partial N_l)$. Now the degree is constant, *i.e.* $d\left(G^{\boldsymbol{\omega}}, N_l, \boldsymbol{\delta}'\right) = d(G^{\boldsymbol{\omega}}, N_l, \boldsymbol{\delta})$ for every $\boldsymbol{\delta}' \in B_\varepsilon(\boldsymbol{\delta})$, where $B_\varepsilon(\boldsymbol{\delta})$ is the ε-ball around $\boldsymbol{\delta}$. By theorem 9 of Shannon, 1994, page 158, $d(G^{\boldsymbol{\omega}}, N_l, \boldsymbol{\delta}) = \mathit{sign} \det DG^{\boldsymbol{\omega}}(\mathbf{r}_l)$. But $\mathit{sign} \det DG^{\boldsymbol{\omega}}(\mathbf{r}_l) \neq 0$ since $\boldsymbol{\delta}$ is a regular value. Thus $d\left(G^{\boldsymbol{\omega}}, N_l, \boldsymbol{\delta}'\right) \neq 0$ for every $\boldsymbol{\delta}' \in B_\varepsilon(\boldsymbol{\delta})$, *i.e.* for every $\boldsymbol{\delta}' \in B_\varepsilon(\boldsymbol{\delta})$ there exists $\mathbf{r}' \in N_l$ such that $\mathbf{r}' \in E^{\boldsymbol{\omega}}_{\boldsymbol{\delta}'}$. This establishes lower-hemicontinuity.

[7] Substantively, the boundary of R is obtained by considering values of the discount factors $\delta_i = 1$ or $\delta_i = 0$. Due to the concavity of u_i, we can certainly extend the definition of continuous functions f^i_ω, hence G^ω to the domain $\overline{R}$, instead of R.

■

Note that lemma 2 also constitutes a proof of existence of pseudo-PSSP for all $\boldsymbol{\omega} \in \Omega$. As a consequence, it also provides a proof of existence of PSSP for oligarchic games when $\xi_i = 1$ for all $i \in N$. We will say that the bargaining game $\Gamma_{\boldsymbol{\delta}}$ is *regular* if $\boldsymbol{\delta}$ is a regular value of $G^{\boldsymbol{\omega}}$, *for every* $\boldsymbol{\omega} \in \Omega$.

We can summarize the implications of our analysis for regular games $\Gamma_{\boldsymbol{\delta}}$ as follows:

Theorem 2 *Consider a regular game $\Gamma_{\boldsymbol{\delta}}$. Then*

(i) $\Gamma_{\boldsymbol{\delta}}$ has a finite number of PSSP equilibria (possibly zero),

(ii) If $X \subset \mathbb{R}^1$, $\Gamma_{\boldsymbol{\delta}}$ has a finite number (at least one) of SSP equilibria, all in pure strategies, and

(iii) if the voting rule $\mathcal{D}$, is oligarchic, $\Gamma_{\boldsymbol{\delta}}$ has an odd number of SSP equilibria and the equilibrium correspondence is lower-hemicontinuous at $\boldsymbol{\delta}$.

Proof. Since Ω is a finite set, part (i) follows directly from lemma 2.

To show part (ii), we shall show that all SSP equilibria of game $\Gamma_{\boldsymbol{\delta}}$ when $X \subset \mathbb{R}^1$ are PSSP equilibria.[8] If this is true, the result follows from part (i) since SSP equilibria for these games exist (Banks and Duggan, 2000, theorem 1). Suppose $\Gamma_{\boldsymbol{\delta}}$ has a no-delay SSP equilibrium in mixed strategies to get a contradiction. Let $\mathbf{r} \in R_{\boldsymbol{\delta}}$ be players' reservation values for this equilibrium. The voting strategies A_i^*, $i \in N$ are stage-undominated and, since u_i are concave, A_i^* and $\bigcap_{h \in C} A_h^*$, are convex for every $C \subseteq N$. Also, $\bigcup_{C \in \mathcal{D}} \left[\bigcap_{h \in C} A_h^*\right]$ is convex, since the average of proposals must belong in $\bigcap_{h \in C} A_h^*$ for every $C \in \mathcal{D}$. By (2), there exists some player i that mixes between distinct agreements $f_\omega^i(\mathbf{r}) < f_{\omega'}^i(\mathbf{r})$. Since $\left[f_\omega^i(\mathbf{r}), f_{\omega'}^i(\mathbf{r})\right] \subseteq \bigcup_{C \in \mathcal{D}} \left[\bigcap_{h \in C} A_h^*\right]$ and u_i is concave, we have $u_i(\mathbf{x}) = c$ for all $\mathbf{x} \in \left[f_\omega^i(\mathbf{r}), f_{\omega'}^i(\mathbf{r})\right]$. Thus, at least one of the two agenda setting plans is a multivalued correspondence, a contradiction.

Lastly, to show part (iii) notice that by (2) and the fact that $\xi_i = 1$ for all $i \in N$, for every reservation values $\mathbf{r} \in R_{\boldsymbol{\delta}}$ there exists a unique optimal agreement for each player. Thus, all SSP equilibria are PSSP equilibria, and the latter coincide with the pseudo-PSSP equilibria of the game.

■

[8] The argument here is identical to the argument in theorem 2, part (ii) of Banks and Duggan, 2000, who assume strictly quasi-concave utilities. Mere concavity suffices in our case because of the added assumption that agenda setting plans are functions.

The theorem provides a sharp description of the PSSP equilibrium set for regular games. With some additional arguments, we could show that lower-hemicontinuity also applies in cases (i) and (ii) of the theorem. Lower-hemicontinuity implies that PSSP equilibria of these games can be expressed as continuous functions of the parameters in a neighborhood of $\boldsymbol{\delta}$. For discounted oligarchic games, coupled with the upper-hemicontinuity result of Banks and Duggan, 2000, theorem 2 provides a complete description of the equilibrium correspondence. In effect, we have shown that regular games $\Gamma_{\boldsymbol{\delta}}$ with oligarchic rules are *essential* (Fort, 1950, Wu and Jiang, 1962) with respect to the SSP equilibrium set.

iii. Sufficient Conditions for Generic Regularity

The obvious next step is to inquire how prevalent are regular games $\Gamma_{\boldsymbol{\delta}}$ in the space of discount factors D? From the example in section 3 we cannot rule out the existence of critical values of the maps $G^{\boldsymbol{\omega}}$. Thus, at most we can hope to show that *almost all* games $\Gamma_{\boldsymbol{\delta}}$ are regular. Assuming $G^{\boldsymbol{\omega}}$ is differentiable, we could proceed to show that the set of critical values of $G^{\boldsymbol{\omega}}$ is of measure zero using Sard's theorem. But the smoothness of $G^{\boldsymbol{\omega}}$ depends on the behavior of agenda setting plans, f_{ω}^{i}. Since the latter are solutions to constrained optimization problems, they will typically not be differentiable due to changes in the set of binding constraints (or due to the non-smoothness of u_i, $i \in N$).

In the context of general equilibrium theory, this problem is analogous to the situation when the excess demand function is not differentiable (e.g. Katzner, 1968). To deal with the luck of smoothness of demand functions in that context, Rader, 1973, using a result by Sard, 1958, established the following:

Theorem 3 *(Rader, 1973) Consider $E \subset R^n$, and $f : E \longrightarrow R^n$ that is a.e. differentiable and maps sets of measure zero into sets of measure zero. Then, the set of critical values of f has measure zero.*

Functions that are a.e. differentiable and map sets of measure zero into sets of measure zero include Lipschitz and locally Lipschitz functions (see Federer, 1969). Rader, 1973, (lemma 3, page 918) showed that these requirements are also met by functions that are pointwise Lipschitz at every point in an open domain. The pointwise Lipschitz property is a stability to perturbations property also known as *calmness* or *local upper Lipschitz* property in the mathematical optimization literature. It is weaker than (local) Lipschitz continuity in that one of the two points of comparison is held fixed:

Definition 3 *A function $f : X \longrightarrow Y$ is pointwise Lipschitz at $x \in X$ if there exists m and a neighborhood $N(x)$ such that $|f(x) - f(z)| \leq m|x - z|$ for all $z \in N(x)$.*

Thus we can state sufficient conditions for almost all games $\Gamma_{\boldsymbol{\delta}}$ to be regular by imposing conditions on the agenda setting plans f_ω^i. In particular:

Theorem 4 *Assume f_ω^i is a pointwise Lipshcitz function at every $\mathbf{r} \in R$, for all $i \in N$, $\omega = 1, ..., \xi_i$. Then, for almost all $\boldsymbol{\delta} \in D$ game $\Gamma_{\boldsymbol{\delta}}$ is regular.*

Proof. By theorem 3, it will suffice to show that $G^{\boldsymbol{\omega}}$ is pointwise Lipschitz for all $\boldsymbol{\omega} \in \Omega$. This is because then the set of critical values of $G^{\boldsymbol{\omega}}$ has measure zero and Ω is a finite set. Since the finite union of sets of measure zero has measure zero, then $\Gamma_{\boldsymbol{\delta}}$ is a regular game for almost all $\boldsymbol{\delta} \in D$.

Since u_i is C^1, the composition $u_i \circ f_\omega^i$ is pointwise Lipschitz for all $i \in N$ and all f_ω^i. Also, the sum of pointwise Lipschitz functions, $\sum_{h=1}^n \pi_h u_i \left(f_{\omega_h}^h(\mathbf{r})\right)$ is also pointwise Lipschitz. Define $g_i(\mathbf{r}) \equiv \sum_{h=1}^n \pi_h u_i \left(f_{\omega_h}^h(\mathbf{r})\right)$, and note that $g_i(\mathbf{r}) > 0$ for all $\mathbf{r} \in R$. Thus, we have $\left|\dfrac{1}{g_i(\mathbf{r})} - \dfrac{1}{g_i(\mathbf{r}')}\right| = \left|\dfrac{g_i(\mathbf{r}') - g_i(\mathbf{r})}{g_i(\mathbf{r})\, g_i(\mathbf{r}')}\right| = \dfrac{1}{g_i(\mathbf{r})\, g_i(\mathbf{r}')} |g_i(\mathbf{r}') - g_i(\mathbf{r})| \leq \dfrac{m}{g_i(\mathbf{r})\, g_i(\mathbf{r}')} |\mathbf{r} - \mathbf{r}'|$ for all $\mathbf{r}' \in N(\mathbf{r})$, for some neighborhood $N(\mathbf{r})$ of $\mathbf{r}$, and some $m > 0$ (since $g_i(\mathbf{r})$ is pointwise Lipschitz). Recall that $\underline{u}_i = \min\{u_i(\mathbf{x}) : \mathbf{x} \in X\} > 0$. Setting $m' = \dfrac{m}{g_i(\mathbf{r})\, \underline{u}_i}$, the function $\dfrac{1}{g_i(\mathbf{r})}$ is pointwise Lipschitz at $\mathbf{r}$ with modulus m'. Then, $G^{\boldsymbol{\omega}}(\mathbf{r}) = \dfrac{r_i}{g_i(\mathbf{r})}$ is pointwise Lipschitz as the product of a pointwise Lipschitz function and r_i, and the proof is complete. ■

Since agenda setting plans typically do not form part of the description of the bargaining games we analyze, it is important to also establish conditions on the primitives of the model, specifically the utility functions u_i, that ensure that almost all games $\Gamma_{\boldsymbol{\delta}}$ are regular. There is a large literature on the stability of solutions to perturbed optimization problems (e.g. Robinson, 1982, Bonnans and Shapiro, 1998, Klatte and Kummar, 1999, Levy, 2000, 2001, Klatte, 2001, etc.) establishing conditions on the smoothness of the objective function and the constraints of the problem (in our case u_i) that guarantee that f_ω^i is pointwise Lipschitz or calm.

Indispensable in all existing results are (a) the Mangasarian-Fromowitz constraint qualification (MFCQ), and (b) a second order sufficient condition that involves generalized notions of derivatives if u_i are not twice continuously differentiable. Since introducing such concepts at this point would detain the reader from the eminent culmination of our investigation, we shall briefly discuss these stronger results at the end of this subsection. We can now show:

Lemma 3 *Assume either*

(A1) u_i is at least C^2 with negative definite second derivative, $D^2 u_i(\mathbf{x})$ for all $\mathbf{x} \in X$, $i \in N$, and the Pareto set of X, $P(X) \subset intX$, or

(A2) $X = \Delta^{n-1}$ and, for all $i \in N$, $u_i(\mathbf{x}) = m_i(x_i)$ where $m_i : [0,1] \longrightarrow \mathbb{R}$, with $m_i'(x) > 0$ and pointwise Lipschitz inverse $m_i^{-1}(x)$ for all $x \in [m_h(0), m_h(1)]$.

Then, for all $i \in N$, $\omega = 1, ..., \xi_i$, $f_\omega^i : R \rightrightarrows X$ is a pointwise Lipschitz function at all $\mathbf{r} \in R$.

Proof. We start with assumption A1 first. Recall that f_ω^i solves the program:

$$\max u_i(\mathbf{x}) \;\; s.t. \tag{AS}$$

$$u_h(\mathbf{x}) \geq r_h,\; h \in C_\omega$$

$$\mathbf{x} \in X$$

Consider the correspondence $A_{C_\omega} : R \rightrightarrows X$ defined by $A_{C_\omega} \equiv \{\mathbf{x} \in X : u_h(\mathbf{x}) \geq r_h, h \in C_\omega\}$. Concavity of u_i, $i \in N$ ensures $A_{C_\omega}(\mathbf{r})$ is convex valued since it is the intersection of convex sets. It is also non-empty for all $\mathbf{r} \in R$, since for every measure $\mu \in \mathcal{P}(X)$ that induces the continuation value $\mathbf{v} \in V$ that corresponds to $\mathbf{r}$, we have that $u_i\left(\int_X \mathbf{x}(d\mu)\right) \geq v_i$, by the concavity of u_i and the fact that $\int_X \mathbf{x}(d\mu) \in X$ by the convexity of X. In fact, since $\delta_i < 1$ for all $i \in N$, we have $v_i > r_i$ and thus A_{C_ω} has non-empty interior. Since u_i, $i \in N$, are continuous without "thick" indifference contours, $A_{C\omega}$ is continuous as a correspondence. Upper-hemicontinuity of f_ω^i then follows by Berge's theorem of the Maximum. By strict concavity of u_i, $i \in N$, we have uniqueness of the maximizer, for every $\mathbf{r} \in R$, thus f_ω^i is a continuous function.

To show that f_ω^i is pointwise Lipschitz at $\mathbf{r} \in R$, note that $P(X) \subset intX$ allows us to ignore the constraint $\mathbf{x} \in X$. For the remaining constraints in program (AS) the MFCQ amounts to Slater's constraint qualification (SCQ): there exists $\mathbf{x} \in X$ such that $u_h(\mathbf{x}) > r_h$ for all $h \in C_\omega$. But we've already argued that $A_{C_\omega}(\mathbf{r})$ has non-empty interior so that there exists $\mathbf{x} \in X$ such that $u_h(\mathbf{x}) > r_h$

for all $\mathbf{r} \in R$. Hence, (SCQ) is satisfied for all $\mathbf{r} \in R$. Let $L = u_i(\mathbf{x}) + \theta_h \sum_{h \in C_\omega} (u_h(\mathbf{x}) - r_h)$ be the Lagrangian of program (AS), ignoring constraint $\mathbf{x} \in X$. The second order sufficient condition: $D^2_{\mathbf{x}} L = D^2 u_i(\mathbf{x}) + \theta_h \sum_{h \in C_\omega} D^2 u_h(\mathbf{x})$ is negative definite, holds *for all* possible Lagrange multipliers of program (AS), since $D^2 u_i(\mathbf{x})$ is negative definite for all $i \in N$, and since $\theta_h \geq 0$ for all $h \in C_\omega$. As a result, for every $\mathbf{r} \in R$ the solution mapping f^i_ω is pointwise Lipschitz by *e.g.* Klatte and Kummer, 1999, corollary 5.5, page 82, (see also Levy, 1999, corollary 5.2, page 439, Levy, 2001, proposition 3.2.1, page 21).

Now consider assumption A2. We have $\sum_{i=1}^n m_i^{-1}(\max\{m_i(0), r_i\}) < 1$, by the concavity of u_i (hence m_i). Then, f^i_ω takes the following form

$$f^i_\omega(\mathbf{r}) = \begin{cases} 0 & if \quad h \notin C_\omega \\ m_h^{-1}(\max\{m_h(0), r_h\}) & if \quad h \in C_\omega \setminus \{i\} \\ 1 - \sum_{l \in C_\omega \setminus \{j\}} m_l^{-1}(\max\{m_l(0), r_l\}) & if \quad h = i \end{cases}, \; i \in N, \, \omega = 1, ..., \xi_i \qquad (6)$$

which is obviously a pointwise Lipschitz function. ■

Note that assumption A1 covers typical spatial models assumed in political applications, while assumption A2 admits divide-the-dollar environments with private goods. In combination, the two assumptions cover most applications in the literature.

We immediately have the following corollary:

Corollary 1 *Assume either (A1) or (A2). Then, for almost all* $\boldsymbol{\delta} \in D$ *game* $\Gamma_{\boldsymbol{\delta}}$ *is regular.*

We have already hinted that we can weaken condition (A1). For instance, condition (A1) does not cover example 1 in section 3, because utilities there are strictly concave and C^1, but not C^2. To accommodate similar cases, we could replace (A1) with:

(A1′) u_i is $C^{1,1}$, and for all $H \in CDu'_i$, H is negative definite...,

where CDu'_i is the *contingent derivative* of the derivative of u_i, u'_i. This alternative assumption (for details see Klatte and Kummer, 1999, and Klatte, 2001) would also secure that f^i_ω is pointwise Lipschitz at $\mathbf{r} \in R$.

We assumed throughout that u_i is C^1, and it seems doubtful whether we can further relax this assumption in general settings. The same applies for the assumption of concavity, since it is necessary to ensure that players do not wish to delay the game. Although we find the assumption that $P(X) \subset intX$ completely natural from casual observation of political interaction, it can also be significantly weakened. The role of this assumption is to ensure that there are no optimal agreements at the boundary of X. If we permit such boundary optima, we can alternatively impose smoothness conditions on the boundary of X.

iv. Uniqueness of PSSP

Lastly, in this subsection we establish sufficient conditions for uniqueness of equilibrium in oligarchic games. We only focus on oligarchic games, since in the remaining cases the fact that there are multiple coalition choices available to players ($|\Omega| > 1$) creates natural possibilities for multiple equilibria even if each combination of coalition choices $\boldsymbol{\omega}$ admits a unique pseudo-equilibrium. To illustrate our uniqueness condition, we apply it on a divide-the-dollar game with linear utilities and recover the uniqueness result of Merlo and Wilson, 1995. Their uniqueness theorem applies for a larger class of games and relies on a contraction mapping theorem, while ours is intimately related to the analysis in subsection (ii).

Since the voting rule $\mathcal{D}$ is oligarchic, we have $\xi_i = 1$ and there is a single agenda setting plan we denote by f^i for each player $i \in N$. For the purposes of this subsection, we work directly with equations (5) instead of the equivalent mapping $G^{\boldsymbol{\omega}}$. We thus define

$$F_i(\mathbf{r}) \equiv r_i - \delta_i \sum_{h=1}^{n} \pi_h u_i\left(f^h(\mathbf{r})\right), \; i = 1, .., n.$$

SSP equilibria are now obtained as the solutions to $F(\mathbf{r}) \equiv \mathbf{0}$, and our uniqueness condition can be stated as follows (see for example Shannon, 1994, corollary 11):

Theorem 5 *If* $\mathbf{0}$ *is a regular value of* $F(\mathbf{r})$*, then the oligarchic game* $\Gamma_{\boldsymbol{\delta}}$ *has a unique SSP equilibrium if* $\det D_{\mathbf{r}} F(\mathbf{r}) > (<) \, 0$ *for all* $\mathbf{r} \in R$.

As an application, consider a divide-the-dollar unanimity game with $X = \Delta^{n-1}$, and $u_i(\mathbf{x}) = x_i$, for all $i \in N$.[9] The agenda setting plans for this game are given by:

$$f^i(\mathbf{r}) = \begin{cases} r_h & if \quad h \neq i \\ 1 - \sum_{h \neq i} r_h & if \quad h = i \end{cases}$$

[9] We allow $u_i(\mathbf{x}) = 0$, which is not consequential for our arguments in this example.

Hence, we formulate $F(\mathbf{r})$ as

$$F_i(\mathbf{r}) \equiv r_i - \delta_i \left[(1-\pi_i) r_i + \pi_i \left(1 - \sum_{h \neq i} r_h \right) \right]$$

We now calculate the Jacobian $D_{\mathbf{r}} F(\mathbf{r})$ as:

$$D_{\mathbf{r}} F(\mathbf{r}) = \begin{bmatrix} 1-\delta_1+\delta_1\pi_1 & \delta_1\pi_1 & \cdots & \delta_1\pi_1 \\ \delta_2\pi_2 & 1-\delta_2+\delta_2\pi_2 & \cdots & \delta_2\pi_2 \\ \vdots & \vdots & & \vdots \\ \delta_n\pi_n & \delta_n\pi_n & \cdots & 1-\delta_n+\delta_n\pi_n \end{bmatrix}$$

Via a series of elementary operations and making use of the properties of the determinant (or by induction) we can obtain

$$\det D_{\mathbf{r}} F(\mathbf{r}) = \prod_{i=1}^{n} (1-\delta_i) + \sum_{i=1}^{n} \delta_i \pi_i \prod_{h \neq i} (1-\delta_h)$$

Hence, a sufficient condition for unique equilibrium is that $(1-\delta_i) > 0$ for each $i \in N$. This is true for each $\boldsymbol{\delta} \in D$. Furthermore, if we allow $\delta_i = 1$ for only one $i \in N$, then $\pi_i > 0$ is also sufficient for a unique equilibrium.

5. CONCLUSIONS

We conclude with a few remarks hinting on interesting avenues for strengthening or weakening the conclusions of our analysis under alternative assumptions. First, under the stronger condition that the mapping $G^{\boldsymbol{\omega}}$, is locally Lipschitz, Shannon, 1994, has shown that the equilibrium correspondence is upper Lipschitzian at regular values $\boldsymbol{\delta}$. Unfortunately, as discussed by Robinson, 1982, page 218-219, solutions to rather unspectacular optimization problems may fail to be locally Lipschitz to canonical perturbations. Thus, to deduce the property we would need stronger conditions, such as the linear independence of the binding constraints at an optimum, which typically fail in our problem. Interestingly, this last remark implies that one environment where such stronger results may be possible is when the dimension of the policy space X is at least as large as the size of the (largest minimum) winning coalition minus one.

Second, there is an intimate connection between the theory we have developed in this paper and studies of the equilibrium set in the theory of general economic equilibrium. For instance, Kleinberg, 1980, Liang, 1993, Pascoa and Ribeiro da Costa Werlang, 1999, have weakened conditions on individual utility to obtain (weaker) properties of the equilibrium set. It appears natural that analogous results can be established in our framework.

Extending their earlier work (Banks and Duggan, 2000), Banks and Duggan, 2003, have developed a discounted bargaining model closely related to the one we analyze that dispenses with the assumption that agreements are desirable. In their model, a status quo remains in place until it is replaced by a new agreement that is implemented in all subsequent periods. It is relevant to inquire whether analogous results can be obtained for this alternative model. A stumbling block appears to be the fact that they require a common discount factor, and our genericity result is obtained from perturbation of these individual parameters. But our analysis only requires existence of equilibrium for an open subset of discount factors, which may obtain in such games with general status quo.

Perhaps the most interesting challenge, which is the subject of our current investigation, is to extend the results of this analysis to the set of mixed strategy equilibria of non-oligarchic games and to games with history dependent recognition probabilities. It appears inescapable that this analysis requires perturbation with respect to a larger class of parameters of this game, such as some form of perturbation of the stage utilities u_i.

APPENDIX: THE CASE OF FIXED DELAY COSTS

In this appendix we show how our arguments extend naturally to the case players incur a delay cost $c_i \in \mathbb{R}_{++}$, $i \in N$. In this case, we define the reservation value $r_i \equiv v_i - c_i$. Then, the definition of agenda setting plans and the equilibrium characterization in theorem 1 hold. We can then derive an analogue to lemma 2, this time setting

$$G_i^{\boldsymbol{\omega}}(\mathbf{r}) \equiv \sum_{h=1}^{n} \pi_h u_i \left(f_{\omega_h}^h(\mathbf{r}) \right) - r_i,\; i = 1, .., n.$$

and letting $G^1(\mathbf{r}) \equiv G^{\boldsymbol{\omega}}(\mathbf{r}) - \mathbf{c}$, $G_i^0(\boldsymbol{v}) \equiv u_i(\mathbf{x}) - r_i - c_i$, $i = 1, ..., n$, for some $\mathbf{x} \in X$, and $H_t(\mathbf{r}) = (1-t) G^0(\mathbf{r}) + tG^1(\mathbf{r})$. Now $\left| d\left(G^0, R, \mathbf{0}\right) \right| = 1$ and the proof simply amounts to showing $H_t^{-1}(\mathbf{0}) \bigcap \partial R = \varnothing$ for all $t \in [0,1]$. For $\mathbf{r} \in H_t^{-1}(\mathbf{0})$ we have $(1-t) G^0 + tG^1 = \mathbf{0} \Longrightarrow r_i = t \sum_{h=1}^{n} \pi_h u_i \left(f_{\omega_h}^h(\mathbf{r}) \right) + (1-t) u_i(\mathbf{x}) - c_i$. Since $t \in [0,1]$, if we set $v_i = t \sum_{h=1}^{n} \pi_h u_i \left(f_{\omega_h}^h(\mathbf{r}) \right) + (1-t) u_i(\mathbf{x})$ for all $i \in N$ we have $\mathbf{v} \in V$, the space of possible continuation values. Thus we also have $\mathbf{r} \in R_{\mathbf{c}} \subset R$ for all $t \in [0,1]$ and the rest of the proof is identical to that in lemma 2. The same applies for theorem 2.

Now the sufficient conditions in subsection (iii) of section 4 apply directly to the agenda setting plans of the game with delay costs, so that for almost all $\mathbf{c} \in \mathbb{R}^n_{++}$ the corresponding game, $\Gamma_{\mathbf{c}}$, is a regular game.

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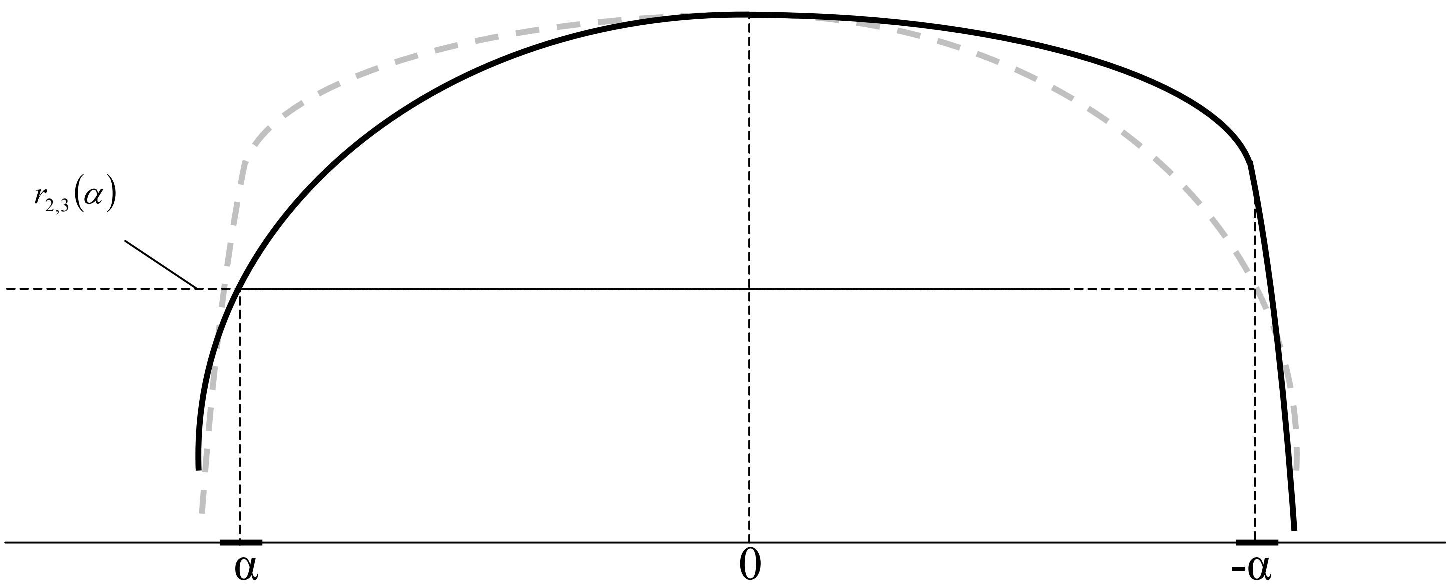
Figure 1: A Four-Player Unanimity Game with a Continuum of Equilibria



Key: There exists a continuum of PSSP where players 2 and 3 propose 0, player 1 who prefers left-wing policies proposes α, and player 4 that prefers right-wing policies proposes –α. The utility of player 2 from α is exactly equal to her reservation value from the lottery of these three policies: $u_2(\alpha) = \delta\left[\frac{1}{4}u_2(\alpha) + \frac{1}{2}u_2(0) + \frac{1}{4}u_2(-\alpha)\right] = r_2(\alpha)$. Also, $u_2(-\alpha) \geq r_2(\alpha)$ when $\alpha \in [-7/6, -1]$.