On the Size and Structure of Group Cooperation

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Abstract

This paper examines characteristics of cooperative behavior in a repeated, n-person, continuous action generalization of a Prisoner's Dilemma game. When time preferences are heterogeneous and bounded away from one, how "much" cooperation can be achieved by an ongoing group? How does group cooperation vary with the group's size and structure? For an arbitrary distribution of discount factors, we characterize the *maximal average cooperation (MAC) likelihood* of this game. The MAC likelihood is the highest average level of cooperation, over all stationary subgame perfect equilibrium paths, that the group can achieve. The MAC likelihood is shown to be increasing in monotone shifts, and decreasing in mean preserving spreads, of the distribution of discount factors. The latter suggests that more heterogeneous groups are less cooperative on average. Finally, we establish weak conditions under which the MAC likelihood exhibits increasing returns to scale when discounting is heterogeneous. That is, larger groups are more cooperative, on average, than smaller ones. By contrast, when the group has a common discount factor, the MAC likelihood is invariant to group size.

JEL Fields: C7, D62, D7.

Keywords: Repeated games, maximal average cooperation likelihood, heterogeneous discount factors, returns to scale

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1 Introduction

Successful cooperation at some level is required in any group. In most contexts, teams, clubs, and partnerships function most effectively when their members get along with one another. When they do not, substantial social conflicts often result.

This paper examines characteristics of group cooperation in a repeated, collective action environment. Specifically, we study an n-person repeated game that resembles a repeated Prisoner's Dilemma game. At each stage, each member of the group can choose some (normalized) level of effort or contribution in the unit interval which indicates how cooperative he is toward other members of the group. An individual is better off the more cooperative are the other members in the aggregate. However, each has a (static) incentive to behave uncooperatively.

How can successful cooperation be determined for this group? Specifically, how "much" cooperation can this group achieve? Moreover, how does the level of achievable cooperation respond to changes in the size and composition of the group?

One standard view of group cooperation, dating at least back to Olson (1965), is that (a) group decisions typically entail free rider problems that limit/prevent cooperative outcomes, and (b) these problems are worse in larger groups than in smaller ones. In the context of a repeated Prisoner's Dilemma game, the view expressed in (a) is more or less correct when discounting is large. That is, when members are uniformly impatient, the incentives of the stage game take over and, consequently, joint cooperation cannot be sustained. On the other hand, a polar view of group cooperation is obtained when the members are uniformly patient. The Folk Theorem, according to which any feasible and individually rational payoff profile is sustainable if all group members are sufficiently and uniformly patient, provides an alternative conclusion to (a).[1] Namely, the collective action problem is easily resolved in the limit as the common discount factor approaches one. Hence, equilibria exist in which all group members are cooperative each period. As for (b), group size plays no role either when players are uniformly patient or uniformly impatient.[2]

These claims are more difficult to evaluate in the intermediate cases where discount factors are heterogeneous and bounded away from either zero or one. Presumably, collective decision problems often occur in these cases. To address the stated questions, we therefore depart from the traditional focus on uniformly patient players in repeated games.[3] Instead, we examine a measure of cooperation in a group whose members possess arbitrary discount factors.

[1] Fudenberg and Maskin (1986) is a standard reference.

[2] To be fair, Olson (1965) outlined several reasons for ineffectiveness of larger groups which are not considered here. See Section 5 for an extended discussion.

[3] Other work which departs from the uniform patience assumption is discussed later in this Section.

Consider a group with n members. Let δ_i denote the discount factor for member, $i = 1, \ldots, n$. Let $\delta = (\delta_1, \ldots, \delta_n)$ denote the profile of discount factors. Consider, in a repeated game of perfect monitoring, the set of stationary, subgame perfect equilibria (SSPE), i.e., equilibria in which the same action profile is taken each period. Let $p_i \in [0, 1]$ indicate the level of cooperation by group member i each period in an SSPE. For instance, p_i may indicate a (normalized) level of effort by member i. A convenient alternative interpretation is that p_i is the probability that member i assigns to a discrete, "cooperative" action in a binary action game.[4] Now consider a profile of actions, $p = (p_1, \ldots, p_n)$, that maximizes the average level of cooperation given by

$$\frac{1}{n} \sum_{i=1}^{n} p_i \tag{1}$$

over all SSPE paths. The value of this program is a level of cooperation, $p^*(n, \delta) \in [0, 1]$, which we call the *maximal average cooperation (MAC) likelihood*. The maximal average cooperation (MAC) likelihood describes the highest average level of cooperation attainable by the group each period . We are interested in it, rather than in group welfare, because it is calculated from behavior directly. A manager of a firm who knows nothing about the details of the payoffs of his divisional workers, may nevertheless measure the effectiveness of the group by examining their average cooperative effort.

On one hand, using the MAC likelihood as a measure of attainable cooperation is restrictive because stationary equilibria entail some loss of generality when discounting is heterogeneous. For one thing, a recent paper by Lehrer and Pauzner (1999) shows that there are equilibrium payoffs possibly outside the convex hull of the stage game payoff set if individuals are sufficiently but heterogeneously patient. The intuition is that relatively impatient players can receive favorable payoffs early on, while relatively patient ones receive rewards later in the game. For another, the MAC likelihood when the interpretation of p is as a *mixed* action profile is problematic: when discount factors are bounded away from one, the assumption that mixed action histories are observable is no longer without loss of generality. For this reason, the interpretation of p as a profile of continuous pure actions may be the more desirable one.[5]

On the other hand, so little is known about equilibria when discounting is bounded away from either one or zero that our approach still extends the literature. Stationary equilibria arguably serve as a useful benchmark by admitting a transparent comparison with standard models. For our purposes, they highlight potential social conflicts generated by the interaction of patient and impatient individuals. Even as an imperfect estimate, the MAC likelihood effectively summarizes these conflicts.[6]

Section 2 describes the case of two players. There the MAC likelihood is explicitly calcu-

[4] The latter interpretation assumes that the history of mixed action profiles are observable.
[5] The mixed action interpretation is maintained in the sequel largely for convenience.
[6] See also Section 5 for an extended discussion and justification of the stationarity assumption.

lated. In the n player case in Section 3, we prove that the MAC likelihood is characterized by a maximal fixed point of a particular function. Properties of this function provide critical information as to how cooperation varies with size and the composition of the group.

Our main results are in Section 4. The findings fall into two categories: (1) the effect on cooperation when the composition of the group changes (holding size fixed); (2) the effect on cooperation when group size changes.

One rather expected finding from the first category is that the MAC likelihood is increasing in monotone shifts in the distribution of time preferences. More patient groups are more cooperative. A less obvious finding is that mean preserving spreads reduce the MAC likelihood. Hence, group heterogeneity is generally detrimental. Greater social conflict is generated by greater conflict in time preferences of the members.

The main result from the second category establishes sufficient conditions under which the MAC likelihood exhibits increasing returns to scale. Suppose an n-person group with discounting profile δ^n doubles in size holding everything else, including the distribution of discount factors, constant. Then the MAC likelihood exhibits increasing returns if $p^*(2n, \delta^{2n}) > p^*(n, \delta^n)$. More generally, for any scale factor α, the MAC likelihood exhibits increasing (decreasing) returns to scale if

$$p^*(\alpha n, \delta^{\alpha n}) \; > \; (<) \; p^*(n, \delta^n) \tag{2}$$

In words, increasing returns to scale exist if the average level of cooperation of the scaled up group exceeds that of the original group. If $p^*(n, \delta^n)$ lies strictly between 0 and 1, we show that increasing (decreasing) returns to scale exist if all individuals' discount factors lie above (below) a certain threshold.

The main result demonstrates, however, an apparent propensity toward increasing, rather than decreasing, returns. Specifically, the MAC likelihood is shown to exhibit increasing returns if two conditions hold. First, the *average* discount factor among interior cooperators (those for whom $p_i < 1$) exceeds the aforementioned threshold. The second condition is a "risk-dominance-like" property on the stage game. The condition requires that in any bilateral relationship, the relative loss from cooperating against an uncooperative member is smaller than the relative gain to deviating against a cooperative one.

The intuition for the result is as follows. Because a player's cooperation likelihood in equilibrium depends on the distribution of discount factors of others, seemingly neutral changes in scale are not neutral from a particular player's point of view. Take for example a 2-person group with a patient and an impatient player. If this group is replicated once, doubling in size, then the patient player faces one other patient player and two impatient players. Whereas before, he faced one impatient player. Hence the proportion of patient to impatient players has changed favorably for the patient player. This explains why the marginal response to a change in scale is positive above a fixed threshold discount factor and is negative below it. What is crucial is that this marginal response is convex in the discount factor. The convexity creates an asymmetry between patient and impatient players in their responses to increases in group size. Impatience has a dampening effect: an impatient player cares less about the future responses of others to his current action. Consequently, the positive response of the patient player to an increase in scale exceeds the negative response of the impatient one.

Significantly, returns to scale critically requires heterogeneity. The assumptions do not hold, and the result fails when discounting is homogeneous. When players' time preferences are the same, the MAC likelihood is invariant to group size.

Section 5 summarizes these findings and contains an extended discussion of the assumptions. Section 6 is an Appendix which contains all the proofs of the results. Most of these are computation-intensive but are otherwise straightforward. The results demonstrate, in limited form, that group size matters. Moreover, it matters in a way quite different than is supposed by the standard logic. In a heterogeneous group, intertemporal incentives for good behavior sometimes improve with size.

Surprisingly, there are few studies of repeated games with heterogeneous discounting. One seminal exception is the aforementioned work by Lehrer and Pauzner (LP) (1999). They examine two-player repeated games with differential discount factors. Their main results characterize equilibrium payoffs in the "heterogeneous limit," meaning that fixed differences between the two discount factors are maintained while discounting approaches one.[7] Heterogeneous limits in discounting also appear in a number of other repeated game studies. Fudenberg, Kreps, and Maskin (1990) prove a Folk Theorem for a subset of the players when the other players' discount factors are 0. Harrington (1989) characterizes the bounds on heterogeneous discount factors required to achieve collusion in oligopolies. Fudenberg and Levine (1989), Celantani, et al. (1995), and Aoyagi (1996) all examine reputation-building by a sufficiently patient, long run player who faces a sequence of short run players in a repeated game. Finally, a similar approach to the present paper was taken in Haag and Lagunoff (2001), which examined stationary trigger strategies in a local interaction network.

2 An Example with Two Group Members

To see how the MAC likelihood is determined, we begin with a simple example. Arguably, the quintessential collective action problem is a Prisoner's Dilemma game. The payoffs are as described in Figure 1 below.

[7]Specifically, a player's discount factor in the LP model is $\delta^{\Delta t}$ where Δ is the time length of a period. LP examine equilibrium payoffs when Δ is close enough to zero while maintaining a fixed log ratio $\frac{\log(\delta_1)}{\log(\delta_2)}$ between the two discount factors.

	C	D
C	c, c	-ℓ, d
D	d,-ℓ	$0, 0$

Figure 1: Prisoner's Dilemma

As is typical of PD games, each individual can choose a cooperative action, C, or an uncooperative one, D, each period. For Prisoner's Dilemma we require that $d > c > 0 > -\ell$, and $2c > d - \ell$. For ease of exposition, we also assume in this Section that $d > c + \ell$.[8]

When the PD game is infinitely repeated, it is a standard exercise to show that the profile (C, C) can be sustained as a subgame perfect equilibrium if both individuals are sufficiently patient. How well can they do if one or both discount factors falls below the "patience threshold?"

Let δ_i denote the discount factor of individual $i = 1, 2$. Each individual uses δ_i to calculate his average discounted sum of payoffs. Let p_i denote the probability that Player i chooses to cooperate in a stationary subgame perfect equilibrium (SSPE) in which mixed action histories are observable by both members.[9] The pair (p_1, p_2) constitutes an SSPE if for each $i = 1, 2$, and $j \neq i$,

$$p_i(p_j c - (1 - p_j)\ell) + (1 - p_i)p_j d \geq (1 - \delta_i)p_j d \tag{3}$$

The left hand side of (3) is Player i's average payoff under the stationary action profile (p_1, p_2). The right hand side is the one shot gain from a deviation to the uncooperative action with the consequence that both members permanently revert thereafter to the one shot Nash equilibrium, (D, D).

Inequality (3) yields this system of inequalities

$$\begin{aligned} \delta_1 p_2 &\geq p_1(p_2\mu + (1 - p_2)\gamma) \\ \delta_2 p_1 &\geq p_2(p_1\mu + (1 - p_1)\gamma) \end{aligned} \tag{4}$$

where $\mu \equiv (d - c)/d$ and $\gamma \equiv \ell/d$.[10] The first inequality is Player 1's incentive constraint against deviating to D. The second inequality is the identical incentive constraint for Player 2. Clearly, full cooperation, i.e., $(p_1, p_2) = (1, 1)$ solves this system whenever $\delta_i \geq \mu \equiv (d - c)/d$ for both $i = 1, 2$. In this case the Folk Theorem property applies.

[8] Roughly, this condition is needed to ensure that symmetric mixed equilibria may be optimal when players are impatient.

[9] Under the stationarity restriction, the set of feasible payoffs is fixed across all profiles of discount factors, (δ_1, δ_2). See Section 5 for further discussion of stationarity.

[10] Note that by our earlier assumptions, $\mu > \gamma$ since $d < c + \ell$.

However, when $\delta_i < \mu$ for either or both players, one can still sensibly ask: how "well" these two can do on average in any SSPE? The maximal average cooperation (MAC) likelihood, defined by the maximal value of $\frac{1}{2}(p_1 + p_2)$ over all SSPE pairs (p_1, p_2), provides one answer. This value, which we denote by p^*, is the highest likelihood that a randomly chosen individual from this pair is cooperative in equilibrium.

Any candidate pair (p_1, p_2) for determining a MAC likelihood must fall into one of four categories. (i) only Player 1's incentive constraint binds; (ii) only Player 2's constraint binds; (iii) both bind; or (iv) neither bind. Notice also that if a constraint does not bind, then for that player i, it must be that $p_i = 1$ since otherwise, his coooperation probability can be increased without violating the incentive constraint of the other player. Solving the system (4) for these four cases, the MAC likelihood is given by

$$p^* = \begin{cases} 1 & if \quad \delta_i \geq \mu, \text{ for } i = 1,2 \\ \frac{1}{2}\left[\frac{\delta_1}{\mu} + 1\right] & if \quad \delta_1 < \mu \text{ and } \delta_1\delta_2 \geq \mu\gamma + \delta_1(\mu - \gamma) \\ \frac{1}{2}\left[1 + \frac{\delta_2}{\mu}\right] & if \quad \delta_2 < \mu \text{ and } \delta_1\delta_2 \geq \mu\gamma + \delta_2(\mu - \gamma) \\ \frac{1}{2}\left[\frac{\delta_1\delta_2 - \gamma^2}{(\delta_2+\gamma)(\mu-\gamma)} + \frac{\delta_1\delta_2 - \gamma^2}{(\delta_1+\gamma)(\mu-\gamma)}\right] & if \quad \gamma^2 \leq \delta_1\delta_2 < \mu\gamma + \min\{\delta_1, \delta_2\}(\mu - \gamma) \\ 0 & if \quad \delta_1\delta_2 < \gamma^2 \end{cases} \tag{5}$$

In expression (5), the MAC likelihood takes on distinct values in each of five regions in discount factor space. These regions are exhibited in Figure 2. For example, the conditions which generate a MAC likelihood of $\frac{1}{2}\left[\frac{\delta_1}{\mu} + 1\right]$ are those in which Player 2's constraint does not bind, and Player 1's does bind. In this case, Player 2 cooperates with certainty, while Player 1 "partially cooperates," choosing C with probability less than one. Solving for Player 1's binding constraint when $p_2 = 1$ gives $p_1 = \delta_1/\mu \equiv \delta_1/\frac{d-c}{d}$. Hence, a player who discounts to 20% of the full cooperation threshold μ is uncooperative 20% of the time.

It is clear from (5) that p^* is (weakly) increasing and continuous in the pair (δ_1, δ_2). It is also straightforward to show that the MAC likelihood is generally decreasing in mean preserving spreads of the original pair (δ_1, δ_2). This means that the larger is the difference in the two players' time preferences, other things equal, the less cooperative is the group on average.

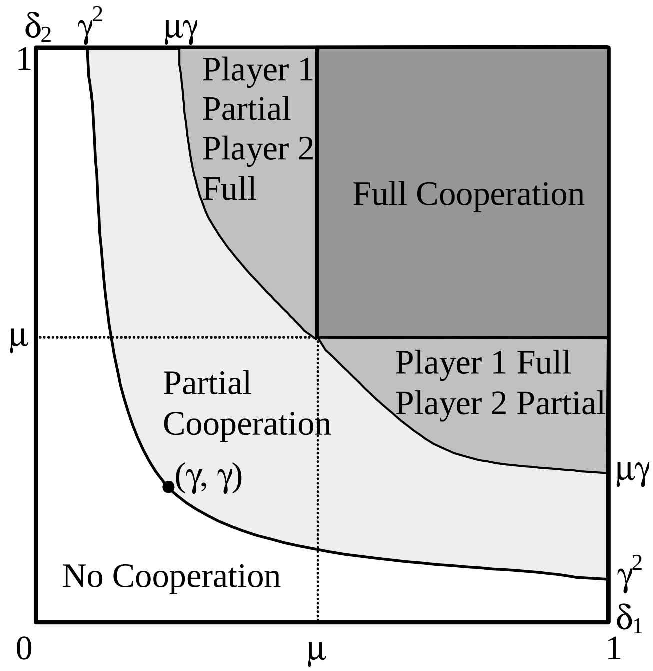


Figure 2: Distribution of Cooperation Across Profiles of Discount Factors

A noteworthy special case is the symmetric case where $\delta_1 = \delta_2 = \bar{\delta}$. In this case (5) reduces to

$$p^* = \begin{cases} 1 & if \quad \bar{\delta} \geq \mu \\ \frac{\bar{\delta}-\gamma}{\mu-\gamma} & if \quad \gamma \geq \bar{\delta} < \mu \\ 0 & if \quad \bar{\delta} < \gamma \end{cases} \tag{6}$$

Many interesting questions pertain to group size. To answer these, the n-group model is developed in the subsequent section. Unfortunately, the simple solution technique — solving a system of inequalities that do not bind at the boundaries of $[0, 1]^2$ — is impractical with $n > 2$ players. The good news is that one need not solve the system to obtain the MAC likelihood. Since only knowledge of the aggregate level of cooperation is required, the MAC likelihood can be characterized by a single, continuous, piecewise-smooth function.

3 A Model of Group Cooperation

3.1 The n-Group

A collection of n individuals play an infinitely repeated game in discrete time $t = 0, 1, \ldots$. We refer to this collection as the "n-group" or, more simply, "the group." In each period, the stage game is as follows. Each member i of the n-group chooses a number, p_i, from a set $[0, 1]$ of feasible actions. The action p_i determines member i's chosen level of cooperation toward the group. An action profile is given by $p = (p_1, \ldots p_n)$.

Each individual's payoff is assumed to increase in the aggregate level of cooperation of other members of the group, but assumed to decrease in his own level of cooperation in the aggregate. If an individual is highly cooperative while other group members are highly uncooperative, the individual incurs a loss. In order to maintain constant returns in the static incentives for cooperation (so as not to bias the results on size later on), we assume that an individual's payoffs have the following co-linear form:

$$\pi_i(p) = \sum_{j \neq i} \left[p_i \left(p_j c + (1 - p_j)(-\ell)\right) + (1 - p_i) p_j d\right] \tag{7}$$

where c is the reward to mutual cooperation, d is the reward to (unilaterally) uncooperative behavior, and ℓ is the loss incurred from others' uncooperative behavior. We assume that $d > c > 0 > -\ell$, and $2c > d - \ell > 0$ (as with Prisoner's Dilemma game). It is easy to check that $p = (0, \ldots, 0)$ is the unique Nash equilibrium of the stage game.

This payoff structure deserves comment. Observe, first, that actions are undirected, i.e, an individual cannot choose different actions toward each of the other $n-1$ group members. If payoffs were separable across directed levels of cooperation for each other member of the group, then each bilateral interaction would be a separate strategic problem, and so the results from the previous Section would suffice.

Second, payoffs are linear in one's own action. This is to maintain "size-neutral" stage game incentives. Since our purpose eventually is to examine the effect of group size on dynamic incentives for cooperation, the stage payoffs themselves should not bias incentives toward either larger or smaller groups. At the same time, the collective action (free rider) problem entails that payoffs are increasing in the aggregate cooperation of others, decreasing in one's own cooperation, and there are nonzero cross effects between the two. With these requirements, one can verify that the payoff in (7) belongs to a "canonical class" defined by

$$\pi_i = [A - p_i] \, F(\sum_{j \neq i} p_j) - B p_i + C \tag{8}$$

where $A, B, C > 0$, and F is increasing in $\sum_{j \neq i} p_j$. The payoff in (7) is a normalization of the case in which F is linear.

Third, the particular normalization used in (7) has a convenient interpretation. Payoffs and actions have been normalized so that the profile p can be interpreted as a profile of mixed strategies in a "generalized version" of Prisoner's Dilemma for n players. To see this, suppose that there are two discrete actions, a "cooperative" action, C, and an "uncooperative" one, D. Let $a_i \in \{C, D\}$, and let $a = (a_1, \ldots, a_n)$ denote the pure action profile. If $n = 2$, so that there are only two members in the group, then the payoffs are described by the (standard) Prisoner's Dilemma game in Figure 1. If $n \geq 2$, the pure strategy stage payoffs to a group member i are given by

$$u_i(a) = \begin{cases} (q-1)c - (n-q)\ell & if \quad a_i = C \\ \\ qd & if \quad a_i = D \end{cases}$$

where $q = |\{j = 1, \ldots, n : \ a_j = C\}|$ describes the number of individuals that choose C. Hence, i's payoff may be interpreted as the sum of the payoffs from each bilateral interaction from all other members of the group. Then the payoff in (7) corresponds to each player's expected payoff when each member $i = 1, \ldots, n$ chooses C with probability p_i.

Because this mixed action interpretion of p requires that we assume mixed action histories are observable, the alternative interpretation in which p_i is the continuous level of some cooperative effort in a team problem, is probably more desirable. Nevertheless, we will occasionally find it more convenient to use the mixed action interpretation.

Letting $p(t)$ denote the action profile taken in period t, an individual's dynamic payoff then is:

$$\sum_{t=0}^{\infty}(1-\delta_i)\delta_i^t \pi_i(p(t))$$

where δ_i denote the discount factor of member i. Let $\delta = (\delta_1, \ldots, \delta_n)$ denote the (arbitrary) profile of discount factors of the group.

3.2 Maximal Average Cooperation

We assume that the histories $(p(1), \ldots, p(t))$ are publically observable and restrict attention to stationary subgame perfect equilibria (SSPE). These are subgame perfect equilibria in which the same vector of actions are chosen each period. For now, we leave the discussion and implications of this restriction to Section 5. Because histories are publically obesrved, any SSPE path in this class of games can be implemented by an SSPE which uses simple "trigger strategies" in which any deviation is met with permanent reversion to the one shot, "uncooperative" equilibrium. Hence, there is no confusion when we refer to the stationary path profile, p, as a *stationary subgame perfect equilibrium (SSPE)*.[11]

Let $E(\delta)$ denote the set of SSPE profiles given δ. Our particular interest is in the profile p that solves

$$\max_{p \in E(\delta)} \frac{1}{n} \sum_{i=1}^{n} p_i \tag{9}$$

The optimal value of (9) is an action $p^* \in [0,1]$ that describes the *maximal average cooperation (MAC) likelihood*. In Section 4, the MAC likelihood will be denoted by $p^*(n,\delta)$ to mark the explicit dependence on group's size, n, and characteristics, δ.

3.3 Characterizing the MAC Likelihood

Fix a type profile δ and a stationary path, $p(t) = p$, $\forall t$. Using (7), profile p describes a SSPE if and only if, for each $i = 1, 2, ...n$, $0 \le p_i \le 1$, and

$$p_i \left(\sum_{j \neq i} p_j c + \sum_{j \neq i} (1 - p_j)(-\ell) \right) + (1 - p_i) \sum_{j \neq i} p_j d \;\ge\; (1 - \delta_i) \sum_{j \neq i} p_j d \tag{10}$$

The right hand side of (10) is the payoff from deviating and behaving uncooperatively. Since the punishment is permanent "uncooperative" behavior by all members, the deviant gains only in the initial period.

One can easily verify that if all other members cooperate fully, i.e., $p_j = 1$ for all $j \neq i$, then member i should cooperate fully if

$$\delta_i \ge (d - c)/d. \tag{11}$$

If (11) holds for all members of the group (which is the case examined by the Folk Theorem), then the problem of optimal cooperation is trivial (the MAC likelihood is 1). Consequently, the problem of optimal cooperation becomes interesting precisely when sufficiently conflicting time preferences exist.

Rewriting the Inequality in (10) and letting $P_{-i} = \sum_{j \neq i} p_j$ yields

$$p_i \left[\left(\frac{d - c}{d} - \frac{\ell}{d} \right) P_{-i} + (n - 1) \frac{\ell}{d} \right] \;\le\; \delta_i P_{-i} \tag{12}$$

[11] The argument for why there is no loss of generality is standard: since the minmax payoff is 0, any equilibrium path remains an equilibrium path if it is enforced by the threat of permanent reversion to the one-shot equilibrium in which everyone in the group chooses "D" thereafter.

Now let $\mu = \frac{d-c}{d}$ and $\gamma = \frac{\ell}{d}$ (as in the previous Section). Parameter μ is the relative gain to choosing the uncooperative action against a cooperative opponent. γ is the relative loss (against payoff d) if one is cooperative against an uncooperative opponent. Using these parameters, Inequality (12) can be combined with the constraint $p_i \leq 1$ to produce

$$p_i \;\leq\; \min\left\{1,\; \frac{\delta_i}{\mu + \gamma\frac{(n-1)-P_{-i}}{P_{-i}}}\right\} \tag{13}$$

The following Lemma asserts that in any equilibrium that achieves the MAC likelihood, the slackness in (12) must come from the boundary condition $p_i = 1$. Another way of saying this is that the second Inequality, (13), must hold with equality.

Lemma 1 *Let p denote a mixed action profile which describes a stationary, subgame perfect equilibrium (SSPE) path. Suppose that $p^* = \frac{1}{n}\sum_{i=1}^{n} p_i$, where p^* is the maximal average cooperation (MAC) likelihood. Then p satisfies (13) with equality for each individual i.*

The proof of this and all other results are in the Appendix. Rewriting Equation (12) with parameters $\mu = \frac{d-c}{d}$ and $\gamma = \frac{\ell}{d}$ we obtain

$$p_i[(\mu-\gamma)P_{-i} + \gamma(n-1)] \;\leq\; \delta_i P_{-i} \tag{14}$$

Let $P = \sum_{i=1}^{n} p_i$ denote the aggregate likelihood of cooperation. Then add and subtract $p_i\delta_i$ to right hand side of (12) and add and subtract $p_i^2\,[\mu-\gamma]$ to the left hand side of (12), to obtain the quadratic inequality

$$Q_i(p_i) \;\equiv\; (\mu-\gamma)\,p_i^2 \;-\; (P(\mu-\gamma) + \gamma(n-1) + \delta_i)\,p_i \;+\; P\delta_i \;\geq\; 0 \tag{15}$$

The graph of the quadratic Q_i takes on one of two forms depending on whether $\mu < \gamma$ or $\mu > \gamma$. Figure 3 exhibits each of the two cases, respectively.

The following Lemma establishes necessary conditions for an equilibrium profile to yield a positive MAC likelihood. It requires from each individual either full cooperation ($p_i = 1$), or partial cooperation as determined by one of the roots of the quadratic expression in (15).

Lemma 2 *Let p denote an SSPE such that $p^* = \frac{1}{n}\sum_i p_i$ is the MAC likelihood, and $P^* = np^*$. If $p^* > 0$ then for each member i,*

$$p_i = \begin{cases} R_i(P^*) & \textit{if} \quad \delta_i \;<\; \mu + \gamma\frac{n-P^*}{P^*-1} \quad \textit{or if} \;\; P^* < 1 \\[2ex] 1 & \textit{if} \qquad\qquad \textit{otherwise} \end{cases}$$

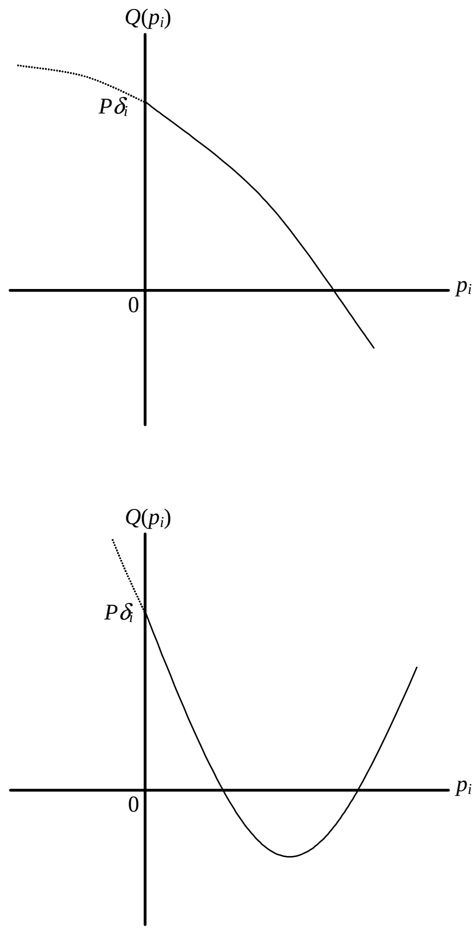


Figure 3: Quadratic Function $Q_i(p_i)$ for cases $\mu < \gamma$ and $\mu > \gamma$, respectively.

where $R_i(P^*)$ is one of the two real roots of the quadratic function Q_i in Expression (15) and is given by

$$R_i(P^*) = \frac{P^*(\mu-\gamma)+\gamma(n-1)+\delta_i - \sqrt{(P^*(\mu-\gamma)+\gamma(n-1)+\delta_i)^2 - 4(\mu-\gamma)P^*\delta_i}}{2(\mu-\gamma)}$$

When $\mu < \gamma$, then $R_i(P)$ is the larger root of the quadratic in (15), and so from the first graph in Figure 3 it is clear that the maximal possible cooperation for i occurs either when $p_i = 1$ or when $p_i = R_i(P)$. When $\mu > \gamma$, then $R_i(P)$ is the smaller root of the quadratic, and so from the second graph, the maximal cooperation once again occurs when either $p_i = 1$ or $p_i = R_i(P)$.

Now define $\psi_i : [0,n] \to [0,1]$ to be the function that determines p_i in Lemma 2, i.e.,

$$\psi_i(P) = p_i = \begin{cases} R_i(P) \quad if \quad \delta_i \; < \; \mu + \gamma\frac{n-P}{P-1} \; \text{ or if } \; P < 1 \\ 1 \quad if \quad otherwise \end{cases} \tag{16}$$

For each member i, ψ_i is the *response likelihood* to aggregate cooperation P. Finally, let $\Psi : [0,n] \to [0,n]$ be the sum, over all members, of the response likelihoods to aggregate cooperation P. Formally,

$$\Psi(P) = \sum_{i=1}^{n} \psi_i(P) \tag{17}$$

Using the Lemmatta, our first result provides a fairly simple characterization of the MAC likelihood in terms of aggregate or average behavior.

Theorem 1 *The MAC likelihood is given by*

$$p^* \; = \; \max\left\{\frac{P}{n} \in [0,1] : \; P = \Psi(P)\right\} \tag{18}$$

where the solution to (18) is nonempty.

Moreover, $p^* > 0$ iff

$$\sum_{i=1}^{n} \frac{\delta_i}{\gamma(n-1)+\delta_i} \; > \; 1 \tag{19}$$

According the result, the MAC likelihood is the largest fixed point of Ψ divided by n. The Proof in the Appendix shows , in fact, that Ψ has at most two fixed points, one of which, $P = 0$, always exists. The other is an interior fixed point, $P > 0$, which exists whenever (19) holds.

While the fixed point characterization is somewhat reminiscent of standard fixed point logic for static games, the derivations are obviously different. Rather than coming from a best response correspondence, Ψ is an aggregate equilibrium likelihood response along the equilibrium path of the repeated game.

4 Size and Structure

The Characterization Result (Theorem 1) will prove useful because it is easier to evaluate parametric changes in the fixed point mapping Ψ, than to evaluate these changes in the MAC likelihood directly. For example, if Ψ is increasing in δ, then it follows that the MAC likelihood is increasing in δ. The former is much easier to establish than the latter directly.

Theorem 2 *Let $\delta' = (\delta_i')$ and $\delta'' = (\delta_i'')$ denote two distinct profiles of discount factors, and let $p^*(n, \delta')$ and $p^*(n, \delta'')$ denote their respective MAC likelihoods.*

Suppose either

1. $\delta' > \delta''$ *or*
2. δ'' *is a mean preserving spread of* δ' *in the sense that*

$$\frac{1}{n} \sum_i \delta_i'' = \frac{1}{n} \sum_i \delta_i'$$

and for each member i, $\delta_i'' \geq \delta_i'$ iff $\delta_i' \geq \frac{1}{n} \sum_i \delta_i'$.

Then $p^*(n, \delta'') \leq p^*(n, \delta')$, with strict inequality if $0 < p^*(n, \delta') < 1$.

The Proof shows that for each i, ψ_i is increasing and concave in δ_i. This establishes the first part. The second part then follows from a standard, second order stochastic dominance argument. Note that one cannot generally infer strict inequality between the two likelihoods since δ'' may already place all individuals in the region of full cooperation (i.e., $p_i = 1, \forall i$), and so further increases in δ have no effect. The fact that average cooperation "increases" in each member's patience is, of course, to be expected. However, the second part is less obvious. Increased heterogeneity increases the degree of social conflict. A group responds poorly to increased variability of time preferences among its members.

The issue of scale of cooperation is more complicated. A corporate merger, for example, requires the merger of two groups of workers, each of a different size and each with different characteristics. It is certainly possible for the merger to increase both the size *and* the social variability relative to each group. Hence, to isolate the effect of size alone, we consider the effect of increasing n while fixing the distribution on δ. The simplest construction replicates the group α times, for some positive integer α.

A useful benchmark is the special case where discounting is homogeneous. In that case, Lemma 1 and (13) imply that SSPE profiles that determine the MAC likelihood are symmetric. Hence, the incentive constraint in (10) reduces to

$$p_i\left((n-1)pc + (n-1)(1-p)(-\ell)\right) + (1-p_i)(n-1)pd \ \geq\ (1-\bar{\delta})(n-1)pd \tag{20}$$

where p is the symmetric probability that each of the other members cooperates, while $\bar{\delta}$ is the uniform discount factor. Obviously, the constraint is invariant to n. Hence, the MAC likelihood is also invariant to n, and is described by (6).

It will prove more helpful if the scale factor α is treated as a continuous, rather than a discrete, variable. First, we write an individual's response likelihood as $\psi(P, n, \delta_i)$ to explicitly express the dependence on group size and on discount factor. Then, a group member's *marginal response to scale* α is given by

$$\frac{\partial\psi(\alpha P_n^*, \alpha n, \delta_i)}{\partial\alpha}.$$

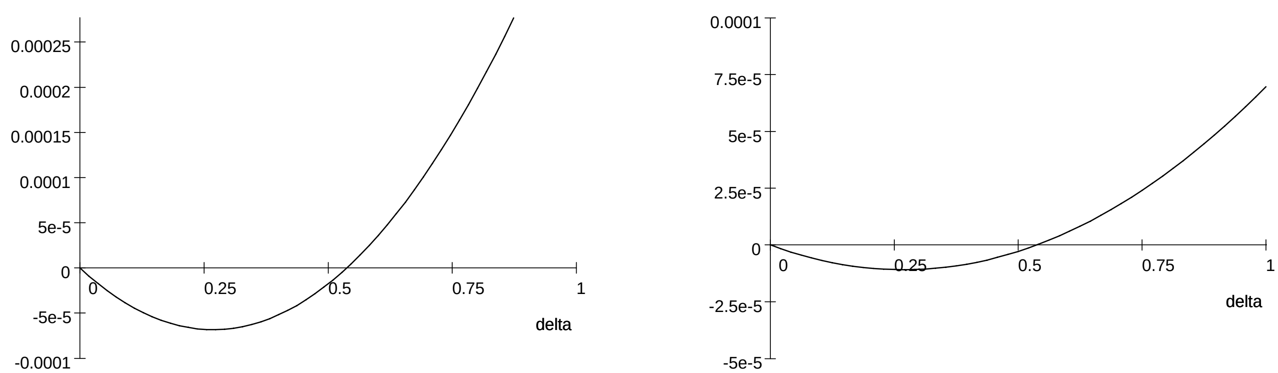
where $P_n^* \equiv np^*(n, \delta^n)$ is the maximal aggregate cooperation in the n-group. If this partial derivative is positive (negative), then member i increases (decreases) his contribution if both group size and its aggregate contribution is scaled up by $\alpha < 1$.

Theorem 3 *Let p be a SSPE such that $\frac{1}{n}\sum_i p_i = p^*(n,\delta)$. Then, for any scale factor $\alpha > 1$, and for any group member $i = 1, \ldots, n$ with $0 < p_i < 1$,*

$$\frac{\partial\psi(\alpha P_n^*, \alpha n, \delta_i)}{\partial\alpha} \begin{cases} > 0 \quad if \quad \delta_i \ > \ p^*(n,\delta)\mu + [1 - p^*(n,\delta)\ ]\gamma \\ = 0 \quad if \quad \delta_i \ = \ p^*(n,\delta)\mu + [1 - p^*(n,\delta)\ ]\gamma \\ < 0 \quad if \quad \delta_i \ < \ p^*(n,\delta)\mu + [1 - p^*(n,\delta)\ ]\gamma \end{cases} \tag{21}$$

In words, the sign of a member i's marginal response to scale α depends on whether or not his discount factor lies above the threshold $p^*(n,\delta)\mu + [1 - p^*(n,\delta)\ ]\gamma$. Since the scale factor α does not appear in (21), an individual's marginal response to scale is positive or negative independent of the change in scale. Of course, the *magnitude* of one's marginal response does depend on the magnitude of the change in scale. Hence, if every member of an n-group is above the threshold (21), then the aggregate response to a change in group size is positive. Similarly, if no group member is above the threshold (21), then the aggregate response is negative.

Figure 4 below displays two individual marginal responses to scale, each graphed as a function of δ_i. The first graph corresponds to a scale factor $\alpha = 2$ while in the second, $\alpha = 10$. Parameter values are $n = 80, P = 30, \mu = (d-c)/d = .6$, and $\gamma = \ell/d = .5$. Notice that each of the graphs cross the δ_i axis at $p^*(n,\delta)\mu + [1 - p^*(n,\delta)\,]\gamma = (30/80).6 + (50/80).5 = .5375$.



Marginal Response to Scale for $\alpha = 2$ Marginal Response to scale for $\alpha = 10$

Figure 4

Recall that the MAC likelihood *exhibits increasing (decreasing) returns at scale factor* α if

$$p^*(\alpha n, \delta^{\alpha n}) \;>\; (<)\; p^*(n, \delta^n) \tag{22}$$

Inequality (22) can be expressed equivalently as $P^*_{\alpha n} \;>\; (<)\; \alpha P^*_n$. Hence, increasing (decreasing) returns to cooperation exist if maximal *aggregate* cooperation is superadditive (subadditive).

An obvious corollary to the previous Theorem is:

Corollary *Suppose* $0 < p^*(n,\delta) < 1$. *Then the MAC likelihood exhibits increasing (decreasing) returns at every scale factor* $\alpha > 1$ *if for every member* $i = 1, \ldots, n$,

$$\delta_i \;>\; (<)\; p^*(n,\delta)\mu + [1 - p^*(n,\delta)\,]\gamma \tag{23}$$

The result would still hold if the hypothesis of the Corollary was weakened so that (23) held only for all "interior cooperators," i.e., all those for whom $0 < p_i < 1$. Even so, the requirement that individuals' discount factors all lie above (or below) some threshold is quite strong. It turns out that a natural asymmetry exists between the increasing an decreasing returns cases. To show increasing returns, one need only compare the *average* discount factor with the threshold.

Theorem 4 *Fix an n-group with MAC likelihood $0 < p^*(n, \delta) < 1$. Order the discount factors in ascending order:*

$$\delta_1 < \delta_2 < \cdots < \delta_n$$

and let i^* be the largest index i for which $\psi(P_n^*, n, \delta_i) < 1$ (i.e., the response likelihood of each member $i = 1, \ldots, i^*$ is less than one).

Then, if $\frac{n}{(n-1)^2} < \gamma < \mu$, and if

$$\frac{1}{i^*} \sum_{i=1}^{i^*} \delta_i \quad > \quad p^*(n, \delta)\mu + [1 - p^*(n, \delta)\ ]\gamma, \tag{24}$$

then the MAC likelihood exhibits increasing returns to scale at every scale factor $\alpha > 1$.

According the Theorem, the MAC likelihood exhibits increasing returns to scale if both Inequality (24) and the inequalities $\frac{n}{(n-1)^2} < \gamma < \mu$ hold. The latter inequalities hold, in turn, if the initial group is large enough and if a "risk-dominance-like" condition holds. This "risk-dominance-like" condition, $\mu > \gamma$, is equivalent to $c + \ell < d$ in the stage game. In words, it entails that in each bilateral interaction, the relative loss from cooperating against an uncooperative member is smaller than the relative gain to deviating against a cooperative one. As for (24), the left hand side of (24) is the average discount factor among those who do not cooperate fully. The right hand side is the same threshold condition as in (21) which determines whether an individual's marginal response to scale is positive or negative. Hence, (24) requires that the conditional average discount factor of the group is above this threshold.

Significantly, the Inequality (24) fails when discount factors are identical. In that case, the MAC likelihood, recall, is given by (6) which is invariant to size. One can check in that case that the left and right hand sides in (24) are equal, and so the marginal response to scale of every member is zero. Figure 5 exhibits returns to scale in the 2-member group. Constant returns occur in the "full cooperation" region and along the 45 degree line where discount factors are identical. Increasing returns occur where (24) holds. Decreasing returns occur where the threshold condition for both individuals fail or where it fails for one individual and the other is no longer an "interior" cooperator.

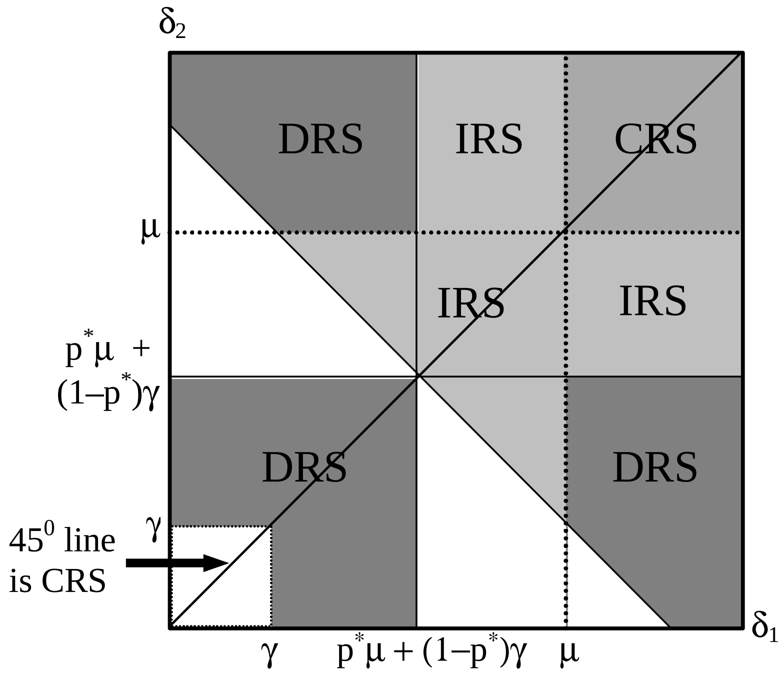


Figure 5: Returns to Scale in the 2-Member Group

The proof combines the previous Theorem 3 with a convexity argument. Roughly, we prove that an individual's marginal response to scale is convex as in Figure 4. A simple Jensen's Inequality argument establishes that if the conditional average discount factor exceeds the threshold $p^*(n,\delta)\mu + [1 - p^*(n,\delta)\ ]\gamma$ then the group's marginal response to a change in scale is positive. Hence, the MAC likelihood exhibits returns to scale.

To get a better idea for why result applies only to increasing returns, consider two individuals, with discount factors δ_H, δ_L, respectively, with $\delta_H > \delta_L$. The high or patient type has a discount factor above the threshold: $\delta_H > p^*(n,\delta)\mu + [1 - p^*(n,\delta)\ ]\gamma$. The impatient type lies below it. From (13) and Lemma 1, we know that an individual's response is

$$p_i \ = \ \min\left\{1,\ \frac{\delta_i}{\mu + \gamma\frac{(n-1)-P_{-i}}{P_{-i}}}\right\} \tag{25}$$

if profile p yields the MAC likelihood. Clearly, p_i increases in one's own discount factor, and decreases in one's "cooperation ratio" $(n - 1 - P_{-i})/P_{-i}$, the ratio of uncooperative to cooperative behavior of others. In this two person case, the patient player faces a "cooperation ratio" of $(1 - p_L)/p_L$ while the impatient player faces $(1 - p_H)/p_H$. In this example, these ratios do not change if each player faces more players of the same opposing type. For example, if the patient player faced two impatient players, then his cooperation ratio, hence his equilibrium response, would not change. Generally, only relative, not absolute, cooperation matters to each player.

However, if the group size, say, doubles, then the patient individual faces another patient type as well as two impatient types. The patient player therefore faces a change in the distribution of types. If all others' responses are scaled accordingly, then the patient player's cooperation ratio is $(3-2p_L-p_H)/(2p_L+p_H)$ which is smaller than before. The patient player therefore responds positively. On the other hand, for the impatient player, his cooperation ratio increases, thereby decreasing his equilibrium response. Returns to scale exist if the positive response of the patient player exceeds the negative response of the impatient one. Indeed, it turns out that an impatient player's marginal response to a change in scale is flatter than that of the patient player. The reason is simple. The incentive constraints measure current gains to deviating against future losses from punishment in the equilibrium continuation. The cooperation ratio $(n-1-P_{-i})/P_{-i}$ is used to determine each member's discounted average payoff in equilibrium. An impatient player is less responsive to changes in this ratio because he is less responsive to the changes in the resulting equilibrium path. Hence, although his cooperation ratio is inferior when the group size doubles, his response (as reflected in (25)) is ultimately dampened by his impatience.

5 Summary and Discussion

This papers studies cooperative behavior in a class of repeated, collective action games in which the stage game incentives are "size-neutral." We define and characterize the MAC likelihood which describes the maximal average level of cooperation of a group in a stationary equilibrium.

The focus on stationary equilibria has costs and benefits. On the cost side, the MAC likelihood may understate the success of the group. Lehrer and Pauzner show that non-stationary equilibria could conceivably give the group members higher payoffs than the stationary ones generated by maximal average cooperation. This is largely due to the fact that the feasible payoffs sets are different than that of the stage game. Improvements in the feasible set can result when time-heterogeneous individuals intertemporally “trade" payoffs.

The benefits to stationarity vary. One is the obvious tractability for the group members. Stationary equilibria facilitates objective measurement of the group by an outside observer. This is especially relevant if the group is externally compensated. Nonstationary equilibria present problems for measurement since it is unclear how cooperation early in the game should be weighted against cooperation later if there is no common discount factor. A second benefit to stationarity is that it allows for comparisons across groups of differing composition. Stationary equilibria allow the members (and the outside observer) to isolate the effect of equilibrium (as opposed to feasible) behavior because the feasible payoff set remains constant across all profiles δ.

With these caveats, the MAC likelihood reveals some unexpected properties of group cooperation in repeated games. In particular it exposes the way in which cooperation depends on the size and structure of the group. More homogeneous groups are, ceteris parabis, shown to be more cooperative, and under certain conditions, larger groups are more cooperative as well.

The latter results do not, of course, imply that social planners should construct groups of arbitrarily large size. Returns to cooperation may be balanced by increasing costs of retaining individuals in larger groups. Moreover, the assumption of perfect monitoring in the model is probably harder to justify with larger groups. Depending on the game, imperfect monitoring may bound the group away from the Pareto frontier. Finally, it may not be possible to keep constant the distribution of types when adding to the group. Our results say little about how mergers of distinctly different groups perform.

Interest in group performance in collective action problems date back at least to Olson (1965). We offer one possible antitode to a commonly expressed view that larger groups are less effective at resolving free rider problems. If this view is correct, then at least the results help to clarify, by process of elimination, why this should be so.

The paper is best viewed as provisional. Presumably, the MAC likelihood is well defined in stage games for which a natural ordering exists for all players' action sets from the less to more "cooperative" actions. Whether the results extend more broadly in this class of games remains an open question.

6 Appendix

Proof of Lemma 1 Define the map $f : [0,1]^n \to [0,1]^n$ by $f(p) = (f_1(p), \ldots, f_n(p))$ where

$$f_i(p_{-i}) = \min\left\{1,\ \frac{\delta_i}{\mu + \gamma\frac{(n-1)-P_{-i}}{P_{-i}}}\right\} \tag{26}$$

for each $i = 1, \ldots, n$. Now fix a profile p according to the hypothesis. Then

$$p \leq f(p)$$

Suppose that (13) holds with slack for individual $i = 1$. Then $p_1 < f_1(p_{-1})$. We now construct an alternative SSPE, p', i.e., a profile satisfying $p' \leq f(p')$, and such that $p'_i \geq p_i$ for all i and $p'_1 > p_1$. First take $\epsilon > 0$ sufficiently small such that $p'_1 = p_1 + \epsilon < f_1(p_{-1})$. Then let $p'_i = p_i$ for each $i \neq 1$. Now since f_i is nondecreasing in p_{-i} for all i, it follows that

$$p'_i < f(p'_{-i})$$

Hence, p' is a SSPE. However,

$$\frac{1}{n}\sum_{i=1}^{n} p'_i > \frac{1}{n}\sum_{i=1}^{n} p_i.$$

which contradicts the assertion that p achieves the MAC likelihood. $\diamondsuit$

Proof of Lemma 2 Fix any $P > 0$. Let $A = \mu - \gamma$, $B = -\left[P(\mu-\gamma) + \gamma(n-1) + \delta_i\right]$ and $C = P\delta_i > 0$. The quadratic in p_i has the form,

$$Ap_i^2 + Bp_i + C \geq 0 \tag{27}$$

We first show that the roots of (27) are real. The discriminant of the root in (27) is $\sqrt{B^2 - 4AC}$ where

$$B^2 - 4AC = \left(-\left[P(\mu-\gamma) + \gamma(n-1) + \delta_i\right]\right)^2 - 4(\mu-\gamma)P\delta_i$$

Case 1: $\mu < \gamma$

$\mu < \gamma \Rightarrow (\mu - \gamma) < 0 \Rightarrow -4(\mu-\gamma)P\delta_i > 0$. Since $B^2 \geq 0$ for all B, we have that $B^2 - 4AC > 0$ and so the roots are real.

Case 2: $\mu > \gamma$

$B^2 - 4AC = \left(-\left[P(\mu-\gamma) + \gamma(n-1) + \delta_i\right]\right)^2 - 4(\mu-\gamma)P\delta_i$

$= P^2(\mu-\gamma)^2 + \gamma^2(n-1)^2 + \delta_i^2 + 2P(\mu-\gamma)\gamma(n-1) + 2P(\mu-\gamma)\delta_i + 2\gamma(n-1)\delta_i - 4(\mu-\gamma)P\delta_i$

$= P^2(\mu-\gamma)^2 + \gamma^2(n-1)^2 + \delta_i^2 + 2P(\mu-\gamma)\gamma(n-1) - 2P(\mu-\gamma)\delta_i + 2\gamma(n-1)\delta_i$

$= \left(P(\mu-\gamma) - \delta_i\right)^2 + \gamma^2(n-1)^2 + 2P(\mu-\gamma)\gamma(n-1) + 2\gamma(n-1)\delta_i > 0$. Therefore, the roots are real.

Now define $Q(p_i) = Ap_i^2 + Bp_i + C$ as the quadratic in Equation (27). By the assumption that $p^* > 0$, it follows that $P > 0$, and so $C > 0$. Then $Q(0) = C > 0$. To begin, let $P \geq 1$ and consider the following two cases: (a) $\mu > \gamma$ and (b) $\mu < \gamma$.[12]

Case (a): in case (a), we have $A > 0$ and $B < 0$. $Q'(p) = 2Ap_i + B$ and therefore $Q'(0) < 0$. Since $A > 0$ and from Proposition (1), we have two positive real roots.

[12]If $\mu = \gamma$, Equation (27) no longer yields a quadratic equation. The solution becomes $\psi_i(\delta_i, P) = \begin{cases} -\frac{C}{B} & \text{if } \delta_i < \frac{d-c}{d} + \frac{\ell}{d}\frac{n-P}{P-1} \\ 1 & \text{otherwise} \end{cases}$. We do not focus on this case, since it occurs only when $c = d - \ell$.

By the previous Lemma, p_i must be the maximum value in $[0,1]$ consistent with $Q(p_i) \geq 0$. That is,

$$p_i = \max\{p_i' \in [0,1] : \; Q(p_i') \geq 0\}$$

Then $p_i < 1$ is a solution iff the lower root of Q is less than 1 while the greater root of Q is greater than 1 .

In other words:

$$\frac{-B - \sqrt{B^2 - 4AC}}{2A} < 1 < \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$

Since $A > 0$, we can write this as:

$-B - \sqrt{B^2 - 4AC} < 2A < -B + \sqrt{B^2 - 4AC}$

$\Leftrightarrow -\sqrt{B^2 - 4AC} < 2A + B < \sqrt{B^2 - 4AC}$

$\Leftrightarrow |2A + B| < \sqrt{B^2 - 4AC}$

$\Leftrightarrow (2A + B)^2 < B^2 - 4AC$

$\Leftrightarrow 4A^2 + 4AB + B^2 < B^2 - 4AC$

$\Leftrightarrow 4A^2 + 4AB < -4AC$

$\Leftrightarrow A + B + C < 0$

$\Leftrightarrow (\mu - \gamma) - (P(\mu - \gamma) + \gamma(n-1) + \delta_i) + P\delta_i < 0$

$\Leftrightarrow \delta_i < \mu - \gamma + \gamma\frac{n-1}{P-1}$

$\Leftrightarrow \delta_i < \frac{d-c}{d} + \frac{\ell}{d}\frac{n-P}{P-1}$ Hence, $p_i = R_i(P)$ whenever $\delta_i < \frac{d-c}{d} + \frac{\ell}{d}\frac{n-P}{P-1}$. This concludes Case (a).

Case (b): in case (b), we have that $\mu < \gamma$, so $A < 0$. Since $Q(0) > 0$ and from the first part of the Proof, we must have one negative real root and one positive real root. In that case, $p_i < 1$ is a solution to

$$\max\{p_i' \in [0,1] : \; Q(p_i') \geq 0\}$$

iff the greater root of Q is less than 1.[13]. In other words,

$$\frac{-B - \sqrt{B^2 - 4AC}}{2A} < 1.$$

[13]Note that when $A < 0$, the greater root is $\frac{-B-\sqrt{B^2-4AC}}{2A}$.

Since $A < 0$, we can write this as:

$-B - \sqrt{B^2 - 4AC} > 2A$

$\Leftrightarrow -\sqrt{B^2 - 4AC} > 2A + B$

$\Leftrightarrow \sqrt{B^2 - 4AC} < -(2A + B)$

$\Leftrightarrow B^2 - 4AC < 4A^2 + 4AB + B^2$

$\Leftrightarrow 4A^2 + 4AB + 4AC < 0$

$\Leftrightarrow A + B + C < 0 \quad (\text{since } A < 0)$

$\Leftrightarrow \delta_i < \frac{d-c}{d} + \frac{\ell}{d}\frac{n-P}{P-1}$. Hence, as with Case (a), $p_i = R_i(P)$ whenever $\delta_i < \frac{d-c}{d} + \frac{\ell}{d}\frac{n-P}{P-1}$. This concludes Case (b).

Finally, we note that when $P < 1$, the constraint that binds is no longer that $p_i \leq 1$, but rather that $p_i \leq P$. $\diamondsuit$

Proof of Theorem 1 The proof proceeds in three steps. First, we show that Ψ is increasing. Second, we prove Ψ is concave. Third, we prove that $\left.\frac{\partial \Psi_i}{\partial P}\right|_{P=0} > 1$ whenever $\sum_{i=1}^{n} \frac{\delta_i}{\gamma(n-1)+\delta_i} > 1$ holds.

(i) Show that Ψ is increasing. Observe, first, that $\mu + \gamma\frac{n-P}{P-1}$ is decreasing in P. Given (16), this means that the cutoff value above which $\psi_i = 1$ is decreasing. Given (16) and (17), it therefore suffices to show that for each i, and each δ_i, $R_i(P)$ is increasing in P. Recall:

$$\begin{aligned} R_i(P) &= \frac{-B - \sqrt{B^2 - 4AC}}{2A} \\ &= \frac{P(\mu-\gamma) + \gamma(n-1) + \delta_i - \sqrt{(P(\mu-\gamma) + \gamma(n-1) + \delta_i)^2 - 4(\mu-\gamma)P\delta_i}}{2(\mu-\gamma)} \end{aligned} \tag{28}$$

Therefore

$$\frac{\partial R}{\partial P} = \frac{\partial(-B/2A)}{\partial P} - \frac{\partial(B^2 - 4AC)^{1/2}(2A)^{-1}}{\partial P} \tag{29}$$

Now observe that A does not vary with P, and $\frac{\partial(-B/2A)}{\partial P} = 1/2$. Hence, it suffices to show $\frac{\partial(B^2-4AC)^{1/2}(2A)^{-1}}{\partial P} < 1/2$, or, equivalently,

$$\frac{1}{A}\frac{\partial(B^2 - 4AC)^{1/2}}{\partial P} < 1. \tag{30}$$

Rewriting (30) gives
$$\frac{1}{A}(B^2 - 4AC)^{-1/2}(B\frac{\partial B}{\partial P} - 2AC\frac{\partial C}{\partial P}) < 1$$
Since $\frac{\partial B}{\partial P} = A$, we have

$$(B - 2\frac{\partial C}{\partial P})^2 < (B^2 - 4AC)$$

or

$$B\frac{\partial C}{\partial P} > AC$$
But substituting for A, B and C gives
$$(P(\mu - \gamma) + \gamma(n-1) + \delta_i)\delta_i > P(\mu - \gamma)\delta_i$$
which is easily confirmed to hold. Hence, (30) holds and so $\partial\Psi/\partial P \geq 0$.

(ii) Show that Ψ is concave. It suffices to show that $\frac{\partial^2 R}{\partial P^2} < 0$. Given (29) and the fact that $\frac{\partial^2 B}{\partial P^2} = 0$ it is necessary to show
$$\frac{\partial}{\partial P}[(B^2 - 4AC)^{-1/2}(B - 2\frac{\partial C}{\partial P}) > 0 \qquad (31)$$

Utilizing the fact that $\frac{\partial B}{\partial P} = -A$ and $\frac{\partial^2 C}{\partial P^2} = 0$, (31) holds iff
$$(B^2 - 4AC) > (B + 2\frac{\partial C}{\partial R})(B + 2\frac{\partial C}{\partial R})$$
Simplifying further:

$$-AC > B\frac{\partial C}{\partial P} + (\frac{\partial C}{\partial P})^2$$
Substituting for A, B, and C, yields
$$-(\mu - \gamma)P\delta_i > -(P(\mu - \gamma) + \gamma(n-1) + \delta_i)\delta_i + \delta_i^2$$
which, finally, simplifies to $\gamma(n-1) > 0$. Since $\gamma = \ell/d > 0$, this condition must hold. Hence, Ψ is concave.

(iii) Verify that $\partial\Psi/\partial P\Big|_{P=0} > 1$ under the stated hypothesis. Observe that a necessary and sufficient condition for this is:
$$\sum_{i=1}^{n} \frac{\partial R_i}{\partial P}\Big|_{P=0} > 1$$

since the derivatives are evaluated in a neighborhood of 0 where the value of each ψ_i coincides with the root R_i. Observe

$$\frac{\partial R_i}{\partial P}\Big|_{P=0} = 1/2 - 1/2[\frac{\gamma(n-1)-\delta_i}{\gamma(n-1)+\delta_i}]$$

Consequently, $\frac{\partial \Psi}{\partial P}\Big|_{P=0} > 1$ holds if and only if

$$\sum_{i=1}^{n}\left(1/2 - 1/2[\frac{\gamma(n-1)-\delta_i}{\gamma(n-1)+\delta_i}]\right) \; > \; 1$$

or, equivalently,

$$\sum_{i=1}^{n}\frac{\delta_i}{\gamma(n-1)+\delta_i} > 1$$

which was assumed in the hypothesis. Since Ψ is increasing, concave, and differentiates above 1 in a neighborhood of 0, it must have a unique, interior fixed point. $\diamondsuit$

Proof of Theorem 2 Write $\Psi(P, n, \delta)$ to express the explicit dependence on size n and structure, δ. By standard results for second order stochastic dominance, it suffices to show that for each i, ψ_i is increasing and concave in δ_i and strictly concave if $\delta'_j < \mu + \gamma\frac{n-P}{P-1}$.

To show ψ_i is increasing, we use the previous construction in (28). Then $\frac{\partial R}{\partial \delta_i} > 0$ iff

$$-C > \frac{\partial C}{\partial \delta_i}B + A(\frac{\partial C}{\partial \delta_i})^2$$

Substituting for A, B, and C gives

$$-P\delta_i > -P(P(\mu-\gamma) + \gamma(n-1) + \delta_i) + P^2(\mu-\gamma)$$

or $\gamma(n-1) > 0$ which always holds. Hence, Ψ is increasing in δ_i.

To show Ψ is concave, observe that $\frac{\partial^2 R}{\partial \delta_i^2} < 0$ iff

$$-C > \frac{\partial C}{\partial \delta_i}B + A(\frac{\partial C}{\partial \delta_i})^2$$

which is exactly the condition determining monotonicity for ψ_i. Just as before, Substituting for A, B, and C gives

$$-P\delta_i > -P(P(\mu-\gamma) + \gamma(n-1) + \delta_i) + P^2(\mu-\gamma)$$

or $\gamma(n-1) > 0$ which always holds. Therefore, ψ_i is strictly concave if $\delta_i < \mu + \gamma\frac{n-P}{P-1}$. But if $p^*(n, \delta') < 1$, it follows from Lemma 2 that for at least one individual i, $\delta_i < \mu + \gamma\frac{n-P}{P-1}$ where $P = np^*(n, \delta')$. In this instance, Ψ is strictly concave, and so $p^*(n, \delta'') < p^*(n, \delta')$.

We conclude that mean preserving spreads of δ decrease the MAC likelihood, and strictly decrease in whenever $p^*(n, \delta') < 1$. $\diamondsuit$

Proof of Theorem 3 From the Proof of Lemma 2, $\psi(\alpha P_n^*, \alpha n, \delta_i) = 1$ iff $\delta_i \geq \mu + \gamma \frac{\alpha(n-P_n^*)}{\alpha P_n^* - 1}$.

Notice that $\mu + \gamma \frac{\alpha(n-P_n^*)}{\alpha P_n^* - 1} > \frac{P_n^*}{n}\mu + (1 - \frac{P_n^*}{n})\gamma$. Hence, it is without loss of generality if we consider δ_i such that

$$\frac{P_n^*}{n}\mu + (1 - \frac{P_n^*}{n})\gamma \; < \; \delta_i < \mu + \gamma \frac{\alpha(n - P_n^*)}{\alpha P_n^* - 1}$$

Then $\psi(\alpha P_n^*, \alpha n, \delta_i) < 1$ so that

$$\frac{\partial \psi(\alpha P_n^*, \alpha n, \delta_i)}{\partial \alpha} \; = \; \frac{\partial R(\alpha P_n^*, \alpha n, \delta_i)}{\partial \alpha}$$

where $R(\cdot)$ is the quadratic root defined in (2) in Lemma 2.

Holding fixed δ_i, we have, for any P,

$$\begin{aligned} R^*(P, n, \alpha) &\equiv R(\alpha P_n, \alpha n, \delta_i) \\ &= \frac{\alpha P(\mu - \gamma) + \gamma(\alpha n - 1) + \delta_i - \sqrt{(\alpha P(\mu - \gamma) + \gamma(\alpha n - 1) + \delta_i)^2 - 4(\mu - \gamma)\alpha P \delta_i}}{2(\mu - \gamma)} \\ &= \frac{-B(\alpha)}{A} - \frac{\sqrt{B(\alpha)^2 - 4AC(\alpha)}}{A} \end{aligned}$$

Then

$$\frac{\partial R^*}{\partial \alpha} = \frac{\partial(-B(\alpha)/2A)}{\partial \alpha} - \frac{\partial((B(\alpha))^2 - 4AC(\alpha))^{1/2}(2A)^{-1}}{\partial \alpha} \tag{32}$$

and so $\frac{\partial R^*}{\partial \alpha} > 0$ iff

$$\frac{1}{2A}\left[-\frac{\partial B}{\partial \alpha} \; - \; (B^2 - 4AC)^{-1/2}(B\frac{\partial B}{\partial \alpha} - 2A\frac{\partial C}{\partial \alpha})\right] \; > \; 0.$$

With some algebra and substitution, this Inequality is equivalent to

$$\frac{1}{2A}\left[B\frac{\partial B}{\partial \alpha} \; - \; \alpha(\frac{\partial B}{\partial \alpha})^2 + A\frac{\partial C}{\partial \alpha}\right] \; > \; 0$$

Substituting for A, B, and C, differentiating with respect to α where appropriate, and re-arranging terms yields,

$$\delta_i > \frac{P}{n}(\mu - \gamma) + \gamma$$

which is the requisite threshold condition whenever P satisfies $\frac{P}{n} = p^*(n, \delta)$. $\diamondsuit$

Proof of Theorem 4 The proof is in several steps. First, we show that $\dfrac{\partial\psi^3(\alpha P^n, \alpha n, \delta_i)}{\partial\alpha\partial\delta_i^2} > 0, \forall\alpha > 1$. In other words, the marginal response $\dfrac{\partial\psi}{\partial\alpha}$ is convex in δ_i. Then, using Jensen's inequality, we can show that if the conditional average of discount factors is above the threshold given, there are returns to scale.

Using Young's Theorem, we can first take the partial derivatives with respect to δ_i and then with respect to α.

$$\frac{\partial\psi}{\partial\delta_i} = \tfrac{1}{2A}\left[1 - (B^2 - 4AC)^{-\frac{1}{2}}\left(B\tfrac{\partial B}{\partial\delta_i} - 2A\tfrac{\partial C}{\partial\delta_i}\right)\right]$$

Since $\frac{\partial B}{\partial\delta_i} = -1$, we can rewrite this as:

$$\frac{\partial\psi}{\partial\delta_i} = \tfrac{1}{2A}\left[1 + (B^2 - 4AC)^{-\frac{1}{2}}\left(B + 2A\tfrac{\partial C}{\partial\delta_i}\right)\right]$$

Then, the second partial with respect to δ_i is:

$$\frac{\partial^2\psi}{\partial\delta_i^2} = \tfrac{1}{2A}\left[(B^2 - 4AC)^{-\frac{3}{2}}\left(B + 2A\tfrac{\partial C}{\partial\delta_i}\right)^2 - (B^2 - 4AC)^{-\frac{1}{2}}\right]$$

Factoring out $(B^2 - 4AC)^{-\frac{3}{2}}$ yields:

$$\frac{\partial^2\psi}{\partial\delta_i^2} = \tfrac{1}{2A}(B^2 - 4AC)^{-\frac{3}{2}}\left[\left(B + 2A\tfrac{\partial C}{\partial\delta_i}\right)^2 - (B^2 - 4AC)\right]$$

Multiplying out the squared term and factoring out $4A$ then gives:

$$\frac{\partial^2\psi}{\partial\delta_i^2} = \tfrac{1}{2A}(B^2 - 4AC)^{-\frac{3}{2}}4A\left[B\tfrac{\partial C}{\partial\delta_i} + A\left(\tfrac{\partial C}{\partial\delta_i}\right)^2 + C\right]$$

or,

$$\frac{\partial^2\psi}{\partial\delta_i^2} = 2(B^2 - 4AC)^{-\frac{3}{2}}\left[B\tfrac{\partial C}{\partial\delta_i} + A\left(\tfrac{\partial C}{\partial\delta_i}\right)^2 + C\right]$$

Finally, taking the partial derivative with respect to α yields:

$$\frac{\partial^3\psi}{\partial\delta_i^2\partial\alpha} = -3(B^2 - 4AC)^{-\frac{5}{2}}\left(2B\tfrac{\partial B}{\partial\alpha} - 4A\tfrac{\partial C}{\partial\alpha}\right)\left[B\tfrac{\partial C}{\partial\delta_i} + A\left(\tfrac{\partial C}{\partial\delta_i}\right)^2 + C\right]$$

$$+2(B^2 - 4AC)^{-\frac{3}{2}}\left[\tfrac{\partial B}{\partial\alpha}\tfrac{\partial C}{\partial\delta_i} + B\tfrac{\partial C}{\partial\delta_i\partial\alpha} + 2A\tfrac{\partial C}{\partial\delta_i}\tfrac{\partial}{\partial\delta_i\partial\alpha} + \tfrac{\partial C}{\partial\alpha}\right]$$

By Young's Theorem, we know that $\dfrac{\partial^3\psi}{\partial\delta^2\partial\alpha} = \dfrac{\partial^3\psi}{\partial\alpha\partial\delta_i^2}$. We want to know when $\dfrac{\partial^3\psi}{\partial\alpha\partial\delta_i^2} > 0$. From the above equation, we can add the first term to the right hand side and factor out a 2.

$$+2(B^2-4AC)^{-\frac{3}{2}}\left[\tfrac{\partial B}{\partial\alpha}\tfrac{\partial C}{\partial\delta_i} + B\tfrac{\partial C}{\partial\delta_i\partial\alpha} + 2A\tfrac{\partial C}{\partial\delta_i}\tfrac{\partial}{\partial\delta_i\partial\alpha} + \tfrac{\partial C}{\partial\alpha}\right]$$

$$> 6(B^2-4AC)^{-\frac{5}{2}}\left(B\tfrac{\partial B}{\partial\alpha} - 2A\tfrac{\partial C}{\partial\alpha}\right)\left[B\tfrac{\partial C}{\partial\delta_i} + A\left(\tfrac{\partial C}{\partial\delta_i}\right)^2 + C\right]$$

Multiplying both sides by $\frac{1}{2}(B^2-4AC)^{-\frac{5}{2}}$ yields:

$$(B^2-4AC)\left[\tfrac{\partial B}{\partial\alpha}\tfrac{\partial C}{\partial\delta_i} + B\tfrac{\partial C}{\partial\delta_i\partial\alpha} + 2A\tfrac{\partial C}{\partial\delta_i}\tfrac{\partial}{\partial\delta_i\partial\alpha} + \tfrac{\partial C}{\partial\alpha}\right]$$

$$> 3\left(B\tfrac{\partial B}{\partial\alpha} - 2A\tfrac{\partial C}{\partial\alpha}\right)\left[B\tfrac{\partial C}{\partial\delta_i} + A\left(\tfrac{\partial C}{\partial\delta_i}\right)^2 + C\right]$$

Substituting in for $A = \mu - \gamma, B = -\left(\alpha P(\mu-\gamma) + \alpha\gamma n + \delta_i - \gamma\right),\ B = \alpha P\delta_i$ and $\frac{\partial B}{\partial\alpha} = -\left(P(\mu-\gamma)+\gamma n\right),\ \frac{\partial C}{\partial\delta_i} = \alpha P,\ \frac{\partial C}{\partial\delta\partial\alpha} = P,\ \frac{\partial C}{\partial\alpha} = P\delta_i$ and rearranging terms yields:

$$\alpha^2(\alpha n-2)\left(P(\mu-\gamma)+\gamma n\right)^2 + \alpha(\alpha n+1)\left(\gamma(P(\mu-\gamma)+\gamma n) + P(\mu-\gamma)\delta_i\right)$$

$$> \alpha(\alpha n+1)\gamma n\delta_i + (2\alpha n-1)(\delta_i-\gamma)^2$$

Squaring the first term and noting that $(\delta_i-\gamma)^2 < \gamma^2$ yields:

$$\alpha^2(\alpha n-2)\left(P^2(\mu-\gamma)^2 + 2P(\mu-\gamma)\gamma n + \gamma^2 n^2\right) + \alpha(\alpha n+1)\left(\gamma P(\mu-\gamma) + \gamma^2 n + P(\mu-\gamma)\delta_i\right)$$

$$> \alpha(\alpha n+1)\gamma n\delta_i + (2\alpha n-1)\gamma^2$$

Since $\mu > \gamma$, we can rearrange terms as:

$$\alpha^2(\alpha n-2)\left(P^2(\mu-\gamma)^2 + 2P(\mu-\gamma)\gamma n\right) + \alpha(\alpha n+1)\left(\gamma P(\mu-\gamma) + P(\mu-\gamma)\delta_i\right)$$

$$+\alpha^2(\alpha n-2)n^2\gamma^2 + \alpha(\alpha n+1)\gamma^2 n - (2\alpha n-1)\gamma^2$$

$$> \alpha(\alpha n+1)\gamma n\delta_i$$

or

$$[\alpha^2(\alpha n-2)n^2 + \alpha(\alpha n+1)n - (2\alpha n-1)]\gamma^2 > \alpha(\alpha n+1)\gamma n\delta_i$$

Dividing by γ and multiplying out the bracketed term yields:

$$[\alpha^3 n^3 - 2\alpha^2 n^2 + \alpha^2 n^2 + \alpha n - 2\alpha n + 1]\,\gamma > \alpha n(\alpha n+1)\delta_i$$

Factoring yields:

$(\alpha n+1)(\alpha n-1)^2\gamma > \alpha n(\alpha n+1)\delta_i$

Dividing by $(\alpha n+1)$ and noting that $\delta_i < 1$, we can write this as:

$\gamma > \dfrac{\alpha n}{(\alpha n-1)^2}$

The derivative with the right-hand side with respect to α is negative for $\alpha \geq 1$, thus we can take as a sufficient condition that $\alpha = 1$ and thus $\dfrac{\partial \psi}{\partial \alpha}$ is convex if $\gamma > \dfrac{n}{(n-1)^2}$.

Using Jensen's Inequality and the threshold conditional average condition, it follows that

$$\begin{aligned}
\frac{\partial \Psi(\alpha P_n^*, \alpha n, \delta^n)}{\partial \alpha} &\equiv \frac{1}{n}\sum_{i=1}^{n}\frac{\partial \psi(\alpha P_n^*, \alpha n, \delta_i)}{\partial \alpha} \\
&= \frac{1}{n}\left[\sum_{i=1}^{i^*}\frac{\partial \psi(\alpha P_n^*, \alpha n, \delta_i)}{\partial \alpha} + \sum_{i=i^*+1}^{n}\frac{\partial \psi(\alpha P_n^*, \alpha n, \delta_i)}{\partial \alpha}\right] \\
&= \frac{1}{n}\sum_{i=1}^{i^*}\frac{\partial \psi(\alpha P_n^*, \alpha n, \delta_i)}{\partial \alpha} + 0 \\
&> \frac{\partial \psi(\alpha P_n^*, \alpha n, \frac{1}{n}\sum_{i=1}^{n}\delta_i)}{\partial \alpha} \\
&> 0
\end{aligned}$$

The first inequality is a straightforward application of Jensen's Inequality. The second follows from Theorem 3. Consequently,

$$\frac{\partial \Psi(\alpha P_n^*, \alpha n, \delta^n)}{\partial \alpha} > 0.$$

Because Ψ is increasing in α, it follows that

$$\Psi(\alpha P_n^*, \alpha n, \delta^n) > \Psi(\alpha P_n^*, \alpha n, \delta^n) = P_n^*$$

Now letting α be an integer,

$$\begin{aligned} 0 &< \alpha\left(\Psi(\alpha P_n^*, \alpha n, \delta^n) - P_n^*\right) \\ &= \alpha\Psi(\alpha P_n^*, \alpha n, \delta^n) - \alpha P_n^* \\ &= \alpha\sum_{i=1}^{n}\psi(\alpha P_n^*, \alpha n, \delta_i) - \alpha P_n^* \\ &= \sum_{j=1}^{\alpha n}\psi(\alpha P_n^*, \alpha n, \delta_j) - \alpha P_n^* \\ &= \Psi(\alpha P_n^*, \alpha n, \delta^{\alpha n}) - \alpha P_n^* \end{aligned}$$

Now since $P_{\alpha n}^*$ is the maximal aggregate likelihood of the αn-group, it follows that

$$\Psi(P_{\alpha n}^*, \alpha n, \delta^{\alpha n}) = P_{\alpha n}^*.$$

Since Ψ is increasing and concave in P (see Proof of Theorem 1), the difference $\Psi(P, \alpha n, \delta^{\alpha n}) - P$ must be decreasing in P. Therefore, $P_{\alpha n}^* > \alpha P_n^*$ which means that the MAC likelihood exhibits increasing returns. We conclude the proof. $\diamondsuit\,\diamondsuit$

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