Rochester Center for Economic Research

Nash Equilibria in Quantum Games

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Working Paper No. 524
February 2006

UNIVERSITY OF ROCHESTER

5/5/05

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by

Steven E. Landsburg

Abstract

When the players in a game $\mathbf{G}$ can communicate with a referee via quantum technology (e.g. by sending emails composed on a quantum computer), their strategy sets naturally expand to include quantum superpositions of pure strategies. These superpositions lead to probability distributions among payoffs that would be impossible if players were restricted to classical mixed strategies. Thus the game $\mathbf{G}$ is replaced by a much larger "quantum game" $\mathbf{G}^Q$. When $\mathbf{G}$ is a 2 x 2 game, the strategy spaces of $\mathbf{G}^Q$ are copies of the three-dimensional sphere $\mathbf{S}^3$; therefore a mixed strategy is an arbitrary probability distribution on $\mathbf{S}^3$. These strategy spaces are so large that Nash equilibria can be difficult to compute or even to describe.

The present paper largely overcomes this difficulty by classifying all mixed-strategy Nash equilibria in games of the form $\mathbf{G}^Q$. One result is that we can confine our attention to probability distributions supported on at most four points of $\mathbf{S}^3$; another is that these points must lie in one of several very restrictive geometric configurations.

A stand-alone Appendix summarizes the relevant background from quantum mechanics and quantum game theory.

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In classical (i.e. "ordinary") game theory, an n-person game is characterized by n strategy spaces $\mathcal{S}_i$ and n payoff functions

$$P_i : \mathcal{S}_1 \times \cdots \times \mathcal{S}_n \to \mathbf{R}$$

Nothing in this formulation attempts to model the process by which the payoffs are actually computed, though in applications there is usually some story to be told about, say, a market mechanism or a referee who observes the strategies and calculates the payoffs.

When the real world imposes limits on what referees can observe and calculate, we can incorporate those limits in the model by restricting the allowable strategy spaces and payoff functions. To take an entirely trivial example, consider a game where each player is required to play one of two pure strategies, say "cooperate" (**C**) and "defect" (**D**).[1] No mixed strategies are allowed. Although such games make perfect sense in the abstract, it's hard to see how they could ever be implemented. Player One announces "I cooperate!" How is the referee to know whether Player One arrived at this strategy through a legal deterministic process or an illegal random one?

So to bring our model more in line with reality, we replace the game with a larger game,

[1] I am using the words "cooperate" and "defect" as generic names for alternative strategies. I do not mean to imply that the strategy "cooperate" has anything to do with cooperation.

abandoning the two- point strategy space $\{\mathbf{C}, \mathbf{D}\}$ for the space of all convex combinations of $\mathbf{C}$ and $\mathbf{D}$, while extending the payoff function in the obvious way.

Quantum game theory[2] begins with the observation that the technology of the near future is likely to dictate that much communication will occur through quantum channels, that is, through the interactions of very small particles. For example, players might communicate their strategies to the referee via email composed on quantum computers. Such communication automatically expands the player's strategy spaces in ways that cannot be prohibited. Instead of declaring either "I cooperate" or "I defect", a prisoner can send a message that is in some quantum superposition of the states "I cooperate" and "I defect". (In Section One, I will be entirely explicit about what this means; for now, I will merely note that a superposition is *not* in general equivalent to playing a mixed strategy.) In the quantum context, there is no way for the referee to detect this kind of "cheating" and hence no way to rule it out.

We can deal with the possibility of quantum strategies just as we deal with the possibility of mixed strategies—by imbedding the original game in a larger one. So for each game we have an associated quantum game—the game that results when players' strategy spaces are expanded to include quantum superpositions, and the payoff function is extended accordingly. (Eventually, we will want to allow for *mixed* quantum strategies, which will require us to expand the strategy spaces still further.) There are in fact several different ways to convert a classical game to a quantum game, depending on exactly how one models the communication between players and referees. In Section One, I will

[2] The idea of using quantum strategies in game theory was introduced by the physicist David Meyer in [M].

introduce one such model, essentially identical to the model used by Eisert, Wilkens and Lewenstein ([EW], [EWL]) for studying the quantum version of the Prisoner's Dilemma. The purpose of this model is to motivate the definition of the quantum game associated to a given classical game. Sufficiently self-motivated readers can skip directly to Definition (2.3). For readers without the requisite background in physics who want a deeper sort of motivation, I've included, as Appendix A, a primer on quantum mechanics for economists.

Section Two gives a formal definition of the quantum game $\mathbf{G}^Q$ associated with a given classical game $\mathbf{G}$. Strictly speaking, $\mathbf{G}^Q$ should be called the *maximally entangled* quantum game associated to $\mathbf{G}$; there are other quantum games that result from alternative assumptions about the communication mechanism. Those other games will play no role in this paper. The definition invokes basic facts about *quaternions*; those basic facts are summarized in Appendix B.

In Section Three, I will recount the results of ([EL] and [EWL]) on the quantum version of the Prisoner's Dilemma. The punch line is that with quantum strategies, there is a Nash equilibrium that Pareto dominates the usual bad equilibrium (but is still Pareto suboptimal).

The class of quantum strategies, and all the moreso the class of mixed quantum strategies, is so huge as to appear intractable. In Section Four, we show that in fact mixed quantum strategies, and Nash equilibria involving mixed quantum strategies, fall naturally into large equivalence classes, which greatly simplifies the classification problem.

Following some quick technical preliminaries in Section Five, Sections Six and Seven contains this paper's main contribution: A complete classification of mixed strategy Nash equilibria in two by two quantum games, in a form that facilitates actual computations. The most powerful theorem is stated in Section Six, with the proof deferred to Section Seven.

Section Eight contains some easy applications of earlier results; the most striking is that in any mixed strategy quantum equilibrium of any two-by-two zero-sum game, each player earns exactly the average of the four possible payoffs.

1. Quantum Games: Physical Description

Consider a classical two by two game where players communicate their strategies as follows: A referee hands each player a penny, say heads up. The player either returns the penny unchanged to indicate a play of strategy **C** or flips it over to indicate a play of strategy **D**. The referee examines each penny, infers the players' strategies, and makes appropriate payoffs.

A quantum penny (and any sufficiently small penny is a quantum penny) need not be either flipped or unflipped; it can, for example, be in a state where it has a 1/4 probability of appearing flipped and a 3/4 probability of appearing unflipped. Immediately upon being observed, the penny *becomes* either flipped or unflipped, destroying all evidence that it was ever in the intermediate state.

Submitting a penny in such a state implements a mixed strategy. Of course, there are plenty of other ways to implement a mixed strategy. So far, then, there's nothing new for game theory.

When players choose mixed strategies, there is an induced probability distribution on the four possible outcomes of the game. But not every probability distribution is possible.

For example, no matter what mixed strategies the players choose, they can never achieve the probability distribution

$$\left.\begin{array}{lclclcl} \text{Prob(unflipped,unflipped)} & = & 1/2 & \quad & \text{Prob(unflipped,flipped)} & = & 0 \\ \text{Prob(flipped,unflipped)} & = & 0 & \quad & \text{Prob(flipped,flipped)} & = & 1/2 \end{array}\right\} \tag{1.1}$$

Quantum mechanics eliminates this restriction. If the pennies are appropriately *entangled*—a physical condition that is easy to arrange—then physical manipulations of the pennies can achieve any probability distribution whatsoever over outcomes, including, for example, distribution (1.1). In fact, more is true: Taking Player One's behavior as given, Player Two can, by manipulating only his own penny, achieve any probability distribution whatsoever among outcomes. (And likewise, of course, with the players reversed.)

Exactly how this comes about is explained (insofar as explanation is possible) in the appendix to this paper. Credulous readers can take this and the next section on faith and skip the appendix.

Starting with a two-by-two game $\mathbf{G}$, I will define a new two-player game $\mathbf{G}^Q$, called the *quantum game associated to* $\mathbf{G}$. The motivating picture is that $\mathbf{G}^Q$ is the same game as $\mathbf{G}$, except that players communicate their strategies by manipulating entangled pennies. Thus in the game $\mathbf{G}^Q$, each player's strategy set is equal to the set of all possible physical manipulations of a penny. Given the players' choices, one gets a probability distribution over the four possible outcomes of the original game $\mathbf{G}$; the expected payoffs in $\mathbf{G}$ are the payoffs in $\mathbf{G}^Q$.

In the next section, I'll make this picture precise.

2. Quantum Games: Mathematical Description

Start with an ordinary two-by-two game **G** represented in matrix form as

		Player Two	
		C	**D**
Player One	**C**	(X_1, Y_1)	(X_2, Y_2)
	D	(X_3, Y_3)	(X_4, Y_4)

(2.1)

where each player chooses a strategy (**C** or **D**), and the pairs (X_i, Y_i) represent payoffs to Players One and Two.

Following a few paragraphs of motivation, I will (in Definition (2.3)) define the associated quantum game $\mathbf{G}^Q$.

First, each player's strategy space should consist of the set of all possible physical manipulations of a penny. According to quantum mechanics (see the appendix for more detail), those manipulations are in a natural one-one correspondence with the *unit quaternions*. (The basic facts about quaternions are summarized in Appendix B.) Therefore, in the game $\mathbf{G}^Q$, each player's strategy space consists of the unit quaternions.

Suppose that Player One chooses the unit quaternion $\mathbf{p}$ and Player Two chooses the unit quaternion $\mathbf{q}$. Write the product $\mathbf{pq}$ as

$$\mathbf{pq} = \pi_1(\mathbf{pq}) + \pi_2(\mathbf{pq})i + \pi_3(\mathbf{pq})j + \pi_4(\mathbf{pq})k \tag{2.2}$$

where the $\pi_i(\mathbf{pq})$ are real numbers. Then according to the laws of physics (see Appendix A for more details) the pennies will appear to be flipped or unflipped according to the following probabilities:

$$\left.\begin{array}{lclclcl} \text{Prob(unflipped,unflipped)} & = & \pi_1(\mathbf{pq})^2 & \quad & \text{Prob(unflipped,flipped)} & = & \pi_2(\mathbf{pq})^2 \\ \text{Prob(flipped,unflipped)} & = & \pi_3(\mathbf{pq})^2 & & \text{Prob(flipped,flipped)} & = & \pi_4(\mathbf{pq})^2 \end{array}\right\} \tag{2.3}$$

Thus we are led to the following definition:

Definition 2.4. Let $\mathbf{G}$ be the game depicted in (2.1). Then the *associated quantum game* (or *quantization)* $\mathbf{G}^Q$ is the two- person game in which each player's strategy space is the unit quaternions, and the payoff functions for Players 1 and 2 are defined as follows:

$$P_1(\mathbf{p},\mathbf{q}) = \sum_{t=1}^{4} \pi_t(\mathbf{pq})^2 X_t \tag{2.4.1}$$

$$P_2(\mathbf{p},\mathbf{q}) = \sum_{\alpha=1}^{4} \pi_t(\mathbf{pq})^2 Y_t \tag{2.4.2}$$

3. Example: The Prisoner's Dilemma

Eisert and Wilkins [EW] analyze the quantum version of the following Prisoner's Dilemma game:

Player Two

Player One		**C**	**D**
	C	$(3,3)$	$(0,5)$
	D	$(5,0)$	$(1,1)$

(3.1)

which has the following Nash equilibrium in mixed quantum strategies:

$$\left.\begin{array}{c} \text{Player 1 plays the quaternions 1 and } k\text{, each with probability } 1/2. \\ \text{Player 2 plays the quaternions } i \text{ and } j\text{, each with probability } 1/2. \end{array}\right\} \tag{3.2}$$

Proposition 3.3 *(3.2) is a Nash equilibrium.*

Proof. I'll give this proof in some detail so the reader can check his understanding of Definition (2.4).

Take Player 1's strategy as given. Suppose Player 2 plays the quaternion $\mathbf{q} = \alpha + \beta i + \gamma j + \delta k$. Then Player 2's expected payoff is

$$\begin{aligned}
\frac{1}{2}P_2(1,\mathbf{q}) + \frac{1}{2}P_2(k,\mathbf{q}) &= \frac{1}{2}\sum_{t=1}^{4}\pi_t(\mathbf{q})^2 Y_t + \frac{1}{2}\sum_{t=1}^{4}\pi_t(k\mathbf{q})^2 Y_t \\
&= \frac{1}{2}\sum_{t=1}^{4}\Big[\pi_t(\alpha+\beta i+\gamma j+\delta k)\Big]^2 Y_t + \frac{1}{2}\sum_{t=1}^{4}\Big[\pi_t(-\delta-\gamma i+\beta j+\alpha k)\Big]^2 Y_t \\
&= \frac{1}{2}\big(3\alpha^2+5\beta^2+\delta^2\big) + \frac{1}{2}\big(3\delta^2+5\gamma^2+\alpha^2\big) \\
&= 2\alpha^2 + \frac{5}{2}\beta^2 + \frac{5}{2}\gamma^2 + 2\delta^2 \qquad (3.3.1)
\end{aligned}$$

Player 2's problem is to maximize (3.3.1) subject to the constraint that $\mathbf{q}$ must be a unit quaternion; i.e.

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$$

Clearly, then, Player 2's optimal response is to choose $\mathbf{q}$ such that $\alpha = \delta = 0$; for example, $\mathbf{q} = i$ and $\mathbf{q} = j$ are optimal responses.

In the same way, one verifies that if Player 2's strategy as given in (3.2), then $\mathbf{p} = 1$ and $\mathbf{p} = k$ are optimal responses for Player 1.

■

Proposition 3.4. *In the Nash equilibrium (3.2), each player has an expected payoff of* $5/2$.

Proof. Consider the following chart:

Probability	Player 1's Strategy($\mathbf{p}$)	Player 2's Strategy($\mathbf{q}$)	$\mathbf{pq}$	Player 1's Payoff	Player 2's Payoff
1/4	1	i	i	0	5
1/4	1	j	j	5	0
1/4	k	i	j	5	0
1/4	k	j	-i	0	5

■

Note in particular that the mixed strategy quantum equilibrium (3.2) is Pareto superior to the unique classical equilibrium in which each player earns a payoff of 1.

4. Notions of Equivalence.

We want to classify mixed-strategy Nash equilibria in quantum games. A mixed strategy is a probability distribution on the space of pure strategies, which in turn is a three-dimensional manifold. Thus the space of mixed strategies is huge and potentially intractable. In this section I will show that in fact mixed strategies fall naturally into equivalence classes with particularly simple representatives.

The definition of equivalent strategies is in (4.2), the main theorem about equivalence classes is (4.6), and the definition of equivalent equilibria is in (4.11).

Definition 4.1. A (pure) *quantum strategy* is a strategy in the game $\mathbf{G}^Q$, i.e. a unit quaternion. We will identify the unit quaternions with the unit ball $\mathbf{S}^3 \subset \mathbf{R}^4$. If $\mathbf{p}$ and $\mathbf{q}$ are quantum strategies chosen by Players One and Two, we write both $P_i(\mathbf{p}, \mathbf{q})$ and $P_i(\mathbf{pq})$ for the payoff to player i; thus P_i is a function of either one or two (quaternionic) variables depending on context.

Definition and Notation 4.2. A *mixed quantum strategy* (or just *mixed strategy* when the context is clear) is a probability distribution μ on the space of unit quaternions. If ν and μ are mixed quantum strategies chosen by Players 1 and 2, we write

$$P_i(\nu, \mu) = \int P_i(\mathbf{pq}) d\nu(\mathbf{p}) d\mu(\mathbf{q})$$

for Player i's expected payoff. We will identify the pure strategy $\mathbf{p}$with the mixed strategy $\nu_{\mathbf{p}}$ which is concentrated at $\mathbf{p}$; thus we will write $P_i(\mathbf{p}, \mu)$ for $P_i(\nu_{\mathbf{p}}, \mu)$.

Definition 4.3. Let μ and μ' be mixed strategies. We will say that μ and μ' are *equivalent* if

$$\int \pi_\alpha(\mathbf{pq}) d\mu(\mathbf{q}) = \int \pi_\alpha(\mathbf{p}, \mathbf{q}) d\mu'(\mathbf{q})$$

for all $\mathbf{p}$ and for $\alpha = 1, 2, 3, 4$, where the π_α are the coordinate functions defined in (2.2).

In other words, μ and μ' are equivalent if for for every quantum game and for every quantum strategy $\mathbf{p}$, $P_1(\mathbf{p}, \mu) = P_1(\mathbf{p}, \mu')$ and $P_2(\mu) = P_2(\mu')$.

Proposition 4.4. The mixed strategies μ and μ' are equivalent if and only if

$$P_i(\mu, \mathbf{q}) = P_i(\mu', \mathbf{q})$$

for all $\mathbf{q}$ and for $i = 1, 2$.

$\blacksquare$

Proposition 4.5. The pure strategy $\mathbf{p}$ is equivalent to the pure strategy $-\mathbf{p}$ and to no other pure strategy.

$\blacksquare$

Theorem 4.6. Every mixed strategy is equivalent to a mixed strategy supported on (at most) four points. Those four points can be taken to form an orthonormal basis for $\mathbf{R}^4$.

Proof. First, choose any orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4\}$ for $\mathbf{R}^4$. For any quaternion $\mathbf{p}$, write (uniquely)

$$\mathbf{p} = \sum_{\alpha=1}^{4} A_\alpha(p)\mathbf{q}_\alpha$$

where the $A_\alpha(p)$ are real numbers.

Define a probability measure ν supported on the four points $\mathbf{q}_\alpha$ by

$$\nu(\mathbf{q}_\alpha) = \int_{\mathbf{S}^3} A_\alpha(\mathbf{q})^2 d\mu(\mathbf{q})$$

For any two quaternions $\mathbf{p}$ and $\mathbf{q}$, define

$$X(\mathbf{p}, \mathbf{q}) = \sum_{\alpha=1}^{4} \pi_\alpha(\mathbf{p})\pi_\alpha(\mathbf{q})X_i \tag{4.6.1}$$

Then for any $\mathbf{p}$ we have

$$\begin{aligned}
P(\mathbf{p}, \mu) &= \int_{\mathbf{S}^3} P(\mathbf{pq})d\mu(\mathbf{q}) \\
&= \int_{S^3} P\left(\sum_{\alpha=1}^{4} A_\alpha(\mathbf{q})\mathbf{pq}_\alpha\right) d\mu(\mathbf{q}) \\
&= \sum_{\alpha=1}^{4} P(\mathbf{pq}_\alpha)\int_{S^3} A_\alpha(\mathbf{q})^2 d\mu(\mathbf{q}) + 2\sum_{\alpha\neq\beta} X(\mathbf{pq}_\alpha, \mathbf{pq}_\beta)\int_{\mathbf{S}^3} A_\alpha(\mathbf{q})A_\beta(\mathbf{q})d\mu(\mathbf{q}) \\
&= P(\mathbf{p}, \nu) + 2\sum_{\alpha\neq\beta} X(\mathbf{pq}_\alpha, \mathbf{pq}_\beta)\int_{\mathbf{S}^3} A_\alpha(\mathbf{q})A_\beta(\mathbf{q})d\mu(\mathbf{q})
\end{aligned}$$

To conclude that μ is equivalent to ν it is sufficient (and necessary) to choose the $\mathbf{q}_\alpha$ so that for each $\alpha \neq \beta$ we have

$$\int_{\mathbf{S}^3} A_\alpha(\mathbf{q})A_\beta(\mathbf{q})d\mu(\mathbf{q}) = 0$$

For this, consider the function $B : \mathbf{R}^4 \times \mathbf{R}^4 \to \mathbf{R}$ defined by

$$B(\mathbf{a}, \mathbf{b}) = \int_{\mathbf{S}^3} \pi_1(\overline{\mathbf{a}}\mathbf{q})\pi_1(\overline{\mathbf{b}}\mathbf{q})d\mu(\mathbf{q})$$

B is a bilinear symmetric form and so can be diagonalized; take the $\mathbf{q}_\alpha$ to be an orthonormal basis with respect to which B is diagonal. Then we have (for $\alpha \neq \beta$)

$$\begin{aligned}\int_{\mathbf{S}^3} A_\alpha(\mathbf{q})A_\beta(\mathbf{q})d\mu(\mathbf{q}) &= \int_{\mathbf{S}^3} \pi_1(\overline{\mathbf{q}_\alpha}\mathbf{q})\pi_1(\overline{\mathbf{q}_\beta}\mathbf{q})d\mu(\mathbf{q}) \\ &= B(\mathbf{q}_\alpha, \mathbf{q}_\beta) = 0\end{aligned}$$

■

Definition (4.3), together with Theorem (4.6), will be our primary tool for dividing Nash equilibria into equivalence classes. In the remainder of this section we develop a secondary tool:

Definition 4.7. Let ν be a mixed quantum strategy and let $\mathbf{u}$ be a unit quaternion. The *right translate of* ν *by* $\mathbf{u}$ is the measure $\nu\mathbf{u}$ defined by

$$(\nu\mathbf{u})(A) = \nu(A\mathbf{u})$$

(Here A is a subset of the unit quaternions and $A\mathbf{u} = \{\mathbf{x}\mathbf{u} | \mathbf{x} \in A\}$.) Similarly, the *left translate of* ν *by* $\mathbf{u}$ is the measure $\mathbf{u}\nu$ defined by

$$(\mathbf{u}\nu)(A) = \nu(\mathbf{u}A)$$

Definition 4.8. A *mixed strategy equilibrium* is exactly what you think it is, namely a pair of mixed strategies (ν, μ) such that ν maximizes $P_1(\nu, \mu)$ and μ maximizes $P_2(\nu, \mu)$. If $\mathbf{u}$ is a unit quaternion and (ν, μ) is a mixed strategy equilibrium, then clearly so is

$$u^*(\nu, \mu) = (\nu\mathbf{u}, \mathbf{u}^{-1}\mu) \tag{4.8.1}$$

We call (4.8.1) the $\mathbf{u}$- *translate* of the equilibrium (ν, μ).

Clearly $\mathbf{u}$-translating an equilibrium does not change either player's payoff.

Proposition 4.9. There exists at least one mixed strategy equilibrium.

Proof. This is proven, using standard methods, in [LJ]. However, there's an even easier argument: Let μ be the uniform probability distribution on the unit quaternions (thought of as the 3-dimensional unit sphere in $\mathbf{R}^4$). Then it's clear that (μ, μ) is an equilibrium.

$\blacksquare$

Remarks 4.10. In view of (4.8) and (4.9), mixed strategy equilibria not only exist; they tend to exist in great profusion: Given an equilibrium (ν, μ), we can always $\mathbf{u}$- translate it to get another. Alternatively, we can replace ν (or μ) with an equivalent strategy (in the sense of (4.3) and get still another equilibrium. This leads to the following definition:

Definition 4.11. Two equilibria (ν, μ) and (ν', μ') are *equivalent* if there exists a unit quaternion $\mathbf{u}$ such that ν' is equivalent to $\nu\mathbf{u}$ and μ' is equivalent to $\mathbf{u}^{-1}\mu$.

Remarks 4.12. There is one additional set of symmetries we can exploit. Let $\mathbf{G}$ be the game (2.1), let σ be any permutation of $(1, 2, 3, 4)$, and let $\mathbf{G}_\sigma$ be the game that results when the payoff pairs (X_t, Y_t) are permuted via σ. In general, the games $\mathbf{G}$ and $\mathbf{G}_\sigma$ are not isomorphic, but the corresponding quantum games are always isomorphic. On strategy spaces, the isomorphism permutes the strategies $1, i, j, k$, at least up to equivalence. (For a full description of the isomorphism see Appendix B.)

Thus, given a Nash equilibrium in $\mathbf{G}^Q$, we can "permute $1, i, j$ and k" to get a Nash equilibrium in $\mathbf{G}_\sigma^Q$, which means that when we classify equilibria we can classify them up to such permutations. This is what we will do in Section 7. Sometimes the results are clearer when we only allow permutations of i, j, k but not 1; this is what we will do in Section 6.

5. Classifying Equilibria: Preliminaries

Remarks 5.1. We want to classify all mixed strategy Nash equilbria (ν, μ) in the game $\mathbf{G}^Q$ associated to the general two by two game (2.1).

By (4.6) we can assume that μ is supported on four points $\mathbf{q}$, $\mathbf{uq}$, $\mathbf{u'q}$, and $\mathbf{uu'q}$ with $u^2 = (u')^2 = -1$ and $\mathbf{uu'} + \mathbf{u'u} = 0$. That is, we can assume that Player Two plays:

$$\begin{cases} \mathbf{q} \text{ with probabilty } \alpha \\ \mathbf{uq} \text{ with probability } \beta \\ \mathbf{u'q} \text{ with probability } \gamma \\ \mathbf{u'uq} \text{ with probability } \delta \end{cases} \tag{6.1.1}$$

where $\alpha + \beta + \gamma + \delta = 1$.

(We can also assume, up to a $\overline{\mathbf{q}}$-translation, that $\mathbf{q} = 1$, but for now it will be convenient to allow $\mathbf{q}$ to be arbitrary.)

Theorem 5.2. Taking Player 2's (mixed) strategy μ as given, Player 1's optimal response set is equal to the intersection of $\mathbf{S}^3$ with a linear subspace of $\mathbf{R}^4$.

(Recall that we identify the unit quaternions with the three-sphere $\mathbf{S}^3$.)

Proof. Player One's problem is to choose $\mathbf{p} \in S^3$ to maximize

$$P_1(\mathbf{p}, \mu) = \int P_1(\mathbf{pq}) d\mu(\mathbf{q}) \tag{5.2.1}$$

Expression (5.2.1) is a (real) quadratic form in the coefficients $\pi_i(\mathbf{p})$ and hence is maximized (over S^3) on the intersection of S^3 with the real linear subspace of $\mathbf{R}^4$ corresponding to the maximum eigenvalue of that form.

■

Definition 5.3. We define the function

$$K : \mathbf{S}^3 \to \mathbf{R}$$

by

$$K(A + Bi + Cj + Dk) = ABCD$$

Thus in particular $K(\mathbf{p}) = 0$ if and only if $\mathbf{p}$ is a linear combination of at most three of the fundamental units $\{1, i, j, k\}$.

Theorem 5.4. Let $\mathbf{p}$ be an optimal response to the strategy μ. Suppose it is not the case that $X_1 = X_2 = X_3 = X_4$. Then $\mathbf{p}$ must satisfy:

$$\begin{aligned}&(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)K(\mathbf{pq})\\ &+(\beta-\alpha)(\beta-\delta)(\beta-\gamma)K(\mathbf{puq})\\ &+(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)K(\mathbf{pu'q})\\ &+(\delta-\alpha)(\delta-\beta)(\delta-\gamma)K(\mathbf{pu'uq})=0\end{aligned} \tag{5.4.1}$$

Proof. Consider the function

$$\begin{array}{rccc}\mathcal{P}: & \mathbf{S}^3 \times \mathbf{R}^4 & \to & \mathbf{R} \\ & (\mathbf{p}, \mathbf{x}) & \mapsto & \sum_{n=1}^4 \mathbf{p}_n^2 \mathbf{x}_n d\mu(\mathbf{q})\end{array}$$

where $\mathbf{p}_n = \pi_n(\mathbf{p})$ (that is, $\mathbf{p}_n$ is defined by $\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 i + \mathbf{p}_3 j + \mathbf{p}_4 k$).

In particular, if we let $X = (X_1, X_2, X_3, X_4)$ then $\mathcal{P}(\mathbf{p}, X) = P_1(\mathbf{p}, \mu)$.

The function $\mathcal{P}$ is quadratic in $\mathbf{p}$ and linear in $\mathbf{x}$; explicitly we can write

$$\mathcal{P}(\mathbf{p}, \mathbf{x}) = \sum_{i,j,k} t_{ijk} \mathbf{p}_i \mathbf{p}_j \mathbf{x}_k$$

for some real numbers t_{ijk}.

Set

$$M_{ij}(\mathbf{x}) = \sum_{k=1}^{4} t_{ijk}\mathbf{x}_k$$

$$N_{ij}(\mathbf{p}) = \sum_{k=1}^{4} t_{ikj}\mathbf{p}_j$$

so that

$$M(\mathbf{x}) \cdot \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \\ \mathbf{p}_4 \end{pmatrix} = N(\mathbf{p}) \cdot \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \mathbf{x}_4 \end{pmatrix} \tag{5.4.2}$$

If $\mathbf{p}$ is an optimal response to the strategy μ, then $(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4)^T$ must be an eigenvector of $M(X)$, say with associated eigenvalue λ. From this and (5.4.2) we conclude that

$$N(\mathbf{p}) \cdot \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \lambda \cdot \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \\ \mathbf{p}_4 \end{pmatrix} = N(\mathbf{p}) \cdot \begin{pmatrix} \lambda \\ \lambda \\ \lambda \\ \lambda \end{pmatrix}$$

where the second equality holds by an easy calculation.

Thus $N(\mathbf{p})$ must be singular. But it follows from a somewhat less easy calculation that the determinant of $N(\mathbf{p})/2$ is given by the left side of (5.4.1).

■

6. Classifying Equilibria: Results.

In this section I will state the classification for Nash equilibria in *generic games* (to be defined below). The proof follows from the detailed results in Section 7, which also classify Nash equilibria in non-generic games.

All results are to be interpreted "up to a permutation of i, j, k" as described in (4.12).

Definition 6.1. The game $\mathbf{G}$ (2.1) is a *generic game* if

a) the X_i are all distinct

b) $X_1 + X_2 \neq X_3 + X_4$, and similarly for any permutation of $1, 2, 3, 4$.

c) the analogues of a) and b) hold for the Y_t as well as the X_t.

Here is the main theorem of this section; italicized words will be defined in 6.3, 6.4 and 6.7.

Theorem 6.2. For a generic game $\mathbf{G}$, every Nash equilibrium in $\mathbf{G}^Q$ is either *induced*, *intertwined*, or *special*.

Proof. Theorem 6.2 follows immediately from the more general results of Section 7, which classify Nash equilibria in both generic and non-generic games.

■

Herewith the definitions:

Definition 6.3. A mixed-strategy Nash equilibrium in the quantum game $\mathbf{G}^Q$ (2.4) is *induced* if both players's strategies are supported on the four-point set $\{1, i, j, k\}$. It is easy to check that any induced equilibrium is already an equilibrium in the 4 by 4 subgame where each player's strategy set is restricted to $\{1, i, j, k\}$. Thus finding induced equilibria in $\mathbf{G}^Q$ is no harder than finding equilibria in 4 by 4 games.

Definition 6.4. A mixed-strategy equilibrium in the quantum game $\mathbf{G}^Q$ (2.4) is *special* if (up to permuting i, j, k) one of the following holds:

i) Player Two plays two strategies 1 and $\mathbf{v}$, each with probability 1/2, and Player One plays two strategies $\mathbf{p}$ and $\mathbf{pv}$, each with probability 1/2.

ii) Player Two's strategy is supported on $\{1, i\}$ and Player One's strategy is supported on three orthogonal points in the linear span of $\{1, i, j\}$, each played with probability 1/3.

Notation 6.5. To define intertwined equilibria, we need some notation. First, it will be convenient to denote the unit quaternion $A + Bi + Cj + Dk$ by the 4-tuple (A, B, C, D), thought of as an element of the unit 3-sphere $\mathbf{S}^3$. Square roots of -1 are represented by unit quaternions of the form $(0, B, C, D) \in \mathbf{S}^2 \subset \mathbf{S}^3$. (That is, $\mathbf{S}^2$ is the unit 2-sphere, thought of as the equator in $\mathbf{S}^3$.)

If (X, Y, Z, W) is any non-zero 4-tuple of real numbers, I will write $\mathbf{p} \sim (X, Y, Z, W)$ to mean that $\mathbf{p}$ is a unit quaternion and a scalar multiple of (X, Y, Z, W); this uniquely determines $\mathbf{p}$ up to a sign.

If $\mathbf{p}$ and $\mathbf{q}$ are orthogonal unit quaternions, I will write $< \mathbf{p}, \mathbf{q} >$ for the circle of unit quaternions they generate; i.e.

$$< \mathbf{p}, \mathbf{q} >= \{\cos(\theta)\mathbf{p} + \sin(\theta)\mathbf{q} \quad | \quad \theta \in (0, 2\pi)\}$$

Remarks 6.6. Next I will define intertwined equilibria. Because the definition itself has several parts, I will start by explaining the main idea. First, in an intertwined equilibrium, each player's strategy is supported on at most two points. More specifically, Player Two's strategy is supported on $\{1, \mathbf{u}\}$ and Player One's strategy is supported on $\{\mathbf{p}, \mathbf{pv}\}$ where $\mathbf{p}, \mathbf{v}$, and $\mathbf{u}$ are unit quaternions satisfyinng

$$\mathbf{u}^2 = \mathbf{v}^2 = -1 \tag{6.5.1}$$

The triple $(\mathbf{p}, \mathbf{v}, \mathbf{u})$ is required to satisfy one of ten quite restrictive conditions, listed in (6.6).

I will elaborate on the phrase "quite restrictive": The triples $(\mathbf{p}, \mathbf{v}, \mathbf{u})$ satisfying (6.5.1) constitute a seven dimensional manifold $\mathbf{S}^3 \times \mathbf{S}^2 \times \mathbf{S}^2$. Each of the ten intertwining conditions picks out a subset of dimension at most 4. In nine out of ten cases, the condition is easy to describe. In those nine cases, there is no difficulty determining whether a given triple $(\mathbf{p},\mathbf{v},\mathbf{u})$ satisfies the condition. The tenth condition is more mysterious, but it applies only to a one- dimensional set of triples. Thus there is a very strong sense in which almost all intertwined equilibria are easy to describe.

Definition 6.7. A mixed-strategy Nash equilibrium in $\mathbf{G}^Q$ is *intertwined* if Player Two's strategy is supported on a two-point set $\{1,\mathbf{u}\}$ and Player One's strategy is supported on a two-point set $\{\mathbf{p},\mathbf{v}\}$ where $\mathbf{u}^2=\mathbf{v}^2=0$ and at least one of the following ten conditions holds (up to permuting i, j and k):

i) $\mathbf{u}=\mathbf{v}\perp\mathbf{p}\perp 1 \qquad \text{or} \qquad \mathbf{u}=\mathbf{v}\perp i\mathbf{p}\perp 1$

ii) $\mathbf{u}=i \quad \mathbf{v}\in<i,\overline{\mathbf{p}}i\mathbf{p}>\cup\{\overline{\mathbf{p}}j\mathbf{p},\overline{\mathbf{p}}k\mathbf{p}\}$

ii′) $\mathbf{v}=\overline{\mathbf{p}}i\mathbf{p} \quad \mathbf{u}\in<i,\overline{\mathbf{p}}i\mathbf{p}>\cup\{j,k\}$

iii) $\mathbf{p}\in<1,j><1,i>, \quad \mathbf{u}=i, \quad \mathbf{v}\in\overline{\mathbf{p}}<i,k>\mathbf{p}$

iii′) $\mathbf{p}\in<1,i><1,j>, \quad \mathbf{v}=\overline{\mathbf{p}}i\mathbf{p}, \quad \mathbf{u}\in<i,k>$

iv) $\mathbf{u}=i$ and $\mathbf{v}$ is (uniquely) determined by one of the following three conditions:

$$\mathbf{pv}\sim\big(-A(C^2+D^2),-B(C^2+D^2),C(A^2+B^2),D(A^2+B^2)\big)$$

$$\mathbf{pv}\sim\big(-C(BC+AD),D(BC+AD),B(AC-BD),A(AC-BD)\big)$$

$$\mathbf{pv}\sim\big(D(AC-BD),C(AC-BD),-A(BC+AD),B(BC+AD)\big)$$

where $\mathbf{p}=(A,B,C,D)$

iv′) $\mathbf{v}=\overline{\mathbf{p}}i\mathbf{p}$ and $\mathbf{u}$ is uniquely determined by one of the following three conditions:

$$\mathbf{pu}\sim\big(-A(C^2+D^2),-B(C^2+D^2),C(A^2+B^2),D(A^2+B^2)\big)$$

$$\mathbf{pu}\sim\big(-D(BD+AC),C(BD+AC),A(AD-BC),B(AD-BC)\big)$$

$$\mathbf{pu} \sim \big(C(AD-BC), D(AD-BC), -B(BD+AC), A(BD+AC)\big)$$

where $\mathbf{p} = (A,B,C,D)$

v) $\mathbf{p} \in < 1, i > \cup < j,k >$ $\quad \mathbf{u},\mathbf{v} \in \{i\} \cup < j,k >$

vi) $\mathbf{p} \in < i,j >< 1,\mathbf{v} >,$ $\quad \mathbf{u} = \mathbf{v} \in < i,j >$

vii) For some (A,B,C) with $A^2+B^2+C^2=1$ we have:

$$\mathbf{p} \sim (A,A,0,2C) \quad \text{or} \quad \mathbf{p} \sim (A,-A,-2B,0)$$

and $\mathbf{u},\mathbf{v}$ determined by one of the following conditions:

$$\mathbf{u} \sim (0, C-B, A, -A) \quad \text{and} \quad \mathbf{v} \sim (0, A^2+2BC, A(B-C), A(C-B))$$

$$\text{or} \qquad \mathbf{u} \sim (0, -B-C, A, A) \quad \text{and} \quad \mathbf{v} \sim (0, A^2-2BC, A(B+C), A(B+C)))$$

viii) $\mathbf{p},\mathbf{u},\mathbf{v}$ are of the form:

$$\mathbf{u} = (0, X, Y, \pm Y) \qquad \mathbf{v}\ (0, 2ABX - Y, 2B^2X, \pm 2B^2X)$$

$$\mathbf{p} = (0, A, B, \pm B) \quad \text{or} \mathbf{p} = (AY - 2BX, 0, BY, \mp BY)$$

ix) $\mathbf{u} = (j \pm k)/\sqrt{2}$ $\quad \mathbf{p} \in < 1, i\mathbf{u} > \cup < i, i\mathbf{u} >$ $\quad \mathbf{v} = i$

ix$'$) $\mathbf{v} = (j \pm k)/\sqrt{2}$ $\quad \mathbf{p} \in < 1, i\mathbf{v} > \cup < i, i\mathbf{v} >$ $\quad \mathbf{u} = \overline{\mathbf{p}} i \mathbf{p}$

x) $(\mathbf{p},\mathbf{v},\mathbf{u})$ is a real point on a certain one- dimensional subvariety of $\mathbf{S}^3 \times \mathbf{S}^2 \times \mathbf{S}^2$

7. Classifying Equilibria: More Results

Remarks, Definitions, and Conventions 7.0. In view of Theorem 5.2, we can classify all mixed strategy equilibria in terms of the dimensions of the players' optimal response sets, each of which is a sphere of dimension $0, 1, 2$ or 3.

We can assume without loss of generality that Player One's optimal response set has at least the same dimension as Player Two's. Thus we say that an equilibrium is of Type (m,n) $(0 \leq n \leq m \leq 3)$ if Player One's optimal response set has dimension m and Player Two's optimal response set has dimension n. Up to renaming the players, every equilibrium is of one of these ten types.

We will classify equilibria up to equivalence (defined in (4.11)) and up to permutations of $1, i, j, k$ (as described in (4.12)). From Theorem (4.6), we can assume that Player One's strategy is supported on $m+1$ orthogonal points and Player Two's strategy is supported on $n+1$ orthogonal points. We can also assume that the quaternion 1 is in the support of Player Two's strategy. (Take any $\mathbf{q}$ in the support of Player Two's strategy and apply a $\mathbf{q}$- transformation.)

Note that the strategy $\mathbf{p}$ is always equivalent to the strategy $-\mathbf{p}$. I will therefore often abuse notation by writing $\mathbf{p} = \mathbf{q}$ to mean $\mathbf{p} = \pm\mathbf{q}$.

Classification 7.1: Equilibria of Type (0,0). Clearly an equilibrium of type (0,0) occurs when and only when there is a $\mathbf{t} \in \{1, 2, 3, 4\}$ that uniquely maximizes both X_t and Y_t. We can assume $t = 1$ and each player plays the quaternion 1.

Classification 7.2: Equilibria of Type (1,0). Up to equivalence, every equilibrium of Type (1,0) is of the following sort:

a) Player Two plays the pure strategy 1.

b) Player Two plays the strategies 1 and i with some probabilities ϕ and $\psi = 1 - \psi$.

c) $\phi \neq 1/2$

Proof. By the remarks in (7.0) we can assume (a). It is clear that Player One's optimal response set is spanned by exactly two of $1, i, j, k$; we can assume it is spanned by 1 and i. By the remarks in (7.0) we can assume Player One's strategy ν is supported on two quaternions of the form $A + Bi$, $-B + Ai$, played with some probabilities ϕ, ψ.

We have

$$P_2(\nu, A - Bi) = \phi Y_1 + \psi Y_2$$

$$P_2(\nu, B + Ai) = \psi Y_2 + \phi Y_1$$

and it is clear that at least one of these is as least as great as

$$P_2(\nu, 1) = (\phi A^2 + \psi B^2)Y_1 + (\phi B^2 + \psi A^2)Y_2$$

The unique optimality of Player Two's response then implies that $1 = A - Bi$ or $1 = B + Ai$; we can assume the former, so that $B + Ai = i$, giving b).

If $\phi = 1/2 = \psi$ then i is also an optimal response for Player Two, violating the uniqueness assumption, so we have $\phi \neq 1/2$, establishing (c).

■

Classification 7.3: Equilibria of Type (2,0). Up to equivalence, every equilibrium of type (2,0) is of the following sort:

a) Player Two plays the pure strategy 1.

b) Player One plays a strategy supported on three mutual orthogonal quaternions $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ played with some probabilities ϕ, ψ, ξ.

c) Each $\mathbf{p}_t$ is a linear combination of $1, i, j$ (but not k).

d) If $Y_1, \ldots, Y_4$ are all distinct, then either $\mathbf{p}_1 = 1$ or $\phi = \psi = \xi = 1/3$.

e) If $Y_1, \ldots, Y_4$ are all distinct and ϕ, ψ, ξ are all distinct, then $\mathbf{p}_1 = 1$, $\mathbf{p}_2 = i$ and $\mathbf{p}_3 = j$.

Proof. We can assume $X_1 = X_2 = X_3 > X_4$; a), b) and c) follow from this and the generalities of (7.0).

Now suppose the Y_t are all distinct. Write

$$\mathbf{p}_1 = A + Bi + Cj$$

$$\mathbf{p}_2 = D + Ei + Fj$$

$$\mathbf{p}_3 = G + Hi + Ij$$

Evaluating the first order conditions for Player Two's maximization problem and setting them equal to zero at his optimal strategy 1 gives:

$$(\phi AB + \psi DE + \xi GH)(Y_1 - Y_2) = 0 \tag{7.3.1a}$$

$$(\phi AC + \psi DF + \xi GI)(Y_1 - Y_3) = 0 \tag{7.3.1b}$$

$$(\phi BC + \psi EF + \xi HI)(Y_2 - Y_3) = 0 \tag{7.3.1c}$$

Because the Y_t are distinct, we can write

$$\begin{pmatrix} AB & DE & GH \\ AC & DF & GI \\ BC & EF & HI \end{pmatrix} \begin{pmatrix} \phi \\ \psi \\ \xi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} AB & DE & GH \\ AC & DF & GI \\ BC & EF & HI \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{7.3.2}$$

Unless $\phi = \psi = \xi = 1/3$, it follows that the matrix

$$\begin{pmatrix} AB & DE & GH \\ AC & DF & GI \\ BC & EF & HI \end{pmatrix}$$

has rank at most one while the matrix

$$O = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}$$

is orthogonal. From this it follows that O contains a row with two zeros. We can permute rows and columns to get

$$O = \begin{pmatrix} 1 & 0 & 0 \\ 0 & E & F \\ 0 & -F & E \end{pmatrix}$$

which establishes d). Moreover, if $\psi \neq \xi$ then (7.3.2) now implies $EF = 0$, establishing (e).

■

Classification 7.4: Equilibria of Type (3,0). Clearly such an equilibrium requires $X_1 = X_2 = X_3 = X_4$. Player One's strategy ν is supported on four mutually orthogonal quaternions $\mathbf{p}, \mathbf{pv}, \mathbf{pv}'$ and $\mathbf{pvv}'$ where $\mathbf{vv}' + \mathbf{v}'\mathbf{v} = 0$. These cannot be played equiprobably. (Otherwise $P_2(\nu, \mathbf{v}) = P_2(\nu, 1)$ so Player Two's response would not be unique.)

Classification 7.5: Equilibria of Type (1,1). Every equilibrium of Type (1,1) is either induced, special or intertwined (as defined in Section 6).

Proof. We can assume the strategies are

Player One		**Player Two**	
Strategy (ν)	Probability	Strategy (μ)	Probability
$\mathbf{p}$	ϕ	1	$\alpha \neq 0$
$\mathbf{pv}$	ψ	$\mathbf{u}$	β

with $\mathbf{u}^2 = \mathbf{v}^2 = -1$.

Proof. First assume $\beta = 0$. Then exactly two of the X's are maximal; we can assume $X_1 = X_2 > X_3, X_4$. It follows that we can write $\mathbf{p} = A + Bi$ and $\mathbf{v} = i$. Because 1 is an optimal response by Player Two, the first order condition

$$AB(\phi - \psi)(Y_1 - Y_2) \tag{7.5.1}$$

must hold. If $AB = 0$, the equilibrium is induced. Otherwise it follows from (7.5.1) that

$P_2(\nu, i) = P_2(\nu, 1)$ so that i is an optimal response for Player Two; thus $\mathbf{u} = i$. Therefore the equilibrium is intertwined by condition (6.7v).

This completes the proof when $\beta = 0$; by symmetry we can assume $\alpha\beta\phi\psi \neq 0$.

Next suppose $\alpha \neq \beta$ and $\phi \neq \psi$. Then (5.4.1) reduces to

$$\alpha^2 K(\mathbf{p}) = \beta^2 K(\mathbf{pu}) \tag{7.5.2}$$

For any real number θ, (5.2) implies that

$$\mathbf{p}(\theta) = \cos(\theta)\mathbf{p} + \sin(\theta)\mathbf{pv}$$

is an optimal response for Player One; therefore we can replace $\mathbf{p}$ with $\mathbf{p}(\theta)$ in (7.5.2). That is, the equation

$$\alpha^2 K(\cos(\theta)\mathbf{p} + \sin(\theta)\mathbf{pv}) = \beta^2 K(\cos(\theta)\mathbf{pu} + \sin(\theta)\mathbf{pvu}) \tag{7.5.3}$$

must hold identically in θ. In [I], I define the quadruple $(\mathbf{p}, \mathbf{pv}, \mathbf{pu}, \mathbf{pvu})$ to be *intertwined* if (7.5.3) holds identically in θ for some fixed nonzero α and β. Thus $(\mathbf{p}, \mathbf{pv}, \mathbf{pu}, \mathbf{pvu})$ is intertwined, and, by reversing the players, so is $(\mathbf{p}, \mathbf{pu}, \mathbf{pv}, \mathbf{pvu})$. When both of these quadruples are intertwined I say that $(\mathbf{p}, \mathbf{pv}, \mathbf{pu}, \mathbf{pvu})$ is *fully intertwined*. The main theorem of [I] shows that all fully intertwined quadruples fit into (at least) one of the families listed in (6.7), so that the equilibrium in question is intertwined.

It remains to consider the case $\alpha = \beta = 1/2$ (a similar argument applies when $\phi = \psi = 1/2$). In this case

$$P_1(\mathbf{pu}, \mu) = \frac{1}{2}P_1(\mathbf{pu}) + \frac{1}{2}P_1(\mathbf{p}) = P_1(\mathbf{p}, \mu)$$

so that the optimality of $\mathbf{p}$ implies the optimality of $\mathbf{u}$, whence $\mathbf{v} = \mathbf{u}$. If $\phi = \psi = 1/2$, we can conclude that the equilibrium is special, so assume $\phi \neq 1/2$. Then as before $(\mathbf{p}, \mathbf{pu}, \mathbf{pv}, \mathbf{pvu})$ is intertwined, and, because $\mathbf{u} = \mathbf{v}$, it is fully intertwined. Hence the argument of the preceding paragraph applies.

∎

Classification 7.6: Equilibria of Type (2,1). A Type (2,1) equilibrium is described (up to equivalence) by

Player One (ν)

Strategy	Probability
$\mathbf{p}_1$	ϕ
$\mathbf{p}_2$	ψ
$\mathbf{p}_3$	ξ

Player Two (μ)

Strategy	Probability
1	α
$\mathbf{u}$	β

where $\mathbf{u}^2 = -1$. We have:

a) $\alpha \neq 1/2$

b) $\mathbf{u} = i$

c) The linear span of $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ is equal to the linear span of $1, i, j$.

d) At least one of the following is true:

   i) $\phi = \psi = \xi = 1/3$

   ii) $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\} = \{1, i, j\}$

   iii) $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\} \cap \{1, i, j\} \neq \emptyset$ and the payoffs Y_t are equal in pairs (e.g. $Y_1 = Y_2$ and $Y_3 = Y_4$).

Proof. On dimensional grounds, Player One's optimal response set overlaps Player Two's optimal response set nonvacuously. That is, Player One has an optimal response of the form $\mathbf{p} = A + B\mathbf{u}$. Then

$$P_1(\mathbf{p}, \mu) = A^2 P_1(1, \mu) + B^2 P_1(\mathbf{u}, \mu)$$

so that 1 and $\mathbf{u}$ must be optimal responses for Player One. Therefore Player One's optimal response set is spanned by 1, $\mathbf{u}$ and $\mathbf{v}$ for some $\mathbf{v}$ such that $\mathbf{v}^2 = -1$ and $\mathbf{uv} + \mathbf{vu} = 0$.

Claim: $\alpha \neq 1/2$. Proof: Otherwise, $P_1(\mathbf{vu}, \mu) = P_1(\mathbf{v}, \mu)$, making $\mathbf{vu}$ an optimal response for Player One. But $1, \mathbf{v}, \mathbf{u}, \mathbf{vu}$ span all of $\mathbf{S}^3$, contradicting the 2-dimensionality of Player One's optimal response set. This proves the claim and establishes a).

Now let X and Y be any real numbers and let $\mathbf{p}(X,Y) = \mathbf{p} + X\mathbf{u} + Y\mathbf{v}$. Then $\mathbf{p}(X,Y)/||\mathbf{p}(X,Y)||$ is an optimal response for Player One. Thus by (5.4) (with $\gamma = \delta = 0$) we have

$$\alpha^2 K(1 + X\mathbf{u} + Y\mathbf{v}) = \beta^2(\mathbf{u} - X + Y\mathbf{vu}) \tag{7.6.1}$$

Write $\mathbf{u} = Pi + Qj + Rk$. Then, setting $Y = 0$, (7.6.1) becomes

$$\alpha^2 PQRX^3 = -\beta^2 PQRX$$

which must hold indentically in X. Thus $PQR = 0$ and we can assume $R = 0$.

The orthogonality of $\mathbf{u}$ and $\mathbf{v}$ implies that $\mathbf{v}$ is of the form $-SQi + SPj + T$. Equating coefficients in (7.6.1) first on XY, then on X^2Y, and then on XY^2, we deduce

$$PQS = PQT = (P^2 - Q^2)ST = 0$$

Together with the requirement that $\mathbf{u}$ and $\mathbf{v}$ have length one (i.e. $P^2 + Q^2 = S^2 + T^2 = 1$) this implies $PQ = ST = -0$. Without loss of generality, $Q = T = 0$ so $\mathbf{u} = i$ and $\mathbf{v} = j$, which establishes b) and c).

Now we can write

$$\mathbf{p}_1 = A + Bi + Cj$$

$$\mathbf{p}_2 = D + Ei + Fj$$

$$\mathbf{p}_3 = G + Hi + Ik$$

The first order conditions for Player Two's maximization problem must be satisfied at both 1 and i. This gives five equations

$$(\phi AB + \psi DE + \xi GH)(Y_1 - Y_2) = 0 \tag{7.6.2a}$$

$$(\phi AC + \psi DF + \xi GI)(Y_1 - Y_3) = 0 \tag{7.6.2b}$$

$$(\phi BC + \psi EF + \xi HI)(Y_2 - Y_3) = 0 \tag{7.6.2c}$$

$$(\phi AC + \psi DF + \xi GI)(Y_2 - Y_4) = 0 \tag{7.6.2d}$$

$$(\phi BC + \psi EF + \xi HI)(Y_1 - Y_4) = 0 \tag{7.6.2e}$$

(Note that (7.6.2a-c) are identical to (7.3.1a-c).)

Claim: We cannot simultaneously have $Y_1 = Y_3$ and $Y_2 = Y_4$. Proof: Otherwise $P_2(\nu, j) = P_2(\nu, 1)$, contradicting the suboptimality of j as a response for Player 2. Similarly, we cannot simultaneously have $Y_1 = Y_4$ and $Y_2 = Y_3$.

From this and equations (7.6.2a-e) we conclude that

$$\begin{pmatrix} AC & DF & GI \\ BC & EF & HI \end{pmatrix} \begin{pmatrix} \phi \\ \psi \\ \xi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} AC & DF & GI \\ BC & EF & HI \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

so that if condition di) fails, then the matrix

$$\begin{pmatrix} AC & DF & GI \\ BC & EF & HI \end{pmatrix}$$

has rank one. Together with the orthogonality of

$$O = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}$$

this implies that O has at least one row with two zeros. That is, $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\} \cap \{1, i, j\} \neq \emptyset$. We can assume $\mathbf{p}_1 = 1$.

Finally, we assume that di) and diii) both fail and establish dii). If $Y_1 \neq Y_2$, then di) follows from the preceding paragraph just as in the proof of 7.3). Thus we can assume $Y_1 = Y_2$ and consequently $Y_3 \neq Y_4$.

From $P_2(\nu, 1) = P_2(\nu, i)$ and $Y_1 = Y_2$ we conclude

$$(\phi C^2 + \psi F^2 + \xi I^2)(Y_3 - Y_4) = 0$$

whence $\phi C^2 + \psi F^2 + \xi I^2 = 0$. From $\mathbf{p}_1 = 1$ we have $C = 0$. We can assume $\phi \neq 1$ (otherwise $\mathbf{p}_2$ and $\mathbf{p}_3$ can be replaced with i and j), so at least one of F and I is zero. This (together with $B = C = 0$) implies that the orthogonal matrix O is in fact a permutation matrix, which is condition dii).

Classification 7.7: Equilibria of Type $(3,1)$**.** Player Two plays

Strategy	Probability
1	α
$\mathbf{u}$	β

where (up to permuting i, j, k) one of the following must hold:

a) $X_1 = X_2 = X_3 = X_4$

b) $\alpha = \beta = 1/2$, $\mathbf{u} = i$, $X_1 + X_2 = X_3 + X_4$

c) $\alpha = \beta = 1/2$, $\mathbf{u} \in < i, j >$, $X_1 = X_4$, $X_2 = X_3$

Proof. Let $\mathbf{p} = (X, Y, Z, W)$ be an arbitrary unit quaternion. Taking Player Two's strategy as given, Player One's payoff is a quadratic form in X, Y, Z, W that must be constant on the unit sphere. Equating the coefficients on the terms X^2, Y^2, Z^2 and W^2, while setting the coefficients on the various cross terms equal to zero, gives the solutions listed.

Classification 7.8: Equilibria of Type (2,2). A Type (2,2) equilibrium is described (up to equivalence) by

Player One (ν)		**Player Two (μ)**	
Strategy	Probability	Strategy	Probability
$\mathbf{p}_1$	ϕ	1	α
$\mathbf{p}_2$	ψ	$\mathbf{u}$	β
$\mathbf{p}_3$	ξ	$\mathbf{w}$	γ

where $\mathbf{u}^2 = \mathbf{w}^2 = -1$ and $\mathbf{uw} + \mathbf{wu} = 0$. If $\alpha\beta\gamma = 0$ then the results of (7.6) hold (by the exact same proof). Otherwise, suppose that α, β, γ all differ from zero, from 1/2 and from each other. Then we have:

a) $\mathbf{u} = i$ and $\mathbf{v} = j$

b) The linear span of $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ is equal either to the linear span of $1, i, j$ or $1, i, k$.

c) At least one of the following is true:

  i) $\phi = \psi = \xi = 1/3$

  ii) $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\} = \{1, i, j\}$

  iii) $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\} = \{1, i, k\}$

  iv) $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\} \cap \{1, i, j, k\} \neq \emptyset$ and the payoffs Y_t are equal in pairs.

Proof. Let $\mathbf{p}$ be any optimal response by Player One. From (5.4.1) with $\delta = 0$ and $\mathbf{u}' = \mathbf{w}$ we get an equation

$$\sigma_1 K_1 + \sigma_2 K_2 + \sigma_3 K_3 + \sigma_4 K_4 = 0 \tag{7.8.1}$$

where $\sigma_1 = (\alpha - \beta)(\alpha - \gamma)\alpha$, $K_1 = K(\mathbf{p})$, etc.

After applying a $\mathbf{u}-$ or $\mathbf{w}-$ translation as needed, we can assume that α lies strictly between β and γ. Thus

$$\sigma_1, \sigma_4 < 0 \qquad \text{and} \qquad \sigma_2, \sigma_3 > 0 \tag{7.8.2}$$

Claim One: Player One's optimal response set contains at least two of $1, \mathbf{u}, \mathbf{w}, \mathbf{uw}$.

Proof of Claim: On dimensional grounds, Player One's optimal response set contains quaternions of the form $Q+R\mathbf{u}$ and $S\mathbf{w}+T\mathbf{uw}$. Applying (7.8.1) to the case $\mathbf{p}=Q+R\mathbf{u}$ and noting that $K(\mathbf{pw})=K(\mathbf{puw})=0$, we get

$$QR(\sigma_1 R^2-\sigma_2 Q^2)K(1+\mathbf{u})=0$$

and similarly

$$ST(\sigma_3 S^2-\sigma_4 T^2)K(1+\mathbf{u})=0$$

Together with (7.8.2), this implies either $QR=0=ST=0$, in which case the claim follows, or $K(1+\mathbf{u})=0$. So we can assume $K(1+\mathbf{u})=0$ and similarly $K(1+\mathbf{w})=0$.

From this and the orthogonality of $\mathbf{u}$ and $\mathbf{w}$, it follows that at least one of $\mathbf{u}, \mathbf{w}, \mathbf{uw}$ is equal to i, j or k. Suppose first thtat $\mathbf{w}=i$. Then we can calculate

$$P_1(Q+R\mathbf{u},\mu)=Q^2P_1(1,\mu)+R^2P_1(\mathbf{u},\mu)$$

$$P_1(S\mathbf{w}+T\mathbf{uw})=S^2P_1(1,\mu)+T^2P_1(\mathbf{u},\mu)$$

Since both responses are optimal, we have either $QR=ST=0$ or $P_1(1,\mu)=P_1(\mathbf{u},\mu)$, and in the latter case we can reset $Q=S=0$. Either way we have proved the claim.

Following an appropriate translation, we can now assume that Player One's optimal response set contains either 1 and $\mathbf{u}$ or 1 and $\mathbf{uw}$. We assume the former, and indicate at the end how to modify the proof in case of the latter. Player One's optimal response set is generated by 1, $\mathbf{u}$ and some $\mathbf{v}\perp\mathbf{u}$ with $\mathbf{v}^2=-1$. Apply (5.4.1) to the case $\delta=0$, $\mathbf{q}=1$, $\mathbf{u}'=\mathbf{w}$ and $\mathbf{p}=1+X\mathbf{u}+Y\mathbf{v}$ where $\mathbf{p}$ is a scalar multiple of some optimal response by Player One. Write the resulting polynomial as

$$\sigma_1 K_1 + \sigma_2 K_2 + \sigma_3 K_3 + \sigma_4 K_4 \tag{7.8.3}$$

and note that the assumptions on α, β, γ imply that the σ_i are all nonzero, as are $\sigma_1 + \sigma_2$ and $\sigma_3 + \sigma_4$.

Note that K_3 and K_4 are both divisible by Y. Thus we can set $Y = 0$ in (7.8.3) to get

$$\sigma_1 K(1 + X\mathbf{u}) = \sigma_2(X - \mathbf{u})$$

The left side is cubic in X and the right side is linear in X; hence both sides are zero. This shows that $\mathbf{u}$ is in the linear span of at most two of i, j, k; without loss of generality write $\mathbf{u} = Ai + Bj$. Then, because $\mathbf{v}$ and $\mathbf{w}$ are orthogonal to $\mathbf{u}$, we can write

$$\mathbf{v} = Rk\mathbf{u} + Ck \qquad \mathbf{w} = Sk\mathbf{u} + D$$

for some real numbers R, C, S, D.

Inserting all this into equation (7.8.3) and examining the coefficients on X^3Y and Y gives

$$ABDS\Big[\sigma_3 D(CD + RS) - \sigma_4 S(CS - DR)\Big] = 0$$

$$ABDS\Big[\sigma_3 S(CD + RS) - \sigma_4 D(CS - DR)\Big] = 0$$

Because $D^2 + S^2 = 1$, this gives

$$ABDS(CD + RS) = ABDS(CS - DR) = 0$$

Now $(CD + RS)^2 + (CS - DR)^2 = 1$ so

$$ABDS = 0$$

I claim that $AB=0$. If not, then either $D=0$ (so $S=1$) or $S=0$ (so $D=1$). If $D=0$ examine the coefficients in (7.8.3) on X^2Y and XY to get

$$(\sigma_1+\sigma_4)ABC=(\sigma_2+\sigma_3)ABR=0$$

Because $C^2+R^2=1$ this gives $AB=0$. If $S=0$, the same argument works. This establishes the claim and establishes that $\mathbf{u}\in\{i,j\}$; without loss of generality $\mathbf{u}=i$.

Now examine the coefficient in (7.8.3) on Y^2 to get

$$(CS-DR)(CD+RS)DS=0 \tag{7.8.4}$$

Thus either $DS=0$ or $(CS-DR)(CD+RS)=0$; either way, the coefficient on X^2Y reveals that $CR=0$ so $\mathbf{v}\in\{j,k\}$. Now with $CR=0$ the left side of (7.8.4) becomes $\pm D^2S^2$ so $DS=0$ and $\mathbf{w}\in\{j,k\}$ also.

We have proven (as always, up to equivalence):

$$\mathbf{u}=i \qquad \text{and} \qquad \mathbf{v},\mathbf{w}\in\{j,k\} \tag{7.8.5}$$

As noted earlier, the proof of (7.8.5) relied on the assumption that $\mathbf{u}$ is in Player One's optimal response set, whereas it is possible that $\mathbf{uw}$ is in Player One's optimal response set instead. But in that case, (7.8.5) still holds, by essentially the same proof: First show that $\mathbf{uw}=Ai+Bj$, then write $\mathbf{u}=Rk\mathbf{uw}+Ck$, $\mathbf{w}=Sk\mathbf{uw}+Dk$. All the remaining calculations work out exactly the same as above. Thus (7.8.5) is proven.

After an appropriate permutation, (7.8.5) implies a) and b). Condition c) then follows exactly as in the proof of (7.6).

■

Remarks 7.8.6. The results of (7.8.6) assume that α, β, γ all differ from zero, from 1/2 and from each other. The motivated reader will have no difficulty modifying the results to cover most of the remaining cases. Suppose, for example that $\alpha = \beta$. Then we can conclude that Player One's response set contains at least one of $1, \mathbf{u}, \mathbf{w}, \mathbf{uw}$.

Proof. On dimensional grounds, there are real numbers σ, τ such that $\sigma\mathbf{w} + \tau\mathbf{uw}$ is an optimal response for Player One. In the following calculation, X is the function defined in (4.6.1):

$$\begin{aligned}
P_1(\mathbf{v}, \mu) &= \alpha P_1(\mathbf{v}) + \alpha P_1(\mathbf{vu}) + \gamma P_1(\mathbf{vuw}) \\
&= \alpha P_1(\sigma\mathbf{w} + \tau\mathbf{uw}) + \alpha P_1(-\sigma\mathbf{uw} + \tau\mathbf{w}) + \gamma P_1(-\sigma - \tau\mathbf{u}) \\
&= \alpha\big(\sigma^2 P_1(\mathbf{w}) + \tau^2 P_1(\mathbf{uw}) + 2\sigma\tau X(\mathbf{w}, \mathbf{uw})\big) \\
&\qquad + \alpha\big(\sigma^2 P_1(\mathbf{uw}) + \tau^2 P_1(\mathbf{w}) + 2\sigma\tau X(-\mathbf{uw}, \mathbf{w})\big) \\
&\qquad\qquad + \gamma\big(\sigma^2 P_1(1) + \tau^2 P_1(\mathbf{u})\big) \\
&= \alpha P_1(\mathbf{w}) + \alpha P_1(\mathbf{uw}) + \gamma\sigma^2 P_1(1) + \gamma\tau^2 P_1(\mathbf{u}) \qquad (7.8.6.4)
\end{aligned}$$

where we've used the facts that $X(\mathbf{w}, \mathbf{uw}) = -X(-\mathbf{w}, \mathbf{uw})$ and $X(1, \mathbf{u}) = 0$ (which follows from $\mathbf{u}^2 = -1$).

It's clear that (7.8.6.4) is maximized at an extreme value where either $\sigma = 1$ (so that $\mathbf{v} = \mathbf{w}$) or $\tau = 1$ (so that $\mathbf{v} = \mathbf{uw}$).

■

Classification 7.9: Equilibria of Type (3,2). Player Two plays

Strategy	Probability
1	α
$\mathbf{u}$	β
$\mathbf{w}$	γ

and one of the following must hold (for exactly the same reasons as in 7.7):

a) $X_1 = X_2 = X_3 = X_4$

a) $X_1 = X_2 = X_3 = X_4$

b) $\alpha = \beta = 1/2$, $\gamma = 0$, $\mathbf{u} = i$, $X_1 + X_2 = X_3 + X_4$

c) $\alpha = \beta = 1/2$, $\gamma = 0$, $\mathbf{u} \in < i, j >$, $X_1 = X_4$, $X_2 = X_3$

Classification 7.10: Equilibria of Type (3,3). Player Two plays a mixed strategy:

Strategy	Probability
1	α
$\mathbf{u}$	β
$\mathbf{w}$	γ
$\mathbf{uw}$	δ

where, up to permuting i, j, k and relabeling $\mathbf{u}, \mathbf{w}, \mathbf{uw}$, one of the following must hold:

a) $X_1 = X_2 = X_3 = X_4$

b) $\mathbf{u} = i$, $\alpha = \beta$, $\gamma = \delta$ $X_1 + X_2 = X_3 + X_4$

c) $\mathbf{u} \in < i, j >$, $\alpha = \beta$, $\gamma = \delta$, $X_1 = X_4$, $X_2 = X_3$

Player One plays a mixed strategy satisfying the analogous conditions.

The proof is as in 7.7 and 7.9.

8. Minimal Payoffs and Zero-Sum Games.

We close by noting some properties shared by all mixed-strategy Nash equilibria.

Theorem 8.1. Consider the game (2.1). In any mixed strategy quantum equilibrium, Player One earns a payoff of at least $(X_1 + X_2 + X_3 + X_4)/4$ and Player Two earns a payoff of at least $(Y_1 + Y_2 + Y_3 + Y_4)/4$.

Proof. Let μ be Player Two's strategy. Then Player One maximizes the quadratic form (5.2.1) over S^3. This quadratic form has trace $X_1+X_2+X_3+X_4$ and hence maximum eigenvalue at least $(X_1+X_2+X_3+X_4)/4$, so Player One is guaranteed a payoff at least that large. Similarly, of course, for Player Two.

Corollary 8.2. In any game, there exists a mixed strategy quantum equilibrium that is Pareto inferior to any other mixed strategy quantum equilibrium.

Proof. The uniform equilibrium always exists.

$\blacksquare$

Corollary 8.3. If the game (2.1) is zero sum, then in any mixed strategy quantum equilibrium, Player One earns exactly $(X_1+X_2+X_3+X_4)/4$ and Player Two earns exactly $(Y_1+Y_2+Y_3+Y_4)/4$.

$\blacksquare$

Appendix A:
A Primer on Quantum Mechanics for Economists

This primer is filled with lies. The first lie is that a complex structure (namely a penny) can be treated as a simple quantum particle. Another lie is that observable properties like color take only two values, so that every penny must be either red or green. And so forth. Some lies are owned up to and corrected shortly after they are told. Every intentional lie is there only to ease the exposition. If there are any unintentional lies, I want to know about them.

For the most part, I've sought to cover just enough quantum mechanics to motivate quantum game theory. I've included a few brief excursions into topics like the uncertainty principle and Bell's theorem which are not strictly necessary for the applications to game theory but seemed like too much fun to omit.

1. States. A classical penny is in one of two states: *heads* (which we denote $\mathbf{H}$) or *tails* (which we denote $\mathbf{T}$). (We ignore the possibility that a penny might stand on end.) I'll use the word *orientation* to refer to this property. (E.g. "What's the orientation of that penny you just flipped?" "Its orientation is heads.")

A quantum penny can be in any state of the form

$$\alpha\mathbf{H} + \beta\mathbf{T} \tag{1.1}$$

where α and β are arbitrary complex numbers, not both zero. State (1.1) is called a *superposition* of the states $\mathbf{H}$ and $\mathbf{T}$; "superposition" means the same thing as "linear combination".

State (1.1) is considered indistinguishable from the state

$$\lambda\alpha\mathbf{H} + \lambda\beta\mathbf{T} \tag{1.2}$$

where λ is an arbitrary nonzero complex number. That is, (1.1) and (1.2) are two different names for the same state.

(Warning: We will abuse language by using the expression (1.1) to represent both a *vector* and the associated *state*. Note that if $\lambda \neq 1$, the *vectors* (1.1) and (1.2) are clearly different whereas the *states* (1.1) and (1.2) are the same.)

Given a penny in state (1.1), one can perform an observation to determine whether the penny is heads-up or tails-up. The act of observation knocks the penny into one of the pure states $\mathbf{H}$ or $\mathbf{T}$, with probabilities proportional to $|\alpha|^2$ and $|\beta|^2$. The observation yields a result of "heads" or "tails" depending on which of the two states the penny jumps into.[1]

1.3. Examples. Consider three pennies in the states

$$2\mathbf{H} + 3\mathbf{T} \tag{1.3a}$$

$$4\mathbf{H} + 6\mathbf{T} \tag{1.3b}$$

$$2\mathbf{H} - 3\mathbf{T} \tag{1.3c}$$

If you observe the orientation of any of these pennies, it will jump to state $\mathbf{H}$ (heads) with probability 4/13 and to state $\mathbf{T}$ (tails) with probability 9/13. However, pennies (1.3a) and (1.3b) are identical (i.e. in the same state) while penny (1.3c) is not. The physical significance of that difference will emerge in what follows.

2. Additional Properties. There is a one-to-one correspondence between states and the values of observable physical properties. For example, the state $\mathbf{H}$ corresponds to the value *heads* of the observable property "orientation". The state $\mathbf{T}$ correspond to the value *tails* of the same observable property.

In the same way, the states

$$\mathbf{R} = 2\mathbf{H} + \mathbf{T} \tag{2.1a}$$

$$\mathbf{G} = -\mathbf{H} + 2\mathbf{T} \tag{2.1b}$$

might correspond to the two values *red* and *green* of the observable property "color".

---

[1] Given a complex number $\alpha = a + bi$, we define the *conjugate* $\overline{\alpha} = a - bi$. The *modulus* $|\alpha|$ is the unique positive real number satisfying

$$|\alpha|^2 = \alpha\overline{\alpha} = a^2 + b^2$$

(We assume for simplicity that color, like orientation, has only two possible values, namely red and green.)

2.2. Computing Probabilities. Color works just like orientation: A penny in the state

$$\alpha\mathbf{R} + \beta\mathbf{G}$$

if examined for color, will either jump into state $\mathbf{R}$ and appear red or jump into state $\mathbf{G}$ and appear green, with probabilities proportional to $|\alpha|^2$ and $|\beta|^2$.

Remarks 2.3. The correspondence between states and observable values must satisfy certain properties which are best described in terms of the *inner product* on a complex vector space. Given two vectors

$$\mathbf{s} = \alpha\mathbf{H} + \beta\mathbf{T} \quad \text{and} \quad \mathbf{t} = \gamma\mathbf{H} + \delta\mathbf{T}$$

we define the inner product

$$< \mathbf{s}, \mathbf{t} >= \alpha\overline{\gamma} + \beta\overline{\delta}$$

We say that $\mathbf{s}$ and $\mathbf{t}$ are *orthogonal* if $< \mathbf{s}, \mathbf{t} >= 0$.

Now we have the following requirements:

2.3a. Two states are orthogonal (or, in more precise language, represented by orthogonal vectors) if and only if they correspond to two opposing values of the same observable property. Note, for example that $\mathbf{H}$ is orthogonal to $\mathbf{T}$ (heads and tails are opposing values of the observable property “orientation”) and that $\mathbf{R}$ is orthogonal to $\mathbf{G}$ (red and green are opposing values of the observable property “color”).

2.3b. Suppose $\mathbf{A}$ and $\mathbf{B}$ are orthogonal vectors associated, say, with the values “bright” and “dull” of the observable property *shininess*. A penny in state

$$\mathbf{s} = \alpha\mathbf{A} + \beta\mathbf{B}$$

if examined for shininess will jump into the shiny state $\mathbf{A}$ or the dull state $\mathbf{B}$ with probabilities proportional to

$$\frac{|\alpha|^2}{< \mathbf{A}, \mathbf{A} >} \quad \text{and} \quad \frac{|\beta|^2}{< \mathbf{B}, \mathbf{B} >}$$

In case $<\mathbf{A},\mathbf{A}>=<\mathbf{B},\mathbf{B}>$ we can ignore the denominators and say that the probabilities are proportional to

$$|\alpha|^2 \qquad \text{and} \qquad |\beta|^2 \tag{2.3.b1}$$

which is exactly what we did in (2.2), where $<\mathbf{R},\mathbf{R}>=<\mathbf{G},\mathbf{G}>=5$.

Throughout the sequel, whenever we choose two orthogonal vectors $\mathbf{A}$ and $\mathbf{B}$, we will always choose them so that $<\mathbf{A},\mathbf{A}>=<\mathbf{B},\mathbf{B}>$ so that we can use (2.3.b1) to compute probabilities.

2.3c. By (2.3a), "red" and "green" must be represented by orthogonal vectors. But why the *particular* orthogonal vectors of (2.1a) and (2.1b)? Physics provides guidelines for determining exactly which states are associated with exactly which observable properties. Here we are ignoring that issue completely and simply taking it as given that $\mathbf{R}$ and $\mathbf{G}$, as defined by the equations (2.1a) and (2.1b), represent red and green.

Example 2.4. Suppose you've just observed a penny's orientation and found it to be heads. Then you know that your observation has knocked the penny into state $\mathbf{H}$. From (2.1a) and (2.1b) we have

$$\mathbf{H} = 2\mathbf{R} - \mathbf{G}$$

(This is because $2\mathbf{R} - \mathbf{G} = 5\mathbf{H}$, which represents the same state as $\mathbf{H}$.) Therefore an observation of color will come out *red* with probability 4/5 and *green* with probability 1/5.

Note that it is *not possible* to observe color and orientation simultaneously. An observation of color must knock the penny into one of the two states $\mathbf{R}$ or $\mathbf{G}$, while an observation of orientation must knock the penny into one of the two states $\mathbf{H}$ or $\mathbf{T}$. Because a penny can only be in one state at a time, it follows that the two observations cannot be simultaneous.

2.5. More on States and Observable Properties. In the above and in everything that follows, we have assumed that all observable properties of interest have only two values. To study three-valued physical properties, we would introduce a three-dimensional complex vector space. To study infinite- valued physical properties (like location or momentum), we would introduce an infinite-dimensional complex vector space.

2.6. The Uncertainty Principle. To each observable property P (such as orientation or color), there is a *complementary* observable property Q defined as follows: Represent the values of P by orthogonal vectors $\mathbf{A}$and $\mathbf{B}$ with $< \mathbf{A}, \mathbf{A} >=< \mathbf{B}, \mathbf{B} >$. Then Q is the observable property whose values are represented by the orthogonal vectors $\mathbf{C} = \mathbf{A} + \mathbf{B}$ and $\mathbf{D} = \mathbf{A} - \mathbf{B}$.

Suppose we are given a penny in state

$$\mathbf{s} = \alpha\mathbf{A} + \beta\mathbf{B} = \gamma\mathbf{C} + \delta\mathbf{D}$$

where $\gamma = \alpha + \beta$ and $\delta = \alpha - \beta$. If the penny is examined for property P, it will be found in state $\mathbf{A}$ with probability

$$p = \frac{|\alpha|^2}{|\alpha|^2 + |\beta|^2}$$

We define the *uncertainty* associated with this observation to be $p(1-p)$, so that the uncertainty is zero when $p = 0$ or 1, and is at a maximum when $p = 1/2$. (Note that the uncertainty is a function *both* of the penny's state $\mathbf{s}$ *and* of the property P which is to be observed.)

If the penny is examined for property Q, it will be found in state $\mathbf{C}$ with probability

$$q = \frac{|\gamma|^2}{|\gamma|^2 + |\delta|^2}$$

and the associated uncertainty is $q(1-q)$. It is an easy exercise in algebra to verify the *uncertainty principle*:

$$p(1-p) + q(1-q) \geq 1/4$$

That is, the sum of the uncertainties of complementary properties is bounded below by 1/4.

2.7. Metaphysics. Consider a penny in the state $\mathbf{R} = 2\mathbf{H} + \mathbf{T}$. If you observe this penny's orientation, it has a 4/5 chance of being heads and a 1/5 chance of being tails.

What exactly do these probabilities mean? Do they describe the limits of our knowledge about the penny or do they describe intrinsic features of the penny itself? Consider two alternative interpretations:

1) Every penny at every moment is either heads or tails. Among pennies in state **R**, 4/5 are heads and 1/5 are tails. The only way to find out the orientation of a *particular* penny is to observe that orientation.

2) A penny in state **R** is neither heads *nor* tails until its orientation is observed. At that moment, the orientation becomes heads with probability 4/5 and tails with probability 1/5.

Theories of the form 1) are called *hidden variable* theories; they suggest that there is a variable (i.e. orientation) which exists, but whose value is sometimes hidden from us. We will see in Section (4.4) that no hidden variable theory can be consistent with the predictions of quantum mechanics; worse yet (worse for the proponents of hidden variable theories, that is), no hidden variable theory can be consistent with the observed outcomes of actual experiments.

Thus we are left with interpretation 2). The quantum state is a full description of the penny. There is no additional, unobservable "truth". A penny in state **R** simply *has* no orientation. It acquires an orientation (and loses its color) when the orientation is observed.

Similarly for any observable property including, for example, location. Quantum pennies have quantum states *instead* of locations. For a penny in a general quantum state, it makes exactly as much sense to ask "where is the penny?" as it does to ask "what is the penny's favorite movie?". Location, like cinematic taste, is simply a concept that does not apply to quantum entities.

3. Physical Operations. Pennies move from one state to another for any of three reasons. First, as we have seen, an observation can (and usually will) knock a penny into a new state. Second, states evolve naturally over time subject to constraints imposed by the *Schrodinger equation*, which is a second order differential equation that the time-path of the state must satisfy. Such evolution will play no role in what follows; we assume that every penny is always on a constant time-path so that its state does not evolve. Third, the penny might be acted on by a force: Someone might spin it, or heat it up, or dip it in paint. In quantum mechanics, each of these physical operations is represented by a *special unitary matrix*, that is, a matrix of the form

$$\begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \tag{3.1}$$

where A and B are complex numbers such that

$$|A|^2 + |B|^2 = 1$$

Start with a penny in state $\alpha\mathbf{H}+\beta\mathbf{T}$ and perform the physical operation represented by the special unitary matrix (3.1). To find the the result, perform the matrix multiplication

$$\begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$$

Then the penny is transformed into the state $\gamma\mathbf{H} + \delta\mathbf{T}$.

Examples 3.2. Consider the physical operation "flip the penny over". Physics provides guidelines for constructing the corresponding special unitary matrix; here, let's just suppose that the matrix is

$$\begin{pmatrix} 0 & \eta \\ -\overline{\eta} & 0 \end{pmatrix} \tag{3.2.1}$$

where $\eta = \exp(i\pi/4)$. Then we can compute what happens to various pennies when they are flipped over.

For a penny in state $\mathbf{H}$, we have

$$\begin{pmatrix} 0 & \eta \\ -\overline{\eta} & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\overline{\eta} \end{pmatrix} \tag{3.2.2}$$

so a penny in state $\mathbf{H}$ is transformed to a penny in state $-\overline{\eta}\mathbf{T}$, which is the same as state $\mathbf{T}$.

For a penny in state $\mathbf{T}$, we have

$$\begin{pmatrix} 0 & \eta \\ -\overline{\eta} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \eta \\ 0 \end{pmatrix} \tag{3.2.3}$$

so a penny in state $\mathbf{T}$ is transformed into state $\mathbf{H}$. The calculations (3.2.2) and (3.2.3) justify the decision to associate the matrix (3.2.1) with the "flipping over" operation.

For a penny in state $\mathbf{R} = 2\mathbf{H} + \mathbf{T}$, we have

$$\begin{pmatrix} 0 & \eta \\ -\overline{\eta} & 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \eta \\ -2\overline{\eta} \end{pmatrix}$$

so that a red penny, when flipped over, is transformed into the state $\mathbf{W} = \eta\mathbf{H} - 2\overline{\eta}\mathbf{T}$ which is the same thing (after multiplying by $\overline{\eta}$) as $\mathbf{H} - 2i\mathbf{T}$. This state, like any state, must be associated with some physical property; suppose that property is *warm* (as opposed to *cold*). Then a red penny, when flipped over, becomes a warm penny. If you measure the temperature of that warm penny, you'll find that it's warm. If you measure its orientation, you'll find that it's either heads or tails with probabilities proportional to $1 = |1|^2$ and $4 = |-2i|^2$. If you measure its color, then you can compute probabilities by first calculating that

$$\mathbf{W} = \mathbf{H} - 2i\mathbf{T} = (2 - 2i)\mathbf{R} + (-1 - 4i)\mathbf{G}$$

so that the measurement yields either red or green with probabilities proportional to $|2 - 2i|^2 = 8$ and $|-1 - 4i|^2 = 17$.

4. Entanglement. Here comes the cool part.

When two pennies interact with each other, they become permanently *entangled*, which means that they no longer have their own individual quantum states. Instead the *pair* of pennies has a single quantum state, represented by a non-zero complex vector of the form

$$\alpha(\mathbf{H} \otimes \mathbf{H}) + \beta(\mathbf{H} \otimes \mathbf{T}) + \gamma(\mathbf{T} \otimes \mathbf{H}) + \delta(\mathbf{T} \otimes \mathbf{T}) \tag{4.1}$$

Upon having their orientation observed, a pair of pennies will fall into one of the four pure states with probabilities proportional to $|\alpha|^2$, $|\beta|^2$, $|\gamma|^2$ and $|\delta|^2$.

Example 4.2 (Spooky Action at a Distance). Consider a pair of pennies in the state

$$\mathbf{H} \otimes \mathbf{H} + \mathbf{T} \otimes \mathbf{T} \tag{4.2.1}$$

These pennies have probability 1/2 of being observed (heads,heads), probability 1/2 of being observed (tails,tails), and probability zero of being observed either (heads,tails) or (tails,heads). Thus an observation on either penny individually is equally likely to yield a measurement of heads or tails. But the instant that observation is made, the outcome of a subsequent observation on the other penny is determined with certainty.

Prior to the first observation, neither penny is either "heads" or "tails". But an observation of *either* penny instantaneously knocks *both* pennies into either the "heads" state or the "tails" state. This effect does not diminish with time or distance; even if the pennies last interacted a millenium ago and one of the pennies has since been transported to another solar system, the outcome of the first observation has immediate consequences for the outcome of the second.

This is the "spooky action at a distance" that so famously distressed Albert Einstein and led Einstein, Podolsky and Rosen [EPR] to argue (fallaciously) that quantum mechanics must ultimately be supplemented by some version of a hidden variable theory. According to [EPR], the phenomenon of entanglement can be explained only by assuming that the pennies acquire identical orientations at the moment of interaction and retain those orientations until they are observed. That way, there's no mystery about how the orientations come to be identical. This is in contrast to the quantum mechanical view that the orientations do not exist until the pennies are observed. It seemed to Einstein and his co-authors that the quantum mechanical view renders the effects of entanglement completely inexplicable. Be that as it may, we shall see in Section (4.4) that no theory of the type that [EPR] envisioned can be consistent with the predictions of quantum mechanics (this is Bell's Theorem). Worse yet for the [EPR] program, the relevant predictions of quantum mechanics have been verified by experiment. Thus not only are [EPR] type theories inconsistent with quantum mechanics; they are inconsistent with reality.

4.3. Tensor Products More Generally. If $\mathbf{s} = \alpha\mathbf{H} + \beta\mathbf{T}$ and $\mathbf{t} = \gamma\mathbf{H} + \delta\mathbf{T}$ are states for single pennies, then we define the *tensor product state*

$$\mathbf{s} \otimes \mathbf{t} = \alpha\gamma(\mathbf{H} \otimes \mathbf{H}) + \alpha\delta(\mathbf{H} \otimes \mathbf{T}) + \beta\gamma(\mathbf{T} \otimes \mathbf{H}) + \beta\delta(\mathbf{T} \otimes \mathbf{T})$$

If two pennies are in a tensor product state then observations of the two pennies are are statistically independent, so that (for example)

$$\text{Prob}\Big(\text{First penny is heads} \;\Big|\; \text{Second penny is heads}\Big) = \text{Prob}\Big(\text{First penny is heads}\Big)$$

But most states (e.g. (4.2.1) are not tensor product states. In fact, the state space for a pair of entangled pennies is three-dimensional (four dimensions for the four basis vectors minus one for the fact that scalar multiples are considered identical) whereas the tensor product states form only a two-dimensional subspace.

Pennies in a tensor product state can be modeled as unentangled. When players in a game are issued unentangled pennies, a quantum strategy is equivalent to a classical mixed strategy. Otherwise (in particular when they are in the *maximally entangled state* (4.1.2)), a quantum strategy is something far more general, as we shall see.

4.4. Bell's Theorem. We've alluded in (2.7) and (4.2) to the expectation of Einstein and others that quantum states describe not the full truth about pennies, but only the full truth about experimenter's knowledge of the pennies. According to such *hidden variable theories*, the pennies themselves have properties that are not fully reflected in the quantum state.

Bell's Theorem [B] says that under extremely general and plausible hypotheses, there can be no hidden variable theory that is consistent with the predictions of quantum mechanics. Experiments performed after [B] was written confirm the relevant predictions of quantum mechanics far beyond any reasonable doubt. Thus under the broad hypotheses of Bell's Theorem, no hidden variable theory can be consistent with reality. To illustrate the theorem, consider a single special case involving the states "red", "green", "shiny" and "dull" represented by the vectors:

$$\mathbf{R} = 2\mathbf{H} + \mathbf{T}$$

$$\mathbf{G} = -\mathbf{H} + 2\mathbf{T}$$

$$\mathbf{S} = \mathbf{H} + 2\mathbf{T}$$

$$\mathbf{D} = 2\mathbf{H} - \mathbf{T}$$

Now use the distributive law to verify that

$$(\mathbf{H}\otimes\mathbf{H}) + (\mathbf{T}\otimes\mathbf{T}) = (\mathbf{H}\otimes\mathbf{S}) + 2(\mathbf{T}\otimes\mathbf{S}) + 2(\mathbf{H}\otimes\mathbf{D}) - (\mathbf{T}\otimes\mathbf{D}) \tag{4.5.1}$$

$$= 2(\mathbf{R}\otimes\mathbf{H}) - (\mathbf{G}\otimes\mathbf{H}) + (\mathbf{R}\otimes\mathbf{T}) + 2(\mathbf{G}\otimes\mathbf{T}) \tag{4.5.2}$$

$$= 4(\mathbf{R}\otimes\mathbf{S}) + 3(\mathbf{R}\otimes\mathbf{D}) + 3(\mathbf{G}\otimes\mathbf{S}) - 4(\mathbf{G}\otimes\mathbf{D}) \tag{4.5.3}$$

Given a pair of pennies in the entangled state $(\mathbf{H}\otimes\mathbf{H})+(\mathbf{T}\otimes\mathbf{T})$, we can use equations (4.5.1)-(4.5.3) to predict the outcomes of the following four experiments:

Experiment A. Observe the orientations of both pennies. From the left side of (4.5.1) one sees that the probabilities of the various outcomes are:

(heads,heads)	1/2
(heads,tails)	0
(tails,heads)	0
(tails,tails)	1/2

Experiment B. Observe the orientations of the first penny and the sheen of the second. From the right side of (4.5.1) one sees that the probabilities of the various outcomes are:

(heads,shiny)	1/ 10
(heads,dull)	2/5
(tails,shiny)	2/5
(tails,dull)	1/10

Experiment C. Observe the color of the first penny and the orientation of the second. From (4.5.2) one sees that the probabilities of the various outcomes are:

(red,heads)	2/5
(green,heads)	1/10
(red,tails)	1/10
(green,tails)	2/5

Experiment D. Observe the color of the first penny and the sheen of the second. From (4.5.3) one sees that the probabilities of the various outcomes are:

(red,shiny)	8/25
(red,dull)	9/50
(green,shiny)	9/50
(green,dull)	8/25

All of these predictions have been confirmed by experiment. Bell's Theorem says in essence that the predictions are not consistent with any model where pennies have properties before those properties are observed.

Here's the proof. If pennies *do* have properties such as orientation, color and sheen, then it makes sense, using the results of Experiments B, C and D, to compute the following conditional proabilities:

$$\text{Prob}\Big(\text{second penny dull} \quad \Big| \quad \text{first penny heads}\Big)) = 4/5 \tag{4.5.4}$$

$$\text{Prob}\Big(\text{first penny red} \quad \Big| \quad \text{second penny heads}\Big) = 4/5 \tag{4.5.5}$$

$$\text{Prob}\Big(\text{second penny dull and first penny red}\Big) = 9/50 \tag{4.5.6}$$

Moreover, from Experiment A, we know that "first penny heads" and "second penny heads" are both equivalent to "both pennies heads", an event we will denote **HH**. Thus we can rewrite the last three equations as

$$\text{Prob}\Big(\text{second penny dull} \quad \Big| \quad \mathbf{HH}\Big) = 4/\ 5 \tag{4.5.4$'$}$$

$$\text{Prob}\Big(\text{first penny red} \quad | \quad \mathbf{HH}\Big) = 4/5 \tag{4.5.5$'$}$$

$$\text{Prob}\Big(\text{second penny dull and first penny red}\Big) = 9/50 \tag{4.5.6$'$}$$

Now from elementary probability theory:

$$\begin{aligned} & \text{Prob}\Big(\text{second penny dull and first penny red} \quad \Big| \quad \mathbf{HH}\Big) \\ \geq \quad & \text{Prob}\Big(\text{second penny dull} \quad \Big| \quad \mathbf{HH}\Big) + \text{Prob}\Big(\text{first penny red} \quad \Big| \quad \mathbf{HH}\Big) - 1 \\ = \quad & \frac{4}{5} + \frac{4}{5} - 1 = \frac{3}{5} \end{aligned}$$

and finally

$$\begin{aligned} \frac{9}{50} &= \text{Prob}\Big(\text{second penny dull and first penny red}\Big) \\ &\geq \text{Prob}\Big(\text{second penny dull and first penny red} \quad \Big| \quad \mathbf{HH}\Big) \cdot \text{Prob}\Big(\mathbf{HH}\Big) \\ &\geq \left(\frac{3}{5}\right) \cdot \left(\frac{1}{2}\right) = \frac{3}{10} \end{aligned}$$

which is false.

The contradiction demonstrates that it never made sense to calculate the conditional probabilities (4.5.4)- (4.5.6) to begin with; that is, it never made sense to talk about the properties of orientation, color and sheen as if they existed independent of observations.

5. Physical Operations on Entangled Pennies. Consider a pair of entangled pennies in the state $\mathbf{s} \otimes \mathbf{t}$. Suppose someone performs a physical operation on the first penny, where said operation is represented by the special unitary matrix

$$U = \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}$$

Then the *pair* of pennies is transformed into the state

$$(U\mathbf{s}) \otimes \mathbf{t}$$

If instead the same operation is performed on the second penny, then the pair is transformed to the state

$$\mathbf{s} \otimes (U\mathbf{t})$$

More generally, if the pennies start in state

$$\sum_{i=1}^{k} \mathbf{s}_i \otimes \mathbf{t}_i$$

they are transformed to state

$$\sum_{i=1}^{k} (U\mathbf{s}_i) \otimes \mathbf{t}_i \tag{5.1a}$$

or

$$\sum_{i=1}^{k} \mathbf{s}_i \otimes (U\mathbf{t}_i) \tag{5.1b}$$

when the physical operation represented by U is performed on the first or second penny.

Example 5.2. Suppose that a pair of pennies starts in the entangled state

$$\mathbf{H} \otimes \mathbf{H} + \mathbf{T} \otimes \mathbf{T} \tag{5.2.1}$$

The pennies are handed to two individuals, named Player One and Player Two.

Each Player sends one of two messages, **C** or **D**. (In the context of game theory, **C** and **D** are strategies, such as "I cooperate" or "I defect".) To send the message **C**, the player returns the penny untouched; to send the message **D** he flips the penny over. These are physical operations and hence represented by special unitary matrices. Leaving the penny untouched is represented by the two-by-two identity matrix. Flipping it over is represented by the matrix

$$\begin{pmatrix} 0 & \eta \\ -\overline{\eta} & 0 \end{pmatrix}$$

as in (3.2).

Then applying rules (5.1a) and (5.1b), we see that the pennies end up in one of four states:

$$\mathbf{CC} = \mathbf{H} \otimes \mathbf{H} + \mathbf{T} \otimes \mathbf{T} \tag{5.2.2a}$$

$$\mathbf{CD} = -\overline{\eta}\mathbf{H} \otimes \mathbf{T} + \eta\mathbf{T} \otimes \mathbf{H} \tag{5.2.2b}$$

$$\mathbf{DC} = \eta\mathbf{H} \otimes \mathbf{T} - \overline{\eta}\mathbf{T} \otimes \mathbf{H} \tag{5.2.2c}$$

$$\mathbf{DD} = i\mathbf{H} \otimes \mathbf{H} - i\mathbf{T} \otimes \mathbf{T} \tag{5.2.2d}$$

6. Quantum Game Theory. Consider a two-by-two classical game in which each player must choose between two strategies, **C** or **D**. The game is implemented as in Section (5.2): A referee starts with two pennies in the maximally entangled state (5.2.1) and hands one to each player. The players return the pennies untouched to indicate a play of **C**, or return the pennies flipped over to indicate a play of **D**. Depending on the players' choices, the pennies end up in one of the four states (5.2.2a)-(5.2.2d).

If the referee examines the orientations of the pennies, it is in general impossible to make appropriate payoffs. For example, if both players choose strategy **C**, then by (5.2.2a) the pennies are certain to be found in identical orientations, but equally likely to be (heads, heads) or (tails, tails). If both players choose strategy **D**, then by (5.2.2d) the exact same statement is true. So it would be impossible to assign different payoffs to the strategy pairs (**C**,**C**) and (**D**, **D**).

Similarly, the plays (**C**, **D**) and (**D**, **C**) both lead to states in which the pennies must have opposite orientations, with (heads, tails) and (tails, heads) equally likely.

So the referee does *not* observe orientation. Instead he observes some *other* physical property whose four possible values correspond to the states **CC**, **CD**, **DC**, and **DD**. Such a property must exist because these four vectors are mutually orthogonal. If the property is, for example, "taste", then the referee tastes the pennies and is certain to register one of the four values (sweet, sweet), (sweet, salty), (salty, sweet) or (salty, salty) depending on which of the four states the pennies have ended up in. Now the referee knows the state, hence knows the player's strategies, and hence can make appropriate payoffs.

6.1. Quantum Strategies.

The above analysis assumes that players follow the rules and return their pennies either unchanged or flipped over, as opposed to, say, dipped in paint or coated with salt.

But suppose one or both of the players does something to his penny other than flip it over. This form of "cheating" is completely undetectable by the referee, whose observation of taste will knock the pair of pennies into one of the four states (5.2.2a)-(5.2.2d), thereby destroying all evidence that anybody ever placed them in any other state.

So we must allow the players to perform any physical operations whatsoever, which means they can apply any special unitary matrices whatsoever. Explicitly, assume the players perform physical operations represented by the matrices

$$\begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} P & Q \\ -\overline{Q} & \overline{P} \end{pmatrix} \tag{6.2}$$

where $A\overline{A} + B\overline{B} = P\overline{P} + Q\overline{Q} = 1$.

The initial state (5.2.1) is then transformed to

$$\begin{aligned} &(A\mathbf{H} - \overline{B}\mathbf{T}) \otimes (P\mathbf{H} - \overline{Q}\mathbf{T}) + (B\mathbf{H} + \overline{A}\mathbf{T}) \otimes (Q\mathbf{H} + \overline{P}\mathbf{T}) \\ &= S(\mathbf{H} \otimes \mathbf{H}) + T(\mathbf{H} \otimes \mathbf{T}) - \overline{T}(\mathbf{T} \otimes \mathbf{H}) + \overline{S}(\mathbf{T} \otimes \mathbf{T}) \\ &= (S + \overline{S})\mathbf{CC} - (T\eta + \overline{T\eta})\mathbf{CD} + i(T\eta - \overline{T\eta})\mathbf{DC} + i(S - \overline{S})\mathbf{DD} \end{aligned} \tag{6.3}$$

where S, T, and α are defined by the equation

$$\begin{pmatrix} S & T \\ -\overline{T} & \overline{S} \end{pmatrix} = \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \cdot \begin{pmatrix} P & Q \\ -\overline{Q} & \overline{P} \end{pmatrix}^T \tag{6.4}$$

The probabilities of outcomes **CC**, **CD**, **DC** and **DD** are then proportional to the squared moduli of the coefficients in expression (6.3).

6.5. Quaternions. Player One's strategy is described by his choice of a pair of complex numbers (A, B) where $A\overline{A} + B\overline{B} = 1$, or equivalently by the unit quaternion $\mathbf{p} = A + B\eta j$. Likewise, Player Two's strategy is defined by his choice of a unit quaternion $\mathbf{q} = P - \eta j Q$.

From (6.4) we have

$$S + T\eta j = \mathbf{pq}$$

and from (6.3) we see that the final state of the pair of pennies is

$$\pi_1 \mathbf{CC} + \pi_2 \mathbf{CD} + \pi_3 \mathbf{DC} + \pi_4 \mathbf{DD}$$

where the π_i are real numbers proportional to the coefficients of

$$S + T\eta j$$

If we switch the names of i and k, then the associated probability distribution is

$$\begin{array}{rclcrcl} \text{Prob}(\mathbf{CC}) & = & \pi_1(\mathbf{pq})^2 & \quad & \text{Prob}(\mathbf{CD}) & = & \pi_2(\mathbf{pq})^2 \\ \text{Prob}(\mathbf{DC}) & = & \pi_3(\mathbf{pq})^2 & \quad & \text{Prob}(\mathbf{DD}) & = & \pi_4(\mathbf{pq})^2 \end{array}$$

Note that this is exactly the probability distribution given in (2.2) of the main paper, which in turn motivates Definition (2.3) for the associated quantum game.

Appendix B: Quaternions

A *quaternion* is an expression of the form

$$p_1 + p_2 i + p_3 j + p_4 k$$

where the p_i are real numbers. The real number p_1 can be identified with the quaternion $p_1 + 0i + 0j + 0k$. The complex number $p_1 + p_2 i$ can be identified with the quaternion $p_1 + p_2 i + 0j + 0k$.

Quaternions are added according to the simple rule

$$(p_1+p_2i+p_3j+p_4k)+(q_1+q_2i+q_3j+q_4k) = (p_1+q_1)+(p_2+q_2)i+(p_3+q+3)j+(p_4+q_4)k$$

Multiplication is defined by the rules

$$i^2 = j^2 = k^2 = -1$$

$$ij = -ji = k \qquad jk = -kj = i \qquad ki = -ik = j$$

together with the distributive property.

For example

$$\begin{aligned}(1+2i)\cdot(3i+4j) &= 1\cdot 3i + 1\cdot 4j + 2i\cdot 3i + 2i\cdot 4j \\ &= 3i + 4j + 6i^2 + 8ij \\ &= 3i + 4j - 6 + 8k \\ &= -6 + 3i + 4j + 8k\end{aligned}$$

The quaternion $p_1 + p_2 i + p_3 j + p_4 k$ is a *unit quaternion* if $\sum_{i=1}^{4} p_i^2 = 1$. Thus unit quaternions can be identified with four-vectors of length 1; that is, the unit quaternions are the points of the unit (three dimensional) sphere in $\mathbf{R}^4$. If $\mathbf{p} = p_1 + p_2 i + p_3 j + p_4 k$ is a quaternion, we write $\pi_i(\mathbf{p}) = p_i$, for $i = 1, \ldots, 4$.

I will often have occasions to write expressions of the form $\pi_i(\mathbf{p})^2$. This means $(\pi_i(\mathbf{p}))^2$, not $\pi_i(\mathbf{p}^2)$. It is not difficult to prove the following facts:

1) The product of two unit quaternions is a unit quaternion.

2) Multiplication of quaternions is associative, so that if $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ are quaternions, then

$$(\mathbf{pq})\mathbf{r} = \mathbf{p}(\mathbf{qr})$$

3) For every quaternion $\mathbf{p} \neq 0$ there is a quaternion $\mathbf{p}^{-1}$ such that $\mathbf{pp}^{-1} = \mathbf{p}^{-1}\mathbf{p} = 1$.
4) If $\mathbf{p}$ is a unit quaternion, then so is $\mathbf{p}^{-1}$.
5) A unit quaternion is a square root of -1 if and only if it is a linear combination of i, j and k.
6) If $\mathbf{u}$ and $\mathbf{v}$ are square roots of -1, then $\mathbf{u}$ is perpendicular to $\mathbf{v}$ (as vectors in $\mathbf{R}^4$) if and only if $\mathbf{uv} + \mathbf{vu} = 0$.

Finally, we provide the isomorphisms alluded to in (4.12). For $\{\alpha, \beta, \gamma\} = \{i, j, k\}$ the map

$$\mathbf{p} \mapsto -\alpha + \beta\mathbf{p}\alpha + \beta/2$$

interchanges α and β while mapping γ to $-\gamma$. Composing these maps, we can construct an isomorphism of the quaternions (and hence of the unit quaternions) that effects any permutation of i, j, k up to the insertion of appropriate signs.

If $\mathbf{G}$ and $\mathbf{G}^\sigma$ are the games described in (4.12) then an isomorphism from $\mathbf{G}$ to $\mathbf{G}^\sigma$ is described (on the level of strategy spaces) as follows: By switching the names of one or both players' strategies, we can assume that $\sigma(1) = 1$. Then use the isomorphisms of the preceding paragraph.

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