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A Constructive Geometric Approach

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# On the existence of consistent rules to adjudicate conflicting claims: a constructive geometric approach

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**Abstract**

For the problem of adjudicating conflicting claims, a rule is consistent if the choice it makes for each problem is always in agreement with the choice it makes for each "reduced problem" obtained by imagining that some claimants leave with their awards and reassessing the situation from the viewpoint of the remaining claimants. We develop a general technique to determine whether a given two-claimant rule admits a consistent extension to general populations, and to identify this extension if it exists. We apply the technique to a succession of examples. One application is to a one-parameter family of rules that offer a compromise between the constrained equal awards and constrained equal losses rules. We show that a consistent extension of such a rule exists only if all the weight is placed on the former or all the weight is placed on the latter. Another application is to a family of rules that provide a compromise between the constrained equal awards and proportional rules, and a dual family that provide a compromise between the constrained equal losses and proportional rules. In each case, we identify the restrictions implied by consistency.

Keywords: claims problems; consistent extensions; proportional rule; constrained equal awards rule; constrained equal losses rule.

JEL classification number: C79; D63; D74

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# 1 Introduction

A group of agents have claims on a resource adding up to more than what is available. How should the resource be divided? For instance, a firm goes bankrupt and its liquidation value has to be allocated among its creditors. How much should each of them receive? The literature on the adjudication of conflicting claims is concerned with the identification of well-behaved general methods, or "rules", of performing such divisions. A rule is "consistent" if the awards vector it recommends for each "claims problem" is in agreement with the awards vector it recommends for each "reduced problem" obtained by imagining that an arbitrary subgroup of claimants leave with their awards: the requirement is that in the reduced problem, each remaining claimant should be awarded the same amount as he was awarded initially. Consistency is a widely applicable principle that has played a central role in a large number of recent axiomatic studies.[1] Our objective is to contribute to the understanding of its implications for the adjudication of conflicting claims, and specifically to develop a technique to determine when a two-claimant rule has a "consistent extension" to arbitrary populations.

Our intuition is stronger in the two-agent case. Indeed the often delicate conceptual issue of how to take coalitions into account does not arise. Also, less sophisticated mathematics are required (e.g. the intermediate value theorem sometimes suffices when fixed point theorems are otherwise needed). Thus, it is natural, and it has been a standard research strategy in various areas, to attempt to solve allocation problems involving an arbitrary number of agents by extending a rule chosen in the two-agent case. Consistency has been key in implementing this strategy.

A result is available that helps considerably in this regard. It additionally involves the property of "converse consistency" of rules: consider a problem and an alternative that is admissible for it. Suppose that this alternative is such that for each two-agent subgroup of the agents the problem involves, its restriction to that subgroup is the choice the rule makes for the associated

[1] The literature, surveyed in Thomson (2006b), now counts several hundred items. A few representative applications are to coalitional games (Davis and Maschler, 1965; Peleg (1985), bargaining theory (Lensberg, 1988), apportionment (Balinski and Young, 1982), various models of resource allocation including the adjudication of conflicting claims (Aumann and Maschler, 1985; Young (1987), classical economies (Thomson, 1988), economies with indivisible goods (Tadenuma and Thomson, 1991), and matching (Sasaki and Toda, 1992).

reduced problem involving these two agents. Then, the requirement is that it should be the choice for the entire group of agents. Now, the "Elevator Lemma" asserts that if a rule is consistent and coincides with a conversely consistent rule for two agents, then it coincides with it for any number of agents. In the context of the adjudication of conflicting claims, converse consistency is generally satisfied by consistent rules, so it is not much of an additional restriction, but to apply the lemma, one needs to have guessed the extension to more than two claimants of the two-claimant rule that is the point of departure.

In some cases, the definition of the two-claimant rule is suggestive of the extension. The proportional rule, which assigns awards proportional to claims, is an obvious example. In other cases, the two-claimant rule has several natural extensions to general populations, one of them being consistent, but the consistent one is easily singled out. An illustration here is "concede-and-divide" (Aumann and Maschler, 1985)[2]. This two-claimant rule first assigns to each claimant the difference between the amount to divide and the other agent's claim (or 0 if this difference is negative); then, it divides what remains, the part that is truly contested, equally. Several well-known rules happen to coincide with concede-and-divide in the two-claimant case. One was proposed to rationalize certain numerical examples in the Talmud, the "Talmud rule" (Aumann and Maschler, 1985);[3] another, also inspired by ancient literature, is the "minimal overlap rule" (O'Neill, 1982), and a third is the "random arrival rule" (O'Neill, 1982).[4] There are others yet, and among all of these rules, one is consistent, namely the Talmud rule, as a simple calculation reveals. In fact, it is the only consistent extension of concede-and-divide. The reason is that if a rule is "resource monotonic", that is, when the amount to divide increases, no claimant ever receives less—and concede-and-divide does satisfy this property—it has at most one consistent extension (Aumann and Maschler, 1985).[5]

Difficult cases are when a two-claimant rule has no obvious extension to general populations, or when a definition seems particularly natural but

[2] The name is proposed by Thomson, (2003).

[3] For that reason concede-and-divide is often referred to as the "contested garment rule". The name concede-and-divide is suggested by Thomson (2003). For references to the ancient literature, see O'Neill (1982), Aumann and Maschler (1985), and Dagan (1996).

[4] Since these rules do not play an essential role here, we omit their formal definitions.

[5] The phrase appears in Thomson (1994) with a different meaning.

fails consistency. How do we go about finding a consistent extension if one exists, or prove that none does if that is the case? Our goal is to develop a general technique to resolve such issues. We proceed by means of a series of examples of increasing complexity, some of which are already understood, and we provide new applications.

The case of "strictly resource monotonic" rules—each claimant whose claim is positive always receives more when the amount to divide increases—is particularly simple and we discuss it first (Example 1). Next, we consider rules that are resource monotonic but perhaps not strictly so. Some violations of strict monotonicity are immaterial (Example 2), but others create complications (Example 3).

To gain further familiarity with our construction, we then apply it to two important rules for which we already know the answer to the extension question. First are the "weighted constrained equal awards rules". The (equal weights) "constrained equal awards rule" has been discussed for centuries (it appears in Maimonides), and it is central to recent theoretical work. It assigns to all claimants equal amounts subject to no one receiving more than his claim. In the fixed-population model, a weighted version is defined by selecting a vector of positive weights for claimants, and for each problem, assigning to all claimants amounts that, when divided by their respective weights, are equal, subject to no one receiving more than his claim. In the variable-population model, we need to specify a weight vector for each group of claimants and apply the definition just given. These vectors could in principle be chosen independently for each population. However, and not surprisingly, for a consistent extension of a two-claimant weighted constrained equal awards rule to exist, the weight vectors chosen for the various two-claimant populations should be related, and these relations come out of our construction (Example 4); the weights for general populations are also determined.

Next, we explain how someone unaware of the Talmud rule would be able to rediscover it as the consistent extension of concede-and-divide (Example 5). The argument is more delicate but its logic is the same.

For our next application, we consider a family of rules that offer a compromise between the constrained equal awards rule and another rule also discussed in medieval literature, the "constrained equal losses rule". This rule divides the amount available in such a way that all claimants experience equal losses subject to no one receiving a negative amount. The constrained equal awards and constrained equal losses rules provide somewhat extreme and opposite ways of reconciling conflicting claims, the former obviously favoring the agents whose claims are the smallest, and the latter favoring the agents whose claims are the largest.[6] To compromise between the rules, and proceeding in a manner that is standard in game theory and the theory of resource allocation, we propose to average them, considering a flexible formulation in which the weights placed on each of them can be freely chosen.[7]

Unfortunately, this operation, which preserves most of the properties the two rules enjoy, does not preserve their consistency.[8] In fact, no weighted average of the rules is consistent, as we first show, unless all the weight is placed on one, or all the weight is placed on the other. This lack of consistency of averages of the constrained equal awards and constrained equal losses rules, however, could simply be a reflection of the fact that their two-claimant versions have not been appropriately extended. If one starts with a certain weighted average of two rules in the two-claimant case, it seems natural enough to consider as an extension to the $n$-claimant case a weighted average of these same rules and to use the same weights. But there are other options. To begin with, instead of placing the same weights on the two component rules independently of how many claimants are present, nothing precludes making these weights depend on the number of claimants. But then, what form should this dependence take? Moreover, averaging may not be the right operation for more than two claimants. But if not, what is? Hence the question: do the weighted averages of the two-claimant constrained equal awards and constrained equal losses rules have consistent extensions? Again, since each of these two rules is resource monotonic, so is any weighted average of them, and such an extension, if it exists, is unique.

What is different about this situation as compared to the previous ones is that we do not have access to a list of candidate rules, one of them being the sought-after extension. If we had this luxury, we would check the consistency of these candidates, a (usually) simple operation. When no obvious candidate consistent extension of a two-claimant rule is available, how does one prove that none exists if that is the case, or uncover one otherwise? The techniques

[6]This informal description of how the two rules differ can be made precise in several ways, as shown by Schummer and Thomson (1997).

[7]By contrast to some of our previous examples, weights are placed on rules here, not on claimants.

[8]For a general study of the "convexity operator" and of the properties it preserves or fails to preserve, see Thomson and Yeh (2001).

we develop here allow us to give constructive answers to such questions.

The answer to the one concerning the weighted averages of the constrained equal awards and constrained equal losses rules is essentially negative. We show that for such a rule to have a consistent extension, all the weight should be placed on the constrained equal awards rule, or all the weight should be placed on the constrained equal losses rule. This is true even if in the two-claimant case, the weights placed on the two rules are allowed to depend on the identity of the two claimants.

Our final application concerns a family of two-claimant rules that offer a compromise between the constrained equal awards and proportional rules. For each two-claimant population and each claims vector, as the amount to divide increases, we first choose equal division, and when some critical point is reached, we choose awards proportional to the difference between the claims vector and that point, allowing the switch of regime to depend on the claims vector. A rule in the family is indexed by a list of functions, one for each two-claimant population, specifying where the switch of regimes occurs. We show that if such a rule has a consistent extension, the critical point should essentially be independent of the claims vector and that the mixture of equality and proportionality assumed in the two-claimant case extends to general populations as follows: there is a non-negative real number such that for each population of claimants and each claims vector, the rule follows the constrained equal awards rule until all claimants whose claims are at least that number have received that number, and it continues in a linear way to the claims vector. We also obtain a characterization of the class of rules that offer a symmetric compromise between the constrained equal losses rule and the proportional rule, and are consistent.

This application is the most interesting because this time, the question of existence of a consistent extension has a non-trivial positive answer provided certain restrictions are met—these restrictions come out of the analysis—and the rules that emerge are certainly not ones known from previous work—they too come out of the construction.

# 2 Other applications

The techniques we develop here have proved very useful in other studies.

One asks whether there exists a consistent extension to the two-claimant rule obtained from the proportional rule by truncating claims at the amount to divide. Dagan and Volij (1997) provide a negative answer. The question can also be solved by our geometric method (Thomson, 2006a).

A similar question concerns the existence of a consistent extension to a two-claimant rule obtained by recursively imposing a certain lower bound on awards. Here too, the answer is no (Dominguez and Thomson, 2006).

Another study addresses the question of the existence of generalizations of the Talmud rule that do not necessarily satisfy "equal treatment of equals" but retain its consistency. In the two-claimant case, a definition is available, and a characterization can be obtained along the lines of Dagan (1996)'s characterization of the original symmetric version by dropping equal treatment of equals and adding "homogeneity", (which says that the awards vector chosen for a problem obtained by multiplying all the data of some initial problem by some number should be obtained from the awards vector chosen for that problem by the same multiplication). When do these weighted versions of the two-claimant Talmud rule have consistent extensions? The answer is that the set of potential claimants has to be partitioned into priority classes to be handled in succession—one can say that equal treatment is "maximally violated" between classes—and otherwise, unequal treatment of equals is an option only within two-claimant classes (Hokari and Thomson, 2003). Within classes with three or more claimants, equal treatment of equals has to prevail.

In the next application, the starting point is a rich family of two-claimant rules characterized on the basis of certain invariance requirements. For a consistent extension to exist, here too, the set of potential claimants should be partitioned into priority classes. For two-claimant classes, consistency imply no further restriction, but within each class with three or more claimants, it forces a considerable reduction in the family of acceptable rules (Moulin, 2000). Our techniques permit developing an alternative proof of this result (Thomson, 2001).

The final application is to the identification of the consistent members of a family of rules proposed by Thomson (2002) under the name of "ICI family". This abbreviation of "Increasing-Constant-Increasing" is a reference to the evolution of a claimant's award as the amount to divide varies from 0 to the sum of the claims. Each member of the family is defined by specifying for each claims vector, the values of the amount to divide at which claimants stop receiving additional compensation and at which they reenter the picture. This family can be understood as a generalization of the Talmud rule. Its richness comes from the freedom in selecting these breakpoints as a function of the claims vector, and in a variable population framework, as a function of the population. The existence of consistent selections from this family is addressed by Thomson (2002). The subfamily of the rules can be completely described.

# 3 The problem of adjudicating conflicting claims

Since our central requirement involves comparing the recommendations made for problems involving different populations of claimants, we need to cast our analysis in a sufficiently general framework for such comparisons to be possible.

There is an infinite set of "potential" claimants indexed by the natural numbers, $\mathbb{N}$. However, at any given time, only a finite number of them are present. Let $\mathcal{N}$ be the class of nonempty and finite subsets of $\mathbb{N}$. A claims problem, or simply a **problem**, is a pair $(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$, where $N \in \mathcal{N}$, such that $\sum_N c_i \geq E$:[9] each agent $i \in N$ has a **claim** $c_i$ over an **amount to divide** $E \in \mathbb{R}_+$, and this amount is insufficient to honor all of the claims. Let $\mathcal{C}^N$ be the class of all problems. A division rule, or simply a **rule**, is a function defined on $\bigcup_{N \in \mathcal{N}} \mathcal{C}^N$, which associates with each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$ a point $x$ of $\mathbb{R}^N_+$. This point should satisfy the inequalities $0 \leqq x \leqq c$ and its coordinates should add up to $E$, a condition to which we refer as "efficiency". Any such point is an **awards vector for $(c, E)$**. Let $S$ be our generic notation for rules. The **path of awards of a rule for a claims vector** is the locus of the awards vector it selects as the amount to divide varies from 0 to the sum of the claims.

The segment connecting two points $x$ and $y$ is denoted seg$[x, y]$, and the broken segment connecting $x$, $y$, and $z$ is denoted bro.seg$[x, y, z]$.

The most prominent rule in practice as well as in the theoretical literature makes awards proportional to claims. Another important rule assigns equal amounts to all claimants subject to no one receiving more than his claim (Figure 1a). A third rule selects the awards vector at which the losses experienced by all claimants are equal subject to no one receiving a negative amount (Figure 1b). The formal definitions are as follows:

**Proportional rule, $P$:** For each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, $P(c, E) \equiv \lambda c$, where $\lambda \in \mathbb{R}_+$ is chosen so as to achieve efficiency.

[9] By the notation $\mathbb{R}^N$ we mean the Cartesian product of $|N|$ copies of $\mathbb{R}$ indexed by the members of $N$. Vector inequalities: $x \geqq y$, $x \geq y$, and $x > y$.

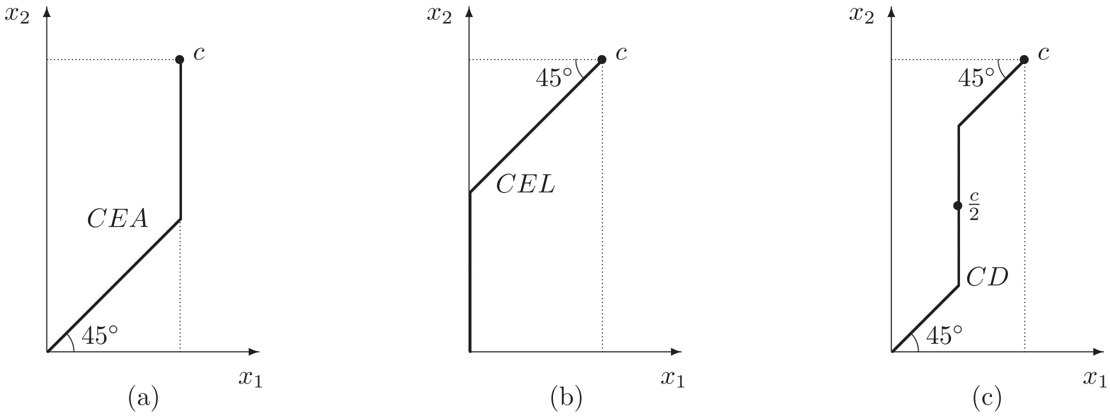


**Figure 1: The constrained equal awards and constrained equal losses rules, and concede-and-divide.** Their paths of awards are depicted here for a claims vector $c \in \mathbb{R}^N_+$, where $N \equiv \{1,2\}$, such that $c_1 < c_2$.

**Constrained equal awards rule, *CEA*:** For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$, $CEA_i(c, E) \equiv \min\{c_i, \lambda\}$, where $\lambda \in \mathbb{R}_+$ is chosen so as to achieve efficiency.

**Constrained equal losses rule, *CEL*:** For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$, $CEL_i(c, E) \equiv \max\{0, c_i - \lambda\}$, where $\lambda \in \mathbb{R}_+$ is chosen so as to achieve efficiency.

The following two rules will also play a role in our analysis. The first one is defined only for the two-claimant case (Figure 1c):

**Concede-and-divide, *CD*:** For $|N| = 2$. For each $(c, E) \in \mathcal{C}^N$ and each $i \in N$, $CD_i(c, E) \equiv \max\{E - c_j, 0\} + \frac{E - \sum_N \max\{E - c_k, 0\}}{2}$.

The next rule can be understood as a hybrid of the constrained equal awards and constrained equal losses rules. In the two-claimant case, it coincides with concede-and-divide:

**Talmud rule, *T*:** For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$,

$$T_i(c, E) \equiv \begin{cases} \min\{\frac{c_i}{2}, \lambda\} & \text{if } E \leq \sum \frac{c_i}{2}, \\ c_i - \min\{\frac{c_i}{2}, \lambda\} & \text{otherwise,} \end{cases}$$

where in each case, $\lambda \in \mathbb{R}_+$ is chosen so as to achieve efficiency.

Finally are weighted versions of the constrained equal awards rule designed to inflect the choice in favor of claimants who are perceived as more deserving. For each $i \in \mathbb{N}$, let $w_i \in \mathbb{R}_{++}$ be claimant $i$'s **weight**, and $w \equiv (w_i)_{i \in \mathbb{N}}$.

**Weighted constrained equal awards rule with weights $\boldsymbol{w} \in \mathbb{R}^{\mathbb{N}}_{++}$, $\boldsymbol{CEA^w}$:** For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$, $CEA^w_i(c, E) \equiv \min\{c_i, w_i\lambda\}$, where $\lambda$ is chosen so as to achieve efficiency.

Many tests have been devised for the evaluation of rules.[10] Our focus is on one that has been important not only in our current context, but in virtually all of the various models of game theory and the theory of resource allocation that have been the object of recent axiomatic analysis. Here, it involves checking whether, when some claimants leave with their awards and the situation is re-evaluated at that point, the rule recommends for each of the remaining claimants the same award as initially. The requirement is that equality of initial and final awards should always hold for them.[11]

**Consistency:** For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $N' \subset N$, if $x \equiv S(c, E)$, then $x_{N'} = S(c_{N'}, \sum_{N'} x_i)$.[12]

A variant of *consistency* that is often considered, **bilateral consistency**, is obtained by requiring that there be only two remaining claimants.

Many rules are *consistent*, the proportional, constrained equal awards, constrained equal losses, and Talmud rules being among them.

We will also refer to the requirements that a rule should assign equal amounts to two claimants with equal claims, and more generally, that it should be invariant under renamings of claimants; finally, that it should assigns amounts that are non-decreasing functions of the amount to divide:

**Equal treatment of equals:** For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each pair $\{i, j\} \subseteq N$, if $c_i = c_j$, then $S_i(c, E) = S_j(c, E)$.

**Anonymity:** For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, each $\pi \in \Pi^N$, and each $i \in N$, $S_{\pi(i)}((c_{\pi(j)})_{j \in N}, E) = S_i(c, E)$.

**Resource monotonicity:** For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $E' > E$, if $\sum c_i \geq E'$, then $S(c, E') \geqq S(c, E)$.

[10] See Thomson (2003) for a survey of this literature.

[11] Its first applications to claims resolution are due to Aumann and Maschler (1995) and Young (1987).

[12] Note that since we require rules to be such that for each $i \in N$, $0 \leq x_i \leq c_i$, then the sum of the claims of the remaining claimants is still at most as large as the amount that remains to divide, and therefore the problem $(c_{N'}, \sum_{N'} x_i)$ is well-defined.

# 4 General technique

As noted in the introduction, the logic underlying our argument bears some familiarity to reasoning encountered in the axiomatic theory of bargaining in connection with the formulation of generalizations of the so-called "egalitarian solution". A brief discussion will be useful. A **bargaining game** with player set $N$ is a compact, convex, and comprehensive[13] subset of utility space $\mathbb{R}^N_+$. A **bargaining solution** selects for each game a point of it. A **monotone path solution** is defined by means of a continuous and monotone path emanating from the origin and unbounded above. The path is fixed in advance and the solution selects for each game its maximal undominated point in the path.[14] In the variable population context, a monotone path should be selected for each population of players, and this list of paths should be such that for each population $N \in \mathcal{N}$ and each subpopulation $N' \subset N$, the projection of the path for $N$ onto the utility space pertaining to $N'$ is a subset of the path for $N'$. This requirement on the projections is what is needed to guarantee certain properties such as "population-monotonicity" and a form of "consistency".[15]

Now, returning to our problem of claims resolution, the *consistency* of a rule says that for each population of claimants $N \in \mathcal{N}$ and each claims vector $c \in \mathbb{R}^N_+$, the path of awards of the rule for $c$, when projected onto the coordinate subspace relative to any $N' \subset N$ should be a subset of its path of awards for $c_{N'}$. If a rule is *resource monotonic* in the two-claimant case, that is, if for each claims vector, it assigns to each claimant an amount that is a nowhere decreasing function of the amount to divide, and if it is *consistent*, then it is *resource monotonic* for any number of claimants (Dagan and Volij, 1997; Hokari and Thomson, 2000): if the paths of awards are monotone in the two-claimant case, they are monotone for any population of claimants.

[13]This means that for each pair $x, y$ of elements of $\mathbb{R}^N$, if $0 \leqq y \leqq x$, and $x$ is in the set, then so is $y$.

[14]See Thomson and Myerson (1980).

[15]A bargaining solution is *population monotonic* if for each population of players $N$, for each game this population may face, and each subpopulation $N'$ of $N$, it selects a payoff vector that is weakly dominated by the payoff vector it selects for the projection of this game onto the subspace relative to $N'$. It is *consistent* if for each population of players $N$, each game this population may face, and each subpopulation $N'$ of $N$, the payoff vector it selects for the reduced game associated with $N'$ and the payoff vector chosen for the game is the restriction to that subpopulation of that payoff vector. See Thomson and Lensberg (1989) for details.

Then, for each profile of claims indexed by the potential claimants, the list of the paths of awards indexed by all finite subpopulations of the set of potential claimants almost defines a monotone path solution as in bargaining theory. We write "almost" because there are two differences. First, in our present context, paths are not unbounded: by definition of a claims problem, the amount to divide is never greater than the sum of the claims, so the path for each claims vector is bounded above by this vector. Second, if the amount to divide is equal to the sum of the claims, every claimant should be fully compensated. Altogether then, and since the two-claimant rules that we consider are *resource monotonic*, the projection of the path relative to any population of claimants onto the subspace relative to any subpopulation coincides with the path for that subpopulation, as opposed to simply being a subset.

Conversely, and supposing that the paths of awards of a *consistent* rule are given for the two-claimant case, how does one construct its paths of awards for larger populations? Our main purpose is to develop a geometric construction to answer such questions. To explain its logic, it is enough to limit our attention to situations in which there are only three potential claimants, indexed by $i \in N \equiv \{1, 2, 3\}$. Let $S$ be a rule defined on $\bigcup_{N' \subseteq N} \mathcal{C}^{N'}$. Let $c \in \mathbb{R}^N_+$. Let $\Pi_3$ be the path of $S$ for $c_{\{1,2\}}$, $\Pi_2$ its path for $c_{\{1,3\}}$, and $\Pi_1$ its path for $c_{\{2,3\}}$. To construct its path for $c$, denoted $\Pi$, let for each $t \geq 0$, $H(t)$ be the plane of equation $x_1 = t$. Let us identify the point(s) of intersection of $H(t)$ with $\Pi_3$ and its point(s) of intersection with $\Pi_2$, and then the point(s) of $\mathbb{R}^N$ whose projections on $\mathbb{R}^{\{1,2\}}$ and $\mathbb{R}^{\{1,3\}}$ are these points of intersection. The path $\Pi$ has to include a subset of these points for its projections on $\mathbb{R}^{\{1,2\}}$ and $\mathbb{R}^{\{1,3\}}$ to be $\Pi_3$ and $\Pi_2$. We then let $t$ increase from 0 to $c_1$, causing the plane $H(t)$ to move parallel to, and away from, the $\{2, 3\}$-coordinate subspace. Depending on the multiplicities of the points of intersection of $H(t)$ with $\Pi_3$ and $\Pi_2$, which depend on the monotonicity properties of these paths, $\Pi$ may or may not be determined at this stage. If not, we turn to its projection onto $\mathbb{R}^{\{2,3\}}$. This projection should be $\Pi_1$. Note that in the process just described, the three coordinates do not play the same role, but we could have just as well started with a plane moving parallel to $\mathbb{R}^{\{1,3\}}$ or $\mathbb{R}^{\{1,2\}}$.

**Example 1** *The paths* $\Pi_3$ *and* $\Pi_2$ *are strictly monotone* (Figure 2).

Then, $\Pi$ can be entirely recovered from $\Pi_3$ and $\Pi_2$. Figure 2a shows the three

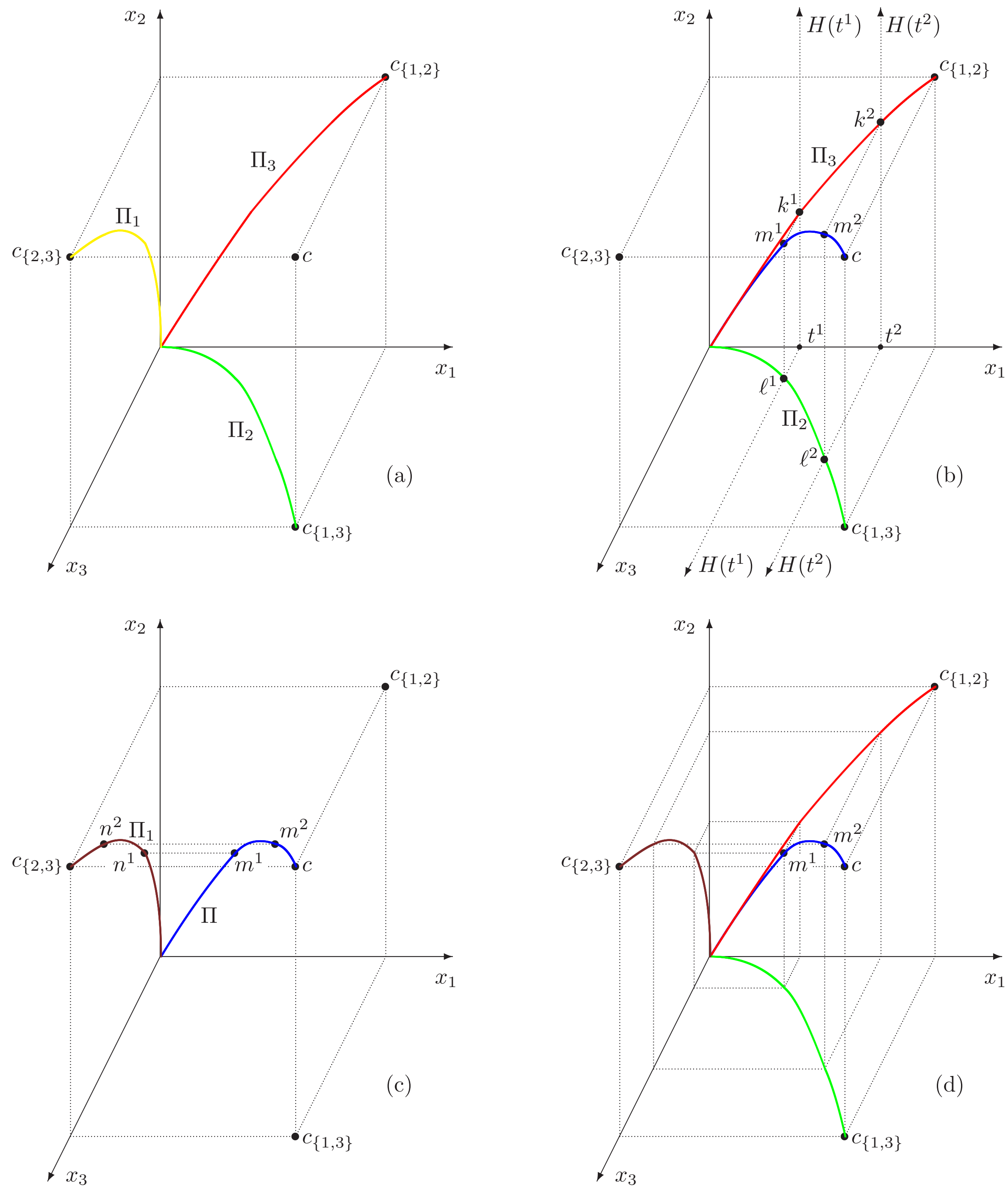


**Figure 2: Constructing the consistent extension of a strictly resource monotonic two-claimant rule** (Example 1). (a) The paths $\Pi_3$, $\Pi_2$, and $\Pi_1$, from which the three-dimensional path for $c$, $\Pi$, is to be constructed. (b) Constructing $\Pi$ from $\Pi_3$ and $\Pi_2$. (c) Projecting $\Pi$ onto $\mathbb{R}^{\{2,3\}}$. (d) The four paths defining the rule for $c$ and its projections.

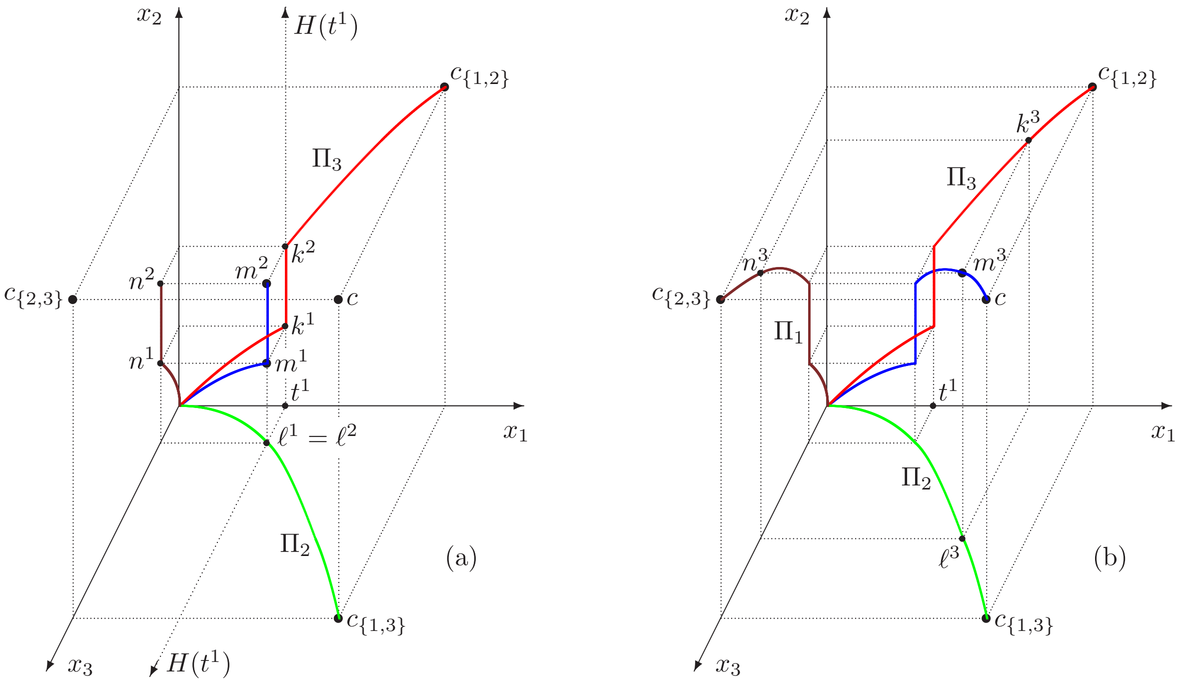


**Figure 3: Constructing the consistent extension of a resource monotonic (but not strictly resource monotonic) two-claimant rule** (Example 2). (a) The path $\Pi$ contains seg$[m^1, m^2]$. (b) Completion of the construction of $\Pi$. Its projection onto $\mathbb{R}^{\{2,3\}}$ is $\Pi_1$.

two-dimensional paths. Figure 2b shows the plane $H(t)$ and its intersections with $\Pi_3$ and $\Pi_2$ for $t = t^1$ and $t = t^2$. These intersections are singletons. For $t = t^1$, they are the points labelled $k^1$ and $\ell^1$. The point in $\mathbb{R}^N$ whose projections onto $\mathbb{R}^{\{1,2\}}$ and $\mathbb{R}^{\{1,3\}}$ are these two points is called $m^1$. It has to be in $\Pi$ and $H(t^1)$ should contain no other point of $\Pi$. For $t = t^2$, we obtain the points $k^2$ and $\ell^2$, and then $m^2$. This point too has to be in $\Pi$, and again, $H(t^2)$ should contain no other point of $\Pi$. As $t$ ranges from 0 to $c_1$, we trace out $\Pi$ in its entirety. Figure 2c shows that the projection of $\Pi$ onto $\mathbb{R}^{\{2,3\}}$ is $\Pi_1$ and Figure 2d shows the four paths for $c$ and its three projections on the three two-dimensional subspaces.

**Example 2** *The path* $\Pi_3$ *is not strictly monotone but* $\Pi_2$ *is* (Figure 3).

The path $\Pi_3$ contains a segment parallel to $\mathbb{R}^{\{2\}}$, seg$[k^1, k^2] \subset H(t^1)$ for some $t^1$. Since $\Pi_2$ is strictly monotone, the intersection of $H(t^1)$ with $\Pi_2$ is a singleton. Thus, we deduce the existence of a segment in $\Pi$ parallel to $\mathbb{R}^{\{2\}}$—seg$[m^1, m^2]$—and then, the existence of a segment in the projection of $\Pi$ onto $\mathbb{R}^{\{2,3\}}$—seg$[n^1, n^2]$ (Figure 3a). (This implies that both $\Pi$ and $\Pi_1$ also fail strict monotonicity.) What is interesting about this example is that the lack of strict monotonicity of $\Pi_3$ in $H(t^1)$ does not prevent recovering $\Pi$ from the two paths (Figure 3b).

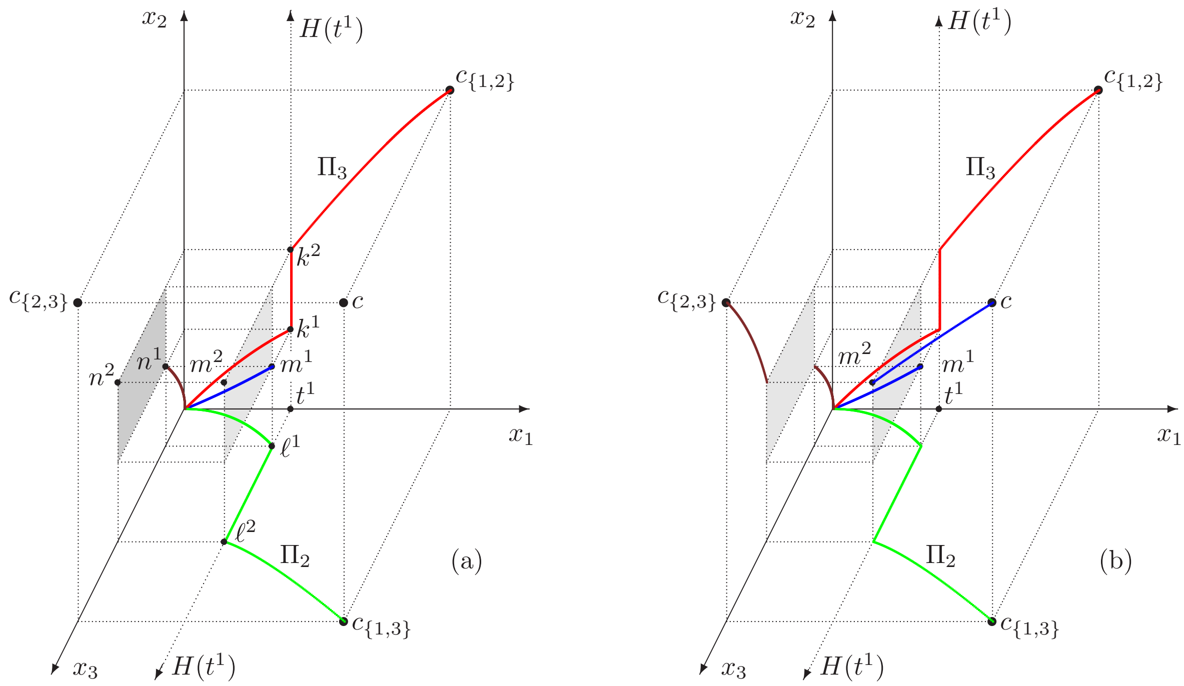


**Figure 4: Constructing the consistent extension of a resource monotonic (but not strictly resource monotonic) two-claimant rule** (Example 3). (a) Here, $\Pi_3$ contains a segment parallel to $\mathbb{R}^{\{2\}}$, seg$[k^1, k^2]$, and $\Pi_2$ contains a segment parallel to $\mathbb{R}^{\{3\}}$, seg$[\ell^1, \ell^2]$. The projection of $\Pi$ onto $\mathbb{R}^{\{2,3\}}$ is contained in the shaded rectangle in $\mathbb{R}^{\{2,3\}}$. This projection connects $n^1$ to $n^2$, the projections of $m^1$ and $m^2$. (b) To complete $\Pi$ in the shaded rectangle in $H(t^1)$, we use the fact that its projection on $\mathbb{R}^{\{2,3\}}$ is $\Pi_1$. The part of $\Pi$ that lies in the shaded rectangle in $H(t^1)$ will simply be a translate of the part of $\Pi_1$ that lies in the shaded rectangle lying in $\mathbb{R}^{\{2,3\}}$ (this section of $\Pi_1$ is not represented).

**Example 3** *The paths* $\Pi_3$ *and* $\Pi_2$ *contain non-degenerate segments that belong to a plane parallel to* $\mathbb{R}^{\{2,3\}}$ (Figure 4).

The paths $\Pi_3$ and $\Pi_2$ contain non-degenerate segments, seg$[k^1, k^2]$ and seg$[\ell^1, \ell^2]$, both of which belong to $H(t^1)$ for some $t^1$. Then $\Pi$ cannot be constructed from $\Pi_3$ and $\Pi_2$ alone. Indeed, for $\Pi$ to satisfy the projection requirements on $\mathbb{R}^{\{1,2\}}$ and $\mathbb{R}^{\{1,3\}}$, it suffices that it contains any continuous curve in $H(t^1)$ connecting $m^1$ and $m^2$. By *resource monotonicity*, the curve should be weakly monotone, so it should lie in the rectangle in that plane that we have shaded. However, there is one more requirement: the projection of $\Pi$ onto $\mathbb{R}^{\{2,3\}}$ should be $\Pi_1$. This requirement determines the shape of $\Pi$ in the shaded rectangle of $H(t^1)$. Since $\Pi_3$ and $\Pi_2$ fail strict monotonicity only in the plane of equation $x_1 = t^1$, the construction of $\Pi$ is otherwise uniquely defined (as in Example 1).

The case of rules whose paths of awards are piece-wise linear is particularly interesting because in applications it is very frequent, as shown by our remaining examples as well as the other applications mentioned earlier. To illustrate, and still supposing that $N \equiv \{1, 2, 3\}$, let $c \in \mathbb{R}^N_+$. If the path for $c_{\{1,2\}}$ has $n_3$ kinks and the path for $c_{\{1,3\}}$ has $n_2$ kinks, the path for $c_N$ has at most $n_3 + n_2$ kinks, and by projection on $\mathbb{R}^{\{2,3\}}$, so does the path for $c_{\{2,3\}}$. If the paths were in general position, the number of kinks in $\Pi$ would actually be $n_3 + n_2$, and so would the number of kinks in its projection on $\mathbb{R}^{\{2,3\}}$. However, there are two reasons why this sum may only be an upper bound. First, a kink in $\Pi_3$ and a kink in $\Pi_2$ may have equal first coordinates. Second, the projections of the first two kinks of $\Pi$ onto $\mathbb{R}^{\{2,3\}}$ may be lined up with the origin, or the projections of three of its successive kinks may be lined up. These possibilities are illustrated by the weighted constrained equal awards and Talmud rules, discussed next.

**Example 4** *Weighted constrained equal awards rules* (Figure 5).

To specify such a rule, we need a positive weight vector for each two-claimant population. For a *consistent* extension to exist, the weights have to be "consistent", and our construction brings out these requirements. Again, let $N \equiv \{1, 2, 3\}$. Then the ratio of (i) the ratio of the weights assigned to claimants 1 and 2 by the component of the rule pertaining to $\mathcal{C}^{\{1,2\}}$, and (ii) the ratio of the weights assigned to claimants 2 and 3 by the component of the rule pertaining to $\mathcal{C}^{\{2,3\}}$ should be equal to the ratio of the weights assigned to claimants 1 and 3 by the component of the rule pertaining to $\mathcal{C}^{\{1,3\}}$.

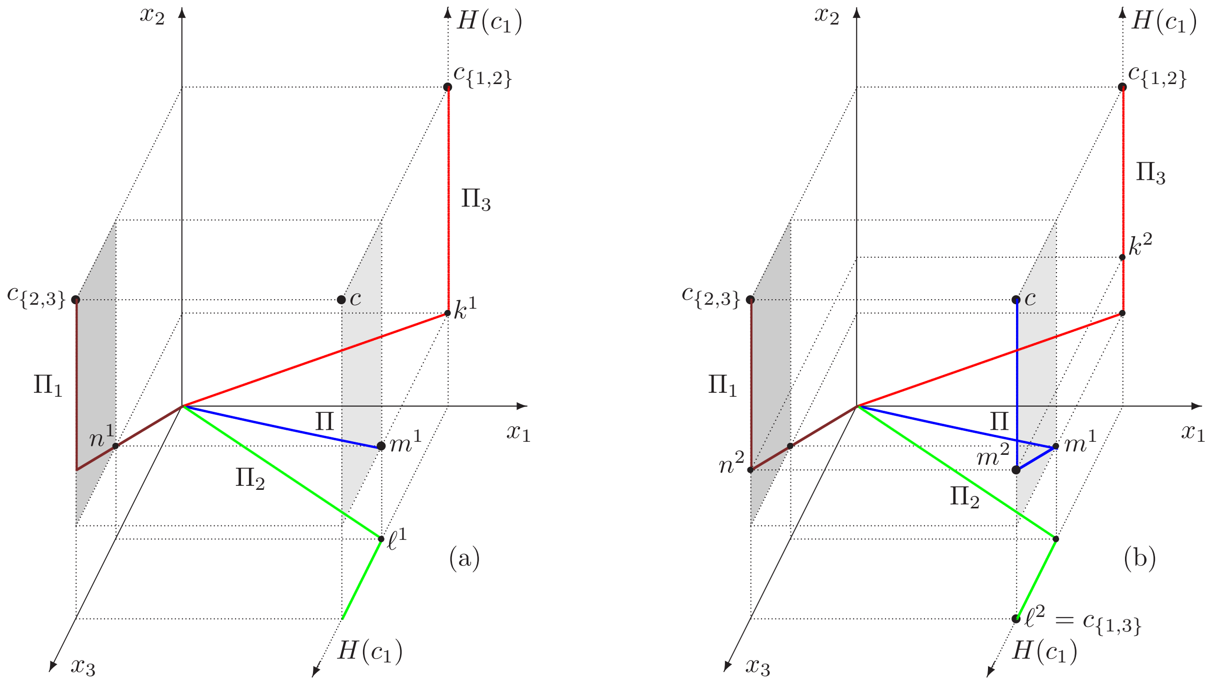


**Figure 5: Constructing the consistent extension of two-claimant weighted constrained equal awards rules** (Example 4). (a) The path $\Pi$ is determined from the origin to the point $m^1$ at which it reaches $H(c_1)$ by its projections onto $\mathbb{R}^{\{1,2\}}$ and $\mathbb{R}^{\{1,3\}}$. (b) From $m^1$ on, it could continue to $c$ in some arbitrary fashion in the shaded rectangle in $H(c_1)$ to meet the projection requirements on these spaces. However, its projection onto $\mathbb{R}^{\{2,3\}}$ should be $\Pi_1$, and this determines the continuation uniquely.

The slope of the initial segment of $\Pi_3$, $\text{seg}[0, k^1]$, is given by the weights assigned to claimants 1 and 2 in solving problems in $\mathcal{C}^{\{1,2\}}$. The slope of the initial segment of $\Pi_2$, $\text{seg}[0, \ell^1]$, is given by the weights assigned to claimants 1 and 3 by the component of the rule pertaining to $\mathcal{C}^{\{1,3\}}$. The point $m^1$ is the point of $\Pi$ whose projections onto $\mathbb{R}^{\{1,2\}}$ and $\mathbb{R}^{\{1,3\}}$ are $k^1$ and $\ell^1$. Since its projection onto $\mathbb{R}^{\{2,3\}}$, $n^1$, is a point of $\Pi_1$, we deduce the weights assigned to claimants 2 and 3 by the component of the rule pertaining to $\mathcal{C}^{\{2,3\}}$. The continuation of $\Pi$ is determined from the knowledge that its projection on $\mathbb{R}^{\{2,3\}}$ is $\Pi_1$.

Weighted constrained equal losses rules can be handled in a similar manner.

**Example 5** *Talmud rule* (Figure 6).

By examining its projections on $\mathbb{R}^{\{1,2\}}$ and $\mathbb{R}^{\{1,3\}}$, we can determine $\Pi$ until the point $m^1$ where it reaches $H(\frac{c_1}{2})$. The continuation of $\Pi$ is determined from the knowledge that its projection on $\mathbb{R}^{\{2,3\}}$ is $\Pi_1$. We only indicate its continuation up to $\frac{c}{2}$. The construction yields the path of awards of the

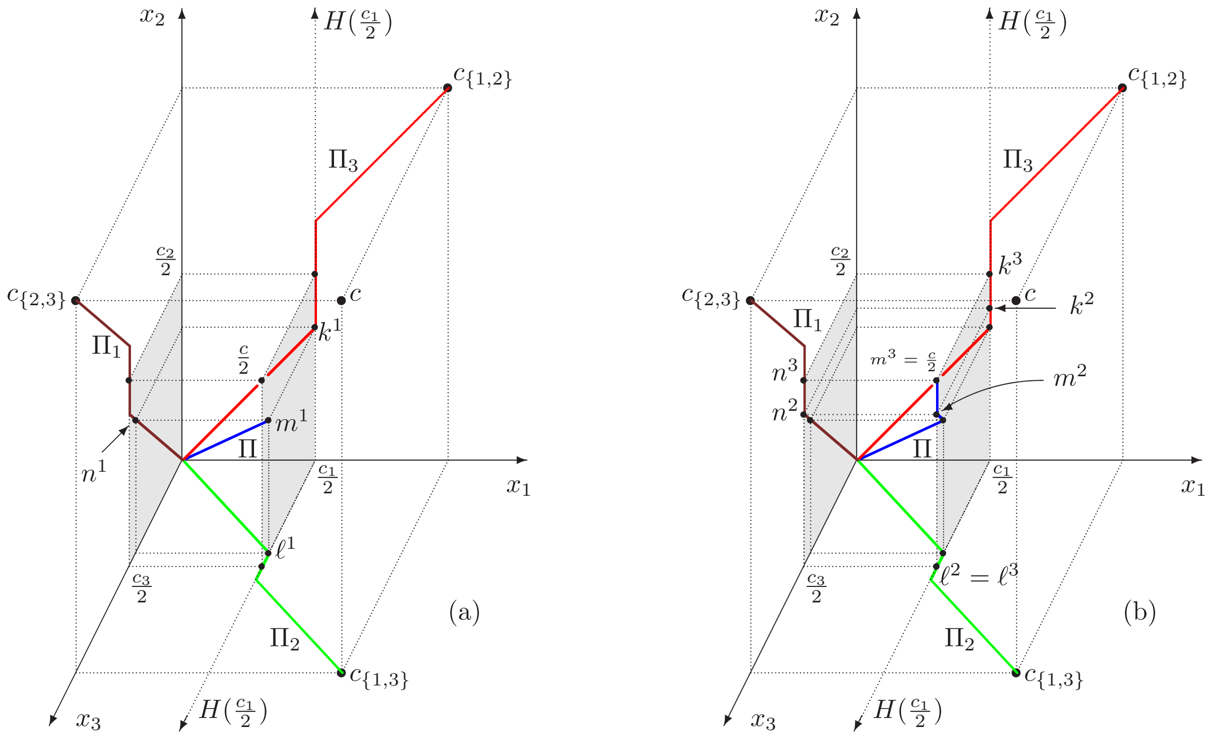


**Figure 6: Constructing the consistent extension of concede-and-divide.** Example 5: Talmud rule. As this simplifies the figure somewhat, we only indicate the construction of $\Pi$ up to $\frac{c}{2}$. (a) Its initial segment, seg$[0, m^1]$, is determined by looking at its projections $\Pi_3$ and $\Pi_2$ onto $\mathbb{R}^{\{1,2\}}$ and $\mathbb{R}^{\{1,3\}}$. (b) It is completed by looking at its projection $\Pi_1$ onto $\mathbb{R}^{\{2,3\}}$.

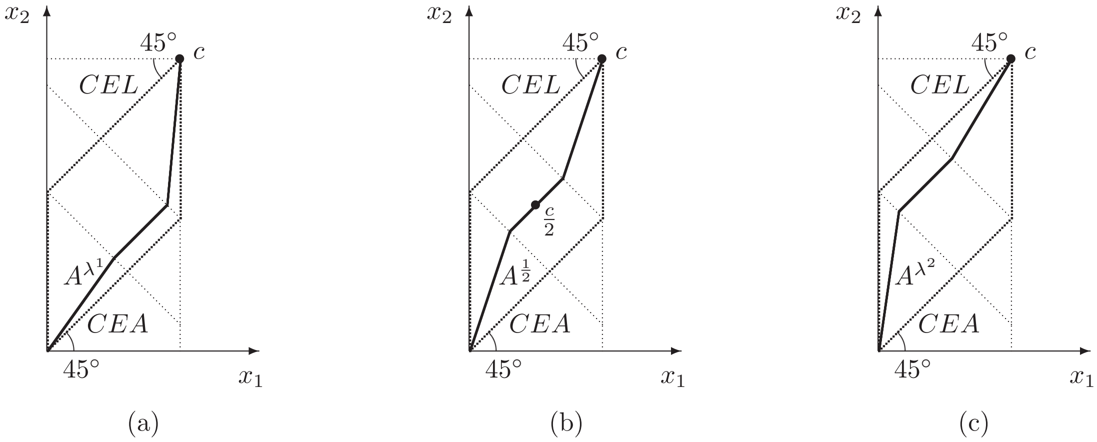


**Figure 7: Three weighted averages of the constrained equal awards and constrained equal losses rules.** (a) The weight placed on the constrained equal awards rule is greater than that placed on the constrained equal losses rule: $\lambda^1 = \frac{4}{5}$. (b) Equal weights are placed on the two rules. (c) A greater weight is placed on the constrained equal losses rule than on the constrained equal awards rule: $\lambda^2 = \frac{1}{4}$.

constrained equal awards rule when the claims vector is $\frac{c}{2}$. The rest of $\Pi$ is obtained by concatenating to it the path of awards of the constrained equal losses rule, also for $\frac{c}{2}$.

# 5 A compromise between the constrained equal awards and constrained equal losses rules

In our next application, we turn to a compromise between the constrained equal awards and constrained equal losses rules. The compromise is defined by averaging them. Given $\lambda \in [0,1]$, let $A^\lambda \equiv \lambda CEA + (1-\lambda)CEL$ denote their weighted average with weights $\lambda$ and $1-\lambda$.[16] The typical path of awards of $A^\lambda$ is shown in Figure 7 for three choices of $\lambda$, including the symmetric case, when $\lambda = \frac{1}{2}$.

First, we assert that no weighted average of these two rules is *consistent* unless all the weight is placed on one or all the weight is placed on the other. This is shown by the following example: Let $N \equiv \{1,2,3\}$ and $(c,E) \in \mathcal{C}^N$

[16] By this notation, we mean the rule that selects for each $N \in \mathcal{N}$ and each $(c,E) \in \mathcal{C}^N$ the award vector $\lambda CEA(c,E) + (1-\lambda)CEL(c,E)$.

be defined by $(c_1, c_2, c_3, E) \equiv (10, 20, 30; 24)$. Then, $A^\lambda(c, E) = \lambda(8, 8, 8) + (1-\lambda)(0, 7, 17) = (8\lambda, 7+\lambda, 17-9\lambda)$. Now, let $N' \equiv \{1, 2\}$. To calculate $y \equiv A^\lambda(10, 20; 8\lambda + 7 + \lambda)$, we distinguish two cases. If $\lambda \leq \frac{1}{3}$, then $y = \lambda(\frac{7+9\lambda}{2}, \frac{7+9\lambda}{2}) + (1-\lambda)(0, 7+9\lambda)$. Claimant 1's award, $\lambda(\frac{7+9\lambda}{2})$, is equal to $8\lambda$, as required by *consistency*, only if $\lambda = 0$, but then $A^\lambda = CEL$. If $\lambda > \frac{1}{3}$, then $y = \lambda(\frac{7+9\lambda}{2}, \frac{7+9\lambda}{2}) + (1-\lambda)(\frac{9\lambda-3}{2}, 10 + \frac{9\lambda-3}{2})$. Claimant 1's award, $\frac{19\lambda-3}{2}$, is equal to $8\lambda$ only if $\lambda = 1$, but then $A^\lambda = CEA$.

Although no weighted average of the constrained equal awards and constrained equal losses rules is *consistent* unless the weights are always 0 or always 1, the two-claimant version of such a rule may have a *consistent* extension for weights other than 0 or 1. However, we show below that this is not the case, and that in fact allowing the weight to depend on the identity of the two claimants does not help. For each $N \in \mathcal{N}$ with $|N| = 2$, let $\lambda^N$ be the weight placed on the constrained equal awards rule for that population (and therefore $1 - \lambda^N$ is the weight placed on the constrained equal losses rule). Note that *anonymity* is violated across two-claimant populations, but is satisfied within each such population.

**Theorem 1** *There is no consistent extension of any weighted average of the two-claimant constrained equal awards and constrained equal losses rules, unless for each two-claimant population, all the weight is placed on the former or all the weight is placed on the latter. In the first case, the constrained equal awards rule is applied for each two-claimant population and the extension is the constrained equal awards rule. In the second case, it is the constrained equal losses that comes out.*

**Proof:** For each $N \in \mathcal{N}$ with $|N| = 2$, let $\lambda^N \in [0, 1]$. Let us suppose that there is a *consistent* rule $S$ such that for each $N \in \mathcal{N}$ with $|N| = 2$, $S = A^{\lambda^N}$. We will show that this is possible only if for each $N \in \mathcal{N}$ with $|N| = 2$, $\lambda^N = 1$, or for each $N \in \mathcal{N}$ with $|N| = 2$, $\lambda^N = 0$. We will do so by considering the population of claimants $N \equiv \{1, 2, 3\}$ when their claims are $c \equiv (2, 4, 6)$.

Since, for each $N \in \mathcal{N}$ with $|N| = 2$, $S = A^{\lambda^N}$, we know the paths of awards of $S$ for $(c_1, c_2)$, $(c_1, c_3)$, and $(c_2, c_3)$. They are illustrated in Figure 8 for $\lambda^{\{1,2\}} = \lambda^{\{1,3\}} = \lambda^{\{2,3\}} = .8$. Using notation that reflects their dependence on the weights, they are:

- $\Pi_3(\lambda^{\{1,2\}}) = \text{bro.seg}[(0, 0), a(\lambda^{\{1,2\}}), b(\lambda^{\{1,2\}}), (c_1, c_2)]$

- $\Pi_2(\lambda^{\{1,3\}}) = \text{bro.seg}[(0,0), d(\lambda^{\{1,3\}}), (c_1, c_3)]$
- $\Pi_1(\lambda^{\{2,3\}}) = \text{bro.seg}[(0,0), e(\lambda^{\{2,3\}}), f(\lambda^{\{2,3\}}), (c_2, c_3)]$

Let us call $\Pi$ the path of awards of $S$ for $c$ and $\Pi'_1$ its projection onto $\mathbb{R}^{\{2,3\}}$.

(i) $\lambda^{\{1,2\}} = 0$ and $\lambda^{\{1,3\}} > 0$. The plane $H(0) = \mathbb{R}^{\{1,2\}}$ contains seg$[(0,0),(0,2)]$, the first segment of the path for $(c_1, c_2)$, and it intersects the path for $(c_1, c_3)$ only at the origin. Thus, the path for $c$ starts with seg$[(0,0,0),(0,2,0)]$ and its projection onto $\mathbb{R}^{\{2,3\}}$ starts with seg$[(0,0),(2,0)]$ (in Figure 9a, these three segments coincide, and in our previous notation, $k^1 = m^1 = n^1$). This is in violation of what we know of the path for $(c_2, c_3)$ (the situation is similar to that encountered in Example 2).

(ii) $\lambda^{\{1,2\}} > 0$ and $\lambda^{\{1,3\}} = 0$. The plane $H(0) = \mathbb{R}^{\{1,2\}}$ intersects the path for $(c_1, c_2)$ only at the origin, and it contains seg$[(0,0),(0,4)]$, the first segment of the path for $(c_1, c_3)$. Thus, the path for $c$ starts with seg$[(0,0,0),(0,0,4)]$ and its projection onto $\mathbb{R}^{\{2,3\}}$ starts with seg$[(0,0),(0,4)]$ (in Figure 9b, these three segments coincide, and in our previous notation, $\ell^1 = m^1 = n^1$). This is in violation of what we know of the path for $(c_2, c_3)$.

(iii) $\lambda^{\{1,2\}} = \lambda^{\{1,3\}} = 0$. This assumption means that $S = CEL$ for problems involving either $\{1,2\}$ or $\{1,3\}$. Analysis similar to that of Example 4 leads us to $S = CEL$.

(iv) $\lambda^{\{1,2\}} > 0$ and $\lambda^{\{1,3\}} > 0$. Figure 10a shows the case when $\lambda^{\{1,3\}}$ is "not too small" in relation to $\lambda^{\{1,2\}}$, so that the kink reached first by $H(t)$, as $t$ increases from 0, belongs to $\Pi(\lambda^{\{1,2\}})$. For $\lambda^{\{1,3\}}$ sufficiently small in relation to $\lambda^{\{1,2\}}$, the kink reached first would be in $\Pi(\lambda^{\{1,3\}})$. As $t$ increases from 0, the plane $H(t)$ intersects $\Pi_3(\lambda^{\{1,2\}})$ and $\Pi_2(\lambda^{\{1,3\}})$ at points that first belong to seg$[0, k^1]$ and seg$[0, \ell^1]$ in Figure 10a. If in fact, $\lambda^{\{1,2\}} = \lambda^{\{1,3\}}$, $\Pi$ begins with a segment that lies in the plane of equation $x_2 = x_3$ (seg$[(0,0,0), m^1]$ in the figure), and its projection onto $\mathbb{R}^{\{2,3\}}$ also begins with a segment of slope 1. If $\lambda^{\{1,2\}} < \lambda^{\{1,3\}}$, this projection lies between the 45° line in $\mathbb{R}^{\{2,3\}}$ and the second axis, but is not contained in the 45° line. However, since $c_3 > c_2$, for each $\lambda^{\{2,3\}}$, the first segment of the path of $S$ for $(c_2, c_3)$ lies between the 45° line in $\mathbb{R}^{\{2,3\}}$ and the third axis. Thus, $\lambda^{\{1,2\}} \geq \lambda^{\{1,3\}}$. Figure 10b shows that the slope of the first segment of the projection $\Pi'_1$ of $\Pi$ onto $\mathbb{R}^{\{2,3\}}$ has moved in the right direction (towards $\mathbb{R}^{\{3\}}$)by choosing $\lambda^{\{1,2\}} > \lambda^{\{1,3\}}$. By considering the claims vector $(2,6,4)$, we conclude in a similar way that

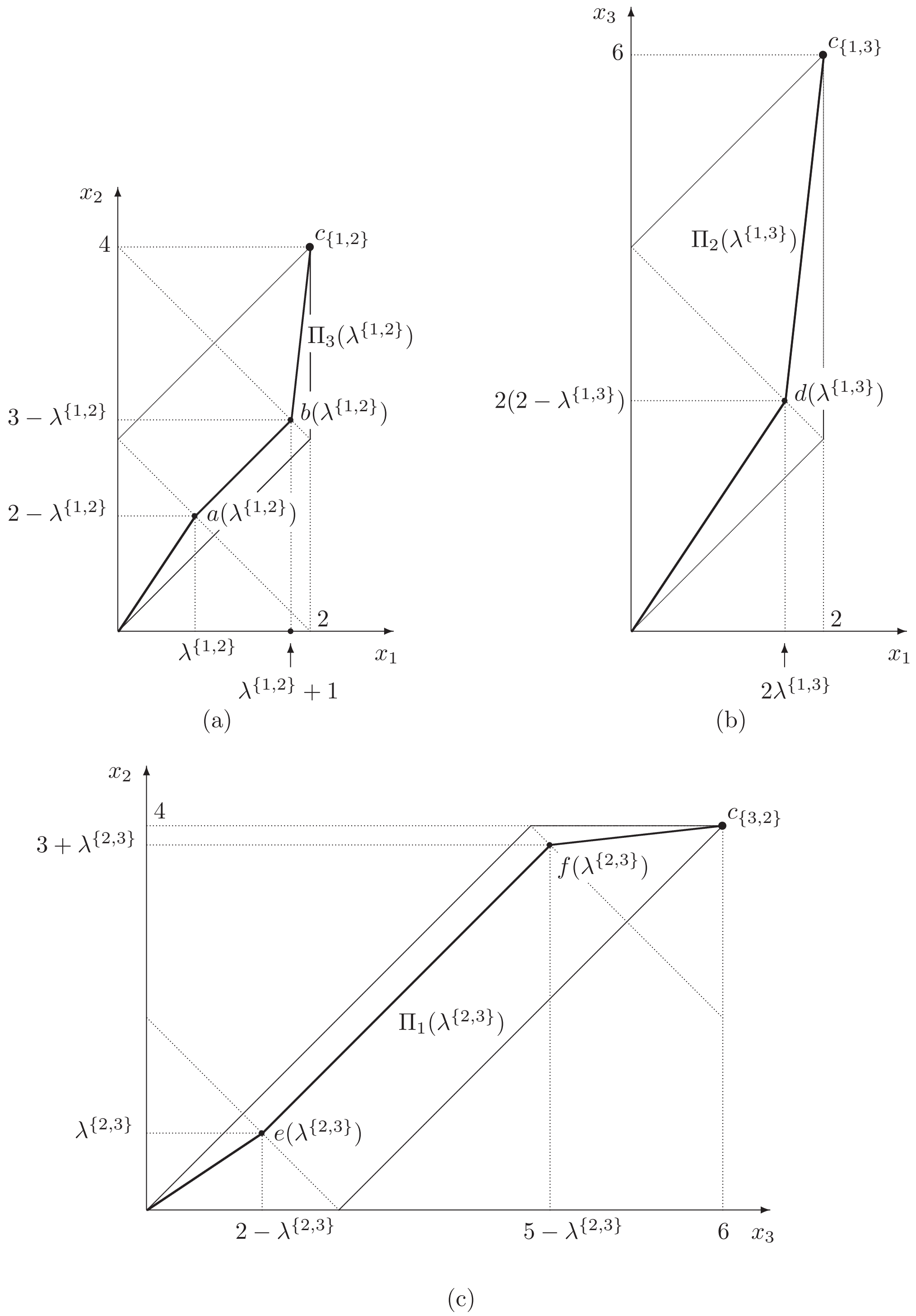


**Figure 8: Proof of Theorem 1. Paths of awards for $c_{\{1,2\}}$, $c_{\{1,3\}}$, and $c_{\{2,3\}}$.** In these illustrations, $\lambda^{\{1,2\}} = \lambda^{\{1,3\}} = \lambda^{\{2,3\}} = .8$. Panel (a) shows the path of $A^{\lambda^{\{1,2\}}}$ for $c_{\{1,2\}}$, panel (b) the path of $A^{\lambda^{\{1,3\}}}$ for $c_{\{1,3\}}$, and panel (c) the path of $A^{\lambda^{\{2,3\}}}$ for $c_{\{2,3\}}$.

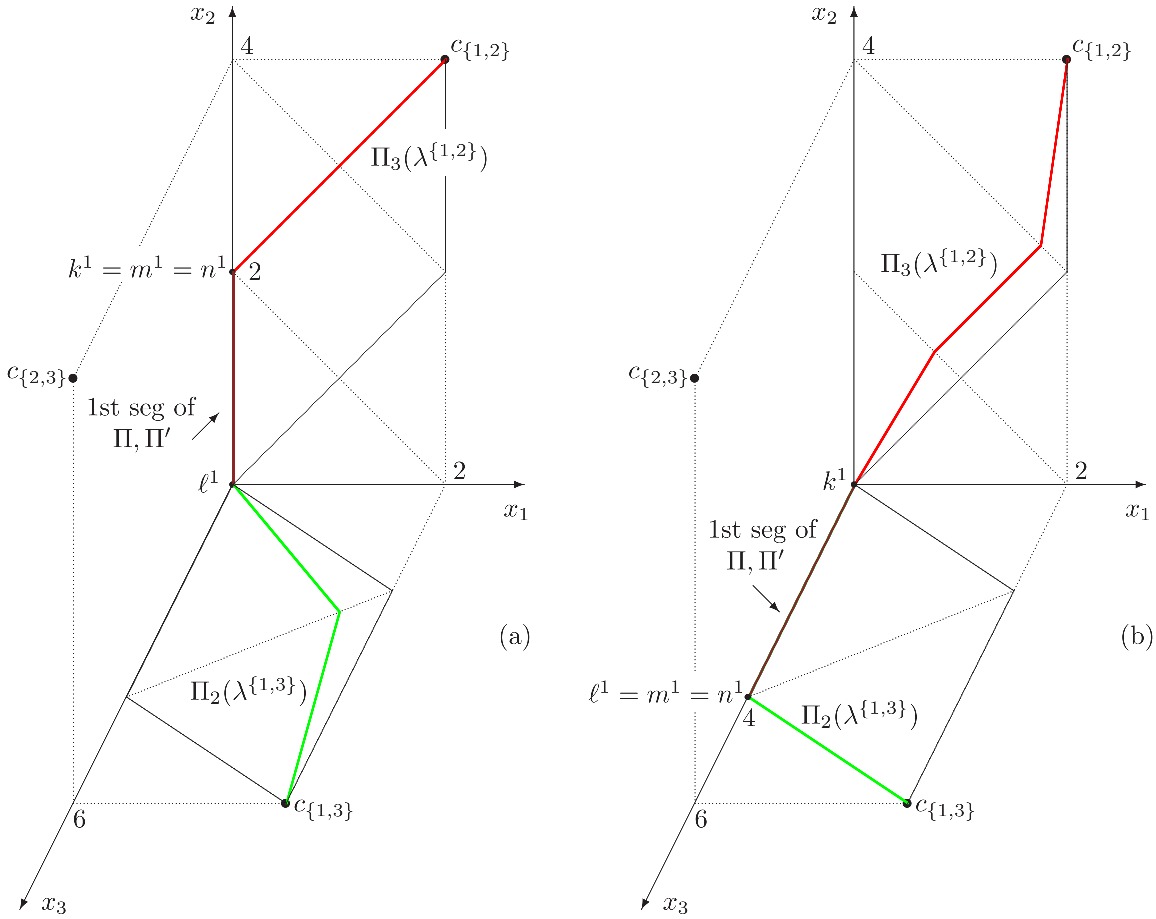


**Figure 9: Proof of Theorem 1; construction of the first segment of Π.** (a) This panel shows the case $\lambda^{\{1,2\}} = 0$ and $\lambda^{\{1,3\}} > 0$. (b) This panel shows the case $\lambda^{\{1,2\}} > 0$ and $\lambda^{\{1,3\}} = 0$.

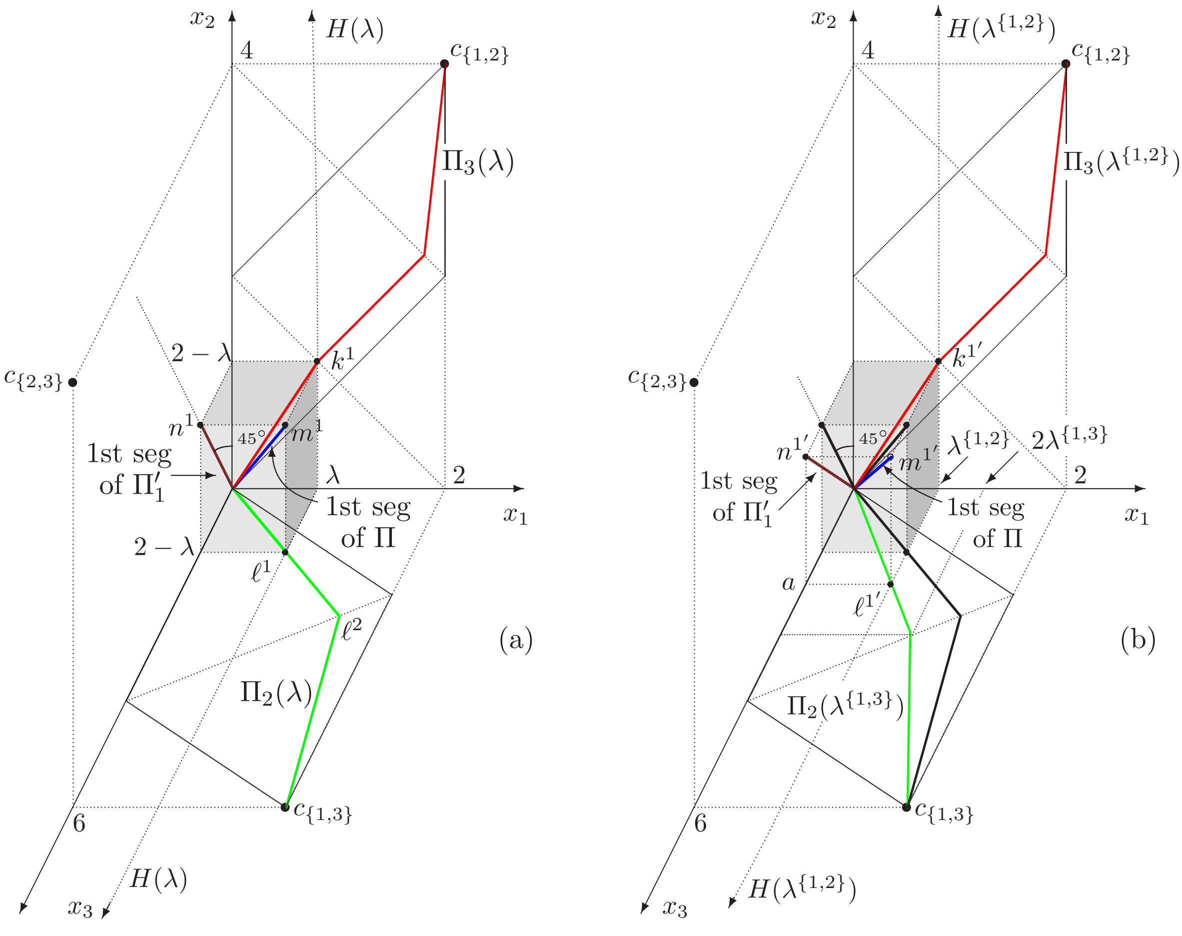


**Figure 10: Proof of Theorem 1; construction of the first segment of Π.** (a) This panel shows the construction if $\lambda^{\{1,2\}} = \lambda^{\{1,3\}} > 0$ (we use $\lambda$ for both). The projection of the first segment of $\Pi$ onto $\mathbb{R}^{\{2,3\}}$ (the first segment of $\Pi'_1$) is different from the first segment of $\Pi_1(\lambda^{\{2,3\}})$ (not represented). (b) This panel shows the case $\lambda^{\{1,2\}} > \lambda^{\{1,3\}} > 0$, the points $\ell^1$, $m^1$, and $n^1$ being replaced by the points $\ell^{1'}$, $m^{1'}$, and $n^{1'}$ ($k^{1'}$ is actually equal to $k^1$). We use the notation $a \equiv \lambda^{\{1,2\}} \frac{2-\lambda^{\{1,3\}}}{\lambda^{\{1,3\}}}$.

$\lambda^{\{1,2\}} \leq \lambda^{\{1,3\}}$. Thus, $\lambda^{\{1,2\}} = \lambda^{\{1,3\}}$. Then, as already noted, $\Pi$ begins with a segment in the plane of equation $x_2 = x_3$ and its projection on $\mathbb{R}^{\{2,3\}}$ has slope 1. This implies that the first segment of the path for $(c_2, c_3)$ has slope 1. This is possible only if $\lambda^{\{2,3\}} = 1$. This argument can be applied to deduce that $\lambda^N = 1$ for each other $N \in \mathcal{N}$ with $|N| = 2$. Indeed, all three weights are positive, and any two of them being positive implies that the third one is 1. Now, the analysis of Example 4 can be invoked. We conclude that $S = CEA$. □

# 6 A compromise between the constrained equal awards and proportional rules, and a compromise between the constrained equal losses and proportional rule

For our final application, we consider a family of two-claimant rules that offer a compromise between equality and proportionality (to claims). Equality of awards is natural if there is little to divide—differences in claims seem irrelevant then—but at some point, when the amount to divide is large enough, one should start recognizing these differences, and proportionality comes to mind. The first component of our definition is also part of the justification offered by Aumann and Maschler (1985) for the Talmud rule. Another ingredient to these authors' definition is duality, whereas here, our second main ingredient is proportionality. The question is where the switch should occur from equality to proportionality. We propose a flexible definition that allows to vary the emphasis on one or the other of the principles.

Given $N \in \mathcal{N}$, let $e_N$ be the vector of all ones in $\mathbb{R}^N$. Let $N \in \mathcal{N}$ be given with $|N| = 2$. For each $c \in \mathbb{R}^N_+$, let $a(c) \in \mathbb{R}_+$ be such that $0 \leq a(c) \leq \min c_i$. Then, let the path of awards for $c$ be bro.seg$[(0,0), a(c)e_N, c]$ (Figure 11a). To each function $a\colon \mathbb{R}^N_+ \to \mathbb{R}_+$ satisfying the inequalities stated above is associated a rule on $\mathcal{C}^N$. Note that if the two coordinates of $c$ are equal, for each $a(c) \in [0, \min c_i]$, the path of awards for $c$ is seg$[(0,0), c]$. Let $\boldsymbol{\mathcal{A}^N}$ be the family of rules so defined.

Of course, one could imagine a smoother way of passing from equality to proportionality, but the two-regime feature of our definition has the merit of simplicity. Also, it produces two important rules as special cases, the proportional rule if, for each $c \in \mathbb{R}^N_+$, $a(c) = 0$, and the constrained equal awards rule if, for each $c \in \mathbb{R}^N_+$, $a(c) = \min c_i$. Finally, it corresponds to the intuition of some experimental subjects.[17]

We also offer a symmetric way of compromising between the proportional and constrained equal losses rules. Again, let $N \in \mathcal{N}$ with $|N| = 2$. For each $c \in \mathbb{R}^N_+$, let $a(c) \in \mathbb{R}_+$ be such that $0 \leq a(c) \leq \min c_i$. Then, let the path of awards for $c$ be bro.seg$[(0,0), c - a(c)e_N, c]$ (Figure 11b). To each function $a\colon \mathbb{R}^N_+ \to \mathbb{R}_+$ satisfying the inequalities stated above is associated a rule on $\mathcal{C}^N$. Here too, if the two coordinates of $c$ are equal, for each $a(c) \in [0, \min c_i]$, the path of awards for $c$ is seg$[(0,0), c]$. Let $\boldsymbol{\mathcal{B}^N}$ be the family of rules so defined. We obtain as special cases the proportional rule if, for each $c \in \mathbb{R}^N_+$, $a(c) = 0$, and the constrained equal losses rule if, for each $c \in \mathbb{R}^N_+$, $a(c) = \min c_i$.

An alternative way to reach this second definition is through the concept of duality alluded to earlier. The dual of a rule $S$ divides what is available in the manner in which $S$ divides "what is missing" (the difference between the sum of the claims and the amount available).

**Dual of rule $\boldsymbol{S}$, $\boldsymbol{S^d}$:** For each $(c, E) \in \mathcal{C}^N$, $S^d(c, E) \equiv c - S(c, \sum c_i - E)$.

It is easy to see that for each $a\colon \mathbb{R}^N_+ \to \mathbb{R}_+$ such that for each $c \in \mathbb{R}^N_+$, $a(c) \in [0, \min c_i]$, the member of $\mathcal{A}^N$ associated with $a$ and the member of $\mathcal{B}^N$ associated with $a$ are dual.

We will show that if a rule coincides, for each $N \in \mathcal{N}$ such that $|N| = 2$, with a member of $\mathcal{A}^N$, and is *consistent*, then for each two-claimant population, the function giving the breakpoints (the function $a$ of the definition) takes a very simple form, as it depends on a single parameter chosen in $\bar{\mathbb{R}} \equiv \mathbb{R} \cup \infty$; that this parameter is the same for each of these populations—let us call it $\alpha$; and finally, that for arbitrarily many claimants, a path of awards first follows the path of the constrained equal awards rule until all agents whose claims are at least $\alpha$ have received $\alpha$, and it concludes with a segment to the claims vector.

[17]Carmen Bevia reported to me that several students in an undergraduate class in which she had presented the subject made a proposal in that spirit. The family we define is a special case of a definition according to which paths of awards consist of two segments. Another special case of the two-segment definition is a family defined and characterized by Moulin (2000) (Also, see Thomson, 2001). However, we have no axiomatic justification for the general two-segment family, nor for the definition discussed here.

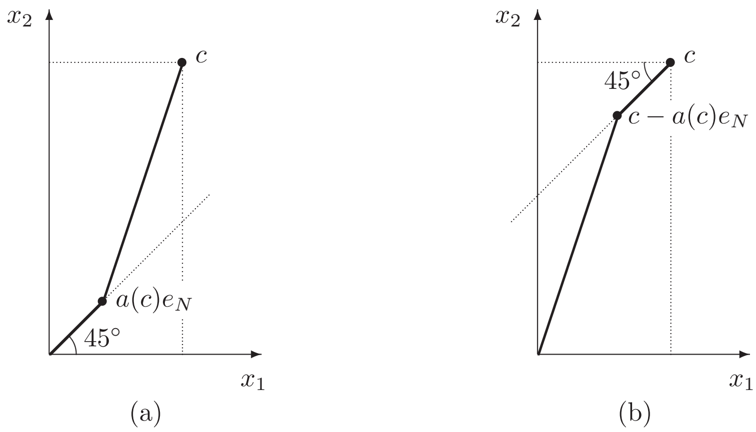


**Figure 11: Two ways of compromising between the constrained equal awards and proportional rules.** (a) To each claims vector $c \in \mathbb{R}^N_+$ is associated a number $a(c) \in [0, \min c_i]$ such that the path of awards for $c$ is bro.seg$[(0,0), a(c)e_N, c]$. (b) Here, to each $c \in \mathbb{R}^N_+$ is associated $a(c) \in [0, \min c_i]$ such that the path for $c$ is bro.seg$[(0,0), c - a(c)e_N, c]$

**Rule $\boldsymbol{S^\alpha}$ associated with $\boldsymbol{\alpha} \in \bar{\mathbb{R}}$:** For each $N \in \mathcal{N}$ and each $c \in \mathbb{R}^N_+$, the path of awards of $S^\alpha$ for $c$ is the path of the constrained equal awards rule until all claimants whose claims are at least $\alpha$ have received $\alpha$. It concludes with a segment to $c$ (if no agent's claim is greater than $\alpha$, this second segment is degenerate).

Let $\mathcal{S} \equiv \{S^\alpha \colon \alpha \in \bar{\mathbb{R}}\}$. Clearly, $S^0 = P$ and $S^\infty = CEA$.

By duality, we obtain the following family:

**Rule $\boldsymbol{R^\alpha}$ associated with $\boldsymbol{\alpha} \in \bar{\mathbb{R}}$:** For each $N \in \mathcal{N}$ and each $c \in \mathbb{R}^N_+$, the path of awards of $R^\alpha$ for $c$, followed down from $c$, is the path of the constrained equal losses rule until all claimants whose claims are at least $\alpha$ have experienced a loss of $\alpha$. It concludes with a segment to the origin (if no agent's claim is greater than $\alpha$, this second segment is degenerate).

Let $\mathcal{R} \equiv \{R^\alpha \colon \alpha \in \bar{\mathbb{R}}\}$. Here, $R^0 = P$ and $R^\infty = CEL$.

Our next theorem spells out when the *consistent* extension question has a positive answer for the first family of two-claimant rules. We obtain the members of $\mathcal{S}$:

**Theorem 2** *A rule coincides, for each two-claimant population $N$, with a member of $\mathcal{A}^N$, and is consistent, if and only if it is a member of $\mathcal{S}$.*

**Proof:** First, it is clear that all rules in $\mathcal{S}$ satisfy the two requirements of the theorem. Conversely, let $S$ be a rule satisfying the requirements. We show that there is $\alpha \in \bar{\mathbb{R}}$ such that $S = S^\alpha$.

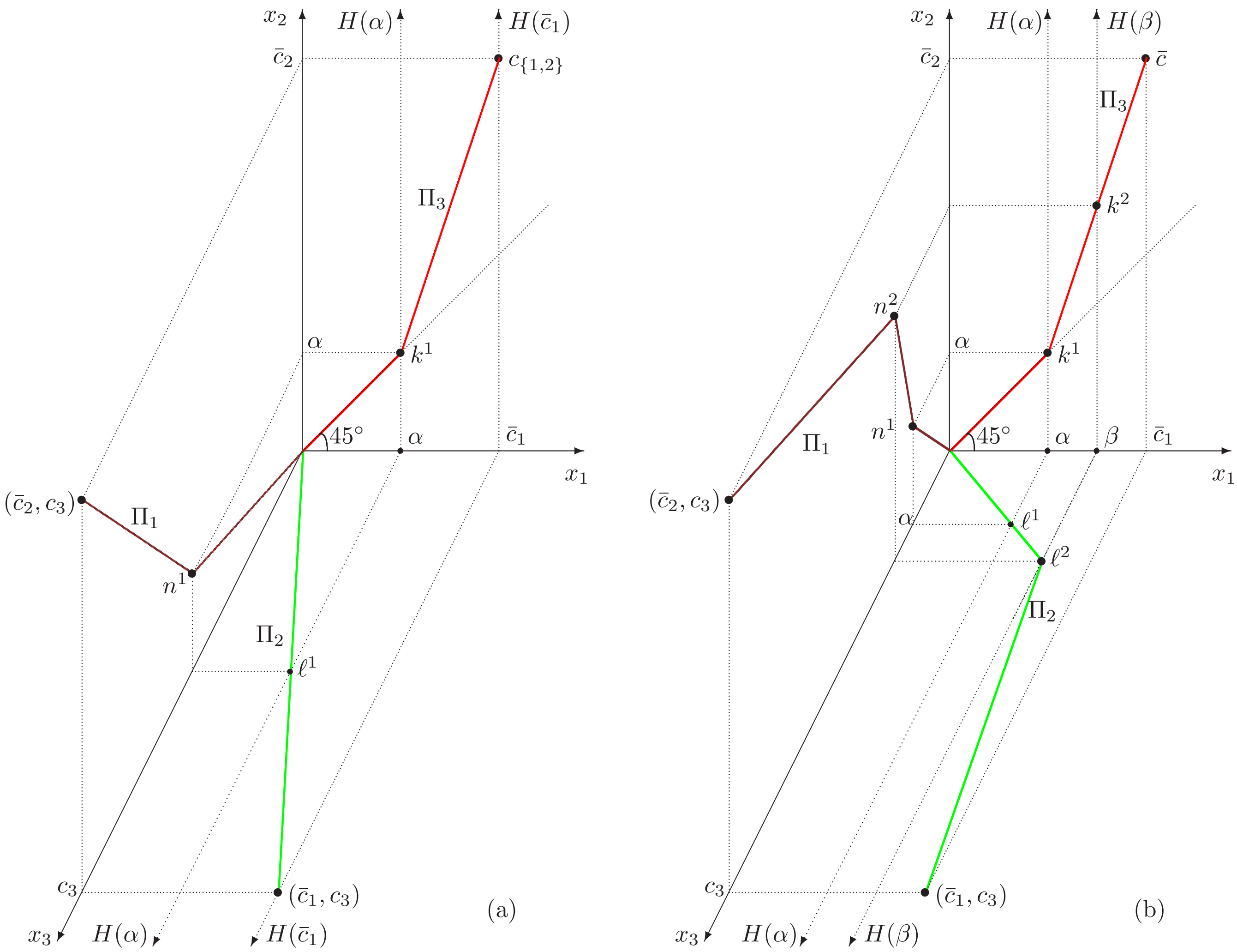


**Figure 12: Proof of Theorem 2: Identifying the parameter $\alpha$ associated with a member of the family $\mathcal{A}$** (Case 2, Step 1, Substep 1-1). (a) Here, (i) holds: $\beta \equiv a(\bar{c}_1, c_3) = 0$. (b) Here, (ii) holds: $0 < \beta < \bar{c}_1$.

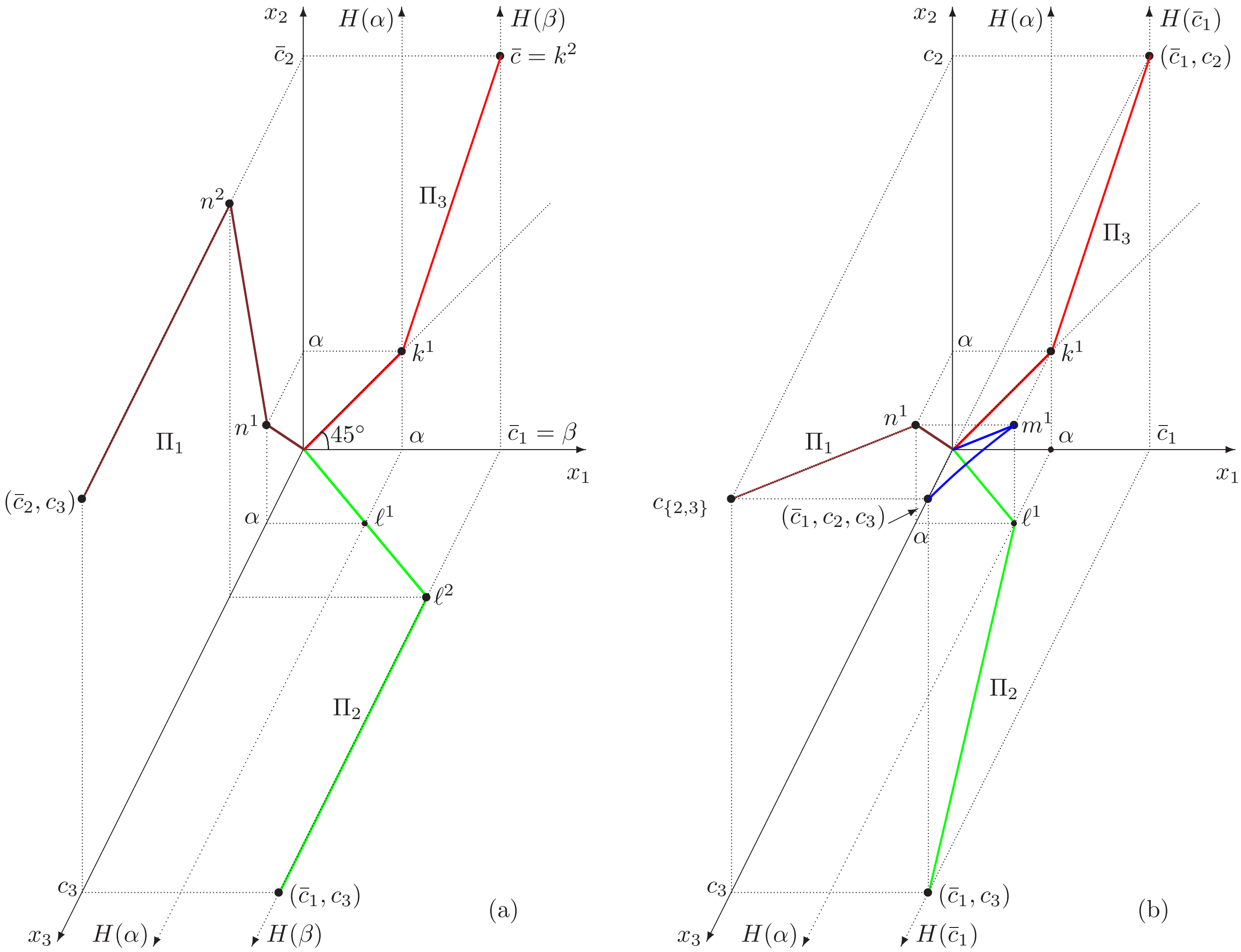


**Figure 13: Proof of Theorem 2: Identifying the parameter $\alpha$ associated with a member of the family $\mathcal{A}$** (Case 2). (a) Substep 1-1 when (iii) holds: $\beta = \bar{c}_1$. (b) Substep 1-2: paths of awards for $c$ and $c_{\{2,3\}}$ when $\alpha = \beta$. Each consists of two segments.

**Case 1: For each $N \in \mathcal{N}$ with $|N| = 2$ and each $c \in \mathbb{R}^N_+$, $a(c) = \min c_i$.** Then, $S$ coincides with the constrained equal awards rule in the two-claimant case. Applying the analysis for Example 4 to the special case when the weights are equal, we conclude that $S = CEA$ in general, that is, $S = S^\infty$.

**Case 2: There is $\bar{N} \in \mathcal{N}$ with $|\bar{N}| = 2$ and $\bar{c} \in \mathbb{R}^{\bar{N}}_+$ such that $a(\bar{c}) < \min \bar{c}_i$.** Let $\alpha \equiv a(\bar{c})$. To simplify notation, suppose that $\bar{N} = \{1, 2\}$.

**Step 1: For each $N \in \mathcal{N}$ with $|N| = 2$ and each $c \in \mathbb{R}^N_+$ with $\alpha \leq \min c_i$, $a(c) = \alpha$.**

We introduce agent 3 and prove the following limited version of Step 1:

**Substep 1-1:** for each $c_3 \in \mathbb{R}_+$ such that $c_3 \geq \alpha$ and $c_3 \neq \bar{c}_1$, $a(\bar{c}_1, c_3) = \alpha$ (recall that if $c_3 = \bar{c}_1$, we can choose $a(\bar{c}_1, c_3)$ arbitrarily in $[0, \bar{c}_1]$. Then, we are done.)

Let $N' \equiv \{1, 2, 3\}$, and $c' \equiv (\bar{c}, c_3) \in \mathbb{R}^{N'}_+$. Let $\beta \equiv a(\bar{c}_1, c_3)$. We show that $\beta = \alpha$. Suppose by contradiction that $\beta \neq \alpha$. Since the path of $S$ for $\bar{c}$ is strictly monotone, we can determine in a unique manner its path for $c'$. Figures 12 and 13 illustrate the argument for $\bar{c}_1 < \bar{c}_2$ and $c_3 > \bar{c}_1$. This time, we do not show $\Pi$.

(i) $\beta = 0$ (Figure 12a). The path for $(\bar{c}_1, c_3)$ is seg$[(0,0), (\bar{c}_1, c_3)]$, a segment whose slope is not 1. The path for $c'$ consists of two segments, the first one of which does not lie in the plane of equation $x_2 = x_3$. Its projection onto $\mathbb{R}^{\{2,3\}}$ consists of two segments, the first one of which has a slope different from 1. (There is one critical position of the moving plane, $H(\alpha)$).

(ii) $0 < \beta < \bar{c}_1$ (Figure 12b). The path for $c'$ consists of three segments. Its projection onto $\mathbb{R}^{\{2,3\}}$ also does. (There are two critical positions of the moving plane, $H(\alpha)$ and $H(\beta)$).

(iii) $\beta = \bar{c}_1$ (Figure 13a). A similar conclusion to that reached in (ii) is obtained (the main difference in the geometry is that the third segment in the path for $c'$ is parallel to the third axis, and that is also the case for the path for its projection onto $\mathbb{R}^{\{2,3\}}$). (There is one critical position of the moving plane, $H(\alpha)$).

In any of the three cases, the projection of the path for $c'$ onto $\mathbb{R}^{\{2,3\}}$ does not coincide with the path for $(\bar{c}_2, c_3)$, in violation of *consistency*.

If $c_3 < \bar{c}_1$, the inequalities defining case (ii) are replaced by $0 < \beta < c_3$ and the equality defining case (iii) is replaced by $\beta = c_3$, but the analysis is otherwise the same.

Thus, Substep 1-1 is proved.

**Substep 1-2:** remaining cases. Instead of starting from $(\bar{c}_1, \bar{c}_2) \in \mathbb{R}_+^{\{1,2\}}$ and introducing claimant 3, we start from $(\bar{c}_1, c_3) \in \mathbb{R}_+^{\{1,3\}}$, $c_3$ being arbitrarily subject to $c_3 \geq \alpha$, and introduce claimant 2. The same argument tells us that for each $c_2 \in \mathbb{R}_+$ such that $c_2 \geq \alpha$ and $c_2 \neq c_3$, $a(\bar{c}_1, c_2) = \alpha$. At the same time, we deduce that the path for $(\bar{c}_1, c_2, c_3)$, constructed from the paths for $(\bar{c}_1, c_2)$ and $(\bar{c}_1, c_3)$, is $\text{bro.seg}[(0,0,0), \alpha e_{N'}, (\bar{c}_1, c_2, c_3)]$, and that its projection on $\mathbb{R}^{\{2,3\}}$ is $\text{bro.seg}[(0,0), \alpha e_{\{2,3\}}, (c_2, c_3)]$, so that $a(c_2, c_3) = \alpha$ (Figure 13b). Thus, we have established Step 1 for $N = \{2,3\}$.

Instead of adding claimant 3 to the population $\bar{N}$, we could have added any $j \in \mathbb{N} \setminus \bar{N}$ and then have substituted any claimant $k$ for claimant 2, thereby establishing Step 1 for each $N \in \mathcal{N}$ with $|N| = 2$. (Alternatively, we could extend the conclusion reached for $\{2,3\}$ to each other two-claimant population by exploiting the fact that *equal treatment of equals* and *consistency* imply *anonymity.*)

**Step 2: For each $N \in \mathcal{N}$ with $|N| = 2$ and each $c \in \mathbb{R}_+^N$ with $\min c_i < \alpha$, $a(c) = \min c_i$.** Suppose by contradiction that the statement fails for some $N^* \in \mathcal{N}$: there is $c^* \in \mathbb{R}_+^{N^*}$ with $\min c_i^* < \alpha$ such that $\beta \equiv a(c^*) < \min c_i^*$. Then, by Step 1, we deduce that for each $c \in \mathbb{R}_+^{N^*}$ such that $\beta \leq \min c_i$, and in particular for each $c \in \mathbb{R}_+^{N^*}$ such that $\alpha \leq \min c_i$, $a(c) = \beta$. However, any such $c$ is also covered by Step 1, which gives us $a(c) = \alpha$. These two conclusions are in contradiction if $c$ has unequal coordinates.

We have now shown that $S$ coincides with $S^\alpha$ on $\bigcup_{N \in \mathcal{N}, |N|=2} \mathcal{C}^N$.

**Step 3: $S = S^\alpha$.** Reasoning analogously as in Example 4, we conclude that, for each $N \in \mathcal{N}$ and each $c \in \mathbb{R}_+^N$, the path of $S$ for $c$ follows the path of the constrained equal awards rule until each claimant $i \in N$ has received $\min\{c_i, \alpha\}$ and continues in a linear way to $c$. This is what $S^\alpha$ recommends. □

By duality, we obtain a parallel characterization of the family $\mathcal{R}$.

**Theorem 3** *A rule $S$ coincides, for each two-claimant population $N$, with a member of $\mathcal{B}^N$, and is consistent, if and only if it is a member of $\mathcal{R}$.*

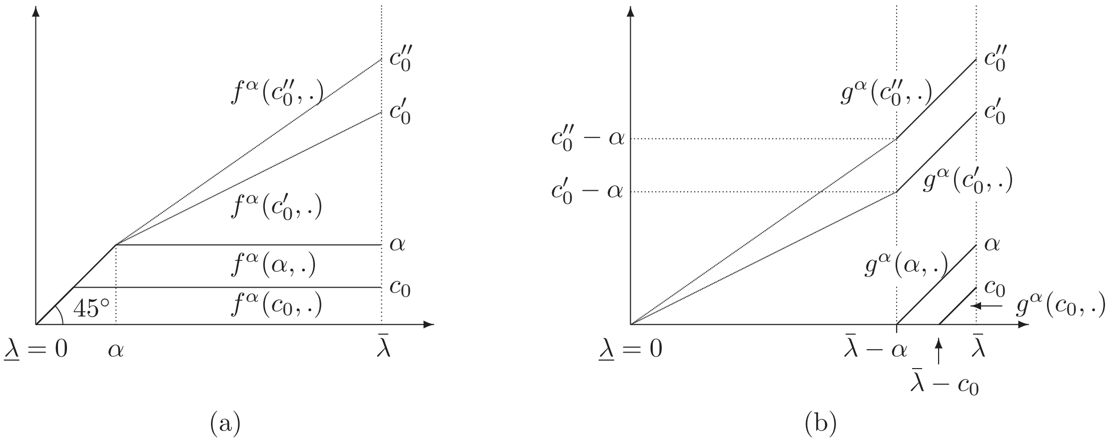


**Figure 14: Parametric representations of the members of $\mathcal{A}$ and $\mathcal{B}$ associated with $\alpha \in \mathbb{R}_+$.** (a) We choose $\bar{\lambda} > \alpha$ and select, for each $c_0 \in \mathbb{R}_+$, the function $f^\alpha(c_0,.)$ whose graph follows the 45° line up to $\min\{c_0,\alpha\}$ and continues in a linear way to $(\bar{\lambda}, c_0)$. (b) If $\alpha < \infty$, we choose $\bar{\lambda} > \alpha$ and select, for each $c_0 \in \mathbb{R}_+$, the function $g^\alpha(c_0,.)$ whose graph is the line segment from the origin to $(\bar{\lambda}-c_0, 0)$ if $c_0 \le \alpha$ and to $(\bar{\lambda}-\alpha, c_0-\alpha)$ otherwise, and the segment (its slope is 1) to $(\bar{\lambda}, c_0)$.

We conclude by giving an alternative description of the members of $\mathcal{S}$ and $\mathcal{R}$. **A rule $S$ has a parametric representation** (Young, 1987) if there is a function $f\colon [\underline{\lambda}, \bar{\lambda}] \times \mathbb{R}_+ \to \mathbb{R}_+$, where $\underline{\lambda}, \bar{\lambda} \in \bar{\mathbb{R}}$, such that for each $c_0 \ge 0$, $f(c_0,.)$ is continuous and nowhere decreasing, $f(c_0, \underline{\lambda}) = 0$ and $f(c_0, \bar{\lambda}) = c_0$, and for each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, the awards vector $S$ selects is $(f(c_i, \lambda))_{i \in N}$, where $\lambda \in [\underline{\lambda}, \bar{\lambda}]$ solves $\sum_N f(c_i, \lambda) = E$. Any rule satisfying *equal treatment of equals*, **continuity** (the requirement that small changes in the data of the problem should not lead to large changes in the awards vector that is selected), and *consistency* has a parametric representation (Young, 1987). Since the rules in $\mathcal{A}$ and $\mathcal{B}$ satisfy these axioms, they have parametric representations.

Rule $S^\alpha$ can be given the representation $f^\alpha$ defined as follows: let $\underline{\lambda} = 0$ and $\bar{\lambda} > \alpha$, and for each $c_0 \ge 0$, let $f^\alpha(c_0,.)$ be the function whose domain is $[\underline{\lambda}, \bar{\lambda}]$ and whose graph is bro.seg$[(0,0),(c_0,c_0),(\bar{\lambda},c_0)]$ if $c_0 \le \alpha$ and bro.seg$[(0,0),(\alpha,\alpha),(\bar{\lambda},c_0)]$ otherwise (Figure 14a).

Rule $R^\alpha$ can be given the representation $g^\alpha$ defined as follows: let $\underline{\lambda} = 0$, $\bar{\lambda} > \alpha$, and for each $c_0 \ge 0$, let $g^\alpha(c_0,.)$ be the function whose domain is $[\underline{\lambda}, \bar{\lambda}]$ and whose graph is bro.seg$[(0,0),(\bar{\lambda}-c_0,0),(\bar{\lambda},c_0)]$ if $c_0 \le \alpha$ and bro.seg$[(0,0),(\bar{\lambda}-\alpha,c_0-\alpha),(\bar{\lambda},c_0)]$ otherwise (Figure 14b).

The most delicate part of the proof of Theorem 2 is identifying the parameter $\alpha$ that characterizes each admissible rule. Once this parameter is identified, and once these representations are available, one can conclude by appealing to the Elevator Lemma.

# 7 Concluding comments

1. Dagan and Volij (1997) observe that the projection implication of *consistency* that is our point of departure can be described as transitivity of a binary relation attached to each awards vector as follows (Kaminski, 2000, 2006, follows the same approach). Let $S$ be a rule. Let $N \in \mathcal{N}$ and $c \in \mathbb{R}^N_+$ and let $x$ be an awards vector of $(c, E)$. Say that "agent $i$ is treated too well in relation to agent $j$ at $x$" if for the problem $(c_i, c_j, x_i + x_j)$, $S$ assigns less to agent $i$ than $x_i$ (and therefore more to agent $j$ than $x_j$). Geometrically, this means that $(x_i, x_j)$ lies between the path of $S$ for $(c_i, c_j)$ and the $i$-th axis. Now, if $S$ is *consistent* and at $x$, (i) agent $i$ is treated too well in relation to agent $j$ and (ii) agent $j$ is treated too well in relation to agent $k$, then (iii) agent $i$ is treated too well in relation to agent $k$. Thus, the relation "to be treated too well in relation to" attached to $x$ is transitive. Conversely, if $S$ is not *consistent*, there is an awards vector $x$ whose projection on $\mathbb{R}^{\{i,j\}}$ falls between its path for $(c_i, c_j)$ and the $i$-th axis, whose projection on $\mathbb{R}^{\{j,k\}}$ falls between its path for $(c_j, c_k)$ and the $j$-th axis, but whose projection on $\mathbb{R}^{\{i,k\}}$ does not fall between its path for $(c_i, c_k)$ and the $i$-th axis. The goal of our geometric technique is to constructively determine when these projections requirements are met, or how to specify rules in the two-claimant case when several choices are available, for the projection requirements to be met. A by-product of our analysis are vectors whose associated relations are not transitive when no *consistent* extension exists.

2. When a two-claimant rule has no *consistent* extension, one can ask whether it can be extended so as to satisfy some weaker notion of *consistency*. Such a notion is proposed by Dagan and Volij (1997): a rule is **average consistent** if for each population of claimants, each problem this population may face, and each claimant in the population, the award it recommends for this claimant is equal to the average of the awards it recommends for the claimant in the associated two-claimant reduced problems relative to all the subpopulations to which he belongs.[18] Dagan and Volij show that if

[18] This idea is initially proposed for coalitional games by Maschler and Owen (1989).

a rule is *resource monotonic*,[19] it has an *average consistent* extension and this extension is unique. Since weighted averages of the constrained equal awards and constrained equal losses rules satisfy this property, this result applies to them. In the light of the negative result we presented for the weighted average of the constrained equal awards and constrained equal losses rules, this extension can therefore be seen as the best means of compromising between the ideas expressed by the two rules while preserving some measure of *consistency*. The observation also applies to the other negative results that we have obtained.

Also, if a two-claimant rule does have a *consistent* extension, *average consistency* provides in principle another route to finding it. Indeed, the *average consistent* extension of the rule would in fact coincide with its *consistent* extension. If not, the *average consistent* extension would violate *consistency*, which we would find by exhibiting a counterexample. A difficulty in implementing this strategy however is that calculating an *average consistent* extension may not be a trivial matter. Also, the method we developed has the advantage of addressing directly the issue at hand, namely whether certain projections requirements can be met. Nevertheless, it would be worthwhile exploring whether the passage through *average consistency* might sometimes lead to answers to the *consistent* extension question.

[19]Their main theorem is stated with the additional requirement of *anonymity* but they observe later that it holds without it.

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