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Abstract

We consider the problem of dividing the cost of a facility when agents can be ordered in terms of the need they have for it, and accommodating an agent with a certain need allows accommodating all agents with lower needs at no extra cost. This problem is known as the "airport problem", the facility being the runway. We review the literature devoted to its study, and formulate a number of open questions.

Key words: airport problem; monotonicity; consistency; core; Shapley value; nucleolus.

Contents

1 Introduction

Our goal is to present the state of the arts concerning the resolution of a very simple class of cost allocation problems, illustrated by the following example. Several airlines are jointly using an airstrip, different airlines having different needs for it. The larger the planes an airline flies, the longer the airstrip it needs. An airstrip that accommodates a given plane accommodates any smaller airplane at no extra cost. The airstrip is large enough to accommodate the largest plane any airline flies. How should its cost be divided among the airlines? We refer to this situation as an "airport problem", and we use this expression to designate all problems with that structure.

Here is another illustration. Ranchers are distributed along an irrigation ditch. The rancher closest to the headgate only needs that the segment from the headgate to his field, the "first segment", be maintained, the second closest rancher needs that the first two segments be maintained, and so on. The cost of maintaining a segment used by several ranchers is incurred only once, independently of how many ranchers use it. How should the total cost of maintaining the ditch be shared?

In general, agents in a group are linearly ordered by their needs for a facility, and accommodating an agent implies accommodating at no extra cost all agents who "come before him". Thus, the facility has the character of a public good.

Our search is for ways of associating with each problem of this type an allocation of the cost among the agents involved. Such a mapping is called a "rule".

Three approaches can be followed to obtain rules. A rule can be defined "directly", by means of a formula, a system of equations, or an algorithm. One makes a case for it on the basis of the attractiveness of the definition, the intuitive appeal of the formula or process leading to it.

The "game-theoretic approach" consists in first associating with each problem a "cooperative game", either a bargaining game or a coalitional game.[1] Then, the game is "solved", that is, a payoff vector is identified that appropriately reflects the "power", "opportunities", or "rights" of each player. Finally, the allocation corresponding to this payoff vector is determined; it is the choice made for the problem under consideration.

For the "axiomatic approach", the point of departure are the properties

[1]A game is a mathematical representation of a conflict situation.

of rules. These properties are the mathematical expressions of intuitive ideas about how certain situations might be handled by agents on their own, or by an impartial arbitrator. They are formally stated as axioms. Properties are used to compare rules. Mainly, one inquires about the implications of the properties, when imposed singly and in various combinations. The goal of the axiomatic program is to trace out the boundary between those combinations that are compatible and those that are not, and when compatible, to give as explicit as possible a description of the family of rules satisfying them.

This survey of the airport problem may also serve as an introduction to these methodologies, all of which have proved useful in handling a variety of other classes of allocation problems.[2]

2 The model

There is a set N of agents for whom a facility they will jointly use is to be built. Agents have different needs for it. The facility should be built so as to accommodate all agents. Each agent $i \in N$ is characterized by the cost c_i of the facility he needs, which we call his "cost parameter". Serving agent i implies serving any other agent j whose cost parameter c_j is at most as large as c_i. The cost of accommodating everyone is $\max_N c_j$. How much should each agent contribute? In our primary application, agents are airlines (or rather plane movements), and c_i is the cost of the airstrip airline i needs.

In summary, an airport problem, or simply a **problem** (Littlechild and Owen, 1973), is a vector $c \in \mathbb{R}^N_+$.[3] Let $\boldsymbol{\mathcal{C}^N}$ denote the domain of all problems. We impose the natural requirement that each agent should bear some of the cost, and should contribute at most what he would have to pay if alone. Thus, a cost allocation for $c \in \mathcal{C}^N$, or simply an **allocation for $\boldsymbol{c}$**, is a vector $x \in \mathbb{R}^N$ such that $0 \leqq x \leqq c$ and $\sum x_i = \max c_i$.[4] Let $\boldsymbol{X(c)}$ be the set of allocations for c. The difference $c_i - x_i$ between agent i's cost parameter and his contribution is the **benefit** he experiences at the allocation x.

Numbering agents in the order of increasing costs, and in the absence of ties, we refer to them as the first, second, ..., and last agents. We say of

[2]For an introduction to cost allocation theory, see Young (1985).

[3]By this notation, we mean the cross-product of $|N|$ copies of $\mathbb{R}$ indexed by the members of N. The superscript N is also used to denoted any object pertaining to the set N, but it will always be clear which usage is intended.

[4]Vector inequalities: $x \leqq y$, $x \leq y$, $x < y$.

an agent whose cost parameter is greater than some other agent's that "he comes after that agent", or, referring to the irrigation application, that he is "downstream" of that agent. We call the differences c_1, $c_2 - c_1$, $c_3 - c_2$, and so on, **segmental costs**. (A number of authors take these segmental costs as primitives and we urge readers of the primary literature to keep that distinction in mind.)

An allocation rule, or simply a **rule**, is a mapping defined on the domain of all problems, which associates with each $c \in \mathcal{C}^N$ a vector in $X(c)$. Let S be our generic notation for rules. Note that rules are **single-valued**. This property is greatly desirable since it implies that the issue of who should pay what has been completely resolved.

The set of allocations of each problem being a convex set, a convex combination of rules is a rule.[5]

Another interpretation of the model is possible, when the intervals of use refer to time. Think of agents who start using a facility at the same time, but each agent stops using it when his needs are satisfied.

What is common to all of these situations is their linear structure and the public good character of the facility.[6] The model can also be interpreted as depicting simple networks, and its analysis may provide principles and techniques helpful in handling general networks. Section 6 shows the relevance of the concepts we first describe in the context of linear networks to the analysis of tree-like networks. This extension requires some adjustments but poses no fundamental conceptual difficulties. Not covered are networks containing cycles. Cycles alter the nature of the problem in ways that require significant changes in techniques.

We close this section of preliminaries by noting the mathematical similarities between an airport problem and a **claims problem**.[7] Such a problem is defined by specifying for each agent a non-negative number interpreted as his claim on some resource, and specifying how much of the resource is available, this amount being smaller than the sum of the claims. An airport problem

[5] Given a family of rules and a list of non-negative weights adding up to one, the "convex combination of these rules with these weights" is the rule that selects for each problem, the weighted average of the contribution vectors chosen by the rules in the family.

[6] It is not a standard public good, as such a good is consumed by everyone at the level at which it is produced, nor an excludable public good, since for such a good, differences in the agents' consumptions of the good are determined endogenously.

[7] See Thomson (2003) for a survey of the literature.

can be seen as a claims problem in which the amount to divide is equal to the largest claim. In the development of the axiomatic theory concerning claims problems, a variety of relational properties involving the amount to divide have played an important role.[8] This amount is not an independent variable here, so there are no counterparts to these properties.[9] Another difference is that the set of allocations from which it is natural to choose when solving an airport problem is a subset of the corresponding set when the problem is interpreted as a claims problem (see below).

In spite of these important differences, we will find that the theory concerning the adjudication of conflicting claims is quite useful in developing a theory of cost allocation for airport problems.

3 Direct approach to defining rules

Here we define a number of rules. They all have intuitive definitions and most of them will come up when we turn to axioms. For simplicity, we set $N \equiv \{1, 2, \ldots, n\}$ and $c_1 \leq c_2 \leq \cdots \leq c_n$.

1. We start with a basic requirement that we will always impose: at the allocation chosen by a rule for a problem, no group N' of agents should contribute more than what it would have to pay on its own, $\max_{N'} c_i$. Otherwise, the members of the group would unfairly "subsidize" the other agents. The group could make all of its members better off by setting up its own facility. In practice, secession is not always an option, but it may still serve as a relevant reference situation in evaluating a proposed allocation.

We require *single-valuedness* of rules, so the mapping that associates with each airport problem its set of allocations satisfying these no-subsidy requirements is not a rule. We refer to it instead as a "correspondence". As we will see, all of the rules that have been discussed in the literature are selections from it.

No-subsidy correspondence, *NoSub*: For each $c \in \mathcal{C}^N$, $NoSub(c) \equiv \{x \in X(c)\colon \text{ for each } N' \subseteq N, \sum_{N'} x_j \leq \max_{N'} c_j\}$.

[8] A "relational" property connects the choices made by a rule for two problems that are related in a certain way. This is in contrast with "punctual" properties, which apply separately to each element in the domain.

[9] Examples of such properties are monotonicity with respect to this amount and "composition" properties.

Since allocations are non-negative vectors, it suffices to write the constraints for all groups of consecutive agents that contain the agent with the lowest cost parameter, ($NoSub(c) = \{x \in X(c)$: for each $i \in N$, $\sum_{j\in N: c_j \le c_i} x_j \le c_i\}$). This significantly reduces the number of constraints.[10] If there are ties, a further reduction is possible. For instance, let $c \in \mathcal{C}^{\{1,2,3,4\}}$. Then, the constraints simplify to $x_1 \le c_1$, $x_1+x_2 \le c_2$, $x_1+x_2+x_3 \le c_3$, the fourth one, $x_1+x_2+x_3+x_4 \le c_4$ being of course implied by the feasibility requirement $x_1+x_2+x_3+x_4 = c_4$. If $c_2 = c_3$, we can also skip the constraint $x_1 + x_2 \le c_2$, which is implied by the constraint $x_1 + x_2 + x_3 \le c_3$ $(= c_2)$.

2. The no-subsidy correspondence places an upper bound on each agent's contribution. The next solution places a lower bound: each agent should contribute at least $\frac{1}{n}$ of his cost parameter. For each agent, imagine a situation in which all agents had his cost parameter. Then, equal division appears as a most reasonable choice:[11]

Identical-cost lower bound solution, *Iclb*: For each $c \in \mathcal{C}^N$, $Iclb(c) \equiv \{x \in X(c)$: for each $i \in N$, $x_i \ge \frac{1}{n}c_i\}$.

This solution is well-defined, and it has a non-empty intersection with the no-subsidy correspondence, as the solution defined next belongs to both.

3. In much of the literature on fair allocation, equality stands as a focal point, but what should be equated is not always clear. Sometimes, more than one choice are available. This is the case here. Several of the rules defined below can indeed be understood as attempts at giving meaning to the goal of equality. Equality of contributions itself is of course not an option given the no-subsidy constraints. Adjustments have to be made to respect them. Our first proposal is to apply equal division to each segment separately, "locally" so to speak: all agents using a given segment contribute equally to its cost. Then, each agent's contribution is a sum of terms, one for each segment he uses. The rule so defined is very natural, and when first exposed to the problem, people often spontaneously come up with it. It has also been used in the real world for many years.[12]

[10] From $2^n - 1$ to n.

[11] This correspondence is defined by Chun, Kayı, and Yeh (2006).

[12] The rule is discussed by Baker and Associates (1965) and Littlechild and Owen (1973). It underlies the "serial" idea that has been the subject of a number of studies by Moulin and various coauthors in several other contexts. See for instance, Moulin and Shenker (1992). Aadland and Kolpin (1998) refer to it as the "serial rule", and explain that it is standard in allocating the cost of irrigation ditches.

Sequential equal contributions rule, *SEC*: For each $c \in \mathcal{C}^N$ and each $i \in N$, $SEC_i(c) \equiv \frac{c_1}{n} + \frac{c_2-c_1}{n-1} + \cdots + \frac{c_i-c_{i-1}}{n-i+1}$.

For each $i \in N$, the group $N' \equiv \{1,\ldots,i\}$ pays the sum $i\frac{c_1}{n} + (i-1)\frac{c_2-c_1}{n-1} + \cdots + 1\frac{c_i-c_{i-1}}{n-i+1}$. Since each of the coefficients of the segmental cost terms appearing in this expression is at most one, the sum $c_1 + (c_2 - c_1) + \cdots + (c_i - c_{i-1}) = c_i$ is an upper bound on what N' contributes in total. Thus, the no-subsidy constraints are met at $SEC(c)$.

4. Our next rule offers a different, this time "global", implementation of the goal of equality. Contributions are chosen equal subject to the no-subsidy constraints being met.[13] Like for several of the other rules defined below, it is convenient to give it an algorithmic definition, collecting progressively more and more from agents until the cost of the project is entirely covered.

Constrained equal contributions rule, *CEC*: Let $c \in \mathcal{C}^N$. Start by requiring equal contributions from all agents in N until there are a quantity $\gamma^1 \in \mathbb{R}_+$ and and agent $k^1 \in N$ such that $k^1\gamma^1 = c_{k^1}$ (if there are several such agents, select the one with the largest index).[14] Then, each $i \in \{1,\ldots,k^1\}$ pays γ^1. Continue by requiring equal contributions from the members of $\{k^1+1,\ldots,n\}$ until there are an amount $\gamma^2 \in \mathbb{R}_+$ and an agent $k^2 \in N$ such that $k^1\gamma^1 + (k^2 - k^1)\gamma^2 = c_{k^2}$ (if there are several such agents, select the one with the largest index). Then, each $i \in \{k^1+1,\ldots,k^2\}$ pays γ^2. Continue until the total amount collected is c_n.

The algorithm just defined is equivalent to finding $k \in N$ for which the ratio $\frac{c_k}{k}$ is the lowest and, denoting by k^1 the largest such k and by γ^1 the corresponding ratio, having each $i \in \{1,\ldots,k^1\}$ pay γ^1; then finding $k \in \{k^1+1,\ldots,n\}$ for which the ratio $\frac{c_k-c_{k^1}}{k-k^1}$ is the lowest and, denoting by k^2 the largest such k and by γ^2 the corresponding ratio, having each $i \in \{k^1+1,\ldots,k^2\}$ pay γ^2; proceeding in this way until the total amount collected is c_n.

Here is a numerical example: let $c \equiv (2,3,6,7) \in \mathcal{C}^{\{1,2,3,4\}}$. Then $\gamma^1 = 1.5$ (the minimum of $\frac{2}{1}$, $\frac{3}{2}$, $\frac{6}{3}$, and $\frac{7}{4}$), and $k^1 \equiv 2$; $\gamma^2 = 2$ (the minimum of $\frac{6-3}{1}$ and

[13] This rule is the counterpart of the "constrained equal awards rule" for the adjudication of conflicting claims. Aadland and Kolpin (1998) refer to it as the "restricted average cost share" rule. They also discuss an equal-contribution rule.

[14] This is actually not needed, but it reduces the number of steps of the algorithm. If we do not, at the next step, we obtain the same γ term.

$\frac{7-3}{2}$) and $k^2 = 4$. Altogether, $CEC(c) = (1.5, 1.5, 2, 2)$. For $c' \equiv (2, 4, 6, 9)$, $\gamma^1 = 2$, and three no-subsidy constraints are reached simultaneously, for $k = 1$, $k = 2$, and $k = 3$, so $k^1 = 3$. One agent is left, who pays what remains of the total cost, so $k^2 = 4$ and $\gamma^2 = 3$. Altogether, $CEC(c') = (2, 2, 2, 3)$.

One can also calculate contributions one at a time, the following being an alternative way of presenting the rule:

$$
\begin{array}{lcllllllll}
x_1 & = & \min\{ & \frac{c_1}{1} & , \frac{c_2}{2} & , \frac{c_3}{3} & , \ldots, & \frac{c_{n-1}}{n-1} & , \frac{c_n}{n} & \} \\
x_2 & = & \min\{ & & \frac{c_2 - x_1}{1} & , \frac{c_3 - x_1}{2} & , \ldots, & \frac{c_{n-1} - x_1}{n-2} & , \frac{c_n - x_1}{n-1} & \} \\
x_3 & = & \min\{ & & & \frac{c_3 - x_1 - x_2}{1} & , \ldots, & \frac{c_{n-1} - x_1 - x_2}{n-3} & , \frac{c_n - x_1 - x_2}{n-2} & \} \\
\ldots & = & \min\{ & & & & , \ldots, & \ldots & , \ldots & \}
\end{array}
$$

It is a direct consequence of its definition that this rule is a selection from the no-subsidy correspondence.[15]

5. Next, we apply an idea that is central to much of the literature on fair allocation, namely proportionality. As for the constrained equal contributions rule, an algorithmic definition is most convenient.[16]

Constrained proportional rule, *CP*: Let $c \in \mathcal{C}^N$. Start by requiring the contributions of all agents in N to be proportional to their components of c until there are a quantity $\rho^1 \in \mathbb{R}_+$ and and an agent $k^1 \in N$ such that $\rho^1 \sum_{1,\ldots,k^1} c_i = c_{k^1}$ (if there are several such agents, select the one with the largest index). Then, each $i \in \{1, \ldots, k^1\}$ pays $\rho^1 c_i$. Continue by requiring the contributions of the members of $\{k^1 + 1, \ldots, n\}$ to be proportional to their components of c until there are a quantity $\rho^2 \in \mathbb{R}_+$ and an agent $k^2 \in \{k^1 + 1, \ldots, n\}$ such that $\rho^1 \sum_{1,\ldots,k^1} c_i + \rho^2 \sum_{k^1+1,\ldots,k^2} c_i = c_{k^2}$ (if there are several such agents, select the one with the largest index). Then, each $i \in \{k^1 + 1, \ldots, k^2\}$ pays $\rho^2 c_i$. Continue until the total amount collected is c_n.

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[15] The rule is equivalent to identifying agent $k^1 \in N$ for whom the average of the first k^1 segmental costs $c_1, c_2 - c_1, \ldots, c_\ell - c_{\ell-1}, \ldots$, is the lowest; then, identifying $k^2 \in \{k^1+1, \ldots, n\}$ for whom the average of the $k^2 - k^1$ next segmental costs $c_{k^1+1} - c_{k^1}, \ldots, c_\ell - c_{\ell-1}, \ldots$ is the lowest, and so on. This description is given by Aadland and Kolpin (1998).

[16] Somewhat surprisingly, given the central role played by the idea of proportionality in the theory of fair allocation and the prominent role it enjoys for the closely related class of claims problems, we are not aware of any previous attempts at applying it to airport problems.

This algorithm is equivalent to finding $k \in N$ such that the ratio $\frac{c_k}{\sum_{1,\ldots,k} c_\ell}$ is the lowest and, denoting by k^1 the largest such k and by ρ^1 the corresponding ratio, having each $i \in \{1, \ldots, k^1\}$ pay $\rho^1 c_i$; then finding $k \in \{k^1+1, \ldots, n\}$ such that the ratio $\frac{c_k - c_{k^1}}{\sum_{k^1+1,\ldots,k} c_\ell}$ is the lowest and, denoting by k^2 the largest such k and by ρ^2 the corresponding ratio, having each $i \in \{k^1+1, \ldots, k^2\}$ pay $\rho^2 c_i$; proceeding in this way until the total amount collected is c_n.

For $c \equiv (2,3,6,7) \in \mathcal{C}^{\{1,2,3,4\}}$, if contributions are set proportional to cost parameters with the constant of proportionality chosen so that their sum is equal to c_4, the no-subsidy constraints are all met and we are done in one step. However, for $c \equiv (2,3,3,7)$, the proportional allocation is $\frac{7}{15}(2,3,3,7)$. Since $\frac{7}{15}(2+3+3) > 3 = c_3$, the no-subsidy constraint is violated for the group $\{1,2,3\}$, so we proceed in steps. We need two steps. At the first step, we obtain the contributions of the first three agents ($\rho^1 = \frac{3}{8}$ and $k^1 = 3$); since only one agent is left, he has to cover what remains of the total cost, so $\rho^2 = 1$ and $k^2 = 4$. Altogether, the allocation is $(\frac{3}{8}2, \frac{3}{8}3, \frac{3}{8}3, \frac{4}{4}(7-3))$.

6. Instead of focusing on the contribution required of an agent, we could focus on the benefit he achieves for not having to take care of his needs on his own, and equate these benefits subject to no one receiving a transfer.[17]

Constrained equal benefits rule, *CEB*: For each $c \in \mathcal{C}^N$ and each $i \in N$, let $CEB_i(c) \equiv \max\{c_i - \beta, 0\}$, where $\beta \in \mathbb{R}_+$ is chosen so that $\sum \max\{c_i - \beta, 0\} = c_n$.

For our earlier example with $c \equiv (2,3,6,7) \in \mathcal{C}^{\{1,2,3,4\}}$, we have $\beta = 3$ and $CEB(c) = (0,0,3,4)$.

This rule is a selection from the no-subsidy correspondence, even though the no-subsidy requirement does not appear in the definition (in contrast with the two previous rules). To see this, let $x \equiv CEB(c)$ and note that for each $i \in N$ such that $c_i \leq \beta$, we have $\sum_{j \in N\setminus\{i\}: c_j \leq c_i} x_j = 0$, and for each $i \in N$ such that $c_i \geq \beta$, we have $\sum_{j \in N\setminus\{i\}: c_j \leq c_i} x_j \leq \sum_{j \in N\setminus\{n\}} x_j = c_n - x_n = \beta \leq c_i$.

7. Imagine agents arriving in the order of increasing cost parameters and hold each of them responsible for the segment he needs beyond the ones that are already covered when he arrives. For this rule, an agent never helps out agents with needs smaller than his, even though he uses the segments they

[17] This rule is the counterpart of the "constrained equal losses rule" for the adjudication of conflicting claims. It is discussed by Potters and Sudhölter (1999).

use, (and of course, he never gets any help in covering his segmental cost from agents with greater needs than his, even though these agents use his segment). If several agents have equal cost parameters, we require that they should share equally their common segmental cost.

Sequential full contributions rule, $\boldsymbol{SFC}$: For each $c \in \mathcal{C}^N$ and each $i \in N$, let $N^i(c) \subseteq N$ be defined by $N^i(c) \equiv \{j \in N\colon c_j = c_i\}$. Then, if $c_i = \min c_j$, $SFC_i(c) \equiv \frac{c_i}{|N^i(c)|}$, and otherwise $SFC_i(c) \equiv \frac{c_i - \max_{j \in N: c_j < c_i} c_j}{|N^i(c)|}$.

8. Next, we define a family of rules that are also based on a simple first-come first-pay scenario, but this time the order of arrival is exogenously given. Given an order on the set of agents, we imagine them arriving in this order, and assign to each one the cost of the extension needed to serve him when he arrives. Let $\mathcal{O}^N$ be the set of orders on N, with generic element $\prec$.[18]

Priority rule relative to $\boldsymbol{\prec \in \mathcal{O}^N}$, $\boldsymbol{D^{\prec}}$: For each $c \in \mathcal{C}^N$ and each $i \in N$, $D_i^{\prec}(c) \equiv \max\{c_i - \max_{j \in N: j \prec i} c_j, 0\}$.

For instance, if $c \in \mathcal{C}^{\{1,2,3\}}$ and agents arrive in the order $2 \prec 1 \prec 3$, agent 2 pays c_2. Agent 1 pays nothing because when he arrives, the segment needed to serve him is already covered. Agent 3 pays the difference $c_3 - c_2$. If they arrive in the order $3 \prec 2 \prec 1$, agent 3 pays c_3 and none of the others pays anything. We also use the notation $D^{3 \prec 2 \prec 1}$ for this rule, and similar notation for the other rules in the family. By construction, each of them satisfies the no-subsidy constraints.

9. Here, we still imagine, as in the previous definition, that agents arrive one at a time, but we assume all orders of arrival to be equally likely and take the average of the allocations associated with all orders. This produces a well-defined rule since the set of allocations is convex:[19]

Random arrival rule, $\boldsymbol{RA}$: For each $c \in \mathcal{C}^N$ and each $i \in N$, $RA_i(c) \equiv \frac{1}{|N|!} \sum_{\prec \in \mathcal{O}^N} \max\{c_i - \max_{j \in N: j \prec i} c_j, 0\}$.

[18] These rules are the counterparts of the rules of the same name in the theory of adjudication of conflicting claims. An important difference is that here the earlier an agent arrives the more he pays, whereas in that theory, the more he receives.

[19] This rule is the counterpart of the rule of the same name in the theory of adjudication of conflicting claims.

This rule is an average of rules satisfying the no-subsidy constraints (the priority rules), and since the set of allocations satisfying these constraints is convex too, it also satisfies them. This property is also a consequence of the fact, established earlier, that the sequential equal contributions rule satisfies the constraints, and of the following equivalence (implicit in Littlechild and Owen, 1973):

Lemma 1 *The random arrival rule coincides with the sequential equal contributions rule.*

10. Our next rule is based on an idea that is familiar to social choice and game theory. When certain constraints have to be met in choosing a payoff vector, a natural objective is to keep as far as possible from violating them. An allocation that is equally far from violating all constraints usually does not exist, and a next best choice is an allocation that is at a greater distance in the lexicographic maximin order from violating the constraints than any other allocation. Given $x \in X(c)$ and $i \in N$, we measure how close x is to violate the non-negativity and no-subsidy constraints. Let us call the ***i*-th slack at *x*** the difference $c_i - \sum_{\{1,\dots,i\}} x_j$.[20]

Slack maximizer rule, *SM*: For each $c \in \mathcal{C}^N$, $SM(c)$ is the allocation $x \in NoSub(c)$ such that for each $y \in NoSub(c)$, the $2n$-dimensional vector consisting of x itself and the slacks at x is greater in the lexicographic maximin order than the vector associated with y in a similar way.

An explicit recursive formula can be given for the rule. It is surprisingly similar to the one for the constrained equal contributions rule. There are two differences however. First, the denominator of each term is equal to the corresponding denominator of that earlier rule incremented by one. Second is the absence of a term associated with the last segment:

Lemma 2 (Sönmez, 1994) *Let $c \in \mathcal{C}^N$. The slack maximizer allocation x of $c \in \mathcal{C}^N$ is given by the following formula:*

[20] The term nucleolus is commonly used and the reason will be clear soon. At this point, we prefer not invoking concepts of game theory.

$$
\begin{array}{rclcccccccl}
x_1 & = & \min\{ & \frac{c_1}{2} & , & \frac{c_2}{3} & , & \frac{c_3}{4} & , \dots, & \frac{c_{n-1}}{n} & \} \\
x_2 & = & \min\{ & & & \frac{c_2-x_1}{2} & , & \frac{c_3-x_1}{3} & , \dots, & \frac{c_{n-1}-x_1}{n-1} & \} \\
x_3 & = & \min\{ & & & & & \frac{c_3-x_1-x_2}{2} & , \dots, & \frac{c_{n-1}-x_1-x_2}{n-2} & \} \\
\dots & = & \min\{ & & & & & & , \dots, & \dots & \} \\
x_{n-1} & = & \min\{ & & & & & & & \frac{c_{n-1}-x_1-\dots-x_{n-2}}{2} & \}
\end{array}
$$

Agent n pays what remains.

For the two-agent case, we obtain $(\frac{c_1}{2}, c_2 - \frac{c_1}{2})$, which is also what the sequential equal contributions and constrained equal benefits rules recommend. For the three- (or more-) agent case, this coincidence does not hold anymore. Let $c \equiv (1,2,3) \in \mathcal{C}^{\{1,2,3\}}$. Then, $CEB(c) = (0,1,2)$, but if $x \equiv SM(c)$, then $x_1 \equiv \min\{\frac{1}{2}, \frac{2}{3}\} = \frac{1}{2}$, $x_2 \equiv \min\{\frac{2-\frac{1}{2}}{2}\} = \frac{3}{4}$, and $x_3 \equiv 3 - (x_1 + x_2) = 1.75$.

When agents are grouped into types having equal cost parameters, with n_j being the number of agents of type j, and $M_i \equiv \sum_{j=1}^{i} n_j$, we obtain the following formula:

Lemma 3 (Littlechild, 1974) *The slack maximizer allocation x of $c \in \mathcal{C}^N$ is given by the following recursive formula:*[21]

$$x_i = \gamma_k, \quad i_{k-1} < i < i_k, \quad k = 1, \dots, k',$$

where γ_k and i_k are defined by:

$$\gamma_k = \min\left[\min_{i_{k-1}+1,\dots,n-1}\{\frac{c_i - c_{i_{k-1}} + \gamma_{k-1}}{M_i - M_{i_{k-1}} + 1}\}, \frac{c_n - c_{i_{k-1}} + \gamma_{k-1}}{M_n - M_{i_{k-1}}}\right];$$

[21] Littlechild also notes that as the number of agents of each type increases without bound, the term "+1" in the denominator of the expression below becomes negligible. Moreover, $\gamma_{k-1}/(M_i - M_{i_{k-1}})$ involves the multiplication of $1/n_i$ and c_i. Hence, under the same assumption, this term can also be ignored. Thus, in this case, the following simpler approximate formula for the slack maximizer rule is available:

$$
\begin{array}{rcl}
\tilde{x}_i & = & \tilde{\gamma}_k, i_{k-1} < i \leq i_k, k = 1, \dots, k', \\
\tilde{\gamma} & = & \min\limits_{i_{k-1}+1,\dots,n}\left[\frac{c_i - c_{i_{k-1}}}{M_i - M_{i_{k-1}}}\right]
\end{array}
$$

where i_k is defined as above.

and i_k is maximal among all indices solving the above minimization problem (that is, first $g_0 = i_0 = c_0 = 0$, proceed until $k = 1, \cdots, k'$, and finally, $i_{k'} = n$).

The similarity between the formulas for the constrained equal contributions rule and the slack maximizer rule suggests that they are members of a single and simple family. Such a family can indeed be defined (Gellekom and Potters, 1997): let $\alpha \in [0,1]$ and set $x_1 \equiv \min\{\frac{c_1}{1+\alpha}, \frac{c_2}{2+\alpha}, \ldots, \frac{c_{n-1}}{n-1+\alpha}, \frac{c_n}{n}\}$, $x_2 \equiv \min\{\frac{c_2-x_1}{1+\alpha}, \ldots, \frac{c_{n-1}-x_1}{n-2+\alpha}, \frac{c_n-x_1}{n-1}\}$, $\ldots$, $x_{n-1} \equiv \min\{\frac{c_{n-1}-x_1-\cdots-x_{n-2}}{1+\alpha}, \frac{c_n-x_1-\cdots-x_{n-2}}{2}\}$, and $x_n \equiv c_n - x_1 - \cdots - x_{n-1}$. (Note how the denominators have been modified as compared to the formulae for the constrained equal contributions and slack maximizer rules.)

11. Finally, we introduce a new family of rules. It is inspired by a concept that has been central for the adjudication of conflicting claims (Young, 1987). Let $\mathcal{F}$ be the family of continuous functions $f: \mathbb{R}_+ \times [a,b] \to \mathbb{R}_+$, where $[a,b] \subseteq \overline{\mathbb{R}}$ (the extended real line), such that for each $\bar{c} \in \mathbb{R}_+$, $f(\bar{c}, \cdot)$ is nowhere decreasing, $f(\bar{c}, a) = 0$ and $f(\bar{c}, b) = \bar{c}$.

Parametric rule associated with $\boldsymbol{f \in \mathcal{F}}$, $\boldsymbol{S^f}$: Let $c \in \mathcal{C}^N$. Start by requiring the contribution of each $i \in N$ to be equal to $f(c_i, \lambda)$ until for some $\lambda^1 \in [a,b]$, there is $k^1 \in N$ such that $\sum_{1,\ldots,k^1} f(c_i, \lambda^1) = c_{k^1}$. (If there are several such k^1, select the largest.) Then, each $i \in \{1, \ldots, k^1\}$ pays $f(c_i, \lambda^1)$. Continue by requiring the contribution of each $i \in \{k^1+1, \ldots, n\}$ to be equal to $f(c_i, \lambda)$ until for some $\lambda^2 \in [a,b]$, there is $k^2 \in N$ such that $c_{k^1} + \sum_{k^1+1,\ldots,k^2} f(c_i, \lambda^2) = c_{k^2}$. (If there are several such k^2, select the largest.) Then, each $i \in \{k^1+1, \ldots, k^2\}$ pays $f(c_i, \lambda^2)$. Continue until the total collected is c_n.

The constrained equal contributions, constrained proportional, and constrained equal benefits rules are members of the family.

4 Game-theoretic approach to defining rules

A standard way of coming up with a recommendation for an allocation problem is to map it into a game; apply the tools of game theory to solve the game, thereby obtaining a payoff vector for the game; then, selecting the allocations whose image is this payoff vector.

Several classes of games have been considered in the literature and allocation problems can meaningfully be mapped into different kinds of games. We will consider two main classes, bargaining games and coalitional games with transferable utility.

4.1 Bargaining games

Let N be a set of "players". A **bargaining game** (Nash, 1950) is a pair (B, d), where B is a subset of $\mathbb{R}^N$ and d is a point of B. The set B, the **feasible set**, is interpreted as the set of utility vectors available to the group N by unanimous agreement, and d, the **disagreement point**, is interpreted as the utility vector that the group obtains if its members fail to reach an agreement. The point chosen by a solution for a specific game is the **solution outcome** of the game. The set B is commonly assumed to be a convex and compact subset of $\mathbb{R}^N$, and to be such that there is $x \in B$ with $x > d$. Let $\boldsymbol{\mathcal{G}^N}$ be a domain of bargaining games. A **bargaining solution on $\boldsymbol{\mathcal{G}^N}$** is a function that associates with each game in $\mathcal{G}^N$ a unique point in the feasible set of the game.[22]

The following are central solutions in the theory of bargaining: The **lexicographic egalitarian solution** outcome of (B, d), $E^L(B, d)$, is the payoff vector at which utility gains from d are maximized in a lexicographic way, starting with the smallest one.[23] Its **Kalai-Smorodinsky solution** outcome (Kalai and Smorodinsky, 1975), $KS(B, d)$, is the maximal payoff vector on the segment from the disagreement point to the "ideal point", $a(B, d)$, where for each $i \in N$, $a_i(B, d)$ is the maximal utility gain achievable by agent i among the feasible points dominating the disagreement point. The **lexicographic Kalai-Smorodinsky solution** outcome of (B, d) (Imai, 1983), $KS^L(B, d)$, is constructed so as to recover efficiency when this property would not be met otherwise (there is no feasible outcome that semi-strictly dominates the chosen vector). It is obtained by first normalizing the problem so that its ideal point has equal coordinates, then applying the lexicographic egalitarian solution, and finally returning to the initial non-normalized problem. The **extended equal loss solution** outcome of (B, d)

[22] For an exposition of the theory of bargaining, see Thomson (1999).

[23] Given x, $y \in \mathbb{R}^\ell$, **$\boldsymbol{x}$ is greater than $\boldsymbol{y}$ in the lexicographic (maximin) order** if, designating by $\tilde{x}$ and $\tilde{y}$ the vectors obtained from x and y by rewriting their coordinates in increasing order, we have either $\tilde{x}_1 > \tilde{y}_1$ or $[\tilde{x}_1 = \tilde{y}_1$ and $\tilde{x}_2 > \tilde{y}_2]$, or $[\tilde{x}_1 = \tilde{y}_1$, $\tilde{x}_2 = \tilde{y}_2$, and $\tilde{x}_3 > \tilde{y}_3$, and so on.

(Bossert, 1993, in a contribution building on the equal loss solution of Chun, 1988), $XEL(B,d)$, is the maximal point at which the utility losses from the ideal point of all agents whose utility gains are positive are equal and the utilities of the others are equal to their disagreement utilities. Finally, the **lexicographic dictatorial solution outcome of $(\boldsymbol{B},\boldsymbol{d})$ associated with the order $\prec$**, $Dic^{\prec}(B,d)$, is the payoff vector at which the utility gain of the player who comes first in that order is maximal if this vector is unique. If not, among all such vectors, it is the vector at which the utility gain of the player who comes second is maximal if this vector is unique; and so on.

Given an airport problem $c \in \mathcal{C}^N$, its **associated bargaining game** is the game whose feasible set is $B(c) \equiv \{y \in \mathbb{R}^N_+ \colon \text{ for some } x \in NoSub(c), y \leqq x\}$ and whose disagreement point is the origin. As the disagreement point is independent of c, we ignore it from the notation.[24] **A rule matches a bargaining solution** if for each problem, the allocation it recommends coincides with the payoff vector assigned by the solution to the bargaining game associated with the problem. Our first theorem describes a number of such correspondences.

Theorem 1 *The following matches between rules and bargaining solutions exist:*

(i) *The constrained equal contributions rule and the lexicographic egalitarian solution.*

(ii) *The constrained equal benefits rule and the extended equal loss solution.*

(iii) *The constrained proportional rule and the lexicographic Kalai-Smorodinsky solution.*

(iv) *The priority rule associated with order $\prec$ and the lexicographic dictatorial solution associated with order $\prec$.*

4.2 Coalitional games

We next turn to a class of games that is richer than the class of bargaining games in that what each group of agents—in this context, they are called **coalitions**—can achieve is specified. Let N be a (finite) set of "players". A (transferable utility) **coalitional game** is a list $v \equiv (v(N'))_{N' \subseteq N} \in \mathbb{R}^{2^{|N|}-1}$, where for each $\emptyset \neq N' \subseteq N$, $v(N')$ is the **worth of coalition $\boldsymbol{N'}$**. This

[24]Other choices of disagreement point are plausible.

number is interpreted as whatN' can achieve on its own, although this is by no means the only possible interpretation. Let $\mathcal{V}^N$ be a domain of coalitional games. A **solution on $\boldsymbol{\mathcal{V}^N}$** is a correspondence that associates with each $v \in \mathcal{V}^N$ a payoff vector in $\mathbb{R}^N$ whose coordinates add up to $v(N)$. One of the most important solutions is the one that selects all the efficient payoff vectors such that no coalition can simultaneously provide a higher payoff to each of its members. Formally, the **core of $\boldsymbol{v \in \mathcal{V}^N}$, $\boldsymbol{C(v)}$,** is the set of payoff vectors $x \in \mathbb{R}^N$ such that $\sum x_i = v(N)$ and for each $\emptyset \neq N' \subseteq N$, $\sum_{N'} x_i \geq v(N')$. The core is multi-valued but the next examples are *single-valued*. First, we imagine agents arriving one at a time and we calculate for each of them the **contribution**[25] he makes to the coalition of agents who arrived before him, that is, the difference between the worth of the coalition after he joins it and before he does so. We calculate the average of these contributions assuming that all orders of arrival are equally likely, thereby obtaining for each $i \in N$, the payoff $\frac{1}{|N|!}\sum_{\prec \in \mathcal{O}^N}[v(\{j \in N | j \prec i\} \cup i) - v(\{j \in N | j \prec i\})]$. Collecting terms, the following is an alternative and more familiar expression for the **Shapley value payoff of player $\boldsymbol{i}$ in the game $\boldsymbol{v \in \mathcal{V}^N}$** (Shapley, 1953):

$$Sh_i(v) \equiv \sum_{N' \subseteq N, i \in N'} \frac{(|N'|-1)!(|N|-|N'|)!}{|N|!}[v(N') - v(N' \backslash \{i\})].$$

The next solution is defined by a lexicographic operation analogous to the one underlying the lexicographic egalitarian solution of bargaining theory. Given $N' \subset N$ and $x \in \mathbb{R}^N$, the difference $v(N') - \sum_{N'} x_i$ is the **dissatisfaction of $\boldsymbol{N'}$ at $\boldsymbol{x}$**. This number indicates how well or how badly a given coalition is treated at x.[26] Now, the **nucleolus of $\boldsymbol{v \in \mathcal{V}^N}$, $\boldsymbol{Nu(v)}$** (Schmeidler , 1969) is the set of payoff vectors $x \in \mathbb{R}^N$ at which the vector of dissatisfactions $(v(N') - \sum_{N'} x_i)_{N' \subseteq N}$ is minimized in the lexicographic order among all efficient payoff vectors, starting with the largest dissatisfaction. The nucleolus is *single-valued*.

For the **modified nucleolus** (Sudhölter, 1997), we perform a parallel exercise using the vector of differences of dissatisfactions between two arbitrary coalitions. The modified nucleolus is also *single-valued*.

[25]Note that here this term has a different meaning from the one we have given it up to this point, but which is intended should be clear from the context.

[26]Note that a payoff vector belongs to the core if all of these differences are non-positive.

Our next definition is based on the Lorenz order.[27] We state the special form it takes for an important class of games, namely games exhibiting strongly decreasing "returns to size": a game is **concave** if for each $i \in N$ and each pair $\{N', N''\} \subseteq N$ such that $N'' \subset N'$ and $i \notin N'$, $v(N' \cup \{i\}) - v(N') \leq v(N'' \cup \{i\}) - v(N'')$. The **Dutta-Ray solution outcome of a concave game $\boldsymbol{v \in \mathcal{V}^N}$, $\boldsymbol{DR(v)}$** (Dutta and Ray, 1989) is the payoff vector in the core of v that Lorenz-dominates every other core payoff vector.

Given an airport problem $c \in \mathcal{C}^N$, its **associated coalitional game** is the game $\boldsymbol{v(c)} \in \mathcal{V}^N$ defined by setting, for each $\emptyset \neq N' \subseteq N$, $v(c)(N') \equiv \max_{N'} c_i$. (Other proposals have been made. One is to set, for each $N' \subset N$, $v'(c)(N') \equiv -\max_{N'} c_i$. Another is to set $v''(c)(N') \equiv \sum_{N'} c_i - \max_{N'} c_i$.)

The no-blocking idea underlying the definition of the core remains meaningful but the inequalities have to be reversed. We still refer to the set of allocations satisfying them as the **core**. The core is non-empty and it can be easily described:

Lemma 4 *Given any airport problem, the core of the coalitional game associated with it is its set of contribution vectors satisfying the no-subsidy constraints.*

Clearly, if $x \in X(c)$ satisfies the core constraints, it satisfies the no-subsidy constraints, which are a subset of them. Conversely, given $N' \subset N$, let $\bar{N}'$ be the coalition consisting of all the agents in N whose cost parameter is at most $\max_{N'} c_i$. Then, since $x \in X(c)$ implies $x \geqq 0$, if $\sum_{\bar{N}'} x_i \leq \max_{N'} c_i$, it follows that $\sum_{N'} x_i \leq \max_{N'} c_i$.

Our next observation is that the game $v(c)$ is concave.

Lemma 5 *Given any airport problem, the coalitional game associated with it is concave.*

When a game is concave, its Shapley value payoff vector belongs to its core. In fact, the vertices of the core (the core is a polyhedron, being defined

[27] Given $x \in \mathbb{R}^N$, we denote by $\tilde{x}$ the vector obtained from x by rewriting its coordinates in increasing order. Given x and $y \in \mathbb{R}^N$ with $\sum x_i = \sum y_i$, we say that **$\boldsymbol{x}$ is greater than $\boldsymbol{y}$ in the Lorenz order** if $\tilde{x}_1 \geq \tilde{y}_1$ and $\tilde{x}_1 + \tilde{x}_2 \geq \tilde{y}_1 + \tilde{y}_2$, and $\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 \geq \tilde{y}_1 + \tilde{y}_2 + \tilde{y}_3$, and so on, with at least one strict inequality.

by a system of inequalities), are the payoff vectors whose average defines the Shapley value payoff vector.[28]

A rule matches a solution to TU games if for each problem, the allocation it recommends coincides with the payoff vector assigned by the solution to the coalitional game associated with the problem.

Theorem 2 *The following matches between rules and solutions to TU games exist:*

(i) *The sequential equal contributions rule (equivalently, according to Lemma 1, the random arrival rule) and the Shapley value.*

(ii) *The constrained equal contributions rule and the Dutta-Ray solution.*

(iii) *The constrained equal benefits rule and the modified nucleolus.*

(iv) *The slack maximizer rule and the nucleolus.*

Assertion (i) is due to Littlechild and Owen (1973), assertion (ii) to Aadland and Kolpin (1998), and assertion (iii) to Potters and Sudhölter (1999). Assertion (iv) is a direct consequence of the fact that for each $x \in X(c)$ and each $N' \subset N$, the slack of N' at x is always at least as large as the slack of $\{1, \ldots, \max_{N'} i\}$ at x (this is the smallest set of consecutive agents starting with agent 1 that contains N'), and therefore can be ignored in the maximization defining the nucleolus (recall that each $x \in X(c)$ is non-negative). The nucleolus is in general difficult to calculate as it involves a sequence of nested maximizations. However, in the present context, it can be given an explicit recursive definition, as we have seen (appealing to (iv)). (If instead of the game $v(c)$, we consider the game $v'(c)$ or the game $v''(c)$, then some of these matches are affected.)

5 Axiomatic approach

We now turn to axioms. We distinguish between fixed-population axioms and variable-population ones (for which we will need to generalize the model). For each axiom, Table 1 shows whether each of the rules introduced in Section 3 satisfies it or not. We also offer characterizations.

[28] Also, its kernel and nucleolus coincide. See Shapley (1971).

5.1 Fixed population

We have incorporated in the definition of a rule three requirements. **Non-negativity** says that for each problem, the rule should only pick a non-negative contribution vector; **cost boundedness** that this vector should be bounded above by the cost vector; **efficiency** that its coordinates should add up to the maximal cost. Here are the new properties:

• Agents with equal cost parameters should pay equal amounts:

Equal treatment of equals: For each $c \in \mathcal{C}^N$ and each pair $\{i, j\} \subseteq N$, if $c_i = c_j$, then $S_i(c) = S_j(c)$.

A stronger requirement is that what agents pay should be independent of their names.

• If agent i's cost parameter is at least as large as agent j's cost parameter, he should pay at least as much as agent j does.[29] For a parametric rule, it is satisfied if its schedules are ordered.

Order preservation for contributions: For each $c \in \mathcal{C}^N$ and each pair $\{i, j\} \subseteq N$, if $c_i \geq c_j$, then $S_i(c) \geq S_j(c)$.

The next property is a counterpart for benefits of the previous one (Littlechild and Thompson, 1977)[30].

Order preservation for benefits: For each $c \in \mathcal{C}^N$ and each pair $\{i, j\} \subseteq N$, if $c_i \geq c_j$, then $c_i - S_i(c) \geq c_j - S_j(c)$.

Properties such as *equal treatment of equals* and the generalizations just stated are very natural in many applications, but not always. Indeed, there may be good reasons not described in the model justifying that agents with equal cost parameters not be treated equally. In the irrigation example, different ranchers may use the ditch to irrigate unequal land areas, or the crops they grow may differ in other respects (Aadland and Kolpin, 1998). In general, the profits agents derive from the project may differ. In such

[29] Some form of the property appears in many domains studied in economics and game theory. For claims problems, it is discussed by Aumann and Maschler (1985).

[30] This property appears in Aadland and Kolpin (1998) under the name of "semi-marginalism".

circumstances, the question arises whether and how rules can be redefined so as to accommodate a perceived need to favor certain agents at the expense of others. One answer is to add to the model this extra information. Another is to specify a vector of weights indexed by agents reflecting the greater or lesser relative importance they should receive. If all weights are positive, it is easy to use them to extend the definitions of our basic rules. We consider this case first.

Let Δ^N be the unit simplex in $\mathbb{R}^N$. For the **weighted sequential equal contributions rule with weights $\boldsymbol{w} \in \mathbf{int}\boldsymbol{\Delta}^{\boldsymbol{N}}$** (the notation "int" denotes the interior of a set), divide the cost of each segment among all agents who use it proportionally to their weights.

For the **weighted constrained equal contributions rule with weights $\boldsymbol{w} \in \mathbf{int}\boldsymbol{\Delta}^{\boldsymbol{N}}$**, set payments proportional to the weights and proceed in steps as before (this choice of weights affects the order in which the no-subsidy constraints are reached). For the **weighted constrained proportional rule with weights $\boldsymbol{w} \in \mathbf{int}\boldsymbol{\Delta}^{\boldsymbol{N}}$**, set payments proportional to the weights multiplied by the cost parameters, and here too, proceed in steps. A similar definition of the **weighted constrained equal benefits rule with weights $\boldsymbol{w} \in \mathbf{int}\boldsymbol{\Delta}^{\boldsymbol{N}}$** is possible, but in contrast to the case when all weights are equal, we now have to keep track of the no-subsidy constraints. To see this, let $c \equiv (1,2,3) \in \mathcal{C}^{\{1,2,3\}}$. Also, let $w \equiv (.1,.1,.8)$. Then, the equations $\frac{c_1-x_1}{.1} = \frac{c_2-x_2}{.1} = \frac{c_3-x_3}{.8}$ and $\sum x_i = 3$ give $x = (\frac{7}{10}, \frac{17}{10}, \frac{3}{5})$, and since $x_1 + x_2 > c_2 = 2$, the no-subsidy constraint for the group $\{1,2\}$ is violated.

Let us now turn to the possibility that weights may have zero components. To see the difficulty zero weights cause, we return to the sequential equal contributions rule and note that the proposal described above cannot be used because some segments may not be covered. For instance, let $N \equiv \{1,2,3,4\}$ and $w \equiv (\frac{1}{4}, \frac{3}{8}, \frac{3}{8}, 0)$. Let $c \equiv (1,2,3,4) \in \mathcal{C}^{\{1,2,3,4\}}$. Then c_1 is divided proportionally to the components of w among all four agents, $c_2 - c_1$ is divided among agents 2, 3, and 4 proportionally to the weights $(\frac{3}{8}, \frac{3}{8}, 0)$ (this means that agents 2 and 3 together pay the entire cost of the second segment), and $c_3 - c_2$ is divided between agents 3 and 4 proportionally to the weights $(\frac{3}{8}, 0)$ (this means that agent 3 pays the entire cost of the third segment). When we get to the last segmental cost $c_4 - c_3$, we are left with only one agent and his weight is zero. This difficulty can be remedied by introducing a second weight vector to be used on such occasions. This second vector can have zero components too, so we add a third weight vector and so on. Formally, a **hierarchy of weights** is an ordered list $\{w^1, w^2, \ldots, w^L\} \in \Delta^N$, for some positive integer L, such that for each $\ell \in \{1, \ldots, L\}$, and each $i \in N$, if $w_i^\ell > 0$, then $w_i^{\ell+1} = 0$. To illustrate, let $N \equiv \{1, 2, 3, 4, 5, 6\}$, $w^1 \equiv (\frac{1}{4}, \frac{3}{8}, \frac{3}{8}, 0, 0, 0)$, and $w^2 \equiv (0, 0, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Let $c \in \mathcal{C}^N$ be such that $c_1 \leq \cdots \leq c_6$. Then, c_1 is divided among the first three agents proportionally to their components of w^1, $c_2 - c_1$ is divided between agents 2 and 3 proportionally to their components of w^1, and $c_3 - c_2$ is entirely covered by agent 3. Then, we switch to the second weight vector. The fourth segmental cost, $c_4 - c_3$ is divided among agents 4, 5, and 6 proportionally to their components of w^2, $c_5 - c_4$ is divided between agents 5 and 6 proportionally to their components of w^2, and $c_6 - c_5$ is covered entirely by agent 6. If c is such that $c_5 < c_1 < c_2 < c_3 < c_6 < c_4$, say, c_5 is divided among agents 1, 2, and 3, according to their components of w^1 (they are the only agents using this segment who have positive weights in w^1), and so is $c_1 - c_5$; $c_2 - c_1$ is divided between agents 2 and 3 proportionally to their components of w^1 (they are the only agents using this segment who have positive weights in w^1); $c_3 - c_2$ is paid entirely by agent 3 (he is the only agent using this segment who has a positive weight in w^1). The penultimate segmental cost $c_6 - c_3$ is divided between agents 4 and 6 proportionally to their weights in w^2; the last segmental cost $c_4 - c_6$ is covered entirely by agent 4, as he is only one left.[31]

Any such rule can alternatively be described in terms of an ordered partition of the set of agents, and a list of positive weights, one for each agent. For each problem, we handle each segment in turn by choosing the contributions of the members of the first component of the induced partition who use it proportionally to their weights. If there is no one in the first component of the induced partition, we divide the cost of the segment among the members of the second component of the induced partition who use it proportionally to their weights, and so on.

We now turn to relational requirements on rules.

• If the cost vector is multiplied by a positive scalar, so should the chosen allocation:[32]

Homogeneity: For each $c \in \mathcal{C}^N$ and each $\alpha \in \mathbb{R}_{++}$, $S(\alpha c) = \alpha S(c)$.

[31] Note that we switch to the second weight vector when we reach a segment that is used by agents who are all assigned zero weights by the first weight vector.

[32] This is the first part of the property Potters and Sudhölter (1999) call "covariance".

• The chosen allocation should vary continuously with the data of the problem.

Continuity: For each sequence $\{c^\nu\}$ of elements of $\mathcal{C}^N$ and each $c \in \mathcal{C}^N$, if $c^\nu \to c$, then $S(c^\nu) \to S(c)$.

• Next is an independence property: what an agent pays should not depend on the costs of the segments he does not use.

Independence of at-least-as-large costs: For each pair $\{c, c'\}$ of elements of $\mathcal{C}^N$ and each $i \in N$, if (i) $c'_i = c_i$, (ii) for each $j \in N \backslash \{i\}$ such that $c_j < c_i$, we have $c'_j = c_j$ and (iii) for each $j \in N \backslash \{i\}$ such that $c_j \geq c_i$, we have $c'_j \geq c_j$, then $S_i(c') = S_i(c)$.

The sequential full contributions rule fails this condition but it satisfies the slightly weaker version of the property obtained by only allowing those cost parameters that were initially greater than agent i's cost parameter to vary, provided they remain greater than agent i's cost parameter (a property that could be called "independence of greater costs").

We are now ready to present our first characterization:[33]

Theorem 3 (Moulin and Shenker, 1992) *The sequential equal contributions rule is the only rule satisfying equal treatment of equals and independence of at-least-as-large costs.*

• Next is a limited version of the above independence property: if the cost parameter of the last agent increases (or if the cost parameter of any one of the agents whose parameter is the largest increases, if there are several of them), the contributions of the other agents should not be affected and his contribution should increase by an equal amount:[34]

Last-agent cost additivity: For each pair $\{c, c'\}$ of elements of $\mathcal{C}^N$, each $\gamma \in \mathbb{R}_+$, and each $i \in N$ with $c_i = \max_N c_j$, if (i) $c'_{N\backslash\{i\}} = c_{N\backslash\{i\}}$ and (ii) $c'_i = c_i + \gamma$, then $S_{N\backslash\{i\}}(c') = S_{N\backslash\{i\}}(c)$ and $S_i(c') = S_i(c) + \gamma$.

[33] *Efficiency* does not appear in any of our characterizations as it is incorporated in the definition of a rule. Remark 5.6 of Potters and Sudhölter (1999) essentially amounts to Theorem 3. They impose an axiom of monotonicity, which implies our independence axiom, and only exploit its independence content.

[34] This is the second part of a property Potters and Sudhölter (1999) call "covariance".

The sequential full contributions rule violates this property but only when there are several agents whose cost parameters are the largest.

A weaker property says that, under the same hypotheses, the payment required of the last agent should increase by an amount equal to the increase in his cost parameter, nothing being said about the payments required of the others. Let us refer to it as **weak last-agent cost additivity**.

• Here is our last property in the series: If two problems for which agents are ordered in the same way are added—of course in the sum problem, their order remains the same—the allocation chosen for the sum problem should be the sum of the allocations chosen for each of them. This second vector may be interpreted as representing cost overruns. Or the cost of each segment may consist of several parts, land, material, maintenance, and the order restriction appearing in the hypotheses means that these component costs are all ordered in the same way, as is the case in many applications. In the case of airports, we may think of two facilities being built at the same time, and all airplanes will use both.

Conditional cost additivity: For each pair $\{c, c'\}$ of elements of $\mathcal{C}^N$ for which agents are ordered in the same way, $S(c + c') = S(c) + S(c')$.

This property implies *last-agent cost additivity*, since the problems c and c' considered there is obtained from the problem c by adding the trivial problem $(0, \ldots, 0, \gamma)$, for which, by definition of a rule, we have to select the allocation $(0, \ldots, 0, \gamma)$.

Our next main result is another characterization of the sequential equal contributions rule:

Theorem 4 (Dubey, 1982) *The sequential equal contributions rule is the only rule satisfying* equal treatment of equals *and* conditional cost additivity.

An "unconditional" additivity requirement would not make sense, since the amount to be collected is not an additive function of the problem. Indeed, let $c \equiv (1, 2) \in \mathcal{C}^{\{1,2\}}$ and $c' \equiv (2, 1) \in \mathcal{C}^{\{1,2\}}$. Then, if $x \equiv S(c)$, we have $\sum S_i(c) = 2$. Similarly, if $x' \equiv S(c')$, we have $\sum S_i(c') = 2$. Note that $\sum(S_i(c) + S_i(c')) = 4$. However, $c + c' = (3, 3)$. Thus, $\sum S_i(c + c') = 3$.

• Our next requirements are monotonicity requirements. First, if the i-th cost parameter increases, agent i should pay at least as much as he did initially. For the application to the airport problem, this increase may be due to the i-th airline switching to larger planes.[35]

Individual cost monotonicity: For each pair $\{c, c'\}$ of elements of $\mathcal{C}^N$ and each $i \in N$, if $c'_i \geq c_i$ and for each $j \in N \backslash \{i\}$, $c'_j = c_j$, then $S_i(c') \geq S_i(c)$.

• Recalling the public good character of the facility, note that if agent i is not the last agent, any increase in his contribution is beneficial to the other agents as a group. Let us require that if an agent's cost parameter increases, each of the others should pay at most as much as what he did initially.

Others-oriented cost monotonicity: Under the hypotheses of *individual cost monotonicity*, for each $j \in N \backslash \{i\}$, $c'_j = c_j$, then $S_i(c') \geq S_i(c)$.

• Suppose the cost vector changes and consider some agent i. In the irrigation application, where $c_i - c_{i-1}$ is interpreted as the cost of maintaining the section of the ditch passing through rancher i's property, an increase in this cost will bring about an equal increase in the cost parameter of each rancher who comes after him. Then, one could ask that he and each of these ranchers should pay at least as much as they did initially.

Downstream cost monotonicity: For each pair $\{c, c'\}$ of elements of $\mathcal{C}^N$, and each $i \in N$, if (i) for each $j \in N$ such that $c_j < c_i$, $c'_j = c_j$ and (ii) for each $j \in N$ such that $c_j \geq c_i$, $c'_j - c_j = c'_i - c_i \geq 0$, then for each $j \in N$ such that $c_j \geq c_i$, $S_j(c') \geq S_j(c)$.

• Under the same hypotheses, we require that each agent who comes before him should pay the same amount as he did initially.

Marginalism: Under the hypotheses of *downstream cost monotonicity*, for each $j \in N$ such that $c_j \leq c_i$, $S_j(c') = S_j(c)$.

• If the cost vector changes in such a way that each segmental cost ends up at least as large as it was initially, each agent should pay at least as much as he did initially.[36] Such an increase in cost can be thought of as coming from the addition of another problem for which agents are ordered in the same way. Thus, the requirement can also be written in the following way:

[35] This property is called "monotonicity in costs" by Potters and Sudhölter (1999).

[36] This property appears in Aadland and Kolpin (1998) under the name of "cost monotonicity".

Weak cost monotonicity: For each pair $\{c, c'\}$ of elements of $\mathcal{C}^N$ such that $c' = c + c''$ for some $c'' \in \mathcal{C}^N$, $S(c') \geq S(c)$.

The next result focuses on the difference between the largest and smallest contributions required of the agents.[37]

Theorem 5 (Combining Definition 3.2 and Theorem 3.1 of Aadland and Kolpin, 1998) *Among all selections from the no-subsidy correspondence satisfying* order preservation for contributions *and* weak cost monotonicity, *the constrained equal contributions rule is the only rule achieving the smallest difference between largest and smallest contributions for each problem.*

The strong solidarity requirement that if all cost parameters increase, each agent should pay at least as much as he did initially is incompatible with the no-subsidy constraints and *equal treatment of equals.* Indeed, let $c \in \mathcal{C}^N$ and $x \equiv S(c)$. Let $c' \in \mathcal{C}^N$ be such that for each $i \in N$, $c'_i = \max c_j$, and $x' \equiv S(c')$. Since the amount to be collected remains the same, the monotonicity requirement implies $x' = x$. By *equal treatment of equals*, all components of x' are equal. Thus, the same statement holds for x. However, the agent with the smallest cost parameter in c may be required to pay more than his cost parameter then.

• Still under the assumption that all cost parameters change in such a way that each segmental cost ends up at least as large as it was initially, the next property places an upper bound on increases in contributions: for each agent, the sum of the increases in the contributions required of him and of all agents who precede him should be no greater than the increase in his cost parameter:[38]

Incremental no subsidy: For each pair $\{c, c'\}$ of elements of $\mathcal{C}^N$ such that $c' = c + c''$ for some $c'' \in \mathbb{R}^N_+$, and for each $i \in N$, $\sum_{j \in N: c_j \leq c_i} (S_j(c') - S_j(c)) \leq c'_i - c_i$.

Theorem 6 (Theorem 3.4 of Aadland and Kolpin, 1998) *The sequential equal contributions rule is the only rule satisfying* order preservation for contributions, weak cost monotonicity, *and* incremental no-subsidy.[39]

[37] Theorem 5 bears some similarity to characterizations of the uniform rule offered by Schummer and Thomson (1997).

[38] This property is introduced by Aadland and Kolpin (1998).

[39] Aadland and Kolpin (1998) also impose *order preservation for benefits*, as in their formulation, this axiom is not implied by the others.

Next, we examine another way of assessing how evenly the total cost is collected from agents with different cost parameters. We focus on the smallest contribution required from anyone, and then on the largest contribution.[40] It is intuitive from their definitions that the constrained equal contributions and sequential equal contributions rules favor agents at opposite ends of the cost distribution. This intuition is confirmed by the following theorems:

Theorem 7 (Theorem 3.3 of Aadland and Kolpin, 1998) *Among all selections from the no-subsidy correspondence satisfying* order preservation for contributions *and* weak cost monotonicity, *the constrained equal contributions rule is the only rule minimizing the largest contribution for each problem.*

Theorem 8 (Theorems 3.5 and 3.6 of Aadland and Kolpin, 1998)
(a) Among all rules satisfying order preservation for contributions, weak cost monotonicity, *and* incremental no-subsidy, *the sequential equal contributions rule is the only rule minimizing the largest contribution for each problem.*
(b) Among all rules satisfying order preservation for contributions, order preservation for benefits, *and* weak cost monotonicity, *the sequential equal contributions rule is the only rule maximizing the largest contribution for each problem.*

5.2 Variable population

In this section, we allow the population of agents to vary and we formulate axioms designed to ensure the good behavior of rules in such circumstances. For this purpose, we need to generalize the model. We imagine that there is an infinite set of "potential" agents, indexed by the natural numbers $\mathbb{N}$. In each given problem, however, only a finite number of them are present. Let $\mathcal{N}$ be the class of finite subsets of $\mathbb{N}$. An **airport problem** is defined by first specifying the population of agents involved, some $N \in \mathcal{N}$, then a cost vector $c \in \mathbb{R}^N_+$. We still denote by $\mathcal{C}^N$ the class of problems with agent set N. A **rule** is a function defined over $\bigcup_{N\in\mathcal{N}} \mathcal{C}^N$, which associates with each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$ a cost allocation for c. We denote the restriction of a rule S to the subdomain of problems with agent set $N \in \mathcal{N}$ by $S_{|\mathcal{C}^N}$.

[40]Aadland and Kolpin (1998) refer to this criterion as Rawlsian.

• We will discuss two main ideas in the context of a variable population, a monotonicity idea and an invariance idea. The first one is the expression of the general objective of solidarity that has already provided the motivation for *strong cost monotonicity*, but this time it is applied to changes in population. Solidarity says that when a change occurs, whether it is socially desirable or not, if no one in particular bears any responsibility for it, the welfares of all agents who are present before and after the change should be affected in the same direction. In the present context, the arrival of a new agent whose cost parameter is no greater than the cost parameter of any agent already present can only be beneficial to them: one more potential contributor to a project whose cost has not changed is good news. If the new agent's cost parameter is greater than the initial greatest cost parameter, the amount to be collected increases, and solidarity implies that each agent initially present should pay at least as much as he did initially, or that each should pay at most as much as he did initially. However, in the presence of the no-subsidy constraints, which imply that the agents initially present should bear none of the additional cost that is incurred for the new agent to be served, it makes sense to require, once again, that each of these agents should pay at most as much as he did initially.[41]

Population monotonicity: For each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $N' \subset N$, we have $S_{N'}(c) \leqq S(c_{N'})$.

It follows directly from their definitions that most of the rules of Section 3 are *population monotonic*. The slack maximizer rule is one of them but this conclusion cannot be obtained by exploiting the known properties of the nucleolus as a solution to TU games. Indeed, this solution is not *population monotonic*, even on the domain of concave games (Sönmez, 1994). Nevertheless, the domain of airport problems is a subdomain of the domain of concave games, and on it, this property holds (Sönmez, 1994).

For the priority rules, we cannot check the property unless they are properly redefined to accommodate variations in populations. What is needed here is an order for each finite set of agents. It is of course natural that these orders be related. Preferential treatment of an agent over some other agent in some group should carry over to any group to which they both belong. The most practical way to achieve this is to specify a "reference" order on the

[41] The property first appears, in the context of bargaining, in Thomson (1983a,b). For a survey of the literature devoted to its study, see Thomson (1995).

set of potential agents; then, for each problem, to work with the order that the reference order induces on the set of agents who are actually present. If this property is imposed, then *population monotonicity* does hold.

• Next, we turn to our invariance property. It says that the recommendation made for any problem should always be in agreement with the recommendation made for any problem obtained by imagining some agents leaving with their payoffs, and "reassessing the situation" from the viewpoint of the remaining agents.[42] By this phrase, we mean defining a new problem involving the remaining agents, and whose cost vector is recalculated so as to take into consideration the fact that some payments have already been made. In contrast to other models of fair allocation, for which a unique definition usually stands out as most natural, there are several ways of defining this reduced problem. Informally, this is because what has to be divided is not a homogeneous whole (such as a social endowment), but it is composed of segments used differently by different agents. When an agent leaves, instead of thinking of his contribution as being an abstract part of the total cost, it is appealing instead to impute it to these various segments. But how should these imputations be defined? We propose several answers. In each of them, the imputation is to the segments the agent uses, which seems very natural. For the first proposal, we only consider the departure of the agent with the lowest cost parameter, so there is nothing else to specify.

For each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $i \in N$ with $c_i = \min c_j$, let $x \equiv S(c)$ and $r^x_{N\backslash\{i\}}(c)$ be the problem with agent set $N\backslash\{i\}$ defined by setting, for each $j \in N\backslash\{i\}$, agent j's cost parameter equal to $\max\{c_j - x_i, 0\}$, which is equal to $c_j - x_i$ since by definition of a rule, $x_i \leq c_i$. Thus, $r^x_{N\backslash\{i\}}(c) \in \mathcal{C}^{N'}$.

First-agent consistency: For each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $i \in N$ with $c_i = \min c_j$, if $x \equiv S(c)$, then $x_{N\backslash\{i\}} = S(r^x_{N\backslash\{i\}}(c))$.

First-agent consistency seems to be a very weak requirement, perhaps too weak to be of much interest. However, if a rule satisfies *first-agent consistency*, then by repeated application of the property, we find that the allocation the rule chooses for any problem is invariant under the departure of an arbitrary group of consecutive agents that includes the first agent. Recall that one of the applications of the model is when the cost parameters correspond to different temporal needs that agents have for a service.

[42] The idea has been the object of a large number of studies, reviewed in Thomson (2005).

In a temporal model, the departure of agents has a particularly appealing interpretation. It means that the agents who are done first leave first.[43,44]

• Next, we consider the departure of an arbitrary agent. First, we think of his contribution as being intended to help cover the part of the project that he (and his successor(s) if any) use but his predecessors do not; unless of course his contribution is larger than that segmental cost, in which case, part of his contribution (the difference between these two numbers) would have to be thought of as intended to help cover the part of the project that he, (his successor(s) if any), and his immediate predecessor use; unless his contribution is larger than these two preceding segmental costs, in which case part of his contribution (the difference between his contribution and the sum of the two preceding segmental costs) would have to be thought of as intended to help cover the part of the project that he, (his successor(s) if any), and his two immediate predecessors use; and so on. In defining the reduced problem, this amounts to decreasing the cost parameters of all agents coming after him by his contribution, and to possibly decrease in succession the cost parameters of the agents coming just before him, starting with the closest ones. Given $N \in \mathcal{N}$, and $c \in \mathcal{C}^N$, let $x \equiv S(c)$. Let $i \in N$. The **downstream-subtraction reduced problem of $\boldsymbol{c}$ with respect to $\boldsymbol{N' \equiv N \backslash \{i\}}$ and $\boldsymbol{x}$**, $d^x_{N'}(c)$, is the problem with agent set N' and cost vector $c' \in \mathbb{R}^{N'}_+$ defined by

1. For each $j \in N'$ such that $c_j < c_i$, $c'_j \equiv \min\{c_j, c_i - x_i\}$,
2. For each $j \in N'$ such that $c_j \geq c_i$ $c'_j \equiv c_j - x_i$.

The requirement is that in the reduced problem, each agent should pay what he was initially asked to pay:[45]

Downstream-subtraction consistency: For each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $N' \subset N$, if $x \equiv S(c)$, then $x_{N'} = S(d^x_{N'}(c))$.

[43] The natural form taken by *consistency* in this context is discussed by Thomson (1992).

[44] All three properties are proposed by Potters and Sudhölter (1999).

[45] The amount to be collected is the largest cost parameter, so one should pay special attention to the last agent leaving. By the no-subsidy constraints, the last agent pays at least the last segmental cost, so if he is the one to leave, the sum of the payments required of the remaining agents is equal to the new largest cost parameter. Thus, there is no restriction as to who can leave, by contrast to the next definition.

For our second definition of *consistency*, we take the opposite viewpoint, and impute an agent's contribution to the initial segments. Then, each agent benefits from it. However, we do not consider the possibility that the agent who leaves is the one whose cost parameter is the largest (unless he is not the only one with the largest cost parameter), as this parameter determines the cost that has to be covered, and the new cost to be covered would have no reason to be related to the sum of the contributions required of the remaining agents. Let $i \in N \setminus \{n\}$. The **uniform-subtraction reduced problem of $\boldsymbol{c}$ with respect to $\boldsymbol{N' \equiv N\setminus\{i\}}$ and $\boldsymbol{x}$**, $ur^x_{N'}(c)$, is the problem with agent set N' and cost vector $c' \in \mathbb{R}^{N'}_+$ defined by

1. For each $j \in N'$ such that $c_j < c_i$, $c'_j \equiv \max\{c_j - x_i, 0\}$,
2. For each $j \in N'$ such that $c_j \geq c_i$, $c'_j \equiv c_j - x_i$.

Here too, the requirement is that in the reduced problem (which is indeed well-defined), each agent should pay what he was initially asked to pay.

Uniform-subtraction consistency: For each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $i \in N$, if agent i is not the unique agent such that $\max c_j = c_i$, if $x \equiv S(c)$, then $x_{N\setminus\{i\}} = S(ur^x_{N'}(c))$.

Our next two results are characterizations of two of our central rules. Remarkably, they differ mainly in which version of *consistency* is imposed:[46]

Theorem 9 (Potters and Sudhölter, 1999) *The constrained equal benefits rule is the only rule satisfying* equal treatment of equals, homogeneity, last-agent cost additivity, *and* uniform-subtraction consistency.

Theorem 10 (Potters and Sudhölter, 1999) *The slack maximizer rule is the only rule satisfying* equal treatment of equals, homogeneity, last-agent cost additivity, *and* downstream-subtraction consistency.

The proofs of these characterizations exploit correspondences between airport problems and solutions to TU games but a version of Theorem 10 is available that is based on axioms that almost entirely focus on the last

[46]Potters and Sudhölter impose "covariance", the conjunction of *homogeneity* and *last-agent cost additivity*, but the former property is redundant.

agent and whose proof involves no concept or technique of the theory of TU games. Recall that *weak last-agent cost additivity* is obtained from *last-agent cost additivity* by dropping from the conclusion the requirement that the contributions made by the agents whose cost parameters remain the same should not change. Now, call **last-agent consistency** the requirement obtained from *downstream-substraction consistency* by applying it only to the departure of the last agent.[47]

Theorem 11 (Yeh, 2003) *The slack maximizer rule is the only rule satisfying equal treatment of equals, weak last-agent cost additivity, and last-agent consistency.*

The final result involve one additional property, which is the weakening of *independence of at least-as-large costs* obtained by applying it only to the first agent.

Theorem 12 (Chun, Kayı, and Yeh, 2006) (a) *The sequential equal contributions rule is the only rule satisfying equal treatment of equals, first-agent independence of at least-as-large costs, and first-agent consistency.*

(b) *It is the only rule satisfying the identical-cost lower bound, others-oriented cost monotonicity, and first-agent consistency.*

• Based on notions of *consistency* for coalitional games, Albizuri and Zarzuelo (2006) propose two alternative definitions of a reduced problem, and state characterizations of the sequential equal contributions and slack maximizer rules. The requirements on a rule are simply coincidence with the two-agent version of these rules (recall that they agree in that case) and either form of *consistency*.

• Another way of applying the idea of consistency involves generalizing the class of problems under investigation. This generalization is an adaptation to the present model of an idea developed in the context of the classical model of exchange when agents have individual endowments. An **airport problem with transfer** is a pair $(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}$ such that $E \leq \max c_i$, interpreted as follows: as before, N is the set of agents, and for each $i \in N$, c_i is the cost of the facility needed to satisfy agent i. In addition, E is a (positive or negative) transfer from an outside source to the group intended to help

[47]We drop the reference to "downstream-subtraction" since no agent is downstream of the last agent.

them cover the cost of the project they choose if positive, and interpreted as a tax if negative. Let $\mathcal{T}^N$ be the domain of all such problems.[48] An **allocation for $(c, E) \in \mathcal{T}^N$** is a vector $x \in \mathbb{R}^N$ such that $0 \leqq x \leqq c$ and $\sum x_i + E = \max c_i$.

When some agents leave with their payoffs, *consistency* now takes a very simple form. In defining the reduced problem, the contribution made by each agent who leaves is simply subtracted from the transfer parameter. Let $\tilde{r}^x_{N'}(c, E) \equiv (c_{N'}, E - \sum_{N \setminus N'} x_i)$ be our notion of a reduced problem.

Consistency for airport problems with transfers: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{T}^N$, and each $N' \subset N$, if $x \equiv S(c, E)$, then $x_{N'} = S(\tilde{r}^x_{N'}(c, E))$.

We see several natural ways of redefining the rules introduced in Section 3 depending upon whether one thinks of the transfer as intended for the entire set of agents, or for any group of agents, or whether it is intended to be distributed equally among all agents (subject to no agent's subsidy exceeding his cost parameter), or by means of some other formula (an alternative that comes to mind is proportional division to the individual costs). Suppose that the first approach is adopted. Then, consider the sequential equal contributions rule. For each order in which agents may arrive, the contribution required of each agent when he arrives is set equal to what it was according to the original definition if the agent is not the last one; otherwise it is equal to the total cost minus the sum of the contributions already made by all other agents *and* the transfer. If the second approach is adopted, the transfer term is subtracted whether or not the agent is the last one to arrive. For the constrained equal benefits rule, two parallel choices are available depending upon how the no-subsidy constraints are revised.

To each choice of a revision of the no-subsidy constraints comes a natural formulation of the reduction operation. For the first choice, we only change the worth of the grand coalition in the reduced problem (subtracting from the cost that has to be covered by a coalition the contributions made by the agents who leave). For the second one, we perform the subtraction for all coalitions.

[48] This concept is the counterpart for airport problems of the concept of a generalized economy (Thomson, 1992). A model of cost allocation in which a production function is explicitly specified includes this model as a special case, by allowing the production function not to take the value 0 at 0 (Kolpin, 1998).

6 Cost allocation on trees.

In this section we generalize the model to trees. Think of a road network linking all agents to a central place where they get supplies. Then, instead of being linear, the network whose cost has to be shared has a "tree structure". A **tree** is a graph that is connected and has no cycle. There is also a distinguished node called the **root** of the tree, the other extremal notes being called **leaves**. Nodes are labelled by agents in the set N. The root is given a separate label. The cost of the direct path from a node to the root is the cost of serving the agent at the node. If an agent is on the path from some other agent to the root, its cost parameter is at most as large as the cost parameter of that second agent. A pair (T, c) as just described is a **tree problem**. The notion of a **segment** is as before. The cost of a tree T, denoted $c(T)$, is the sum of the costs of the segments of which it is composed. A **subtree** of a tree T is a subset of T that is also a tree. A **rooted subtree** of T is a subtree of T that includes the root. The cost of servicing a group of agents is the cost of the smallest rooted subtree that contains all the agents in the group, namely the costs of the segments of which this subtree consists. An **allocation for a tree problem $(\boldsymbol{T}, \boldsymbol{c})$** is a vector $x \in \mathbb{R}^N$ whose coordinates, indexed by agents, satisfy $0 \leqq x \leqq c$ and $\sum_N x_i = c(T)$.

Gellekom and Potters (1997) and Koster, Molina, Sprumont, and Tijs (2001) consider such a formulation and calculate for it the counterparts of the sequential equal contributions and constrained equal benefits rules. The logic is transparent once the counterparts of the no-subsidy constraints themselves are defined. An allocation satisfies the **no-subsidy constraints** for $(T, c) \in \mathcal{T}^N$ if no group of agents pays in total more than what is required to satisfy its needs. We consider in turn each of the rules defined for airport problems and discuss how to generalize its definition to trees.

• **Sequential equal contributions rule** (Koster *et al.*, 2001). We still divide equally the cost of each segment among all the agents who use it. The contribution required of each agent is the sum of these partial payments. The **priority rules** and the **random arrival rule** are defined as for airport problems. The sequential equal contributions rule is still an average of the priority rules and it still coincides with the random arrival rule.

• **Constrained equal contributions rule** (Koster *et al.*, 2001). For each rooted subtree, calculate the per capita cost of the subtree. Find the largest rooted subtree with minimal per capita cost. Remove it and assign to each agent in the subtree this per capita cost. What remains of the original tree is a union of disjoint subtrees. The root of each of them is a leaf of the subtree that has been removed. Also, the per capita cost of any rooted subtree of the second stage is greater than the minimal per capita cost identified in the first stage. (If a second-stage rooted subtree had a lower per capita cost, by concatenating it to the rooted path from which it emerges, we would obtain a rooted subtree of the original tree whose per capita cost is smaller than the per capita cost of the first stage, which therefore would not be the smallest). Repeat the operation for each of these subtrees.

Koster *et al* (2001) also define extensions of the **sequential equal contributions** and **constrained equal contributions rules** designed to accommodate asymmetric treatments of agents, by introducing weights. They show that as the weights vary, the set of allocations that result is equal to the core. The characterizations of these rules presented above extend.

• **Constrained proportional rule**. For each rooted subtree, the process is similar to the definition of the previous rule except that at each stage, we calculate the proportional contributions for the subtree. That is, if there is one branch that splits up into two and the three costs are 5, 7, and 10, the proportions required of individual costs to cover the three subtrees are $\frac{5}{5}$, $\frac{5}{5+7}5$, and $\frac{5}{5+7}7$. The smallest of these numbers is the second one.

• **Constrained equal benefits rule**. Let $\lambda \in \mathbb{R}_+$ be such that $\sum \max\{c_i - \lambda, 0\} = c(T)$. For each $i \in N$, let $x \equiv \max\{c_i - \lambda, 0\}$. The no-subsidy constraints are met at x. This can be shown by induction on the number of branches of the tree.

Van Gellekom and Potters (1997) introduce axioms pertaining to possible changes in the structure of the tree, such as deleting links between nodes whose cost is zero, or deleting agents. They also study *consistency* issues and base on these various axioms a characterization of a one-parameter family of rules. This family, previously discussed in the context of linear facilities, connects the constrained equal contributions and slack maximizer rules (see the paragraph following the introduction of this rule).

7 Empirical studies

Situations close to the ideal theoretical model studied in the foregoing pages exist in the real world and it is useful to understand the sort of arrangements that people have made to deal with them. An interesting example is irrigation. Aadland and Kolpin (2004) survey ranchers distributed along a large number of irrigation ditches in Montana, in an attempt to determine whether environmental factors help anticipate the rules they use. They argue that a central property that distinguishes between rules is whether what a given agent pays is affected by the cost parameters of downstream agents, and identify criteria that can be used to predict which of these rules will tend to prevail.

8 Extensions of the model

We conclude this survey by indicating a number of directions for future research. Some have been the object of some initial work but in most cases, many interesting questions remain. They concern extending the rules defined for the basic model, reformulating the axioms for these extensions, introducing new axioms that reflect additional considerations emerging from the generalization, and understanding their implications.

1. Several airports. Suppose that several airports have to be built, each airline possibly using several of them. When defining *additivity* we have essentially enlarged the problem in this way, but an assumption there was that all agents use all facilities. A more general formulation would allow each agent to use only some of the facilities. Dubey (1982) considers such an extension of the model and characterizes the sequential equal contributions rule along the line of Shapley's (1953) original characterization of the Shapley value. This result constitutes a slight generalization of Theorem 4.

2. Accommodating profits. In addition to the cost parameters, we may also have information about the profit that each agent derives from the facility. How should this information be taken into account? This extension of the model is proposed by Littlechild and Owen (1976) and further studied by Brânzei, Iñarra, Tijs, and Zarzuelo (2006) (for a discussion of the relation, see Arin, 2004). They develop an algorithm to calculate the nucleolus of the associated game.

3. Accommodating unions. Another way to enrich the model is to imagine that planes belong to airlines, and to look for a division of the cost between airlines first, then planes. This approach is taken by Vásquez-Brage, van den Nouweland, and García-Jurado (1997). They show that if the value for coalitional games defined for games with "coalition structures" defined by Owen (1977) is used to allocate cost, then airlines gain by merging. They also provide a characterization of the Owen value for the model they formulate.

4. Accommodating transfers from outside sources. This concept was already discussed in Section 5.2 in connection with *consistency*, but additional properties with respect to change in this parameter can also be formulated, such as monotonicity with respect to this transfer, or composition properties (for instance, invariance with respect to whether the transfer is made in one or several installments).

5. Accommodating crowding effects. Suppose that the cost of each segment depends on the number of agents who use it. For each $n \in \mathbb{N}$, we specify a function giving the cost of a runway of length ℓ when n agents use it. This cost may also be written as a function of the numbers of planes of each type using it, as opposed to their sum. How should such external effects be taken into account?

6. Accommodating incentives. The length of the service demanded by an agent depends on how much he will be charged. So the choice of which rule to use has an impact on the cost parameters defining the problem to be solved. A recent strategic analysis of the problem is due to Arin, Iñarra, and Luquin (2006).

7. Accommodating general networks. Instead of assuming that the facility whose cost has to be covered is a line or a tree, a realistic and more general case is when it is a general graph, and the formal model includes cost information on the link between any two agents. Although efficiency dictates that the links that are used to connect all agents to the source will have a tree structure, the possibility now exists of assessing agents as a function of the costs of links that are not used, and relational axioms can be formulated involving all links. The literature on the subject is too extensive to be described here. Recent contributions are by Dutta and Kar (2004) and Bergantiños and Vidal-Puga (2004).

8. Min problems. Consider a group of agents (municipalities) considering building a facility that they will jointly use. For each agent, there is a cost of building it. Efficiency dictates that the facility be built where it is the cheapest. Now, the problem is to divide the minimal coordinate of the cost vector. The class of games is defined in Thomson (2006).

9. Sequential rules. Let us generalize the concept of a rule, as a mapping that specifies, for each agent, an itemized list of contributions, one for each of the segment he uses. His total contribution is the sum of these numbers. The sequential equal contributions rule is defined in this way, but the general concept seem to be worth studying. The concept is proposed by Thomson (2006).

Prop \rules	Seq eq cont	Seq full cont	Constr eq cont	Constr eq benef	Constr prop	Slack max'zer	Priority rule
	SEC	SFC	CEC	CEB	CP	SM	$D^{\prec}$
Cost boundedness	+	+	+	+	+	+	+
No-subsidy	+	+	+	+	+	+	+
Id-cost lower bound	+	−	+	−	−	−	−
Equal treat equal	+ 3 , 4	+	+	+ 9	+	+ 10 , 11	−
Order pres for contrib	+ 6	−	+	+	+	+	−
Order pres for benefits	+ 6	+	+	+	+	+	−
Homogeneity	+	+	+	+ 9	+	+ 10	+
Continuity	+	− (t)	+	+	+	+	+
Ind of at-least-as-large costs	+ 3	− (t)	−	−	−	−	+
Last-agent cost add	+	− (t)	−	+ 9	−	+ 10	+
Weak last-agent cost add	+	− (t)	−	+	−	+ 11	+
Cond cost additivity	+ 4	− (t)	−	−	−	−	+
Individual cost mon	+	− (t)	+	+	+	+	+
Others-oriented cost mon	+	− (t)	+	+	+		+
Downstream cost mon	+	+	+	+	+	+	+
Weak cost mon	+ 6	+	+	−	−	+	+
Incremental no-subsidy	+ 6	− (t)	+	+	+	+	+
Population mon	+	+	+	+	+	+	+
First-agent cons	+	+	+	+	−	+	+
Uniform sub-cons	−	−	−	+ 9	−	−	−
Downstream-sub cons	−	+	+	−	−	+ 10	+
Last-agent cons	−	+	+	−	−	+ 11	+
Barg game version			*Lexi E*	XEL	*Lexi KS*		$D^{\prec}$
TU game version	*Shapley*		$Dutta - Ray$	*Mod Nuc*		$Nucleolus$	$Dic^{\prec}$

Table 1: Showing which properties the main rules satisfy. A "+" in a cell means that the property in the row is satisfied by the rule indexing the colum. A "−" means the opposite. The numbers refer to characterizations. The number 3, for instance, appearing in the first column marks the axioms appearing in Theorem 3, a characterization of the sequential equal contributions rule. The notation t next to a $-$ sign in the column for the sequential full contributions rule indicates that the property is only violated when there are ties between cost parameters.

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