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Abstract

Working in a complete-markets setting, a property of asset demands is identified that is inconsistent with the investor's preference being based on probabilities. In this way, a market counterpart of the Ellsberg Paradox is provided.

1. INTRODUCTION

The Ellsberg Paradox and related evidence (see [1] for a survey) shows that decision-makers may not use probabilities to represent likelihoods and guide choice in settings where information is imprecise. In such environments, where ambiguity prevails in addition to risk, nonindifference (aversion, for example) to ambiguity is inconsistent with the global reliance on probabilities. Though this evidence is confined to experimental settings, it is highly intuitive. For this reason and because it is intuitive also that the phenomenon of ambiguity aversion may be important in market settings, the Ellsberg Paradox has stimulated the development of models of preference that are not based on probabilities as well as applications of these models to asset market settings.

However, this literature has not addressed the following question that is the focus here: Is there behavior in a market setting that is inconsistent with the decision-maker's preference being based on probabilities? The intention is to provide a *market counterpart of the Ellsberg Paradox*.

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To be more precise, define 'based on probabilities' in the sense of Machina and Schmeidler [5], a property that they term probabilistic sophistication. Consider an investor who chooses a preference maximizing portfolio given a complete set of asset markets. I identify properties of her asset demand function that rule out probabilistic sophistication of preference, just as the typical choices in the Ellsberg experiment rule out probabilistic sophistication for that setting.

2. ASSET DEMANDS AND PROBABILITIES

Let $S = \{1, ...s, ...S\}$ denote a finite state space. Securities have real-valued payoffs. Given a complete set of security markets (and the absence of arbitrage), there is no loss of generality in supposing that trades are in elementary Arrow-Debreu securities with corresponding prices $\pi = (\pi_s)_{s \in S}$, a vector in R^S_{++}. The investor has unit wealth and preference order $\succeq$ on R^S. Her problem is to find x in R^S such that $\pi \cdot x \leq 1$ and, for all $y \in R^S$,

$$\pi \cdot y \leq 1 \quad \Longrightarrow x \succeq y. \tag{2.1}$$

Denote by $D(R^1)$ the set of lotteries (probability distributions on R^1) with finite support. Say that $\succeq$ is *probabilistically sophisticated* if there exist: (i) a probability measure p on S and (ii) a real-valued functional W defined on $D(R^1)$ and strictly monotone there in the sense of first-order stochastic dominance, such that

$$x \succeq y \quad \text{iff} \quad W(\Psi_{x,p}) \geq W(\Psi_{y,p}),$$

where $\Psi_{x,p}$ and $\Psi_{y,p}$ are the lotteries induced respectively by x and y. In other words, the probability measure p is used to translate any prospect x into a lottery. The latter purely risky prospect is then evaluated via the functional W, which is unrestricted (except for monotonicity).

An implication of probabilistic sophistication for the asset demands can now be described. Suppose that for some specific pair of states s and t, that[1]

$$\pi_s \geq \pi_t \quad \text{and } x_s > x_t. \tag{2.2}$$

Thus the demand for the state s security is larger even though it has the higher price. Presumably, this is because state s is viewed as more likely than state t.

[1]Below, x and x' denote demands at prices π and π' respectively. For any two states s and t, x_{-s-t} is the vector in R^{S-2} obtained from x by deleting the s and t components.

Accordingly, if state prices change to π' with $\pi'_s \leq \pi'_t$, then the demand for the state s security should again be larger.

A more precise statement and proof follow.

Theorem 2.1. *Let π and π' be in R^S_{++} and let x and x' be corresponding solutions in (2.1) for the preference $\succeq$. Suppose that there exist states s and t such that both*

(a) $\pi_s \geq \pi_t$ and $\pi'_s \leq \pi'_t$, with at least one inequality strict, and

(b) $x_s > x_t$ and $x'_s < x'_t$.

Then $\succeq$ is not probabilistically sophisticated.

Proof. Given prices π, it is feasible to reverse the demands for states s and t and to choose (x_{-s-t}, x_t, x_s), (demanding x_t is state s), rather than $x = (x_{-s-t}, x_s, x_t)$ (demanding x_s is state s). That is because

$$\pi_s x_t + \pi_t x_s \leq \pi_s x_s + \pi_t x_t. \tag{2.3}$$

Therefore,

$$x = (x_{-s-t}, x_s, x_t) \succeq (x_{-s-t}, x_t, x_s). \tag{2.4}$$

In other words, it is preferable to have the larger outcome in state s rather than state t. If probabilistic sophistication prevails (relative to p and W), then the latter preference is invariant to the specific outcomes in states s and t and to changes in the common component x_{-s-t}. This is axiom $P4^*$ in Machina and Schmeidler's axiomatization of probabilistic sophistication; alternatively, it follows from $p(s) > p(t)$ and the strict monotonicity of W. In particular, it follows that

$$\left(x'_{-s-t}, x'_t, x'_s\right) \succeq \left(x'_{-s-t}, x'_s, x'_t\right) = x'. \tag{2.5}$$

If $\pi_s > \pi_t$, then the inequality in (2.3) is strict. Therefore the same is true for the rankings in (2.4) and (2.5) and the latter case contradicts the optimality of x'. Similarly if $\pi'_s > \pi'_t$. ■

Remark 1. *The proof does not use the full force of the existence of a single budget constraint as delivered by complete markets. The critical requirement is that π, s and t satisfy: If x is feasible given π and if $x_{-s-t} = y_{-s-t}$, $\pi_s y_s + \pi_t y_t \leq \pi_s x_s + \pi_t x_t$, then also y is feasible given π.*

The prices π' and π can differ arbitrarily in states other than s and t. This broadens the scope of the Theorem and also permits intuition for the behavior described there violating probabilistic sophistication. The intuition is as follows: In a thinly disguised translation of the setting in the 3-color Ellsberg Paradox, suppose that $S = \{s_1, s_2, s_3\}$, that the investor is confident that the probability of the first state is 1/3, but that she has only a vague idea of the relative likelihoods of states 2 and 3, that is, these states are ambiguous. Aversion to this ambiguity is expressed in a complementarity between the contingent wealths for states 2 and 3; for example, in the typical Ellsberg choices,

$$(100,\ 0,\ 0) \succ (0,\ 100,\ 0) \quad \text{but}$$

$$(100,\ 0, 10\,0) \prec (0,\ 100, 10\,0)\,.$$

To relate to the Theorem, take $s = 1$ and $t = 2$. If π_3 is large, then it would be too costly to purchase very much of the state 3 security. Because of the ambiguity of state 2, this leaves the state 1 security attractive relative to that for state 2. Accordingly, it is plausible that $x_1 > x_2$ even if $\pi_1 \geq \pi_2$. However, under π', if π'_3 is small, the investor can afford to buy more of the state 3 security and the complementarity between states 2 and 3 acts to increase the appeal of state 2 relative to the situation under π. Therefore, $x'_1 < x'_2$ is plausible even if $\pi'_1 \leq \pi'_2$.

The refutation of probabilistic sophistication requires that one observe demands at two distinct price vectors where the investor has the same beliefs. In other words, the observations described in the Theorem rule out the *conjunction* of probabilistic sophistication and constant beliefs. The reliance on two price situations is not surprising; for example, the Ellsberg Paradox also involves two choices. In addition, Lo [4] has shown recently that when the choice from only one feasible set is observable, then one cannot distinguish between subjective expected utility (*a fortiori*, probabilistic sophistication) and any preference satisfying a form of monotonicity (Savage's $P3$). Nevertheless, the applicability of the Theorem is limited by the assumption that the same beliefs prevail at the two price situations.

It is natural to wonder about the converse of the Theorem, or alternatively, about the power of the implied test of probabilistic sophistication. Is the behavior described in the Theorem so restrictive that it would contradict also many models of nonprobabilistically sophisticated preferences? In this connection, it is readily seen that the noted behavior does *not* contradict state-dependent expected utility theory, Choquet-expected utility theory [7] or the multiple-priors model [3]. Thus the implied test is powerful against these alternatives.

Conclude with an extension to aggregate data that also involves a joint hypothesis. If one observes only the aggregate demand for securities of a group of agents, then one can refute the *joint* hypothesis of probabilistic sophistication and common (across agents) beliefs. For each investor $i = 1, ..., I$, denote preference by $\succeq_i$ and wealth by M_i. Let $x_i = (x_{s,i})_{s \in S}$ be the optimal portfolio for i given M_i and prices π, that is, $\pi \cdot x_i \leq M_i$ and, for all $y \in R^S$,

$$\pi \cdot y \leq M_i \quad \Longrightarrow x \succeq_i y. \tag{2.6}$$

Similarly for demands x_i' given π' and wealth M_i'.

Theorem 2.2. *Let $(\pi, (M_i)_{i \in I})$ and $(\pi', (M_i')_{i \in I})$ be two price-wealth situations and let x_i and x_i' be corresponding solutions as in (2.6) for the preference $\succeq_i$, $i = 1, ..., I$. Suppose that there exist states s and t such that both*

(a) $\pi_s \geq \pi_t$ and $\pi_s' \leq \pi_t'$, with at least one inequality strict, and

(b) $\Sigma_i x_{s,i} > \Sigma_i x_{t,i}$ and $\Sigma_i x_s' < \Sigma_i x_t'$.

Then it is impossible that every $\succeq_i$ be probabilistically sophisticated with a common probability measure.

Proof. Roughly, $\Sigma_i x_{s,i} > \Sigma_i x_{t,i} \Longrightarrow x_{s,i} > x_{t,i}$ for some $i \Longrightarrow p(s) > p(t)$ for the common probability measure $p \Longrightarrow$ everyone prefers the higher outcome in state s rather than in state t. This is feasible given π'. Thus $x_{s,i}' \geq x_{t,i}'$ for every i. (Details are as in the proof of the previous theorem.) ■

Finally, one could view prices as endogenous and take endowments $\omega_i \in R^S$ as exogenous, with $M_i = \pi \cdot \omega_i$. In this way, the Theorem delivers restrictions across equilibria of two exchange economies $(\succeq_i, \omega_i)_{i \in I}$ and $(\succeq_i, \omega_i')_{i \in I}$ that contradict probabilistic sophistication of all agents relative to a probability measure that is common to all agents and both economies. (Simply replace aggregate demands in (b) by aggregate endowments.)

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