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Abstract

We study the behavior of rules for the adjudication of conflicting claims when there are a large number of claimants with small claims. We model such situations by replicating some basic problem. We show that under replication, the random arrival rule (O'Neill, 1982) behaves like the proportional rule, the rule that is the most often recommended in this context. Also, under replication, the minimal overlap rule (O'Neill, 1982) behaves like the constrained equal losses rule, the rule that selects a division at which all claimants experience equal losses subject to no-one receiving a negative amount.

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1 Introduction

A "claims problem" consists of a group of agents having claims on a resource that add up to more than is available. A "rule" associates with each problem a division of the resource among the claimants. An important example is when the liquidation value of a firm has to be divided among its creditors. In his formulation and study of the model, O'Neill (1982) proposed a number of rules and suggested a variety of ways of evaluating rules. This paper is a contribution to the literature spawned by this pioneering work.[1]

Our purpose is to study the behavior of rules under "replication" of problems. Given a natural number k, in the k-replica of a problem, each of the claimants initially present has $k-1$ "clones", that is, agents whose claims are equal to his, and the amount to divide is k times what it was initially. The replication exercise allows us to obtain insight into situations in which there are a large number of claimants whose claims are small in relation to the amount to divide (small investors in a local savings bank that has gone bankrupt, say). Replicating economies is a standard technique of general equilibrium theory to gain understanding of the behavior of the Walrasian rules and other rules in situations where traders are negligible relative to the entire market (Debreu and Scarf, 1963). In the recent fairness literature, it has also become common to include a replication test in the battery of tests to which allocation rules are subjected to evaluate them (Thomson, 1988).[2]

A rule is "invariant under replication" if what each claimant is awarded in any problem is also what he and each of his clones is awarded in any replica of the problem. Several important rules are invariant under replication. They include the "proportional rule", the rule that is the most often recommended in this context, the "constrained equal awards rule", which selects the division at which all agents receive equal amounts subject to no-one receiving more than his claim, the "constrained equal losses rule", which selects the division at which all claimants experience equal losses subject to no-one receiving a negative amount, and the "Talmud rule", a rule proposed by Aumann and Maschler (1985) in order to rationalize certain numerical examples described in the Talmud. Some rules are not replication invariant however. The two main ones are the "random arrival" rule and the "minimal overlap" rule, both introduced by O'Neill (1982): The random arrival rule is based on a

[1] For a survey of the literature devoted to this subject, see Thomson (2003).

[2] Replicating is not the only technique to model problems with a large number of agents with small claims. In the conclusion, we suggest alternative ones.

first-come first-serve scenario; for each possible order in which claimants may arrive to be compensated, assign to each of them an amount equal to his claim if that is possible, and whatever is left otherwise; then, take the average of the resulting awards vectors, assuming that all orders of arrival occur with equal probabilities. Informally speaking, the minimal overlap rule chooses awards vectors that minimize "extent of conflict" over each unit available. (The formal definition is given below.)

Although the random arrival and minimal overlap rules are not replication invariant, we ask whether anything interesting can be said about the awards vectors they recommend for replicated problems. The answer is yes, and our main findings relate the rules to two of the rules mentioned earlier. As the order of replication increases, the random arrival rule "behaves like" the proportional rule, and the minimal overlap rule "behaves like" the constrained equal losses rule.

Of course these convergence results cannot be interpreted as the core convergence results of general equilibrium theory, but they reveal connections between rules that are helpful in understanding the subject. Of particular relevance is that, to the extent that our interest in rules largely stems from the properties they enjoy, one can deduce from the convergence of some rule to another that in some approximate sense, the former will inherit the properties of the latter when the number of claimants is large enough. The proportional rule is known to satisfy a large number of desirable properties, many of which are not satisfied by the random arrival rule (the reverse is true for only a few properties). The random arrival rule will therefore satisfy these properties approximately when the number of claimants is large. Similarly, the constrained equal losses rule satisfies several properties that the minimal overlap rule violates (and here too, the reverse is true in only a few cases). One can deduce from our analysis that the minimal overlap rule satisfies these properties in some approximate sense, once again, in situations in which the number of claimants is large enough.

2 The model

There is an infinite set of "potential" claimants, indexed by the natural numbers $\mathbb{N}$. At any given time however, only a finite number of them are present. Let $\mathcal{N}$ be the class of finite subsets of $\mathbb{N}$. A ***claims problem*** is a pair $(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$, where $N \in \mathcal{N}$, such that $\sum_N c_i \geq E$. For each $i \in N$,

c_i is the ***claim*** of agent i; c is the ***claims vector***, and E is the ***amount to divide***. Let $\mathcal{C}^N$ be the class of all such problems. A ***rule*** is a function defined on the union of all $\mathcal{C}^N$'s, where N ranges over $\mathcal{N}$, which associates with each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$ a point of $\mathbb{R}^N_+$ whose coordinates add up to E—we refer to this property as "efficiency"—and at which no agent receives more than his claim. A vector satisfying these conditions is an ***awards vector for*** $\boldsymbol{(c, E)}$. Our generic notation for a rule is the letter S. In a Euclidean space of dimension equal to the number of claimants, and for each claims vector, we refer to the path followed by the awards vector chosen by a rule as the amount to divide varies from 0 to the sum of the claims, as the ***path of awards of the rule for the claims vector***.

Next, we introduce the two rules whose behavior under replication we will study. The first rule, defined by O'Neill (1982), is obtained by imagining claimants arriving one after the other and being fully compensated until money runs out; then, taking the average of the resulting awards vectors when all orders of arrival are equally likely. Let Π^N designate the class of bijections on N.

Random arrival rule, *RA*: For each $(c, E) \in \mathcal{C}^N$ and each $i \in N$,

$$RA_i(c, E) \equiv \frac{1}{|N|!} \sum_{\pi \in \Pi^N} \min\{c_i, \max\{E - \sum_{j \in N, \pi(j) < \pi(i)} c_j, 0\}\}.$$

The second rule, also defined by O'Neill (1982), generalizes a proposal made by Rabad (12-th Century): imagine that the amount available consists of distinct parts, and that each agent, instead of expressing his claim in some abstract way, claims specific parts of total amount equal to his claim; now arrange which parts agents claim so as to "minimize conflict" in the following way: first, make the part of the amount to divide claimed by exactly one agent as large as possible; this maximization may have many solutions, so narrow them down by turning attention to the part that is claimed by exactly two agents and rearrange claims so as to make it as large as possible; this second maximization may still have several solutions, so perform a further narrowing down by turning attention to the part that is claimed by exactly three agents, and so on; finally divide each part equally among all agents claiming it, and have each claimant collect the compensations he gets from the various parts he claimed.

Minimal overlap rule, *MO*: For each $(c, E) \in \mathcal{C}^N$, claims on specific parts of E are arranged so that the part claimed by exactly one claimant is maximized, and for each $k = 2, \ldots, |N| - 1$ successively, subject to the previous k maximizations being solved, the part claimed by exactly $k + 1$ claimants is maximized. Once claims are arranged in this way, for each part, equal division prevails among all agents claiming it. Each agent receives the sum of the compensations he gets from the various parts he claimed.

For the proof of Theorem 2 below, we rely on the following "computational" characterization of the rule (O'Neill, 1982).[3]

Proposition 1 *Up to relabelling parts of the amount available, there is a unique arrangement of claims achieving minimal overlap. It is obtained as follows:*

Case 1: *There is $j \in N$ such that $c_j \geq E$. Then, each claimant $i \in N$ such that $c_i \geq E$ claims $[0, E]$ and each other claimant i claims $[0, c_i]$ (claims are nested).*

Case 2: $\max c_j < E$. *Then, there is $t \in [0, E]$ such that*

- *(a) each claimant $i \in N$ such that $c_i \geq t$ claims $[0, t]$ as well as a part of $[t, E]$ of size $c_i - t$, with no overlap between these claims;*
- *(b) each other claimant i claims $[0, c_i]$.*

It is convenient to picture the amount to divide as a horizontal segment of length E, with each claimant "covering" part of it with his claim. Case 2-b of Proposition 1 states that for some $t \in \mathbb{R}_+$, each part of the amount to divide exceeding t is covered by only one claimant: thus, t should be such that $\sum_{i \in N | c_i > t} c_i - t = E - t$. For instance, consider the problem with claimant set $N \equiv \{1, 2, 3\}$ and $(c, E) \equiv ((2, 5, 6), 8)$. No claim is larger than E so indeed Case 2-b applies. The "equilibrium" t lies between the smallest and second smallest claim, and it is defined by the equation $5 - t + 6 - t = 8 - t$ (then $t = 3$). Claimant 1 receives $x_1 \equiv \frac{2}{3}$ because whatever he claims is also claimed by the other two agents. Claimant 2 receives $x_1 + \frac{t-2}{2} + 5 - t$ because he also claims the interval $t - 2$, an interval that is claimed by agent 3 as well, the last term $(5 - t)$ coming from his claiming (alone) an interval of size $5 - t$.

[3] The notation $[x, y]$ refers to the interval from x to y.

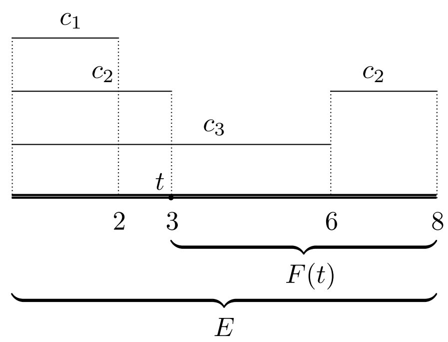

Figure 1: Example illustrating the definition of the minimal overlap rule.

Claimant 3 receives $x_1 + \frac{t-2}{2} + 6 - t$, the first two terms being calculated as for claimant 2, and the last term $(6-t)$ coming form his claiming (alone) an interval of size $6-t$. In Figure 1, the interval $F(t) \equiv [t, E] = [3, 8]$ is covered by parts of the claims of agents 2 and 3: agent 2 "covers" the subinterval $[6, 8]$ and agent 3 the subinterval $[3, 6]$.

Next, we define the replication operation. Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$. Also, let k be a natural number. By a ***k*-*replica of* (c, E)** we mean a problem in which each of the members of N has $k-1$ "clones"—these agents have claims equal to his—and in which the amount available is k times what it is initially. Formally, if N' designates the claimant set in the replicated problem, we have $N' \supset N$, $|N'| = k|N|$, and there is a partition of N' into $|N|$ groups of k agents indexed by $i \in N$, $(N^i)_{i \in N}$, such that for each $i \in N$ and each $j \in N^i$, $c_j = c_i$. The property of a rule formulated next is that the amount it assigns to each $i \in N$ in (c, E) is also the amount it assigns to him and to each of his clones in any k-replica of (c, E).

Replication invariance: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, each $N' \supset N$, and each $(c', E') \in \mathcal{C}^{N'}$, if (c', E') is a k-replica of (c, E) with associated partition $(N^i)_{i \in N}$, then for each $i \in N$ and each $j \in N^i$, $S_j(c', E') = S_i(c, E)$.

The following two rules are *replication invariant*: for the proportional rule, awards are proportional to claims. For the constrained equal losses rule, the losses agents experience are equal subject to no-one receiving a negative amount.

Proportional rule, $\boldsymbol{P}$: For each $(c, E) \in \mathcal{C}^N$, $P(c, E) \equiv \lambda c$, where $\lambda \in \mathbb{R}_+$ is chosen so as to achieve feasibility.[4]

[4]Then, $\lambda = E / \sum c_i$. For this expression to be well-defined, we exclude the degenerate case $\sum c_i = 0$, for which, by definition of a claims problem $E = 0$, and the only choice is $(0, \dots, 0)$.

Constrained equal losses rule, *CEL*: For each $(c, E) \in \mathcal{C}^N$ and each $i \in N$, $CEL_i(c, E) \equiv \max\{0, c_i - \lambda\}$, where $\lambda \in \mathbb{R}_+$ is chosen so as to achieve feasibility.

On the other hand, neither the minimal overlap rule nor the random arrival rule is *replication invariant*, as can be seen by means of simple examples.

3 The results

Our results are convergence results for the random arrival and minimal overlap rules. They are made possible by the fact that the rules satisfy *equal treatment of equals*: two agents with equal claims should be awarded equal amounts. Therefore, the awards vector of a replicated problem is the replica of an awards vector for the problem that is replicated, and its behavior can be studied in a space of dimension equal to the number of claimants involved in the problem subject to the replication.

Our first result pertains to the random arrival rule. In its proof, we use the notation $k * N$ to indicate a k replica of the population N.

Theorem 1 *As the order of replication increases, the random arrival awards vector of a replicated problem is the replica of an awards vector for the problem that is replicated that converges to the proportional awards vector of that problem.*

Proof: Without loss of generality, let $N \equiv \{1, \ldots, n\}$ and $(c, E) \in \mathcal{C}^N$. If for some $i \in N$, $c_i = 0$, then trivially $RA_i(c, E) = P_i(c, E) = 0$. From now on, we assume that $c > 0$. Let $k \in \mathbb{N}$. To calculate the random arrival awards vector of a k-replica of (c, E), we need to find out how much each agent is awarded at each permutation. At each permutation, an agent receives either his claim, or some positive amount stritly less than his claim, or nothing. Given $i \in k * N$, let $n_1(i, k)$ be the number of permutations at which he receives c_i, $n_2(i, k)$ the number of permutations at which he receives some positive amount strictly less than c_i, and $n_3(i, k)$ the number of permutations at which he receives 0. For each $\ell = 1, 2, 3$, let $p_\ell(i, k)$ be the corresponding probability, that is, $p_\ell(i, k) = n_\ell(i, k)/(kn)!$.

At each permutation, in the k-replica of (c, E), agent i receives c_i if the sum of the claims of the agents coming before him plus c_i is equal to, or less than, kE. For each $\pi \in \Pi^{k*N}$, the class of bijections on $k * N$, and each $i \in k * N$, let $b^\pi(i)$ be the set of agents who arrived before i. For each $\{i, j\} \subset k * N$ and each $\pi \in \Pi^{k*N}$, let $\pi' \in \Pi^{k*N}$ be the permutation obtained from π by exchanging i and j.

For each $\{i, j\} \subset k * N$,

$$\begin{aligned} p_1(i,k) &= Pr[\pi \in \Pi^{k*N} | \sum_{m \in b^\pi(i)} c_m + c_i \le kE] \\ &= Pr[\pi \in \Pi^{k*N} | j \in b^\pi(i), \sum_{m \in b^\pi(i)} c_m + c_i \le kE] \\ &\quad + Pr[\pi \in \Pi^{k*N} | j \notin b^\pi(i), \sum_{m \in b^\pi(i)} c_m + c_i \le kE] \\ &= Pr[\pi \in \Pi^{k*N} | j \in b^\pi(i), \sum_{m \in b^\pi(i), m \neq j} c_m \le kE - c_i - c_j] \\ &\quad + Pr[\pi \in \Pi^{k*N} | j \notin b^\pi(i), \sum_{m \in b^\pi(i)} c_m \le kE - c_i] \\ &= Pr[\pi \in \Pi^{k*N} | j \in b^\pi(i), \frac{\sum_{m \in b^\pi(i), m \neq j} c_m}{k} \le E - \frac{c_i}{k} - \frac{c_j}{k}] \\ &\quad + Pr[\pi \in \Pi^{k*N} | j \notin b^\pi(i), \frac{\sum_{m \in b^\pi(i)} c_m}{k} \le E - \frac{c_i}{k}]. \end{aligned}$$

Similarly,

$$\begin{aligned} p_1(j,k) &= Pr[\pi \in \Pi^{k*N} | i \in b^\pi(j), \frac{\sum_{m \in b^\pi(j), m \neq i} c_m}{k} \le E - \frac{c_j}{k} - \frac{c_i}{k}] \\ &\quad + Pr[\pi \in \Pi^{k*N} | i \notin b^\pi(j), \frac{\sum_{m \in b^\pi(j)} c_m}{k} \le E - \frac{c_j}{k}]. \end{aligned}$$

Let us compare $p_1(i,k)$ and $p_1(j,k)$.

$$\begin{aligned} & Pr[\pi \in \Pi^{k*N} | j \in b^\pi(i), \frac{\sum_{m \in b^\pi(i), m \neq j} c_m}{k} \le E - \frac{c_i}{k} - \frac{c_j}{k}] \\ &= Pr[\pi' \in \Pi^{k*N} | i \in b^{\pi'}(j), \frac{\sum_{m \in b^{\pi'}(j), m \neq i} c_m}{k} \le E - \frac{c_j}{k} - \frac{c_i}{k}] \\ &= Pr[\pi \in \Pi^{k*N} | i \in b^\pi(j), \frac{\sum_{m \in b^\pi(j), m \neq i} c_m}{k} \le E - \frac{c_j}{k} - \frac{c_i}{k}]. \end{aligned}$$

Moreover, if $c_i \leq c_j$,

$$\begin{aligned} & Pr[\pi \in \Pi^{k*N} | j \notin b^\pi(i), \frac{\sum_{m \in b^\pi(i)} c_m}{k} \leq E - \frac{c_i}{k}] \\ \geq \quad & Pr[\pi \in \Pi^{k*N} | i \notin b^\pi(j), \frac{\sum_{m \in b^\pi(j)} c_m}{k} \leq E - \frac{c_j}{k}]. \end{aligned}$$

Therefore, if $c_i \leq c_j$, then $p_1(i, k) \geq p_1(j, k)$.

Also, if $c_i \leq c_j$, then

$$\begin{aligned} p_3(i,k) = & Pr[\pi \in \Pi^{k*N} | \sum_{m \in b^\pi(i)} c_m \geq kE] \\ \geq \quad & Pr[\pi \in \Pi^{k*N} | \sum_{m \in b^\pi(j)} c_m \geq kE] = p_3(j,k). \end{aligned}$$

At each permutation, at most one agent receives some positive amount strictly less than his claim, so that $\sum_{j \in k*N} n_2(j, k) \leq (kn)!$. Moreover, since the random arrival rule satisfies *equal treatment of equals*, if i and i' are of the same type, then for each $\ell = 1, 2, 3$, $n_\ell(i, k) = n_\ell(i', k)$. Since there are k agents of the same type, $\sum_{j \in k*N} n_2(j, k) \leq (kn)!$ implies that for each $i \in k * N$, $n_2(i, k) \leq \frac{(kn)!}{k}$, or equivalently, $0 \leq \frac{n_2(i,k)}{(kn)!} \leq \frac{1}{k}$. As $k \to \infty$, $\frac{n_2(i,k)}{(kn)!} \to 0$. Consequently, as $k \to \infty$, for each $i \in k * N$, $p_2(i, k) \to 0$.

Let i, $j \in k * N$ be such that $c_i \leq c_j$. Since $p_1(i, k) + p_2(i, k) + p_3(i, k) = p_1(j, k) + p_2(j, k) + p_3(j, k) = 1$, $\lim_{k \to \infty} p_2(i, k) = \lim_{k \to \infty} p_2(j, k)$ implies that $\lim_{k \to \infty}\{p_1(i, k) + p_3(i, k)\} = \lim_{k \to \infty}\{p_1(j, k) + p_3(j, k)\}$.

As discussed above, since the random arrival rule satisfies *equal treatment of equals*, $\sum_{i \in N} RA_i(k * (c, E)) = E$. Since $\sum_{i \in N} c_i p_1(i, k) \leq E \leq \sum_{i \in N} c_i p_1(i, k) + \sum_{i \in N} c_i p_2(i, k)$, as $k \to \infty$, $\lim_{k \to \infty} \sum_{i \in N} c_i p_1(i, k) = E$. It follows that for each $\varepsilon > 0$, there is $K_1(\varepsilon) > 0$ such that for each $k > K_1(\varepsilon)$, we have $|E - \sum_{i \in N} c_i p_1(i, k)| < \varepsilon$.

Also, for each $\{i, j\} \subset k*N$, from $\{p_1(i, k) - p_1(j, k)\}\{p_3(i, k) - p_3(j, k)\} \geq 0$,

$$\begin{aligned} |p_1(i,k) - p_1(j,k)| & \leq |p_1(i,k) - p_1(j,k) + p_3(i,k) - p_3(j,k)| \\ & \leq |p_1(i,k) + p_3(i,k) - 1| + |p_1(j,k) + p_3(j,k) - 1|. \end{aligned}$$

Since $\lim_{k\to\infty}\{p_1(i,k)+p_3(i,k)\} = \lim_{k\to\infty}\{p_1(j,k)+p_3(j,k)\} = 1$, it follows that for each $\varepsilon > 0$, there is $K_2(\varepsilon,i,j) > 0$ such that for each $k > K_2(\varepsilon,i,j)$, we have $|p_1(i,k) - p_1(j,k)| < \varepsilon$. Let $K_2(\varepsilon) \equiv \max_{i,j\in N} K_2(\varepsilon,i,j)$. Then, for each $\varepsilon > 0$, each $\{i,j\} \subset k * N$, and each $k > K_2(\varepsilon)$, we have $|p_1(i,k) - p_1(j,k)| < \varepsilon$.

Now, we show that for each $i \in N$, $\lim_{k\to\infty} p_1(i,k) = \frac{E}{\sum_{j\in N} c_j}$. Indeed,

$$\begin{aligned}
|\frac{E}{\sum_{j\in N} c_j} - p_1(i,k)| &\leq \frac{1}{\sum_{j\in N} c_j}|E - \sum_{j\in N} c_j p_1(i,k)| \\
&\leq \frac{1}{\sum_{j\in N} c_j}|E - \sum_{j\in N} c_j p_1(j,k)| + \frac{1}{\sum_{j\in N} c_j}|\sum_{j\in N} c_j p_1(j,k) - \sum_{j\in N} c_j p_1(i,k)| \\
&\leq \frac{1}{\sum_{j\in N} c_j}|E - \sum_{j\in N} c_j p_1(j,k)| + \frac{1}{\sum_{j\in N} c_j}\sum_{j\in N}\{c_j|p_1(j,k) - p_1(i,k)|\}.
\end{aligned}$$

By setting $K(\varepsilon) = \max\{K_1(\frac{\sum_{j\in N} c_j\varepsilon}{2}), K_2(\frac{\varepsilon}{2})\}$, we deduce that for each $\varepsilon > 0$, and each $k > K(\varepsilon)$, $|\frac{E}{\sum_{j\in N} c_j} - p_1(i,k)| < \varepsilon$. Thus, as $k \to \infty$, for each $i \in k * N$, $RA_i(kc,kE) \to \frac{c_i}{\sum_{j\in N} c_j}E$, as announced.

□

Let us note that the convergence described in Theorem 1 will typically not occur in a finite number of steps. For each claims vector, there are values of E for which the awards vector is invariant under replication. For instance, invariance holds when the amount to divide is equal to the half-sum of the claims. The path of awards of the random arrival rule is piece-wise linear, and an easy way to see that for each claims vector, and each $k \in \mathbb{N}$, there are values of the amount to divide for which convergence has not occurred in k steps is that the path of awards of the proportional rule is a segment from the origin to the claims vector whereas the path of the random arrival rule always starts with a non-degenerate segment of slope 1. This is because, if the amount to divide is smaller than the smallest claim, the rule recommends equal division; indeed, whoever arrives first takes everything then, and all agents arrive first with equal probabilities. The path ends with a segment of slope 1, for "symmetric" reasons.[5]

[5]These properties can be deduced from the fact that the rule is *self-dual* (Aumann and Maschler, 1985).

Our second result pertains to the minimal overlap rule. Its proof relies on Proposition 1.

Theorem 2 *As the order of replication increases, the minimal overlap awards vector of a replicated problem is the replica of an awards vector for the problem that is replicated that converges to the constrained equal losses vector of the problem.*

Proof: Without loss of generality, let $N \equiv \{1, \ldots, n\}$ and $(c, E) \in \mathcal{C}^N$. Let $k \in \mathbb{N}$. In Proposition 1, two cases are distinguished depending upon whether the largest claim is larger than the amount available. Since the largest claim in any replica of (c, E) is equal to the largest claim in (c, E), whereas the amount available increases without bound with the order of replication (if $E > 0$, which we assume since otherwise the conclusion holds trivially), it follows that for each k larger than some critical value, no claim is larger than the amount available, kE, and Case 2 applies. Without loss of generality, assume that this inequality holds for (c, E) itself, and that $c_1 \leq \cdots \leq c_n$. Then, as noted when we defined the rule, $MO(c, E)$ is obtained by identifying $t \geq 0$ such that $\sum_N \max\{c_i - t, 0\} = E - t$. Let t^k be the solution to the corresponding equation in the k-replica of (c, E), namely $k \sum_N \max\{c_i - t^k, 0\} = kE - t^k$.

We claim that $\{t^k\}$ is a decreasing sequence. Indeed, in the passage from a k-replica of (c, E) to a $(k+1)$-replica, the amount to be covered increases by E but if we kept the same cut-off point (the same t^k), the additional parts of claims made available by the arrival of an additional copy of N to cover it would be $\sum_N \max\{c_i - t^k, 0\} = E - \frac{t^k}{k} \leq E$. Let $t^* \equiv \lim t^k$ and $i^* \equiv \max\{i \in N \colon c_i \leq t^*\}$. For each $i \in N$, let $N^i(k)$ be the set of claimants who are clones of claimant i in the k-replica of (c, E). Now, note that there is $i^k \in N$ such that what each agent $j \in k * N$ receives can be expressed as

$$x_j \equiv \begin{cases} \frac{c_1}{nk} & \text{if} \quad j \in N^1(k) \\ \frac{c_1}{nk} + \frac{c_2 - c_1}{(n-1)k} & \text{if} \quad j \in N^2(k) \\ \vdots \; + \; \vdots & \vdots \\ \frac{c_1}{nk} + \frac{c_2 - c_1}{(n-1)k} + \cdots + \frac{c_{i^k} - c_{i^k - 1}}{(n - i^k + 1)k} & \text{if} \quad j \in N^{i^k}(k) \\ \frac{c_1}{nk} + \frac{c_2 - c_1}{(n-1)k} + \cdots + \frac{c_{i^k} - c_{i^k - 1}}{(n - i^k + 1)k} + \frac{t^k - c_{i^k}}{n - i^k} + c_i - t^k & \text{if} \quad j \in N^i(k) \text{ for each } i > i^* \end{cases}$$

As $k \to \infty$, each of the ratios goes to 0, so that at the limit, for each $i \leq i^*$ each j who is a clone of i, agent j receives 0, and for each $i > i^*$, and each j who is a clone of i, agent j receives $c_j - t^*$. Thus, the limit awards vector is the one chosen by the constrained equal losses rule. $\square$

Here too, we ask whether convergence occurs in a finite number of steps. The answer is no. For each problem in which at least two claims differ, there is always an entire interval of values of the amount to divide for which convergence has not occurred.

To see this, let us consider a generic two-claimant example. Let $N \equiv \{1, 2\}$ and $(c, E) \in \mathcal{C}^N$ with $c_1 \leq c_2$. Then, since the minimal overlap rule satisfies *equal treatment of equals*, we can represent the path of awards of the rule in a Euclidean space of dimension 2 (an observation made earlier).

For each $k \in \mathbb{N}$, there are values of E for which $MO(k*(c, E)) = CEL(k*(c, E))$. For instance, if $k = 1$, for each $E \in [c_2, \sum c_i]$, this equality holds (Figure 2a). Indeed, in the two-claimant case, the minimal overlap rule coincides with the contested garment rule, the two-claimant version of the "Talmud rule".[6] The path of awards of the contested garment rule coincides with that of the constrained equal losses rule for amounts to divide in the interval $[c_2, \sum c_i]$. Under replication, the equality $MO(k*(c, E)) = CEL(k*(c, E))$ is preserved.

As k increases, the interval of values of E for which coincidence occurs enlarges. Figure 2b shows the projection onto the $\{1, 2\}$-space of the paths of awards of the minimal overlap rule for $k = 1, 2$ and 3. In general, for the k-th replica, this projection consists of a segment of slope 1 containing the origin (the segment to the point $(\frac{c_1}{2k}, \frac{c_1}{2k})$), a vertical segment of length $c_2 - c_1$, and a segment of slope 1 to the claims vector.

The limit of the path of awards is $[(0, 0), (0, c_2 - c_1)] \cup [(0, c_2 - c_1), c]$, namely the path of awards of the constrained equal losses rule (Figure 2c). Figure 2 shows that for any value of the amount to divide in $[0, c_2 - c_1]$, there is no finite k after which coincidence with the awards vector chosen by the constrained equal losses occurs.

More can be said. Thomson (2001) defines a simple (parametric) family of rules that connects a number of important rules (the constrained equal awards, constrained equal losses, Talmud, and minimal overlap rules), the "ICI family". For each claims vector $c \in \mathbb{R}^N_+$ (not only for the two-claimant case), for each natural number k, the path of awards of the minimal overlap

[6]The rule defined by Aumann and Maschler (1985) to rationalize some examples in the Talmud.

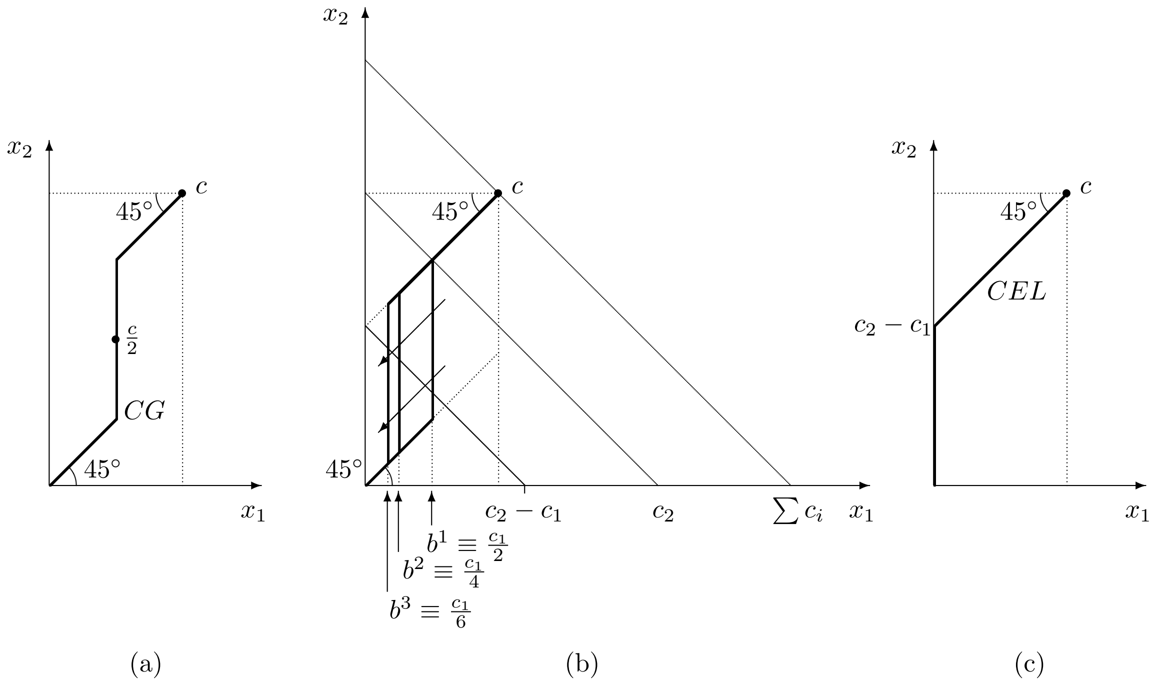

Figure 2: Convergence for the replica of a two-claimant example. (a) For $k = 1$, the minimal overlap rule coincides with contested garment rule. (b) As the order of replication increases, the vertical segment contained in the path slides down the 45° line. (c) At the limit, we obtain the path of awards of the constrained equal losses rule.

rule for the k replica of c, when projected onto the $|N|$-dimensional awards space, is the path of awards of a member of the family for c. One can calculate the break points in the schedules giving for each type of claimants their common award as a function of the amount to divide, and show that each of them converges to the corresponding breakpoint of the schedule for the constrained equal losses rule.

4 Concluding comments

• We focused on two rules that are not *replication invariant* and studied their behavior under replication, but we should note that many rules are *replication invariant*. This is in particular the case for a large subclass of the class of rules satisfying the following important property: a rule is "consistent" if the recommendation it makes for each problem "agrees" with the recommendation it makes for each of the associated "reduced problems" obtained by imagining some agents leaving with their awards and reassessing the situation faced by the remaining agents. A fundamental theorem due to Young (1987) states that if a rule satisfies *equal treatment of equals*, and if it is *continuous* and *consistent*, then it belongs to a certain family of "parametric rules". This family includes many of the rules that have been considered in the literature. It is therefore of much interest that each member of this family satisfies *replication invariance*, as this property is a consequence of *equal treatment of equals* and *consistency*.

• A more general way of modeling problems with a large number of small claimants is by drawing them at random from some distribution with a finite support. It is intuitively clear that parallel results would be obtained.[7] Alternatively, one could model the set of claimants as a continuum. Such a formulation was recently proposed by Chambers and Thomson (2002).

• One can associate to each rule S new ones by subjecting S to certain operators.[8] The "duality operator" associate to S the rule that treat losses in the way S treats awards. The "claims truncation operator" associates to S the rule defined for each problem by first truncating claims at the amount available and then applying S. The "minimal rights operator" associates with S the rule that calculates the awards vector of each problem in two steps, as follows: first, each claimant is attributed the difference between the amount to divide and the sum of the claims of the other agents (or 0 if this difference is negative); this difference, the agent's "minimal right" is an obvious minimum to which he is entitled; second, S is applied to allocate what remains, the part that is truly contested, claims being adjusted down by the "minimal rights" of the first step.

It is easy to see that for each rule S, each $N \in \mathcal{N}$, and each problem $(c, E) \in \mathcal{C}^N$, and under sufficient replication, the awards vector selected by S for the replica problem is the same as the awards vector selected by the rules obtained by subjecting S to the claims truncation operator, the attribution of minimal rights operator, or their composition. This is obvious if $E = 0$ or $E = \sum c_i$. If neither of these equalities hold, the statement follows from the fact that under sufficient replication, the amount to divide is larger than any claim, and that the minimal right of each agent is 0. Curiel, Maschler and Tijs (1987) have proposed the rule obtained from the proportional rule by subjecting it to the composition of these operators (Thomson and Yeh, 2003, show that the order of composition does not matter). An application of the argument just made is that under sufficient replication, this rule coincides with the proportional rule.

[7]Some care would have to be exercised however so as to guarantee that the amount to divide is no greater than the sum of the claims.

[8]For a study of such operators, see Thomson and Yeh (2003).

As far as the duality operator is concerned, it is clear that the behavior under replication of any rule is mirrored by the behavior of its dual under that operation. If a rule S behaves like a certain rule R under replication, the dual of S behaves likes the dual of R under replication. The random arrival rule being invariant under the duality operator (it is ***self-dual***), nothing else needs to be said about it, but the minimal overlap rule is not *self-dual*. Thus, one can say that its dual under replication behaves likes the dual of the constrained equal losses rule, namely the constrained equal awards rule. (This is the rule that makes awards as equal as possible subject to no one receiving more than his claim.)

• Our final comment pertains to the connection between rules for the adjudication of conflicting claims and certain solutions developed in the theory of "coalitional games with transferable utility". Such a game is defined by specifying for each set of agents, or "coalition", a number that is usually interpreted as the total utility the coalition can obtain without the assistance of the members of the complementary coalition. This total utility is the "worth" of the coalition. A "solution" associates with each game a payoff vector whose coordinates add up to the worth of the grand coalition. Two central solutions in this theory are the "core", which selects the set of payoff vectors such that the worth of no coalition is greater than the sum of the payoffs assigned to its members (Gillies, 1953), and the "Shapley value", which assigns to each player a payoff equal to the average of his contributions[9] to the coalition consisting of all the players who have arrived before him when all orders of arrival are equally likely (Shapley, 1953). To be able to apply these concepts to construct rules, one needs to convert each claims problem into a game. O'Neill (1982) suggested the following definition, which has been central in the subsequent literature (Aumann and Maschler, 1985): set the worth of each coalition of claimants equal to whatever is left when all other claimants have been fully compensated if that is possible, and nothing otherwise. It turns out that for each claims problem, the random arrival rule produces the same awards vector as the Shapley value applied to the associated transferable utility game (O'Neill, 1982). Our Theorem 1 can therefore be understood as providing a convergence theorem for a Shapley-value-type concept analogous to the convergence results known for this concept when applied to allocation problems. In exchange economies, the Shapley value

[9] A player's contribution to a coalition is the difference between the worth of the coalition after he has joined and he worth of the coalition before he joins.

allocations are known to converge to competitive allocations (Aumann and Shapley, 1974), as the core does (Debreu and Scarf, 1963). One may think that the convergence of the random arrival rule is due to a similar shrinking of the core. This is not the case however. In our context, the core does not shrink. In fact, the core is nothing else than the set of vectors satisfying the non-negativity and claims boundedness requirements (the set of awards vectors). Therefore (as a correspondence), it is *invariant under replication*.

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