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A Characterization of a Family of Rules for the Adjudication of Conflicting Claims

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Contents

Abstract

We consider the problem of adjudicating conflicting claims, and characterize the family of rules satisfying four standard invariance requirements, homogeneity, two composition properties, and consistency. It takes as point of departure the characterization of the family of two-claimant rules satisfying the first three requirements, and describes the restrictions imposed by consistency on this family and the further implications of this requirement for problems with three or more claimants. The proof, which is an alternative to Moulin's original proof (*Econometrica*, 2000), is based on a general method of constructing consistent extensions of two-claimant rules (Thomson, 2001), which exploits geometric properties of paths of awards, seen in their entirety.

Keywords: claims problems; consistent extensions; proportional rule; constrained equal awards rule; constrained equal losses rule.

JEL classification number: C79; D63; D74

1 Introduction

We consider the problem of allocating an infinitely divisible and homogeneous resource among agents having claims on it that cannot be jointly honored. A primary example is when the liquidation value of a bankrupt firm has to be divided among its creditors. A "division rule" is a function that associates with each situation of this kind, or "claims problem", a division of the amount available, an "awards vector" for the problem. This division is interpreted as the choice that a judge or an arbitrator could make. The literature devoted to the analysis of this sort of situations originates in a path-breaking paper by O'Neill (1982).[1] We offer here a geometrically intuitive proof of a characterization, due to Moulin (2000), of the family of rules satisfying four standard invariance requirements. We aim to make this important theorem more easily accessible and to illustrate a method of proof that has provided answers to an entire class of similar questions. This method is presented in a didactic way and applied to a series of examples in Thomson (2001), but a compact presentation of it is in Subsection 3.3.

The requirements we impose on a rule are the following. First, multiplying all the data of a problem by the same positive number results in a new problem for which the chosen awards vector should be related by the same operation to the awards vector initially chosen. Second, if the amount to divide decreases from some initial value, the awards vector chosen for the final problem should be equivalently obtainable in two ways: either the awards vector initially chosen is ignored and the rule is reapplied to divide the final amount; or this initial awards vector is used as claims vector when dividing the final amount. Third is a counterpart of this condition pertaining to possible increases in the amount to divide. The fourth requirement is a variable-population condition: the awards vector chosen for each problem should be in agreement with the awards vector chosen for each "reduced problem" obtained by imagining that an arbitrary subpopulation of claimants leave the scene with their awards, and re-evaluating the situation at that point.

Note that no symmetry or anonymity requirement is imposed. Although such requirements are often natural, in many situations it is desirable to have the option of treating agents differently even when their claims are equal. This allows taking into account characteristics they have other than

[1] For a survey, see Thomson (2003).

their claims. An obvious example of such a characteristic is income. Others are family responsibility, health status, record of service, and so on. Then, fairness may require that agents with equal claims not receive equal amounts.

The main result is a characterization of the family of rules satisfying the four axioms. Any such rule is obtained by first partitioning the set of potential claimants into ordered "priority classes". For each problem, one calculates the ordered partition of the set of agents involved induced by this "reference partition". The components of the induced partition are handled in succession: one fully satisfies the claims of all agents in a class before giving anything to any class with a lower priority. This is common practice (for instance, secured claims have priority over unsecured claims). More interesting is how each class is handled. There are two cases. For a class that coincides with a two-claimant member of the reference partition, a rule in a rich family of rules that "link" the proportional, constrained equal awards, and constrained equal losses rules, is applied (illustrations are in Subsection 3.1). For a class with more than two claimants, one of the following rules is applied: the proportional rule, a weighted constrained equal awards rule, or a weighted constrained equal losses rule. In each of the last two cases, the weights are proportional to a vector of positive weights assigned once and for all to the members of the component of the reference partition of which this induced class is a subset (Subsection 3.4).

The proof is in two parts. First, the family of two-claimant rules satisfying the three fixed-population axioms is characterized. Second, the family of rules defined over the entire domain and satisfying consistency is identified. We focus on that second, more delicate step, exploiting a technique to pass from two claimants to more than two claimants whose usefulness extends much beyond the question addressed here; it can be used very generally to demonstrate the existence of a "consistent extension" of an a priori given two-claimant rule if such an extension exists, and to identity this extension, or to show that none exists if that is the case.

Other applications of this technique are given in several papers. As already mentioned, a didactic presentation is in Thomson (2001), where the technique is also used to determine whether there exists a consistent extension of the two-claimant weighted averages of the constrained equal awards and constrained equal losses rules (the set of awards vectors of a problem is a convex set, which makes this attractive operation possible). The result is that consistency requires that in fact all the weight should always be placed on the constrained equal awards rule, or that all the weight should always be placed on the constrained equal losses rule.

This paper also identifies the consistent extensions of a two-claimant family defined as another kind of compromise between the proportional and constrained equal awards rules: for each claims vector, and as the amount to divide increases, one begins with equal division and at some point, one switches to proportional division. A symmetric proposal is a compromise between the proportional and constrained equal losses rules. In each case, we show that these rules have consistent extensions if the point where the switch occurs between equality and proportionality satisfies certain conditions, which we spell out, and we describe these extensions.

Next is a characterization of the family of weighted generalizations of the so-called Talmud rule (Hokari and Thomson, 2003). The goal there was to maintain the properties that make the Talmud rule attractive, in particular its consistency, but as in the present contribution, not impose symmetry so as to allow favoring claimants who are thought of as being more deserving. The result is that skewing awards towards particular claimants can be done only in some limited way.

Another study starts from a simple family that collects a number of diverse rules that have been central in the theory (Thomson, 2002), including the constrained equal awards, constrained equal losses, Talmud, and minimal overlap rule, the latter being a rule proposed by O'Neill (O'Neill, 1982) to rationalize the resolution in a Medieval text for a particular example. The technique can be used to identify its consistent members.

A final paper shows that the version of the proportional rule obtained by truncating claims at the amount to divide has no consistent extension (Thomson, 2006), as first proved by Dagan and Volij (1997).

In the concluding section, we relate our approach to two approaches proposed by Dagan and Volij (1997) to solve extensions questions.

2 Preliminaries

There is a population of "potential" claimants indexed by $\mathbb{N}$, the set of natural numbers. Alternatively, we could assume the population to be finite and at least equal to 3. (If there are at most two agents, the variable population axiom has no bite.) Let $\mathcal{N}$ be the class of finite subsets of $\mathbb{N}$. A **claims problem with agent set $N \in \mathcal{N}$** is a pair $(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$ where $c \equiv (c_i)_{i \in N}$

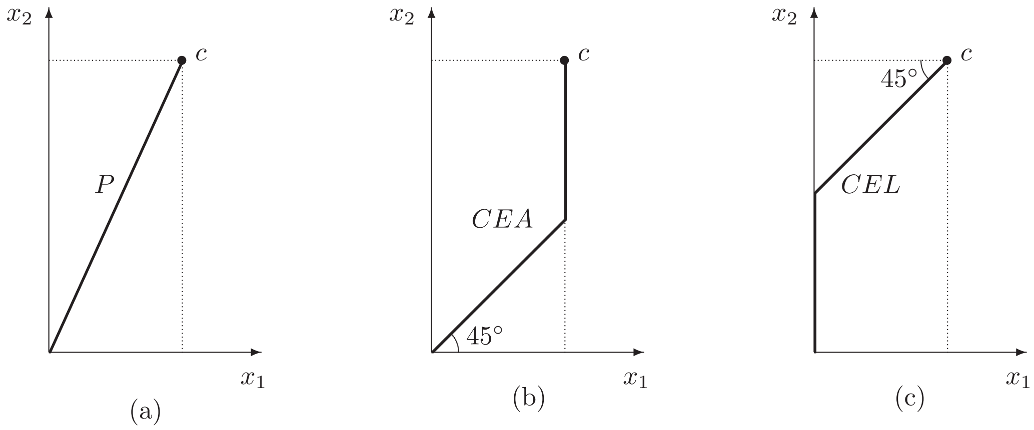

Figure 1: Three rules illustrated in the two-claimant case. (a) Proportional rule. (b) Constrained equal awards rule. (c) Constrained equal losses rule.

and $\sum_N c_i \geq E$. Each agent $i \in N$ has a **claim** c_i over the **amount to divide** $E \in \mathbb{R}_+$: this amount is insufficient to honor all the claims. Let $\mathcal{C}^N$ denote the class of all such problems. An **awards vector for (c, E)** is a vector $x \in \mathbb{R}^N_+$ satisfying the claims boundedness inequalities $x \leqq c$ and the efficiency equality $\sum x_i = E$.[2] Let $X(c, E)$ be the set of awards vectors for (c, E). A **rule** is a mapping defined over $\bigcup_{N \in \mathcal{N}} \mathcal{C}^N$ that associates with each problem an awards vector for it. Let S be our generic notation for rules.

For the two-(or-three-)claimant case, a rule S is conveniently described in a two-(or-three-)dimensional Euclidean space by fixing the claims vector c and determining the path $p^S(c)$ followed by $S(c, E)$ as E increases from 0 to $\sum c_i$. We refer to this path as the **path of awards of S for c**. Our proofs mainly consist in uncovering how these geometric objects should be related, in particular as the claimant set changes.

The notation (c'_i, c_{-i}) designates the vector c with its i-th component replaced by c'_i. Given $x \in \mathbb{R}^N_+$, $B(0, x) \equiv \{y \in \mathbb{R}^N_+ : 0 \leqq y \leqq x\}$. Given $x^1, \ldots, x^k \in \mathbb{R}^N$, and $\ell \in \{1, \ldots, k-1\}$, $\text{seg}[x^\ell, x^{\ell+1}] \equiv \{y \in \mathbb{R}^N$: there is $\lambda \in [0, 1]$ with $y = \lambda x^\ell + (1-\lambda)x^{\ell+1}\}$ and $\text{bro.seg}[x^1, \ldots, x^k] \equiv \text{seg}[x^1, x^2] \cup \ldots \cup \text{seg}[x^{k-1}, x^k]$. Given $A \subset \mathbb{R}^N$, $\text{int}\{A\}$ is the relative interior of A.

Next, we introduce several important rules and illustrate three of them in the two-claimant case by means of their paths of awards for a typical claims vector (Figure 1). We give the definitions for a fixed $N \in \mathcal{N}$. To extend them to $\bigcup_{N \in \mathcal{N}} \mathcal{C}^N$, it suffices to add a universal quantification over $N \in \mathcal{N}$. For the last three, we need to specify a list of weight vectors indexed by $N \in \mathcal{N}$. In general, no special relation has to hold between the vectors chosen for different N's, but our variable-population condition will generate such relations.

[2] Conventions for vector inequalities: given $x, y \in \mathbb{R}^N$, $x \geqq y$ means $x_i \geq y_i$ for each $i \in N$; $x \geq y$ means $x \geqq y$ but $x \neq y$; $x > y$ means $x_i > y_i$ for each $i \in N$.

Our first rule chooses awards proportional to claims. Our second rule assigns equal amounts to all claimants subject to no one receiving more than his claim. Our third rule assigns amounts such that the losses experienced by all claimants are equal subject to no one receiving a negative amount. (O'Neill, 1982, Aumann and Maschler, 1985, Young, 1987, and Dagan, 1996, give references to ancient literature in which these rules are mentioned.) In the formulas, $(c, E) \in \mathcal{C}^N$ is arbitrary, $i \in N$ is arbitrary, and $\lambda \in \mathbb{R}_+$ is chosen so as to achieve efficiency. For the **proportional rule**, $P_i(c, E) \equiv \lambda c_i$; for the **constrained equal awards rule**, $CEA_i(c, E) \equiv \min\{c_i, \lambda\}$; for the **constrained equal losses rule**, $CEL_i(c, E) \equiv \max\{c_i - \lambda, 0\}$. Now, let $w \in \text{int}\{\Delta^N\}$. For the **weighted proportional rule with weights $\boldsymbol{w}$**, $P_i^w(c, E) \equiv \min\{\lambda w_i c_i, c_i\}$; for the **weighted constrained equal awards rule with weights $\boldsymbol{w}$**, $CEA_i^w(c, E) \equiv \min\{c_i, w_i \lambda\}$; for the **weighted constrained equal losses rule with weights $\boldsymbol{w}$**, $CEL_i^w(c, E) \equiv \max\{c_i - \frac{\lambda}{w_i}, 0\}$.

We impose four requirements on rules. First, if claims and amount to divide are multiplied by the same positive number, then so should awards:

Homogeneity: For each $(c, E) \in \mathcal{C}^N$ and each $\lambda > 0$, $S(\lambda c, \lambda E) = \lambda S(c, E)$.

Consider the following situation: after an awards vector has been chosen for a problem, the amount to divide is found to be smaller than originally thought. In dealing with this change, we have two options. (i) We cancel the initial division and recalculate the awards for the revised amount to divide. (ii) We take the initial awards as claims in dividing the revised amount. The next requirement is that both options should result in the same awards vector:

Composition down: For each $(c, E) \in \mathcal{C}^N$ and each $E' < E$, we have $S(c, E') = S(S(c, E), E')$.

The next requirement pertains to the symmetric situation, when an upwards revision of the amount to divide is needed. It states the equivalence of two parallel options. (i) We ignore the initial division and recalculate the awards for the revised amount to divide. (ii) We let agents keep their initial awards and give them as second installments the awards obtained by solving the problem of dividing the increment, after having revised their claims down by the first installments:

Composition up: For each $(c, E) \in \mathcal{C}^N$ and each $E' > E$ such that $\sum c_i \geq E'$, we have $S(c, E') = S(c, E) + S(c - S(c, E), E' - E)$.[3]

Next is the variable-population requirement that is central to our analysis. Consider some problem and solve it by applying the chosen rule. Now, imagine that some claimants leave with their awards and address the problem of dividing what is left among the remaining claimants. The requirement is that in this problem (by definition of a rule, it is well-defined), the same awards should be recommended for them as initially (Aumann and Maschler, 1985; Young, 1987).[4]

Consistency: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $N' \subset N$, if $x \equiv S(c, E)$, then $x_{N'} = S(c_{N'}, \sum_{N'} x_i)$.

3 Characterizations

This section is devoted to the characterizations.

3.1 A family of two-claimant rules

Our starting point is the following family of rules defined for two-claimant populations. They are axiomatized by Moulin (2000) on the basis of *homogeneity*, *composition down*, and *composition up*. Let $N \in \mathcal{N}$ be such that $|N| = 2$. The definition involves "partitioning" $\mathbb{R}^N_+$ into closed cones. For each claims vector in each cone, the path of awards looks like a "compressed" version of the path of the constrained equal awards or constrained equal losses rules (it consists of two segments whose directions are those of the boundary rays of the cone). If the cone is a ray, the path is the segment from the origin to the claims vector. Given two adjacent non-degenerate cones, we find it convenient to also include these rays as cones in the partition. Thus, it is indeed an abuse of language to speak of a "partition" of $\mathbb{R}^N_+$ into cones.

Family $\boldsymbol{\mathcal{D}}$: Let $N \in \mathcal{N}$ be such that $|N| = 2$. A rule S on $\mathcal{C}^N$ in the family $\mathcal{D}$ can be described as follows. Awards space $\mathbb{R}^N_+$ is partitioned into closed cones. For each non-degenerate cone, a boundary ray is chosen as the **first**

[3]Note that the problems appearing in this equality are well-defined since by definition, rules satisfy *claims boundedness*.

[4]For a survey of the literature on *consistency* and its *converse*, see Thomson (1999).

ray; the other is the **second ray**. Now, let $c \in \mathbb{R}^N_+$. If c belongs to a degenerate cone, $p^S(c)$ is the segment from the origin to c. If c belongs to a non-degenerate cone, $p^S(c)$ consists of a segment containing the origin and contained in the first boundary ray of the cone, and a segment parallel to the second boundary ray containing c.[5,6]

A cone with labelled boundary rays is an **oriented cone**. A ray in a partition is **interior** if it is not an axis.[7] A non-degenerate cone in a partition is **interior** if neither of its boundary rays is interior.

A special case is when the partition contains the entire awards space as a cone. Then, a typical path of awards consists of a segment containing the origin and contained in one of the axes (the first ray of the cone) and a segment parallel to the other axis (the second ray).

Figures 2 and 3 illustrate Definition $\mathcal{D}$. The left panel of each row represents the partition of awards space into oriented cones defining a rule. Let $N \equiv \{1, 2\}$. For each non-degenerate cone in the partition, the first boundary ray is labelled "1" and the other one "2" (the labels are printed inside the cone). The cone whose first boundary ray is in the direction $a \in \mathbb{R}^N_+ \setminus \{0\}$ and second boundary ray in the direction $b \in \mathbb{R}^N_+ \setminus \{0\}$ is denoted $C(a, b)$. The right panel of the row represents sample paths of awards. Row 1 of Figure 2 shows the constrained equal awards rule, Row 2 a weighted constrained equal awards rule, and Row 3 the constrained equal losses rule. Note that the partition of $\mathbb{R}^N_+$ for that rule is the same as that for the constrained equal awards rule but the orientation of the cones is reversed. Row 4 shows a general example.

The first two rules depicted in Figure 3 are **anonymous** (the paths of awards of two claims vectors that are symmetric with respect to the 45° line

[5]If c belongs to a boundary ray, one of these segments reduces to a point.

[6]This description differs slightly from that given by Moulin (2000). Alternatively, we could include in each non-degenerate cone its first ray, and add to the partition any ray that is the common boundary ray of two adjacent non-degenerate cones, if it is second for both cones. In any case, no description can be at the same time minimal and reflect the *anonymity* of rules that are *anonymous* (a formal definition of this property is given below). For instance, for a description of a rule such as the constrained equal awards rule to reflects its *anonymity*, the 45° line would have to be included in both of the non-degenerate cones that are needed (then, all claims vectors on the 45° line are covered twice), or in neither (then, the description has to include the 45° line as cone and it is not minimal), or in only one (then the *anonymity* of the rule is not shown). Our description is not minimal but it shows *anonymity*.

[7]Here, we really mean the non-negative part of an axis.

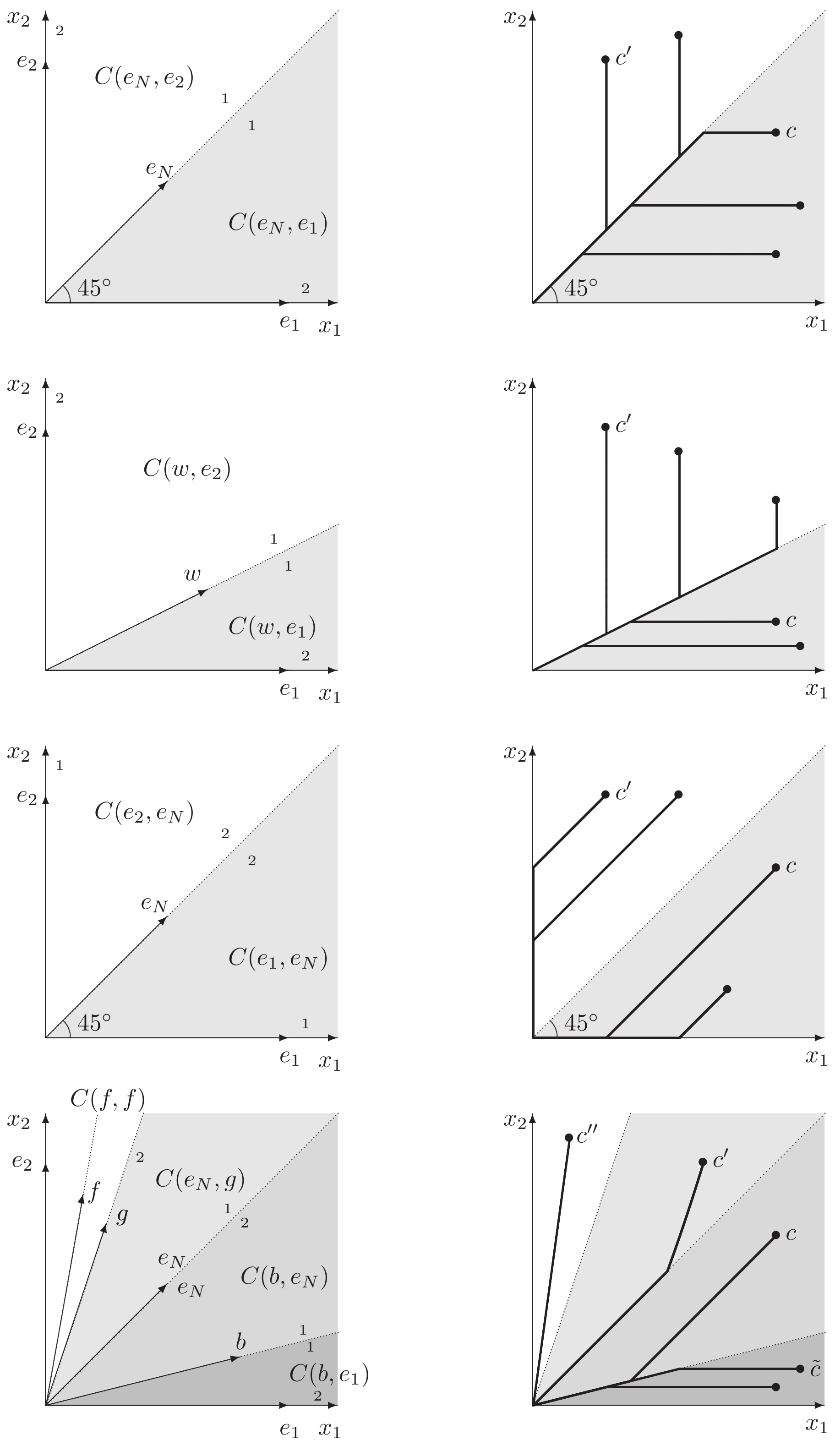

Figure 2: Four members of the family $\mathcal{D}$. Row 1: Constrained equal awards rule. Row 2: A weighted constrained equal awards rule. Row 3: Constrained equal losses rule. Row 4: General example for which the partition consists of three non-degenerate cones and a cone of rays (the unshaded area).

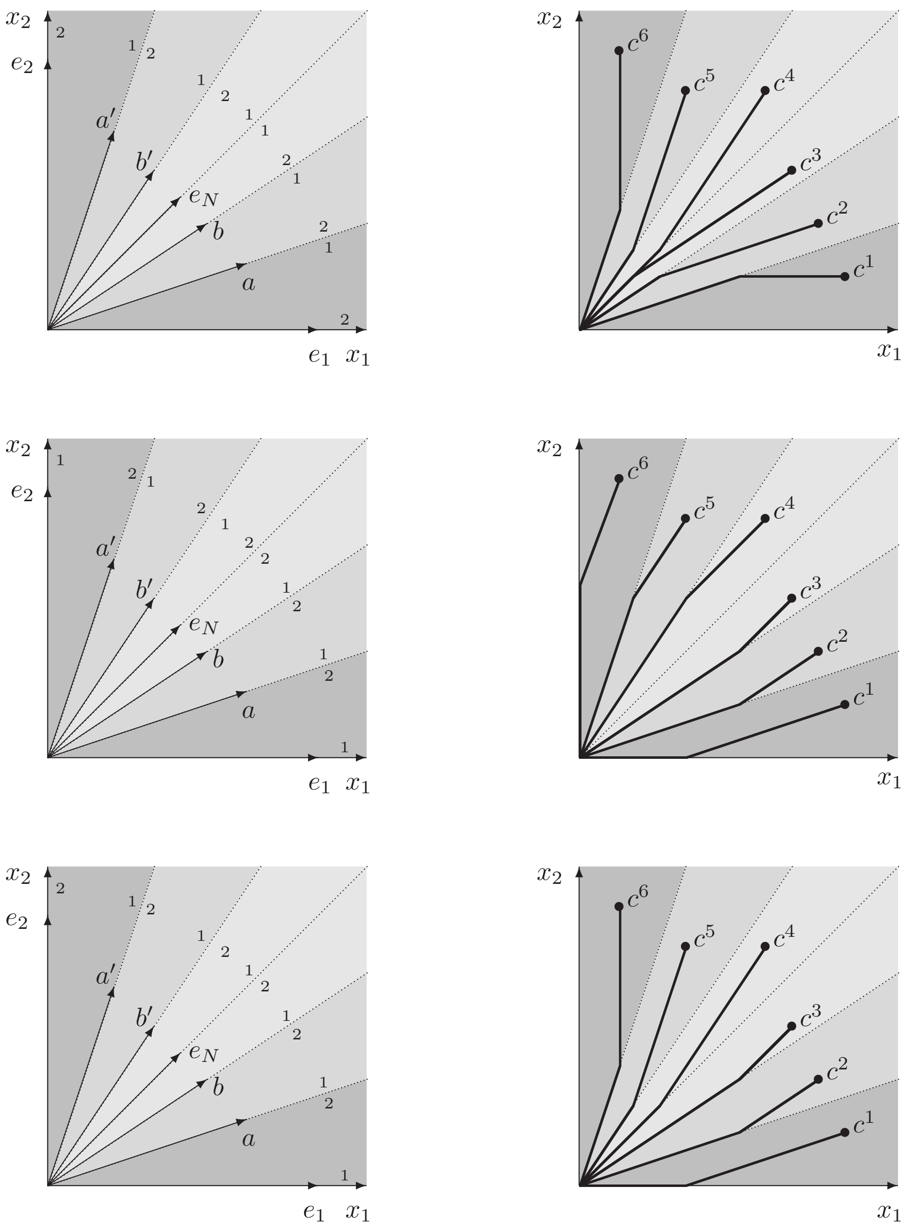

Figure 3: Three additional members of the family $\mathcal{D}$. Row 1: For this example, in each of the cones into which awards space is partitioned, the path is "pulled" to the 45° line. Row 2: Here, the partition of awards space is the same but the orientation of each cone is reversed. Thus, in each cone, paths of awards are pulled to the coordinate axis that is the closest. Row 3: Here, paths are all pulled to the first axis.

also have that symmetry), and they illustrate how the family under discussion provides a way of "linking" the proportional rule to the constrained equal awards and constrained equal losses rules. In Row 1, the orientation of each cone favors the agent with the smaller claim. In Row 2, the partition is the same but each cone is given the reverse orientation; the resulting rule favors the agent with the larger claim. In Row 3, the partition is still the same but the orientation of all of the cones favors claimant 1, independently of the relative values of the claims. If in Row 1, we reversed the orientation of the non-degenerate cone containing $\mathbb{R}^{\{1\}}_+$, we would obtain a rule that maximally favors claimant 1 subject to the **order preservation** conditions being met, namely that awards be ordered as claims are, and that so be losses (Aumann and Maschler, 1985).

If the partition is fine, paths of awards are close to the paths of awards of the proportional rule.

It may be surprising that the weighted proportional rules are not members of the family, but in fact they satisfy neither *composition down* nor *composition up*. To see this, consider the following examples:

Example 1 Let $N \equiv \{1, 2\}$ and $w \equiv (\frac{1}{3}, \frac{2}{3})$. Let $c \equiv (6, 2)$ and note that $p^{P^w}(c) = \text{bro.seg}[(0,0),(3,2),(6,2)]$. Let now $c' \equiv (4, 2)$ and observe that $p^{P^w}(c') = \text{bro.seg}[(0,0),(2,2),(4,2)]$. Since $c' \in p^{P^w}(c)$ but $p^{P^w}(c')$ is not a subset of $p^{P^w}(c)$, we have a contradiction to *composition down*.

Example 2 Let $N \equiv \{1, 2\}$ and $w \equiv (\frac{1}{3}, \frac{2}{3})$. Let $c \equiv (4, 2)$ and note that $p^{P^w}(c) = \text{bro.seg}[(0,0),(2,2),(4,2)]$. Let now $x \equiv (1, 1) \in p^{P^w}(c)$. Note that $p^{P^w}(c - x) = \text{bro.seg}[(0,0),(1.5,1),(3,1)]$. Since $\{x\} + p^{P^w}(c - x)$ is not a subset of $p^{P^w}(c)$, we have a contradiction to *composition up*.

3.2 Adding consistency

At this point, variable-population considerations enter the scene. We impose *consistency* and identify the restrictions on the partitions of two-dimensional awards spaces associated with rules in $\mathcal{D}$ that are implied by this property. We also show what the rules look like for more than 2 claimants.

It will be useful to relate the first step of the proof (Lemma 1), in which a partition of the set of potential agents into priority classes is derived, to similar constructions in other branches of the theory of resource allocation. When no symmetry-type requirement is imposed, a standard way to obtain *consistency* is by means of so-called "dictatorial rules". For such a rule, and for each group of agents, there is an agent, the "dictator", whose preferred alternative is chosen for each problem this group may face (for the classical problem of allocating privately appropriable and infinitely divisible goods among agents with standard preferences, he receives the entire social endowment). The dictator does not vary arbitrarily from group to group however. Given two groups containing two given agents, the first agent cannot be the dictator in one group and the other agent a dictator in the other group. The required relations between the identities of the dictators are achieved by specifying a strict "reference order" on the set of potential agents, and for each group, selecting as dictator the agent who is first in the order induced on that group by the reference order.

In certain situations, simply maximizing a dictator's preferences may violate requirements of interest, a prime example being efficiency (for the classical problem, assigning the entire social endowment to one agent may result in an inefficient allocation if his preferences are not monotonic). To prevent this possibility, one designates, for each group, a "second-order" dictator, a "third-order" dictator, and so on. If the first-order dictator's preferences are maximized at several alternatives, the second-order dictator is brought in to break the tie, and if his preferences are maximized at several alternatives, the third-order dictator is called upon to break this second-order tie, and so on. For the resulting "lexicographic dictatorial" rules to be *consistent*, and given two groups that overlap, the restrictions to their intersection of the orders specified for them should coincide. Here too, this is achieved by specifying a strict reference order on the set of potential agents, and for each group, using the order induced on that group by the reference order.

More generally, one can partition each group into subgroups, choose a strict order on the subgroups, and when solving any problem, not give anything to any of the subgroups until the preferences of all agents in subgroups with higher priorities have been maximized. Let us call these subgroups "priority classes". *Consistency* is achieved by inducing the orders on subgroups from a weak reference order on the set of potential agents (this order need not be strict any more). To describe a rule, it suffices, for each priority class separately, to describe the restriction of the rule to this class. Priority classes are "insulated" from each other in the sense that, provided that each such restriction is *consistent*, then *consistency* holds overall. The restrictions do not have to be related in any particular way.

In the situation considered here, a constraint exists on how much an agent (or a group of agents) can receive, namely his claim (or the sum of the claims of its members). If the amount to divide is greater than his claim (or the sum of their claims), and this claimant has priority (or this group of claimants has priority), one has to know what to do with the remainder once he has (they have) been fully compensated. Let S be a rule that coincides with a member of $\mathcal{D}$ for each two-claimant group and satisfies *consistency*. Lemma 1 states that a partition of the set of potential claimants into priority classes can be associated with S. Let $\tilde{N}$ be such a class. If $|\tilde{N}| = 1$, then of course, the restriction of S to $\mathcal{C}^{\tilde{N}}$ is uniquely defined: there is only one awards vector for a one-claimant problem. If $|\tilde{N}| = 2$, *consistency* has no bite: Definition $\mathcal{D}$ describes the kind of two-claimant rules with which S has to coincide on $\mathcal{C}^{\tilde{N}}$. If $|\tilde{N}| \geq 3$, we will show that the following possibilities emerge:

(i) The awards space pertaining to each two-claimant subgroup of $\tilde{N}$ is partitioned into rays. Then, S coincides on $\bigcup_{N \subseteq \tilde{N}} \mathcal{C}^{\tilde{N}}$ with the proportional rule (Lemma 3). (ii) The awards space pertaining to each two-claimant subgroup of $\tilde{N}$ is partitioned into two non-degenerate cones (Lemma 4). Moreover, either (iia) the boundary ray shared by the two cones in the partition of each such awards space is always first, or (iib) it is always second. In either case, the directions of these shared boundary rays are "consistent" (Lemma 5). (iia) amounts to saying that in each two-dimensional space, a weighted constrained equal awards rule is applied, and (iib) to saying that in each two-dimensional space, a weighted constrained equal losses rule is applied. The direction of the central ray in the $\{i, j\}$-partition indicates the relative "importance" the rule gives to claimants i and j. By "consistency" of the directions of the central rays, we mean for example that if in $\mathbb{C}^{\{i,j\}}$, the rule gives to claimant j twice the importance it gives to claimant i, and in $\mathbb{C}^{\{j,k\}}$, it gives to claimant k twice the importance it gives to claimant j, then in $\mathbb{C}^{\{k,i\}}$, it gives to claimant k four times the importance it gives to claimant i. Finally, these conclusions for two-claimant problems extend to all populations: either for each $N \subset \tilde{N}$, S coincides on $\mathcal{C}^N$ with a weighted constrained equal awards rule, or for each $N \subset \tilde{N}$, S coincides on $\mathcal{C}^N$ with a weighted constrained equal losses rule; in each case, the weights used for each $N \subset \tilde{N}$ are derived from a single set of weights for $\tilde{N}$.

Family $\mathcal{M}$ (Moulin, 2000) *A rule S belonging to the family can be described as follows. There is an ordered "reference" partition of the set of potential claimants into priority classes. Let $\tilde{N}$ be such a class. • If $|\tilde{N}| = 2$, awards space $\mathbb{R}^{\tilde{N}}_+$ is partitioned as in Definition $\mathcal{D}$. • If $|\tilde{N}| \geq 3$, three possible labels* *are attached to* $\tilde{N}$:

(i) *"P"*,

(ii) *"weighted CEA", a vector* $\tilde{w} \in \mathbb{R}^{\tilde{N}}_{++}$ *of weights being given,*

(iii) *"weighted CEL", a vector* $\tilde{w} \in \mathbb{R}^{\tilde{N}}_{++}$ *of weights being given.*

For each $N \in \mathcal{N}$ *and each* $(c, E) \in \mathcal{C}^N$*, the reference partition induces an ordered partition of* N*, whose components are handled in succession. Let* $\bar{N}$ *be such a component and* $\tilde{N}$ *be the class from which* $\bar{N}$ *is induced. • If* $|\tilde{N}| = 1$*,* S *selects the unique awards vector for each problem in* $\mathcal{C}^{\bar{N}}$*. • If* $|\tilde{N}| = 2$*,* S *coincides on* $\mathcal{C}^{\bar{N}}$ *with the rule associated with the partition specified for* $\tilde{N}$*. • If* $|\tilde{N}| \geq 3$*, we distinguish three cases according to the labels attached to* $\tilde{N}$*:*

(i) *"P":* S *coincides on* $\mathcal{C}^{\bar{N}}$ *with* P*.*

(ii) *"weighted CEA":* S *coincides on* $\mathcal{C}^{\bar{N}}$ *with* $CEA^{\tilde{w}_{\bar{N}}}$*.*

(iii) *"weighted CEL":* S *coincides on* $\mathcal{C}^{\bar{N}}$ *with* $CEL^{\tilde{w}_{\bar{N}}}$*.*

Here is an example: using the left-to-right direction to indicate lower and lower priorities, let the reference ordered partition be $\{\{1, 2, 3\}, \{4, 6\}, \{8, 10, 12, \ldots\}, \{5, 7, 9, \ldots\}\}$, the first class being labelled P, an ordered partition of $\mathbb{R}^{\{4,6\}}_{+}$ as in Definition $\mathcal{D}$ being specified for the second class, the third class being labelled CEA with weights $(1, 1, 1, \ldots)$, and the last class being labelled CEL with weights $(1, 2, 3, \ldots)$. Let $N \equiv \{2, 3, 4, 6, 7, 8, 9, 10, 12\}$. The ordered partition of N induced by the reference partition is $\{\{2, 3\}, \{4, 6\}, \{8, 10, 12\}, \{7, 9\}\}$. Thus, in succession: the proportional rule is applied for $\{2, 3\}$; the rule corresponding to the partition of $\mathbb{R}^{\{4,6\}}_{+}$ specified for that space is applied for $\{4, 6\}$; the weighted constrained equal awards rule with weights proportional to $(1, 1, 1)$ is applied for $\{8, 10, 12\}$; the weighted constrained equal losses rule with weights proportional to $(2, 3)$ is applied for $\{7, 9\}$.

Our goal is to present a proof of the following result (Moulin, 2000).

Theorem 1 *A rule on* $\bigcup_{N \in \mathcal{N}} \mathcal{C}^N$ *satisfies homogeneity, composition down, composition up, and consistency, if and only if it belongs to the family* $\mathcal{M}$*.*

3.3 A tutorial on constructing consistent extensions

The strategy we follow to prove that only members of $\mathcal{M}$ satisfy the axioms exploits the fact that *consistency* of a rule S implies that for each $N \in \mathcal{N}$, each $c \in \mathbb{R}^N$, and each $N' \subset N$, $p^S(c)$, when projected on $\mathbb{R}^{N'}$, is a subset of $p^S(c_{N'})$. Moreover, if S is **resource monotonic**, that is, assigns to each claimant an award that is a nowhere decreasing function of the amount to divide, coincidence actually occurs.[8] Since the two-claimant rules for which we investigate the existence of *consistent* extensions are indeed *resource monotonic*, these extensions share this property (in the language of Hokari and Thomson, 2002, *resource monotonicity* is "lifted" by *consistency*, as follows from a result of Dagan and Volij, 1997). Thus, $p^S(c)$ is a continuous curve from the origin to c, which implies that its projection on $\mathbb{R}^{N'}$ is a continuous curve from the origin to $c_{N'}$.

Let $N \in \mathcal{N}$ with $|N| = 3$, say $N \equiv \{1, 2, 3\}$ (the logic is the same for more agents), and let $c \in \mathbb{R}^N_+$. Then, $p^S(c)$ is such that its projections on the three two-dimensional subspaces $\mathbb{R}^{\{1,2\}}$, $\mathbb{R}^{\{1,3\}}$, and $\mathbb{R}^{\{2,3\}}$, are $p^S(c_{\{1,2\}})$, $p^S(c_{\{1,3\}})$, and $p^S(c_{\{2,3\}})$ respectively. Thus, the question is whether there is indeed a path in $\mathbb{R}^N$ that projects on these three paths. As the rule is kept fixed and in each step of the proof, a claims vector is chosen once and for all, we use the notation Π_3 for $p^S(c_{\{1,2\}})$, Π_2 for $p^S(c_{\{1,3\}})$, Π_1 for $p^S(c_{\{2,3\}})$, Π for $p^S(c)$, and Π'_1 for the projection of Π on $\mathbb{R}^{\{2,3\}}$. In most applications, one is able to uniquely determine a path in $\mathbb{R}^N$ by exploiting this projection requirement on only two two-dimensional subspaces, say $\mathbb{R}^{\{1,2\}}$ and $\mathbb{R}^{\{1,3\}}$. This is the case if at least one of these two paths is strictly monotone. If Π_3 contains a segment parallel to $\mathbb{R}^{\{2\}}$ and Π_2 contains a segment parallel to $\mathbb{R}^{\{3\}}$, the path for c can also be recovered, provided these segments do not lie in a common plane parallel to $\mathbb{R}^{\{2,3\}}$. If either Π_3 or Π_2 contains a segment parallel to the first axis, or if they both do, the recovery can also proceed. Once Π is obtained, projecting it on $\mathbb{R}^{\{2,3\}}$ should yield Π_1. If violations of strict monotonicity occur for both Π_3 and Π_2 in a common plane parallel to $\mathbb{R}^{\{2,3\}}$, one recovers Π by invoking the requirement that its projection on $\mathbb{R}^{\{2,3\}}$ has to be Π_1.

The method is illustrated in Figure 4 by means of four examples that are representative of all of the configurations that we will encounter. To obtain Π from Π_3 and Π_2, we intersect the cylinder spanned by Π_3 whose generators are parallel to $\mathbb{R}^{\{3\}}$ with the cylinder spanned by Π_2 whose generators are parallel to $\mathbb{R}^{\{2\}}$. We construct this intersection "plane by plane": for each $t \in [0, c_1]$, we identify the intersections of the plane $H(t)$ of equation $x_1 = t$

[8] It is essentially this projection property that guarantees *population monotonicity* of the monotone path solutions of bargaining theory, namely the property that the arrival of additional agents unaccompanied by an expansion of opportunities should make all agents initially present at most as well off as they were initially (Thomson, 1987).

with Π_3 and Π_2, and use the facts that (i) each point in either intersection has to be the projection on $\mathbb{R}^{\{1,2\}}$ and $\mathbb{R}^{\{1,3\}}$ respectively of a point of Π, and (ii) conversely, the projections of each point of $\Pi \cap H(t)$ has to be in these intersections.

Each of the remaining figures should be scanned row by row: panel (a) shows Π_3 and Π_2, panel (b) the construction of Π and panel (c) the projection Π'_1 of Π on $\mathbb{R}^{\{2,3\}}$. We have shaded $H(t)$ when it reaches critical positions: when it contains a kink in either Π_3 or Π_2 or in both (because this produces a kink in Π, and in turn and in most cases, a kink in Π'_1), and when it contains a segment in either of these paths (because then, Π contains a segment parallel to either $\mathbb{R}^{\{2\}}$ or $\mathbb{R}^{\{3\}}$, and thus, so does Π'_1).

Row 1: both Π_3 and Π_2 are strictly monotone. Then, for each $t \in [0, c_1]$, the plane $H(t)$ intersects each of these paths at a single point; from these two points, we deduce a point of Π. (In this row, the shaded plane is a generic plane.) The whole of Π can be obtained.

Row 2: Π_3 is piecewise linear in two pieces and Π_2 is linear. Then, Π is piecewise linear in two pieces and so is Π'_1.

Row 3: Π_3 and Π_2 are strictly monotone; they are piecewise linear in two pieces, and the first coordinates of their kinks differ. Then, Π is piecewise linear in three pieces and so is Π'_1.

Row 4: Π_3 starts with a segment in $\mathbb{R}^{\{2\}}$ (the vertical axis), and therefore is not strictly monotone; however, the intersection of $H(0)$ with Π_2 is a singleton. Thus, Π also starts with a segment in $\mathbb{R}^{\{2\}}$, and of course Π'_1 does too.

In many applications—it is the case here—paths of awards are piecewise linear, and the issue is often whether kinks in a three-dimensional path are lost in the projection on a two-dimensional space, contradictions being derived by exhibiting configurations for which a projection has too many kinks. Row 3 shows two two-dimensional paths, Π_3 in $\mathbb{R}^{\{1,2\}}$ and Π_2 in $\mathbb{R}^{\{1,3\}}$, from which one can derive a path in $\mathbb{R}^{\{2,3\}}$ that has two kinks. However, if the second coordinate of the kink in Π_3 were c_2, and the first coordinate of the kink in Π_2 were c_1, (or the first coordinates of both kinks were c_1), Π would still have two kinks, but Π'_1 would only have one. These situations are important. It is precisely because kinks in their paths are located "in the right place" that the two-claimant weighted constrained equal awards or constrained equal losses rules can have *consistent* extensions: they do if their weight vectors are appropriately related. The last step of our main theorem

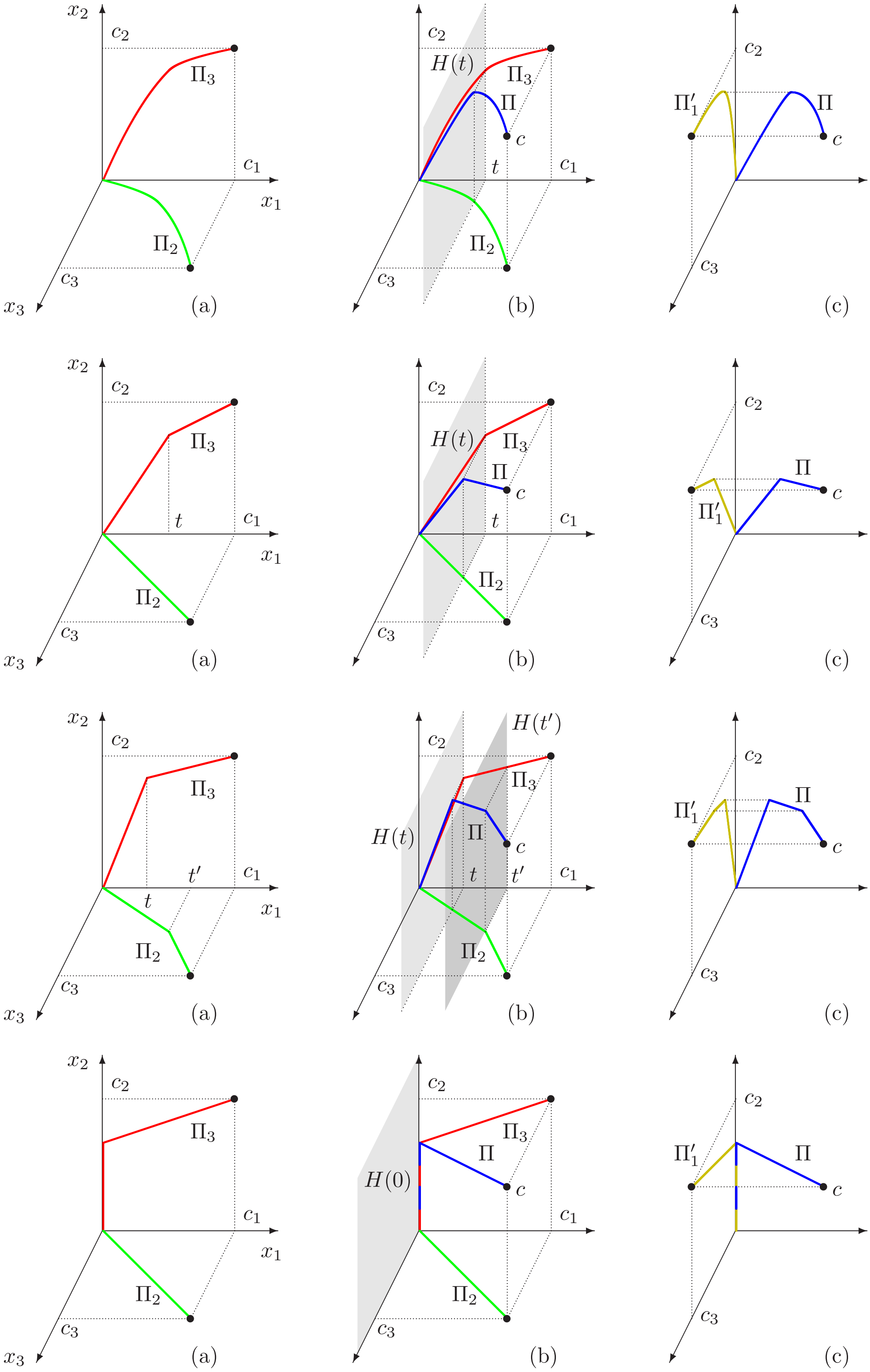

Figure 4: Four typical configurations. Row 1: Π_3 and Π_2 are strictly monotone. Row 2: Π_3 has a kink but Π_2 does not. Row 3: Π_3 has a kink and so does Π_2; these kinks have unequal first coordinates. Row 4: Π_3 contains a segment in $\mathbb{R}^{\{2\}}$.

relies crucially on this fact.

Note: Paths are color-coded according to the space to which they belong. When a segment belongs to several paths, such as the segment in $\mathbb{R}^{\{2\}}$ of panel (a) of Row 4, it appears on a monitor as a series of dashes whose alternating colors are those assigned to these paths. (On a black-and-white printer, different colors come out as different shades of grey, which unfortunately are not sufficiently distinct to show when two paths have a segment in common, but common segments can of course be deduced from the fact that a path has to be a continuous curve from the origin to the claims vector.)

3.4 Proof of the main result

We identify which of the partitions of awards space allowed by Definition $\mathcal{D}$ are acceptable if a *consistent* extension is to exist. We eliminate unacceptable partitions by forcing two kinds of contradiction. Let $c \in \mathbb{R}^{\{1,2,3\}}$ say, and let us refer to the partition of the awards space of a two-claimant group $N \subset \{1,2,3\}$ as the ***N*-partition**.

1. When the projection on $\mathbb{R}^{\{2,3\}}$ of $p^S(c)$ consists of two segments, these segments "reveal" the directions of the boundary rays of a non-degenerate cone in the $\{2,3\}$-partition (as in panels (c) of Rows 2 and 4 of Figure 4). By performing the operation for $c' \neq c$, one may obtain a second cone that overlaps with the first cone. We then have an **overlap contradiction**.

2. The path for c may have two kinks, neither of which is lost in the projection on $\mathbb{R}^{\{2,3\}}$. Since, by Definition $\mathcal{D}$, paths in that space have at most one kink, we have a **two-kink contradiction** (as in Row 3 of Figure 4).

In the figures, all paths are piecewise linear. To save space, we do not explicitly indicate the coordinates of their kinks.

Proof: (of the main theorem) We omit the easy proof that the members of $\mathcal{M}$ satisfy the four axioms listed in the theorem. Conversely, let S be a rule satisfying the four axioms. The proof that $S \in \mathcal{M}$ is in four lemmas (one of which, Lemma 2, being a general model-free lemma).

Lemma 1 *The set of potential claimants is partitioned into priority classes.*

Proof: We defined a relation on the set of claimants. Given $i, j \in \mathbb{N}$, ***i* has priority over *j*** (for S), written as $i \prec j$, if for each $c \in \mathbb{R}_+^{\{i,j\}}$,

$p^S(c) = \text{bro.seg}[(0,0),(c_i,0),c]$. If neither $i \prec j$ nor $j \prec i$, we write $i \sim j$. If either $i \prec j$ or $i \sim j$, we write $i \preceq j$.

Step 1: $\preceq$ is complete and transitive. Completeness is trivial. For transitivity, it suffices to show that if $i \prec j$ and $j \preceq k$, then $i \prec k$. To fix the ideas, suppose that $i = 2$, $j = 1$, and $k = 3$: thus $2 \prec 1$ and $1 \preceq 3$. We need to show that $2 \prec 3$. Let $(c_2, c_3) \in \mathbb{R}_+^{\{2,3\}}$. Let $N \equiv \{1,2,3\}$.

Case 1: $1 \prec 3$ (Row 1 of Figure 5). Let $c_1 > 0$ and $c \equiv (c_1, c_2, c_3)$. Then, $\Pi_3 = \text{bro.seg}[(0,0),(0,c_2),(c_1,c_2)]$ and $\Pi_2 = \text{bro.seg}[(0,0),(c_1,0),(c_1,c_3)]$. The only path in $\mathbb{R}^N$ whose projections on $\mathbb{R}^{\{1,2\}}$ and $\mathbb{R}^{\{1,3\}}$ are Π_3 and Π_2 respectively is $\text{bro.seg}[(0,0,0),(0,c_2,0),(c_1,c_2,0),c]$: by *consistency*, Π is this path.

Case 2: $1 \sim 3$ (Row 2 of Figure 5). Then, the $\{1,3\}$-partition contains at least one interior ray λ. Let $c_1 \in \mathbb{R}_+$ be such that $(c_1, c_3) \in \lambda$ and $c \equiv (c_1, c_2, c_3)$. Then, $\Pi_3 = \text{bro.seg}[(0,0),(0,c_2),(c_1,c_2)]$ and $\Pi_2 = \text{bro.seg}[(0,0),(c_1,c_3)]$. The only path in $\mathbb{R}^N$ whose projections on $\mathbb{R}^{\{1,2\}}$ and $\mathbb{R}^{\{1,3\}}$ are Π_3 and Π_2 respectively is $\text{bro.seg}[(0,0,0),(0,c_2,0),c]$: by *consistency*, Π is this path.

In either case, the projection Π_1' of Π on $\mathbb{R}^{\{2,3\}}$ is $\text{bro.seg}[(0,0),(c_2,0),(c_2,c_3)]$. By *consistency*, $\Pi_1 = \Pi_1'$. This conclusion holds for each $(c_2,c_3) \in \mathbb{R}_+^{\{2,3\}}$. Thus, $2 \prec 3$.

Step 2: for each $N \in \mathcal{N}$, the components of the ordered partition of N induced by $\preceq$ are handled in succession. Indeed, suppose that there are $i, j \in \mathbb{N}$, $N \in \mathcal{N}$ with $\{i,j\} \subseteq N$, and $(c,E) \in \mathcal{C}^N$ such that $(*)$ $x_i < c_i$ and $x_j > 0$, where $x \equiv S(c,E)$. Let $N' \equiv \{i,j\}$. By *consistency*, $x_{N'} = S(c_{N'}, \sum_{N'} x_k)$. By $(*)$, it is not the case that $i \prec j$. □

Having partitioned the population of potential claimants into priority classes, we now study each class separately. There is only one way to solve a one-claimant problem. For a two-claimant class, *consistency* has no bite; then, an element of the family $\mathcal{D}$ has to be applied. Let then $\tilde{N}$ be a class with more than two claimants. In fact, let us assume $|\tilde{N}| = 3$; if $|\tilde{N}| > 3$, our conclusion easily extends by induction. To simplify notation, set $\tilde{N} \equiv \{1,2,3\}$. We identify restrictions that the partitions for the two-claimant subsets of $\tilde{N}$ have to satisfy. The proof of the theorem concludes by invoking the structural lemma[9] below relating *consistency* and the following requirement on a

[9]It is stated in this form in Thomson (1999).

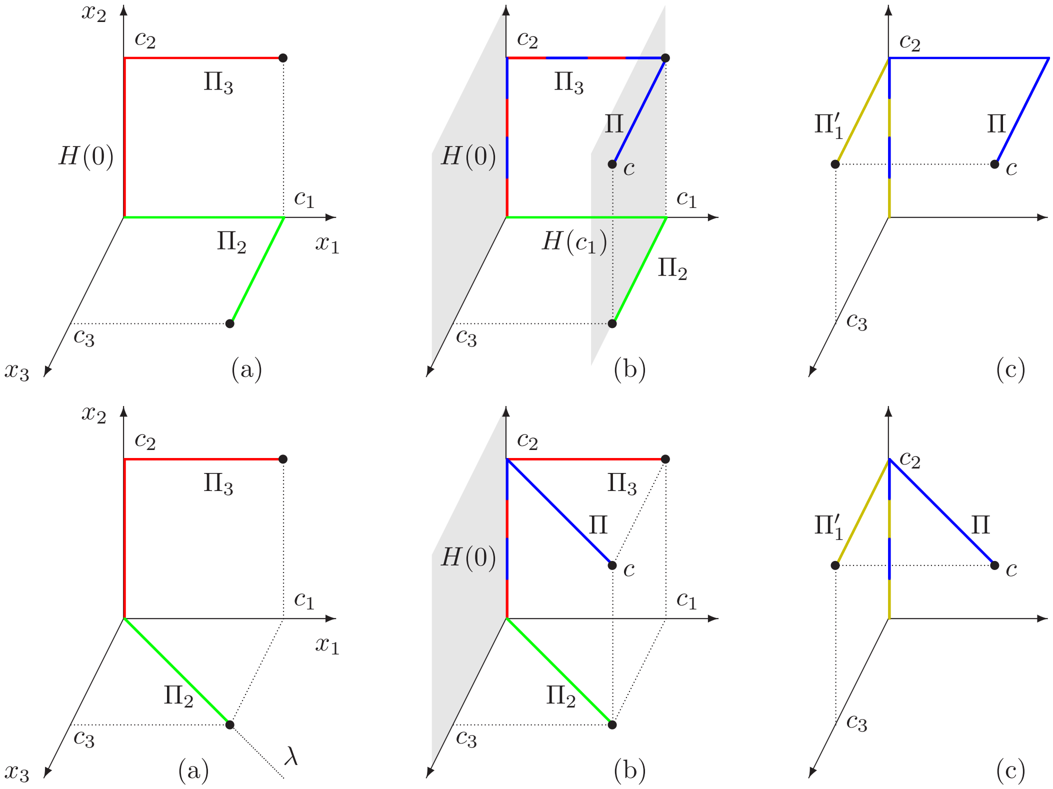

Figure 5: Lemma 1 says that the set of potential claimants is partitioned into priority classes. In each row, claimant 2 has priority over claimant 1. Row 1: claimant 1 has priority over claimant 3. Row 2: neither one of claimant 1 or claimant 3 has priority over the other.

rule: whenever an awards vector for some problem is such that its restriction to each two-claimant group is the choice the rule makes for the associated reduced problem, then it is the choice it makes for the initial problem:

Converse consistency: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $x \in X(c, E)$, if for each two-claimant group $N' \subset N$, $x_{N'} = S(c_{N'}, \sum_{N'} x_i)$, then $x = S(c, E)$.

Lemma 2 *(Elevator Lemma) Given two rules on $\bigcup_{N \subseteq \mathbb{N}} \mathcal{C}^N$, if one is consistent, the other conversely consistent, and they coincide for each two-claimant group, then they coincide in general.*

The lemma holds for rules defined on $\bigcup_{N \subseteq \tilde{N}} \mathcal{C}^N$ for any $\tilde{N} \subseteq \mathbb{N}$. Thus, the following is an immediate consequence of the Elevator Lemma and of the fact that the proportional rule and, if the weights used for different groups are consistent, all weighted constrained equal awards and constrained equal losses rules, are both *consistent* and *conversely consistent.*

Corollary 1 *Let S be a rule on $\bigcup_{N \subseteq \tilde{N}} \mathcal{C}^N$ that is consistent.*

(i) If S coincides for each two-claimant group with the proportional rule, then it is the proportional rule.

(ii) If there is $w \in \mathbb{R}^{\tilde{N}}_{++}$ such that for each $\{i,j\} \subset \tilde{N}$, S coincides with $CEA^{\{w_i,w_j\}}$, then for each $N \subset \tilde{N}$, it coincides on $\mathcal{C}^N$ with CEA^{w_N}.

(iii) A statement parallel to (ii) holds for the constrained equal losses rule.

Given a ray $\rho \in \mathbb{R}^{\{1,2\}}_+$ of direction (α_1, α_2) and a ray $\lambda \in \mathbb{R}^{\{1,3\}}_+$ of direction (β_1, β_3) such that $\alpha_2\beta_3 > 0$ and $\alpha_1\beta_1 > 0$, we designate by $d(\rho, \lambda)$ the ray in $\mathbb{R}^{\{2,3\}}_+$ of direction $(\alpha_2\beta_1, \alpha_1\beta_3)$. Next, we return to the proof of the main theorem. There are two possibilities, one covered by Lemma 3, and the other by Lemmas 4-5.

Lemma 3 *If the partition for each pair of claimants in $\tilde{N}$ consists entirely of rays, then $S = P$.*

Proof: This is part (i) of Corollary 1. □

Lemma 4 *If there is a two-claimant group $\bar{N} \subset \tilde{N}$ such that the $\bar{N}$-partition contains a non-degenerate cone K, then, for each two-claimant group $N \subset \tilde{N}$ (including $\bar{N}$), the N-partition consists of exactly two non-degenerate cones and their boundary rays (or equivalently, the N-partition has exactly one interior ray).*

Proof: Without loss of generality, suppose $\bar{N} \equiv \{1, 2\}$.

Step 1: K is not an interior cone. To show this, we suppose otherwise, calling its boundary rays ρ and ρ', ρ being its steeper ray.

Substep 1-1: the $\{2,3\}$-partition contains an interior non-degenerate cone. To prove the assertion, we use the fact that since $1 \sim 3$, the $\{1,3\}$-partition contains at least one interior ray λ. Let $c \in \mathbb{R}^{\tilde{N}}_+$ be such that $c_{\{1,2\}} \in \text{int}\{K\}$ and $c_{\{1,3\}} \in \lambda$ (Row 1 of Figure 6). This choice of c produces a configuration of the type illustrated by Row 2 of Figure 4. Panel (a) of Row 1 of Figure 6 differs from panel (a) of that earlier row only in the labelling that is added to show the oriented cone K from which the path for (c_1, c_2) is derived. The cone in $\mathbb{R}^{\{2,3\}}$ revealed by the path for $c_{\{2,3\}}$ that we obtain, denoted M, is also indicated in panel (c). Its boundary rays are $d(\rho, \lambda)$ and $d(\rho', \lambda)$, written $\rho\lambda$ and $\rho'\lambda$ in the figure (we use this more compact notation

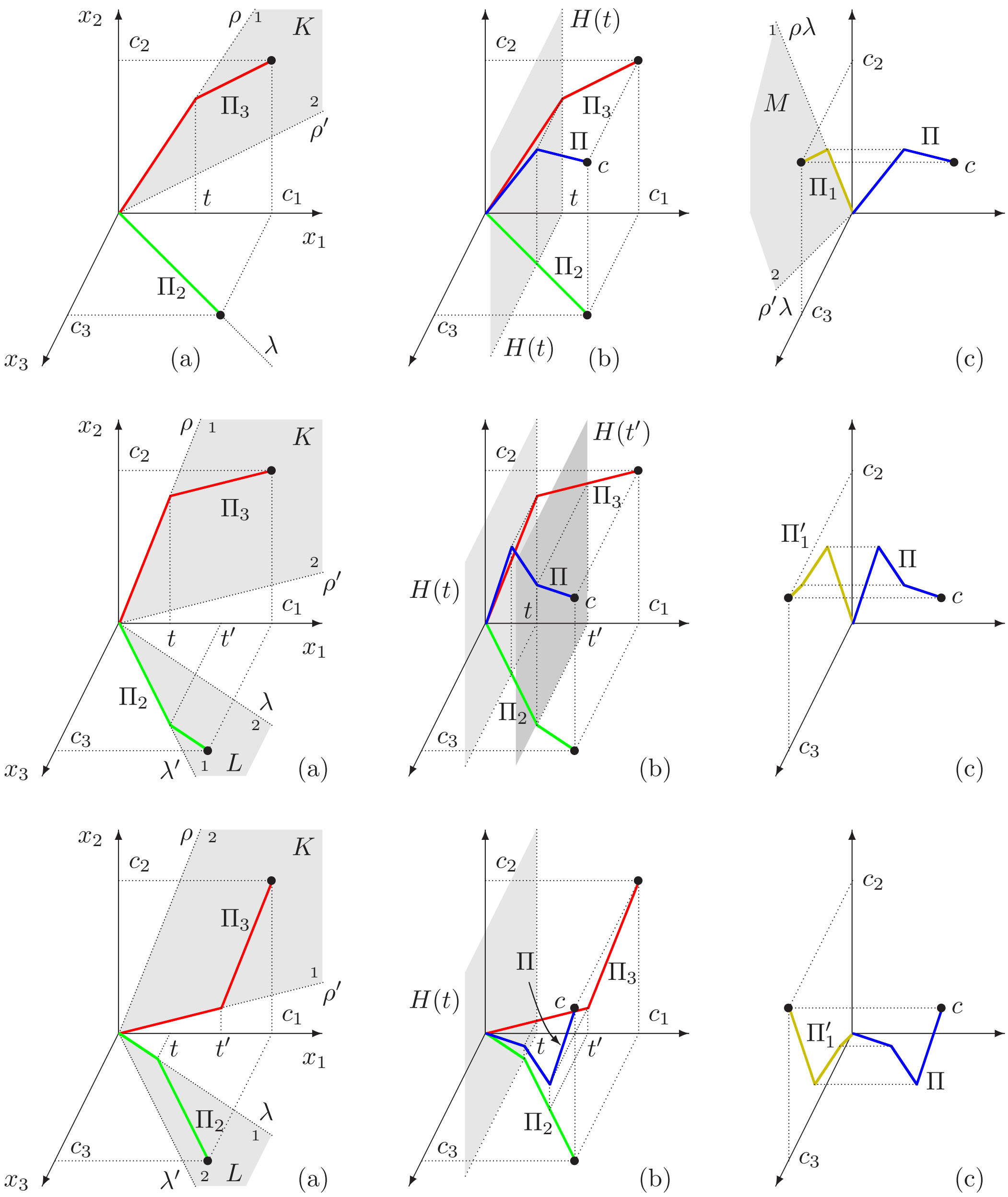

Figure 6: Lemma 4, Substeps 1-1 and 1-2. Row 1: the $\{2,3\}$-partition contains a non-degenerate interior. Rows 2 and 3: the $\{1,2\}$- and $\{1,3\}$-partitions cannot both have non-degenerate interior cones K and L. In Row 2, the boundary rays that are first for both cones are the furthest from the axis in common to the spaces in which these cones lie, $\mathbb{R}^{\{1\}}$. In Row 3, the boundary rays that are first are the closest to $\mathbb{R}^{\{1\}}$ for both cones.

in the remaining figures too). Since ρ, ρ', and λ are interior rays, so are $d(\rho, \lambda)$ and $d(\rho', \lambda)$. Also, $d(\rho, \lambda)$ is first in M if and only if ρ is first in K (this is the case represented in Row 1 of Figure 6).

Substep 1-2: in fact, the $\{1, 2\}$- and $\{2, 3\}$-partitions cannot both contain interior non-degenerate cones oriented as just discovered. For convenience, we illustrate the proof using the $\{1, 2\}$- and $\{1, 3\}$-partitions instead, calling L the cone in the $\{1, 3\}$-partition. (The reason for replacing $\{2, 3\}$ by $\{1, 3\}$ is that in all of our other diagrams, we start from paths in $\mathbb{R}^{\{1,2\}}$ and $\mathbb{R}^{\{1,3\}}$.) Because the orientation of the two cones are related (last line of the previous paragraph), there are only two cases to consider. In Row 2 of Figure 6, the first rays of K and L are both further from the axis shared by the spaces in which these cones lie, $\mathbb{R}^{\{1\}}$, than their second rays. In Row 3, they are both closer. In each case, we choose $c \in \mathbb{R}^{\tilde{N}}_{+}$ such that $c_{\{1,2\}} \in \text{int}\{K\}$, $c_{\{1,3\}} \in \text{int}\{L\}$, and the first coordinates of the kinks in the paths for $c_{\{1,2\}}$ and $c_{\{1,3\}}$ differ. (This last restriction can be met precisely because K and L are non-degenerate.) We obtain a configuration of the type illustrated by Row 3 of Figure 4, namely a two-kink contradiction.[10]

This completes the proof of Step 1. It follows from this step that one boundary ray of K is an axis.

Step 2: completing the proof of the lemma for the $\{1, 3\}$-partition, that it has exactly one interior ray. We assume, by way of contradiction, that the $\{1, 3\}$-partition contains at least two interior rays, say λ and λ'. Let $c \in \mathbb{R}^{\tilde{N}}_{+}$ be such that $c_{\{1,2\}} \in \text{int}\{K\}$ and $c_{\{1,3\}} \in \lambda$ (Figure 7 only shows the construction for λ). This choice of c produces a configuration of the type illustrated by Row 4 of Figure 4. The path for $c_{\{2,3\}}$ that we obtain reveals the existence in the $\{2, 3\}$-partition of a cone M having as boundary ray one axis. Substituting λ' for λ, the existence in the $\{2, 3\}$-partition of a cone M' having that same axis as boundary ray is similarly revealed. Since the direction of the second boundary rays of M and M' differ, these cones overlap. There are four possibilities.

Case 1: the axis that is a boundary ray of K is $\mathbb{R}^{\{2\}}_{+}$. Then, $d(\rho, \lambda) =$

[10] Except that there, the boundary ray that is first for one cone (K) is closer to $\mathbb{R}^{\{1\}}$ than the boundary ray that is second, and the opposite holds for the other cone (L). Together, Row 3 of Figure 4 and Rows 2 and 3 of Figure 6 show that, more generally, the partitions of two adjacent two-dimensional spaces cannot both contain non-degenerate interior cones, independently of the orientations of these cones.

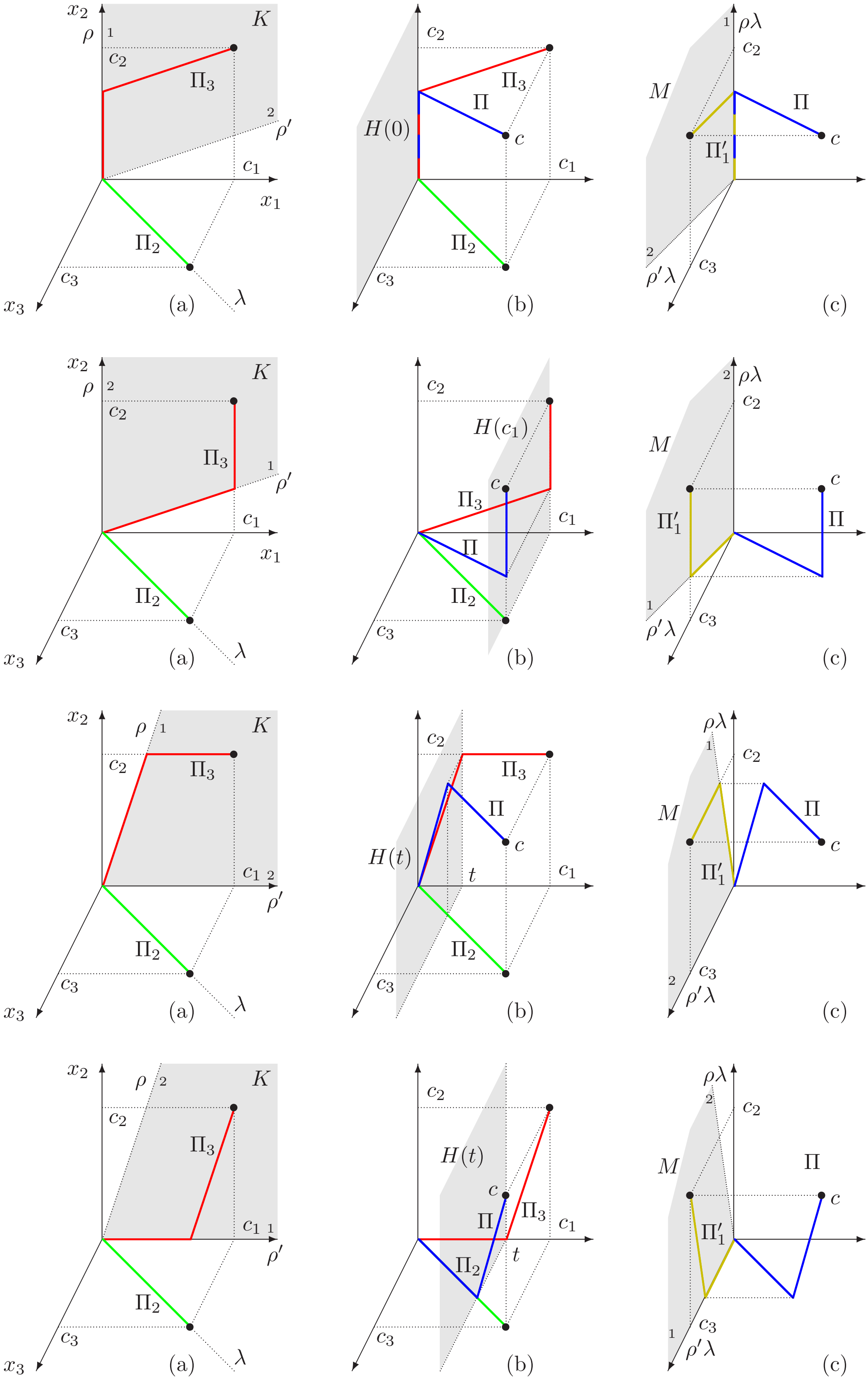

Figure 7: Lemma 4, Step 2. A non-degenerate cone K in the $\{1,2\}$-partition having an axis as boundary ray, and an interior ray λ in the $\{1,3\}$-partition, reveal the existence of a non-degenerate boundary cone in the $\{2,3\}$-partition. In Rows 1 and 2, $\mathbb{R}_+^{\{2\}}$ is a boundary ray of K. Row 1: This ray is first. Row 2: this ray is second. In Rows 3 and 4, $\mathbb{R}_+^{\{1\}}$ is a boundary ray of K. Row 3: This ray is first. Row 4: this ray is second.

$d(\rho, \lambda') = \mathbb{R}_+^{\{2\}}$ and $d(\rho', \lambda) \neq d(\rho', \lambda')$.

(i) $\mathbb{R}_+^{\{2\}}$ is first in K (Row 1 of Figure 7).

(ii) $\mathbb{R}_+^{\{2\}}$ is second in K (Row 2 of Figure 7).

Case 2: the axis that is a boundary ray of K is $\mathbb{R}_+^{\{1\}}$. Then, $d(\rho', \lambda) = d(\rho', \lambda') = \mathbb{R}_+^{\{1\}}$ and $d(\rho, \lambda) \neq d(\rho, \lambda')$.

(i) $\mathbb{R}_+^{\{1\}}$ is second in K (Row 3 of Figure 7).

(ii) $\mathbb{R}_+^{\{1\}}$ is first in K (Row 4 of Figure 7).

Thus, the $\{1, 3\}$-partition consists of exactly two non-degenerate cones and their boundary rays.

We now exchange the roles played by $\{1, 2\}$ and $\{1, 3\}$ in the above proof. Since the $\{1, 3\}$-partition contains at least one non-degenerate cone (an implication of the conclusion just reached), the $\{1, 2\}$-partition consists of exactly two non-degenerate cones and their boundary rays.

The same conclusion can of course be obtained for the $\{2, 3\}$-partition.

□

The two non-degenerate cones of the partition of a two-dimensional awards space have a common boundary ray (this ray is an interior ray), which we call the **central direction** of the space.

Lemma 5 *Either (i) the central direction of each two-dimensional awards space is first for both cones, or (ii) it is second for both. The central directions are consistent (they are all defined by projection of a direction in the three-dimensional space).*

Proof: Case 1: there is a two-claimant group $\bar{N} \subset \tilde{N}$ and a non-degenerate cone K in the $\bar{N}$-partition whose interior boundary ray is first.

Step 1: for each other two-claimant group $N' \subset \tilde{N}$, the central direction of the N'-partition is first. Without loss of generality, set $\bar{N} \equiv \{1, 2\}$. Also, suppose that $\mathbb{R}_+^{\{2\}}$ is a boundary ray of K.

First, we show that for both non-degenerate cones in the $\{1, 3\}$-partition, the first ray is the central direction of $\mathbb{R}^{\{1,3\}}$. Let L be one of these two cones, and suppose by contradiction, that its interior boundary ray is second. Let $c \in \mathbb{R}_+^N$ be such that $c_{\{1,2\}} \in \text{int}\{K\}$ and $c_{\{1,3\}} \in \text{int}\{L\}$.

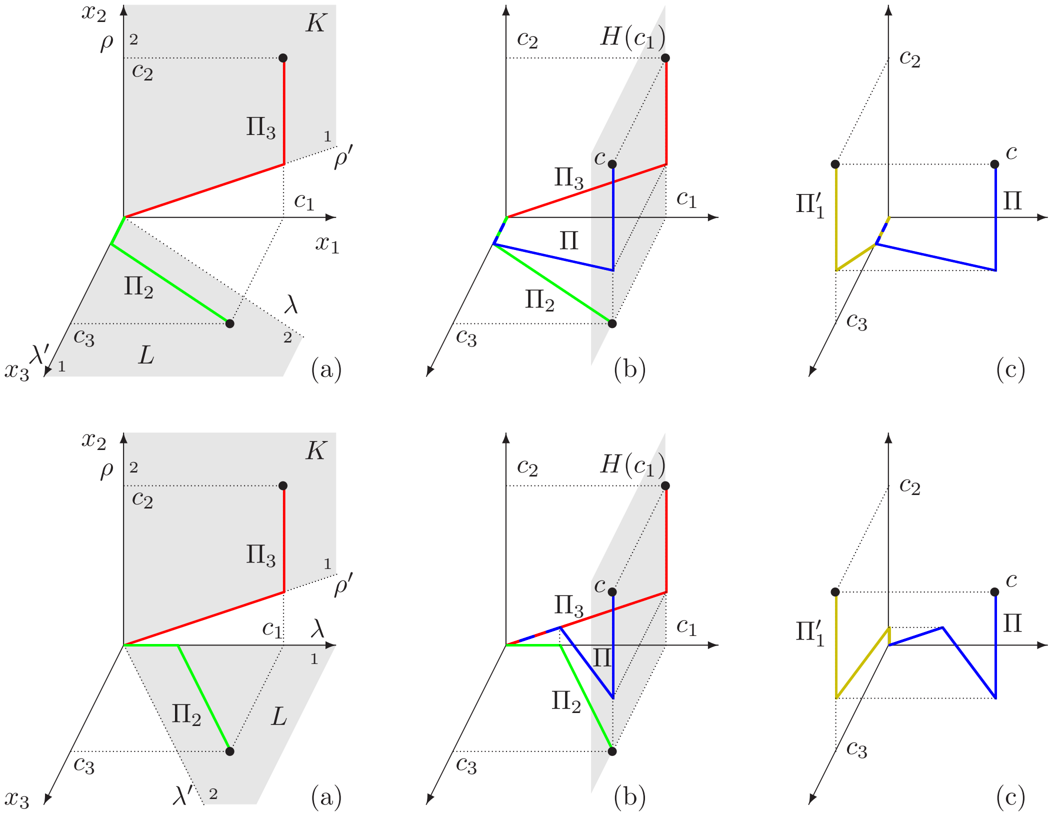

Figure 8: Lemma 5 says that either the central directions all come first or that they all come second. Row 1: $\mathbb{R}_+^{\{1\}}$ is a boundary ray of neither K nor L. Row 2: $\mathbb{R}_+^{\{1\}}$ is a boundary ray of one of these two cones.

(i) $\mathbb{R}_+^{\{3\}}$ is a boundary ray of L. Then, Π has two kinks and so does Π'_1, resulting in a two-kink contradiction (Row 1 of Figure 8).

(ii) $\mathbb{R}_+^{\{1\}}$ is a boundary ray of L. Here too, we obtain a two-kink contradiction (Row 2 of Figure 8).

The lesson to be drawn from the argument just made is that if the interior boundary ray of a non-degenerate cone K in the partition for some two-claimant group N is first, and some other two-claimant group N' does not contain the agent in N whose axis is a boundary ray of K, then the interior boundary rays of both non-degenerate cones in the partition for N' are also first. Thus, since the central direction of the non-degenerate cone in the $\{1,3\}$-partition that has $\mathbb{R}_+^{\{3\}}$ as boundary ray is first, the interior boundary rays of both non-degenerate cones in the $\{1,2\}$-partition are first (K if one of these cones, and the statement we reach for it was in fact our hypothesis). Also, since the central direction of the non-degenerate cone in the $\{1,3\}$-partition that has $\mathbb{R}_+^{\{1\}}$ as boundary ray is first, then the interior rays of both non-degenerate cones in the $\{2,3\}$-partition are first.

Step 2: the central directions are consistent. This follows from the fact that the central direction of the space associated with each two-claimant group $N \subset \tilde{N}$ is a (degenerate) cone in the N-partition. From the existence of interior rays ρ and λ in the $\{1,2\}$- and $\{1,3\}$-partitions, we deduce the existence of an interior ray in the $\{2,3\}$-partition whose direction is $d(\rho, \lambda)$.

Case 2: The central direction of K is second. Then, it follows from Case 1 that the central direction of each other cone in each partition is second. The proof of the consistency of the central directions also applies (Step 2). Altogether, we obtain that for each two-claimant group $N \subset \tilde{N}$, S coincides on $\mathcal{C}^N$ with a weighted constrained equal losses rule, and that the weights assigned to these two-claimant groups are consistent.

□

We return to the proof of the theorem. Case 1 of Lemma 5 amounts to saying that for each two-claimant group $N \subset \tilde{N}$, S coincides on $\mathcal{C}^N$ with a weighted constrained equal awards rule, and that the list of weight vectors assigned to these two-claimant groups are consistent (they are all derived from a single set of weights for the members of $\tilde{N}$). The proof concludes by (ii) of Corollary 1. Case 2 of the Lemma leads to weighted constrained equal losses rules instead, invoking (iii) of Corollary 1. □

4 Concluding comments

Dagan and Volij (1997) give conditions on two-claimant rules guaranteeing that they admit *consistent* extensions: to each awards vector, one should be able to attach a binary relation having certain properties. (Kaminski, 2000; 2005, takes up this approach.) By contrast to Dagan and Volij's approach, which is existential, algebraic, and focused on awards vectors, our line of reasoning is mainly constructive, geometric, and it deals with properties of paths of awards seen in their entirety. For a more extensive discussion, we refer to Thomson (2001).

As an alternative, Dagan and Volij suggest (i) calculating the "average consistent" extension of the two-claimant rule that is the point of departure (such an extension exists under very mild conditions on the two-claimant rule), and (ii) determining whether this rule is in fact consistent. Our method is more direct as it does not pass through average consistency; mainly, it avoids the difficult calculations required for the first step. When the issue is not that of determining whether a particular rule chosen for all two-claimant groups has a *consistent* extension, but rather whether, for each two-claimant group, one can select one member of a family of admissible rules in such a way that the resulting way of solving all two-claimant problems has a *consistent* extension, a question of the type examined here, the calculations may be ever more prohibitive. For further discussion, we also refer to Thomson (2001).

Appendix

Figure 9 further illustrates Lemma 5. It shows that if the central directions are first in both K and L, no contradiction occurs, and it also shows how the path of S for c can be constructed without invoking the Elevator Lemma (Row 2 of Figure 9). This construction is one of the illustrations developed in Thomson (2001).

The path Π can be constructed from Π_3 and Π_2 only up to the point where it reaches $H(c_1)$. In that plane, it can continue from that point in any monotone way up to c: the projection on $\mathbb{R}^{\{1,2\}}$ of any curve in the darkly shaded rectangle connecting the lowest corner to c would be the desired vertical segment containing $c_{\{1,2\}}$ and the projection on $\mathbb{R}^{\{1,2\}}$ of any such curve would be the desired segment parallel to $\mathbb{R}^{\{1,3\}}$ containing $c_{\{1,2\}}$. It is by exploiting the projection requirement on $\mathbb{R}^{\{2,3\}}$ that one can deduce the continuation of Π: this path has to have two kinks but its projection Π_1 loses one (the projections of its kinks on $\mathbb{R}^{\{2,3\}}$ are lined up with the origin of that space. In Row 1, $\mathbb{R}^{\{1\}}_+$ is a boundary ray of neither K nor L. In Row 2, it is the boundary ray of one of these cones.

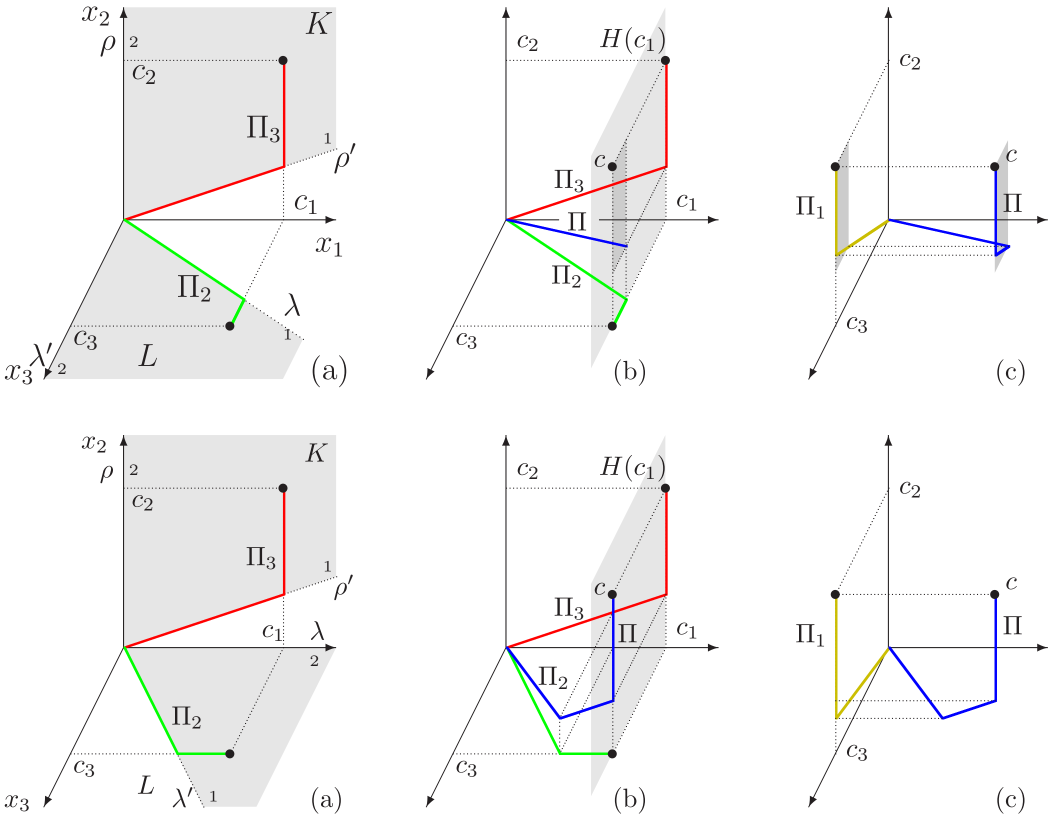

Figure 9: Lemma 5 says that the central directions are "consistent". Row 1: $\mathbb{R}_+^{\{1\}}$ is a boundary ray of neither K nor L. Row 2: it is the boundary ray of one of these cones.

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