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A New Solution to the Problem of Adjudicating Conflicting Claims

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Abstract

For the problem of adjudicating conflicting claims, we consider the requirement that each agent should receive at least $1/n$ his claim truncated at the amount to divide, where n is the number of claimants (Moreno-Ternero and Villar, 2004a). We identify two families of rules satisfying this bound. We then formulate the requirement that for each problem, the awards vector should be obtainable in two equivalent ways, (i) directly or (ii) in two steps, first assigning to each claimant his lower bound and then applying the rule to the appropriately revised problem. We show that there is only one rule satisfying this requirement. We name it the "recursive rule", as it is obtained by a recursion. We then undertake a systematic investigation of the properties of the rule.

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1 Introduction

When a group of agents have claims over a resource that add up to more than is available, how should the resource be divided? A "rule" selects for each situation of this kind a division among the claimants of what is available. Much of the literature devoted to the study of rules has been axiomatic.[1] A variety of tests of good behavior of rules have been formulated and the existence of rules passing these tests, singly and in various combinations, investigated. Among them are tests designed to guarantee agents certain minimal amounts. A recent suggestion along these lines is that each agent should receive at least the minimum of (i) his claim divided by the number of claimants and (ii) the amount available divided by the number of claimants (Moreno-Ternero and Villar, henceforth MTV, 2004a). This requirement is not very demanding, being satisfied by many of the rules that have been central in the literature. Yet, when combined with a dual lower bound on the losses agents incur, and the requirement of consistency, which expresses a form of robustness of the choice with respect to variations of populations, only one rule remains admissible, the so-called Talmud rule (again, see MTV, 2004a).

This lower bound on awards is our point of departure. We first identify two ways of finding out whether a rule respects it. These results cover all examples previously known to do so, and infinitely many others.

Next, we formulate the following invariance requirement on a rule: for each problem, suppose that we first award the lower bounds, revise claims down accordingly, and apply the rule to divide what remains; the requirement is that the resulting awards vector should be the same as when the rule is applied directly to the problem. We show that there is a unique rule satisfying it. As the rule is defined by means of a recursion, we call it the "recursive rule".

We then undertake a systematic evaluation of the rule. We first establish a number of basic properties it satisfies. We then show that it is well behaved from the viewpoint of monotonicity. In particular, when the amount available increases, all agents receive at least as much as they did initially. Moreover, when an agent's claim increases, he receives at least as much as he did initially, and each of the others receives at most as much as he did initially.

[1] For surveys, see Herrero and Villar (2001), Moulin (2002), and Thomson (2003a, 2003b).

Next, we show that the rule is invariant under truncation of claims at the amount to divide. We finally turn to the behavior of the rule in the context of variable populations. One central property here is replication invariance. The rule violates this property, but asymptotically, as the order of replication increases, there is a sense in which it behaves "like" the proportional rule, as we show next. Also, it fails the consistency requirement alluded to in the opening paragraph of this introduction. So we ask whether there is any consistent rule that coincides with it in the two-claimant case. Consistency has indeed provided a very useful means of extending, to general populations, rules chosen in the conceptually and mathematically more transparent two-claimant case. Here, the answer is unfortunately negative. Although the bound itself is compatible with consistency since many consistent rules satisfy it, pursuing its logic recursively is not.

2 The model of adjudication of conflicting claims

There is a set N of ***claimants*** having claims over a resource, the amount of the resource available being insufficient to honor all of these claims. For each $i \in N$, let c_i denote agent i's claim and E the amount to divide. A claims problem, or simply a ***problem***, is a pair $(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$ such that $\sum_N c_i \geq E$.[2] Let $\mathcal{C}^N$ denote the domain of all problems. A division rule, or simply a ***rule***, is a function defined on $\mathcal{C}^N$, which associates with each $(c, E) \in \mathcal{C}^N$ a vector $x \in \mathbb{R}^N_+$. This vector should satisfy the ***non-negativity*** and ***claims boundedness*** inequalities $0 \leqq x \leqq c$, and its coordinates should add up to E, a condition to which we refer as ***efficiency***. Any such vector is an ***awards vector for*** $\boldsymbol{(c, E)}$. Let $X(c, E)$ denote the set of these vectors. Let S be our generic notation for rules. For each $c \in \mathbb{R}^N_+$, the locus of the awards vector a rule selects as the amount to divide varies from 0 to $\sum c_i$ is its ***path of awards for c***.

In the variable-population version of the model, there is an infinite population of "potential" claimants indexed by the natural numbers, $\mathbb{N}$. However, at any given time, only a finite number of them are present. Let $\mathcal{N}$ be the class of finite subsets of $\mathbb{N}$. A claims problem is defined by first specifying

[2] By the notation $\mathbb{R}^N$ we mean the Cartesian product of $|N|$ copies of $\mathbb{R}$ indexed by the members of N. Vector inequalities: $x \geqq y$, $x \geq y$, and $x > y$.

some population $N \in \mathcal{N}$, and then $(c, E) \in \mathcal{C}^N$. A rule is a function defined on $\bigcup_{N \in \mathcal{N}} \mathcal{C}^N$, which associates with each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$ an awards vector of (c, E).

The segment connecting a and b is denoted seg$[a, b]$ and the broken segment connecting $a, b, \ldots, f$ is denoted bro.seg$[a, b, \ldots, f]$. Given a and b such that $a \leqq b$, the set of vectors x such that $a \leqq x \leqq b$ is denoted box$[a, b]$. The interval $[a, b[\subset \mathbb{R}$ contains a but not b.

3 A lower bound on awards

The axiomatic study of any class of problems usually includes lower or upper bound requirements on assignments, welfares, or utilities. These requirements are motivated by fairness, participation, or incentive considerations, the desire to restrict inequalities in incomes or the range of welfare levels agent reach, and often by combinations of the above. In the context of the present model, several requirements of this type have been proposed, and our starting point is one such requirement: for each problem in $\mathcal{C}^N$, each agent should receive at least $\frac{1}{|N|}$ of his claim if his claim is at most as large as the amount to divide, and $\frac{1}{|N|}$ of the amount to divide otherwise (MTV, 2004a).[3] This lower bound on an agent's award is nothing other than $\frac{1}{|N|}$ of his claim truncated at the amount to divide. The idea of truncating claims in this manner is central in the literature. It underlies the definition of several rules (Aumann and Maschler, 1985), and the property of a rule that it be invariant with respect to truncation appears in a number of axiomatic characterizations of well-known rules (Dagan, 1996; Hokari and Thomson, 2003).

Also, if a rule "corresponds" to a solution defined on a domain of coalitional games, it satisfies this invariance requirement (Curiel, Maschler, and Tijs, 1987).[4]

For a formal statement, for each $i \in N$, let $t_i(c, E) \equiv \min\{c_i, E\}$, $t(c, E) \equiv (t_i(c, E))_{i \in N}$, and $\mu(c, E) \equiv \frac{1}{|N|} t(c, E)$.

[3] They refer to it as "securement". A further analysis is in Moreno-Ternero and Villar (2004b).

[4] A rule corresponds to a solution for coalitional games if for each problem, the awards vector it recommends is also the payoff vector obtained by first converting the problem into a coalitional game, and then applying the solution to the game.

Reasonable lower bounds on awards: For each $(c, E) \in \mathcal{C}^N$, $S(c, E) \geqq \mu(c, E)$.

In classical models, agents' individual endowments are often used in the definition of lower bounds, underlying the commonly imposed condition of "individual rationality". In the theory of fairness, equal division is used instead. Reference hypothetical situations in which all agents have the characteristics of a particular agent (his endowment, his preferences, his productivity) have also provided the basis for lower bounds and upper bounds (depending upon whether goods are private or public). One can imagine basing the bound(s) imposed on an agent's welfare on the characteristics of all agents, or basing them on his own characteristics and on the collective variable. Here, individual characteristics are not endowments, but claims. An agent's claim is already used in the definition of a rule as an upper bound on what he should receive. We are now proposing to use his truncated claim as the basis for a lower bound: the bound is a pre-specified proportion, the inverse of the number of agents, of his claim. One could think of using as lower bound a pre-specified proportion of his claim itself, but that is not a meaningful option. Indeed, the bounds so obtained are compatible for all values of the parameters of the problem only if that proportion is 0, but then all rules qualify. Using truncated claims is a natural and meaningful alternative, and in fact, the proportion we choose is the highest that preserves compatibility. Consider for example a problem in which all claims are equal to the amount to divide. Then all truncated claims are equal to that amount, and the requirement that each agent should receive a proportion α of his truncated claim, when imposed on each agent, can be met only if $\alpha \leq \frac{1}{|N|}$.

Let $N \equiv \{1, 2\}$, and $c \in \mathbb{R}^N_+$ with $c_1 \leq c_2$. Let $E \leq c_1$. Then, if S satisfies *reasonable lower bounds on awards*, and since awarding each agent at least half of the amount to divide is possible only at equal division, its path of awards for c contains $\text{seg}[(0, 0), (\frac{c_1}{2}, \frac{c_1}{2})]$. The view is widely held that if the amount to divide is small in relation to claims, equal division should prevail.[5] The path continues in a region in $\text{box}[(\frac{c_1}{2}, \frac{c_1}{2}), c]$ whose boundary is defined by the 45° line and the horizontal line of ordinate $\frac{c_2}{2}$. There are two cases depending upon whether or not $c_1 \leq \frac{c_2}{2}$. They are illustrated in

[5]Carmen Bevia (oral communication) reported to us that, once presented to the subject in class, her undergraduates have often spontaneously expressed it. It would be interesting to conduct formal experiments to measure its prevalence.

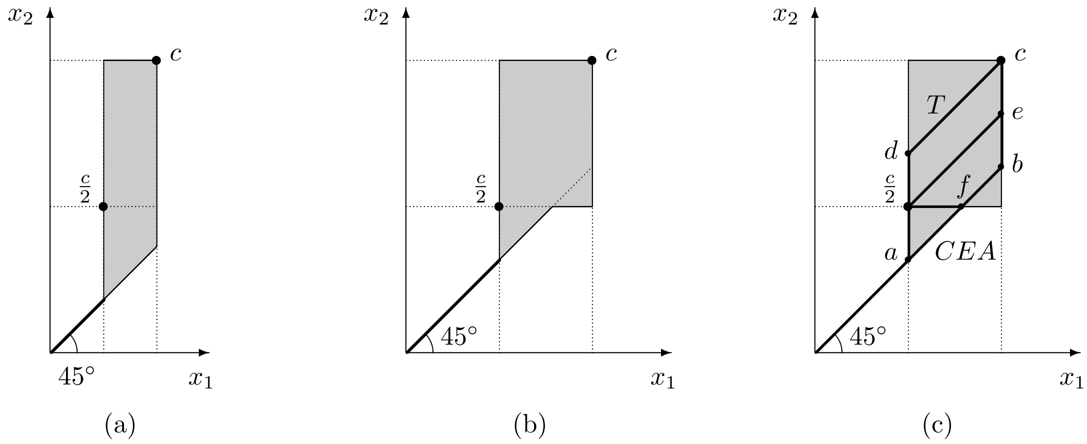

Figure 1: Guaranteeing a minimal amount to each claimant. This figure illustrates *reasonable lower bounds on awards* for $N \equiv \{1,2\}$ and $c \in \mathbb{R}^N_+$ with $c_1 < c_2$. A rule satisfies the requirement if its path of awards for c lies in the region consisting of the thick segment from the origin to $\frac{1}{2}(\min\{c_i\}, \min\{c_i\}) = (\frac{c_1}{2}, \frac{c_1}{2})$ and the shaded area. (a) Here, $c_1 < \frac{c_2}{2}$ and the constraint $x_2 \geq \frac{c_2}{2}$ is binding for no amount to divide. (b) Here, $c_1 > \frac{c_2}{2}$. (c) The path of the constrained equal awards rule is bro.seg$[(0,0), b, c]$, that of the Talmud rule is bro.seg$[(0,0), a, d, c]$, that of Pineles' rule is bro.seg$[(0,0), a, \frac{c}{2}, e, c]$, and that of the constrained egalitarian rule is bro.seg$[(0,0), a, \frac{c}{2}, f, b, c]$. All of these rules satisfy *reasonable lower bounds on awards* for arbitrarily many claimants.

Figure 1. If $c_1 \leq \frac{c_2}{2}$, the constraint that agent 2 should get at least half of his claim is binding for no amount to divide, in the sense that if an awards vector x satisfies the bound for claimant 1, then $x_2 \geq \frac{c_2}{2}$ whenever $c_2 \leq E$ (Figure 1a). If $c_1 > \frac{c_2}{2}$, it is binding over the non-empty interval $]c_2, 2c_1[$ of amounts to divide (Figure 1b).

A number of important rules satisfy *reasonable lower bounds on awards*. Here are the primary ones. Let $(c, E) \in \mathcal{N}$. The ***constrained equal awards*** rule selects $x \in X(c, E)$ such that for some $\lambda \in \mathbb{R}_+$, $x = (\min\{c_i, \lambda\})_{i \in N}$ (O'Neill, 1982; the rule also appears in Maimonides). ***Piniles' rule*** (Piniles, 1861) selects $x \in X(c, E)$ such that for some $\lambda \in \mathbb{R}_+$, $x = (\min\{\frac{c_i}{2}, \lambda\})_{i \in N}$ if $E \leq \sum c_i$ and $x = \frac{c}{2} + (\min\{\frac{c_i}{2}, \lambda\})_{i \in N}$ otherwise. The ***Talmud rule*** (Aumann and Maschler, 1985) selects $x \in X(c, E)$ such that for some $\lambda \in \mathbb{R}_+$, $x = (\min\{\frac{c_i}{2}, \lambda\})_{i \in N}$ if $E \leq \frac{\sum c_i}{2}$ and $x = \frac{c}{2} + (\max\{\frac{c_i}{2} - \lambda, 0\})_{i \in N}$ otherwise. Define the ***minimal right of claimant i in*** $\boldsymbol{(c, E)}$ as the difference between E and the sum of the claims of the other agents, or 0 if this difference is negative: $m_i(c, E) \equiv \max\{E - \sum_{N \setminus \{i\}} c_j, 0\}$.[6] Now, the ***adjusted proportional rule*** selects $m(c, E) + P(t(c - m(c, E), E - \sum m_j(c, E)), E - \sum m_j(c, E))$ (Curiel, Maschler

[6]This quantity, generalized to groups of claimants, underlies O'Neill's proposal to associate with each problem a coalitional form game, providing the ground for the application of the solution concepts developed in that theory to solve claims problem

and Tijs, 1987). All of these rules pass the test (MTV, 2004a).

Others do too. One is the ***random arrival rule*** (O'Neill, 1982), which selects the average of the awards vectors obtained by imagining claimants arriving one at a time and fully compensating them until money runs out, under the assumption that all orders of arrival are equally likely. Indeed, since the proportion of orders in which a given claimant is first is $\frac{1}{|N|}$, and that for each such order, he is either fully compensated or receives the entire amount available, a lower bound on his award is the quantity specified by *reasonable lower bounds on awards.* The ***minimal overlap rule*** (O'Neill, 1982) and the ***constrained egalitarian rule*** (Chun, Schummer, and Thomson, 2001) also satisfy *reasonable lower bounds on awards.*[7] Since the only major rules in the literature that violate the property are the ***proportional rule***, which selects $x \in X(c, E)$ such that for some $\lambda \in \mathbb{R}_+$, $x = \lambda c$, and the ***constrained equal losses rule***, which selects $x \in X(c, E)$ such that for some $\lambda \in \mathbb{R}_+$, $x = (\max\{c_i - \lambda, 0\})_{i \in N}$, one can say that the property is not very restrictive. The lower bounds it places on awards are indeed "reasonable".

Next, we present two general ways of identifying rules satisfying *reasonable lower bounds on awards.*

• First, consider the following variable-population invariance requirement. Let $N \in \mathcal{N}$, $(c, E) \in \mathcal{C}^N$, and $x \equiv S(c, E)$. Now, imagine some claimants leaving with their awards (their components of x), and reassess the situation at that point. The requirement is that in the revised problem faced by the remaining claimants, the rule should attribute to each of them the same amount as initially. (A survey of the various applications that have been made of the consistency principle is Thomson, 2003c.)

Consistency: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $N' \subset N$, if $x \equiv S(c, E)$, then $x_{N'} = S(c_{N'}, \sum_{N'} x_i)$.

Certain properties of a rule, if satisfied in the two-claimant case, are automatically satisfied for more than two claimants if the rule is *consistent.* We say that these properties are "lifted" by *consistency* (this expression is proposed by Hokari and Thomson, 2003b). Our first lemma states that "lifting" occurs for the property that interests us here:

Lemma 1 *Reasonable lower bounds on awards is lifted from the two-claimant case to the general case by* consistency.

[7] We omit the proof for these rules, as their definitions are more involved.

Proof: Let $N \in \mathcal{N}$, $(c, E) \in \mathcal{C}^N$, and $x \equiv S(c, E)$. Suppose by contradiction, that there is $i \in N$ such that $x_i < \frac{1}{|N|} \min\{c_i, E\}$, which implies that (i) $x_i < \frac{c_i}{2}$. By *efficiency*, there is $j \in N$ such that $x_j > \frac{1}{|N|}E$, and thus (ii) $x_i < \frac{x_i+x_j}{2}$. Let $N' \equiv \{i, j\}$, and consider the problem $(c_i, c_j, x_i + x_j)$. By *consistency*, $(x_i, x_j) = S(c_i, c_j, x_i + x_j)$. Since S satisfies *reasonable lower bounds on awards* in the two-claimant case, $x_i \geq \frac{1}{2} \min\{c_i, x_i+x_j\}$. This is incompatible with (i) and (ii). □

The constrained equal awards, Talmud, Piniles', and constrained egalitarian rules all satisfy *reasonable lower bounds on awards* in the two-claimant case: Figure 1c shows that their paths of awards indeed lie in the required region. Also, they are *consistent* (see Aumann and Maschler, 1985; Chun, Schummer and Thomson, 2001). It then follows from Lemma 1 that they satisfy *reasonable lower bounds on awards* in general.

For an interesting family of rules, one can say more. First, a rule ***has a parametric representation*** if there are $[a, b] \subset \overline{\mathbb{R}}$ and a continuous and nowhere decreasing function $f \colon [a, b] \times \mathbb{R} \to \mathbb{R}$ such that for each $(c, E) \in \mathcal{C}^N$, it selects $x \in X(c, E)$ such that for some $\lambda \in [a, b]$, $x = (f(c_i, \lambda))_{i \in N}$. (The class is characterized by Young, 1987, on the basis of ***continuity***, the requirement that small changes in problems should not be accompanied by large changes in the recommended awards vector, ***equal treatment of equals***, the requirement that claimants with equal claims should receive equal amounts, and *consistency*). We will consider the generalization of this notion obtained by allowing the function f to depend on claimants, as follows: there are $[a, b] \subset \overline{\mathbb{R}}$, and for each $i \in N$, a continuous and nowhere decreasing function $f_i \colon [a, b] \to \mathbb{R}$ such that for each $(c, E) \in \mathcal{C}^N$, the rule selects $x \in X(c, E)$ such that for some $\lambda \in [a, b]$, $x = (f_i(c_i, \lambda))_{i \in N}$. We refer to such a rule as "generalized parametric". The following straightforward lemma tells us when a rule of this type satisfies the *reasonable lower bounds on awards.*

Lemma 2 *Let S be a generalized parametric rule of representation $(f_i)_{i \in N} \colon [a, b] \times \mathbb{R} \to \mathbb{R}$, where $[a, b] \subset \overline{\mathbb{R}}$. Then, S satisfies* reasonable lower bounds on awards *if $(*)$ for each $i \in N$, the upper envelope C_i of the schedules $\{f_i(c_i, .)\}_{c_i \in \mathbb{R}_+}$ is well-defined and independent of i, and $(**)$ for each $i \in N$ and each $c_i \in \mathbb{R}_+$, the schedule $f_i(c_i, .)$ follows C_i from $(a, 0)$ up to a point of ordinate at least $\frac{c_i}{2}$.*

Proof: Let $(c, E) \in \mathcal{C}^N$ and $\lambda \in [a, b]$ be such that $\sum f_i(c_i, \lambda) = E$. Then, for each $i \in N$, $x_i = f_i(c_i, \lambda)$. Now, for each $i \in N$, either $x_i = f_i(c_i, \lambda) < \sup_{c_0 \in \mathbb{R}_+} f_i(c_0, \lambda)$, in which case $x_i \geq \frac{c_i}{2} \geq \frac{c_i}{|N|}$, or $f_i(c_i, \lambda) = \max_{j \in N} x_j$, in which case, $x_i \geq \frac{E}{|N|}$. Thus, the reasonable lower bounds on awards is met. $\square$

Since the constrained equal awards, Talmud, Piniles', and constrained egalitarian rules are parametric rules whose representations all meet requirement $(*)$, they satisfy *reasonable lower bounds on awards*.

• A second way of identifying rules that satisfy *reasonable lower bounds on awards* is obtained by exploiting the notion of an "operator" on the space of rules, that is, a mapping from the space of rules into itself. Given any rule S, consider the rule S^m that selects for each problem the awards vector obtained by first assigning to each claimant his minimal right, revising claims down by these amounts, and then applying S to divide the remainder: formally, for each $(c, E) \in \mathcal{C}^N$, $S^m(c, E) \equiv m(c, E) + S(c - m(c, E), E - \sum m_j(c, E))$. We say that S^m is obtained from S by subjecting it to the ***attribution of minimal rights operator***. Also, given any rule S, consider the rule S^t that associates with each problem $(c, E) \in \mathcal{C}^N$, the awards vector obtained by first truncating claims at the amount to divide: $S^t(c, E) \equiv S(t(c, E), E)$. We call this operator the ***claims truncation operator***. (A systematic investigation of this operator and others is found in Thomson and Yeh, 2003.) We now assert that if a rule S satisfies *reasonable lower bounds on awards*, so do S^m and S^t.

Lemma 3 *Reasonable lower bounds on awards is preserved under the attribution of minimal rights operator and by the claims truncation operator.*

Proof: We first consider the attribution of minimal rights operator. We need to show that $S^m(c, E) \geqq \frac{1}{|N|} t(c, E)$. Our hypothesis on S implies that $S(c - m(c, E), E - \sum m_j(c, E)) \geqq \frac{1}{|N|} t(c - m(c, E), E - \sum m_j(c, E))$. Using the relation $t(c - m(c, E), E - \sum m_j(c, E)) = t(c, E) - m(c, E)$ (Thomson and Yeh, 2003), this inequality can be simplified to $S^m(c, E) \geqq m(c, E) + \frac{1}{|N|}[t(c, E) - m(c, E)] \geqq \frac{1}{|N|} t(c, E)$. After canceling $\frac{1}{|N|} t(c, E)$ from both sides, it further simplifies to $(1 - \frac{1}{|N|}) m(c, E) \geqq 0$, which trivially holds since $|N| \geq 1$ and $m(c, E) \geqq 0$.

Next, we consider the claims truncation operator. We have that $S^t(c, E) \equiv S(t(c, E), E) \geqq \frac{1}{|N|} \min\{t(c, E), E\} = \frac{1}{|N|} \min\{c, E\}$, where the first inequality comes from the fact that S satisfies *reasonable lower bounds on awards*, and the equality follows trivially from the definition of the truncation. $\square$

We use the first part of Lemma 3 to give a very simple proof that the adjusted proportional rule satisfies *reasonable lower bounds on awards* (as established directly by MTV, 2004a). Indeed, this rule can be described as the result of subjecting the proportional rule to the attribution of minimal rights operator and then to the claims truncation operator. Equivalently, it is obtained by subjecting the proportional rule to these operators in reverse order (Thomson and Yeh, 2003, show that they commute). Now, we assert that P^t satisfies *reasonable lower bounds on awards*. Indeed, to show that $P_i^t(c, E) \geqq t(c_i, E)$, we write $P_i^t(c, E) \equiv P_i(t(c, E), E) = \frac{t(c_i,E)}{\sum t(c_j,E)} E \geqq \frac{1}{|N|} t(c_i, E)$, which holds since for each $i \in N$, $E \geq t(c_i, E)$ and thus $|N|E \geq \sum t(c_i, E)$.

Reasonable lower bounds on awards is defined by focusing on what claimants receive. By switching attention to the losses they incur, we obtain the requirement that if agent i's claim is at most as large as the deficit $\sum c_j - E$, he should receive at most $c_i - \frac{1}{|N|} c_i$, and otherwise, he should receive at most $c_i - \frac{1}{|N|}(\sum c_j - E)$. The formal statement is as follows (MTV, 2004a):

Reasonable lower bounds on losses: For each $(c, E) \in \mathcal{C}^N$ and each $i \in N$, $c_i - S_i(c, E) \geq \frac{1}{|N|} \min\{c_i, \sum c_j - E\}$.

Two rules S and S^d are dual if one of them divides what is available in the same way as what the other divides what is missing: formally, for each $(c, E) \in \mathcal{C}^N$, $S^d(c, E) \equiv c - S(c, \sum c_i - E)$. Also, ***two properties are dual*** if whenever a rule satisfies one of them, the dual of the rule satisfies the other. *Reasonable lower bounds on awards* and *reasonable lower bounds on losses* are dual properties (MTV, 2004a). Thus, two families of rules satisfying *reasonable lower bounds on losses* can be identified by duality from the families of rules satisfying *reasonable lower bounds on awards*.[8]

[8]Given the geometric interpretation of *reasonable lower bounds on awards*, it is easy to see (for a formal proof see MTV, 2004a), that in the two-claimant case, the Talmud rule is the only rule satisfying both of these properties. Indeed, in the two-claimant case, for the path of awards of a rule to belong to the admissible area identified in Figure 1 as well as to the symmetric image of that area with respect to the half-claims vector,

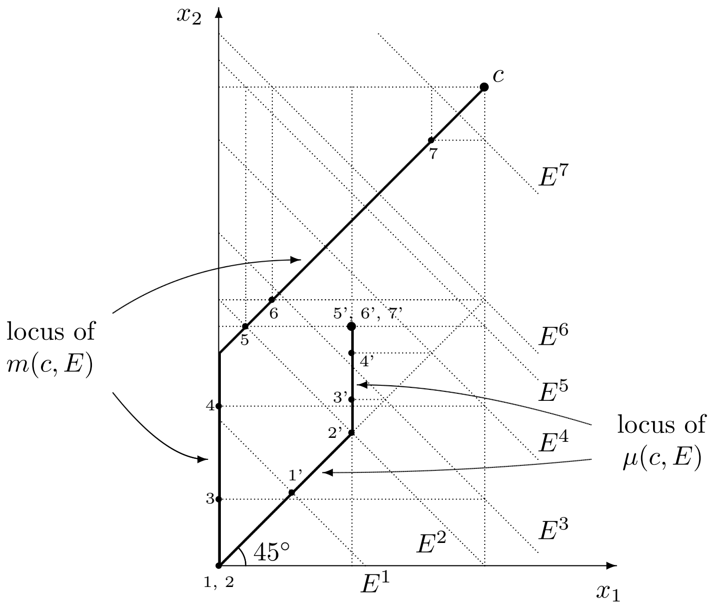

Figure 2: Comparing two lower bounds. The loci of the vectors of minimal rights and the vector of reasonable awards are plotted as a function of the amount to divide, seven values being indicated explicitly. For each $k = 1, \ldots, 7$, the former vector for (c, E^k), $m(c, E^k)$, is labelled "k" whereas the latter, $\mu(c, E^k)$, is denoted k'.

The *reasonable lower bounds on awards* should be compared to another lower bound that has been extensively studied in the literature. We have already defined the "minimal right" of an agent in a problem. This alternative bound is the claimant's minimal right (Curiel, Maschler and Tijs, 1987). It is illustrated in the two-claimant case and compared to *reasonable lower bounds on awards* in Figure 2 where the loci of the vectors $m(c, E)$ and $\mu(c, E)$ are plotted as a function of E. The range of the amount to divide can be divided into three intervals. For $E \in [0, c_1 + \frac{c_2}{2}]$, if an awards vector x satisfies $x \geqq \mu(c, E)$, then it satisfies $x \geqq m(c, E)$; for $E \in]c_1 + \frac{c_2}{2}, \frac{c_1}{2} + c_2[$, the two bounds are not comparable; and for $E \in [\frac{c_1}{2} + c_2, \sum c_i]$, if $x \geqq m(c, E)$, then $x \geqq \mu(c, E)$.

By contrast to the lower bound appearing in *reasonable lower bounds on*

as required by the dual property of *reasonable lower bounds on losses*, it has to be the path of the Talmud rule. It then follows from the Elevator Lemma (Thomson, 2003c), as MTV note, that the Talmud rule is the only rule to satisfy the two bounds together with *consistency*. This is because the Talmud rule is *consistent* and *conversely consistent*. (The Elevator Lemma asserts that if a *consistent* rule coincides in the two-claimant case with a *conversely consistent* rule, then coincidence occurs in general.)

Note that the random arrival rule is *self-dual*, and so it too satisfies *reasonable lower bounds on losses*. Since both of these properties are preserved under convex operations, and the Talmud and adjusted proportional rules also satisfy both, we obtain a whole family of rules satisfying *self-duality*, *reasonable lower bounds on awards*, and *reasonable lower bounds on losses*.

awards, the "minimal right lower bound" just defined on a claimant's award depends on all components of a problem.

The two bounds differ significantly in their implications. Indeed, it is a consequence of the definition of a rule that it always selects a vector that weakly dominates the vector of minimal rights, whereas we have seen that a rule may or may not select an awards vector that dominates the vector of reasonable awards. Characterizations involving *reasonable lower bounds on awards* can be found in Yeh (2003).

4 An invariance requirement on rules

Next, we formulate an invariance requirement on rules based on the reasonable lower bounds: for each problem, the awards vector chosen for it should be obtainable in either one of the following two ways: directly or in two steps, first assigning to each claimant his reasonable lower bound, and second, dividing the remainder, after having revised claims down by these amounts.

Reasonable awards first: For each $(c, E) \in \mathcal{C}^N$,

$$S(c, E) = \mu(c, E) + S(c - \mu(c, E), E - \sum \mu_i(c, E)).$$

This requirement is inspired by one based on minimal rights that has been important in the literature. It says that the awards vector should be obtainable in two ways: directly or in two steps, first assigning to each claimant his minimal right, and second dividing the remainder, after having revised claims down by these amounts (Curiel, Maschler and Tijs, 1987). Quite a few rules satisfy this property—let us call it ***minimal right first***—the Talmud and random arrival rules being examples, but as we now show, only one satisfies *reasonable awards first*:

Theorem 1 *There is a unique rule satisfying* reasonable awards first*.*

Proof: Let $(c, E) \in \mathcal{C}^N$. The proof is based on the observation that in the problem obtained from (c, E) by assigning reasonable awards, reasonable awards may still be positive, justifying a second round of awards. Once these second-round awards are made, a third problem is obtained in which once again, reasonable awards may be positive. So, the process can continue. Let

$(c^1, E^1) \equiv (c, E)$ and for each $k \geq 2$, let (c^k, E^k) be the problem obtained at the k-th step, namely

$$(c^k, E^k) \equiv \left(c^{k-1} - \mu(c^{k-1}, E^{k-1}), E^{k-1} - \sum_{i \in N} \mu_i(c^{k-1}, E^{k-1})\right).$$

Note that no agent's claim ever increases from one step to the next and that the same statement applies to the amount to divide. Since all claims and amount to divide are bounded below by 0, they have limits. Let these limits be denoted $\bar{c}$ and $\bar{E}$. We will show that $\bar{E} = 0$. Suppose, by way of contradiction, that $\bar{E} > 0$. Let $k \in \mathbb{N}$ be such that $E^k - \bar{E} \leq \frac{\bar{E}}{|N|^2}$. Since (c^k, E^k) is a well-defined problem, there is $i \in N$ such that $c_i^k \geq \frac{E^k}{|N|}$. At the $(k+1)$-th step, agent i receives $\frac{1}{|N|} \min\{c_i^k, E^k\}$, and since all agents receive non-negative amounts, the amount to divide decreases by at least this expression. Thus, $E^{k+1} < E^k - \frac{1}{|N|} \min\{c_i^k, E^k\} < \bar{E}$, in contradiction with the definition of $\bar{E}$. □

Let $\mu^1(c, E) \equiv \mu(c, E)$, and for each $k > 1$,[9]

$$\mu^k(c, E) \equiv \mu\left(c - \sum_{l=1}^{k-1} \mu^l(c, E), E - \sum_{i \in N} \sum_{l=1}^{k-1} \mu_i^\ell(c, E)\right).$$

It follows from Theorem 1 that the unique rule satisfying *reasonable awards first*—the name we choose for it reflects the construction—can be defined as follows:

Recursive rule, *R*: For each $(c, E) \in \mathcal{C}^N$,

$$R(c, E) \equiv \sum_{k=1}^{\infty} \mu^k(c, E).$$

Alternatively, for each $(c, E) \in \mathcal{C}^N$, let $(\bar{c}, \bar{E}) \equiv \lim_{k \to \infty}(c^k, E^k)$. Then, $R(c, E) = c - \bar{c}$.

If all claims are positive, there is $k \in \mathbb{N}$ at which there is nothing left to divide, and conversely, finite convergence requires that all claims be positive.

[9] Note that $\mu_i^k(c, E)$ depends on the other agents' claims (c_{-i}) since the resources available at step k depend on the entire claims vector.

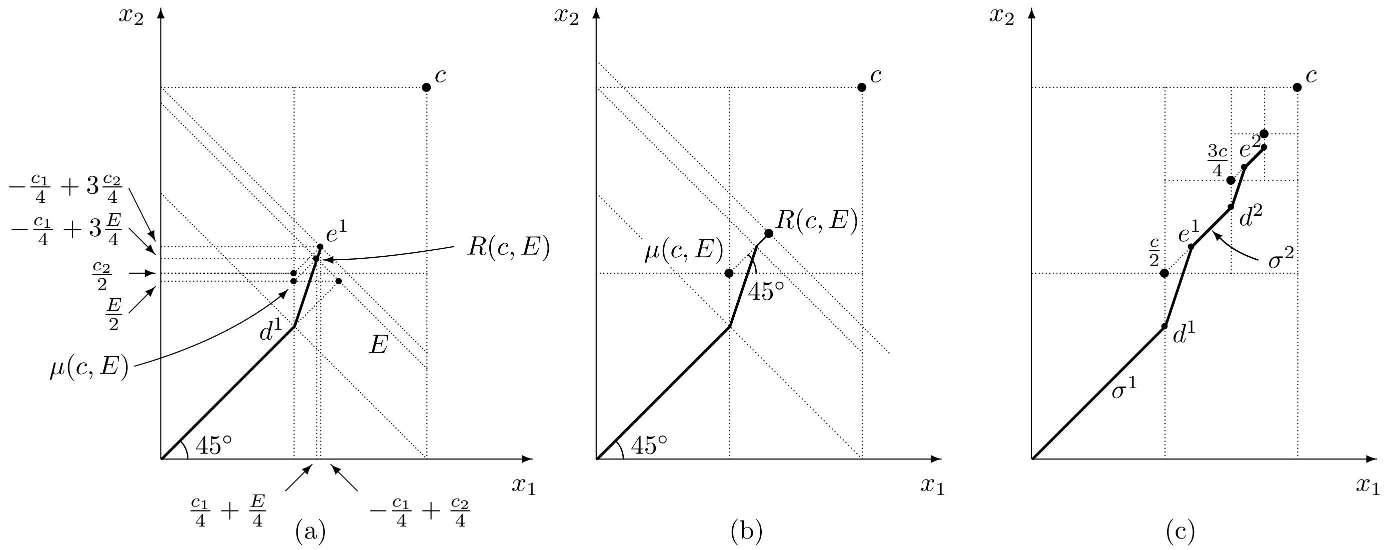

Figure 3: Defining the recursive rule for $c \in \mathbb{R}^N_+$ with $N \equiv \{1, 2\}$ and $c_2 \leq 2c_1$. (a) The first and second segments of its path of awards are obtained by letting E vary in $[0, c_2]$. They are seg$[(0,0), d^1]$ and seg$[d^1, e^1]$. (b) Repeating the construction when E varies in $[c_2, \frac{c_1}{2} + c_2]$. (c) The next two segments are seg$[e^1, d^2]$ and seg$[d^2, e^2]$.

One may wonder why the parallel property of *minimal rights first* does not give us a unique rule. The reason is that for each problem, after minimal rights are assigned, we obtain a revised problem in which minimal rights are zero (Thomson, 2003b). Thus, there is no reason to repeat the process, and the property cannot serve directly as the basis for the definition of a rule.

Incidentally, the order of claims is never reversed by the attribution of reasonable awards. Let $i, j \in N$ be such that $c_i < c_j$. If $E \leq c_i$, both claims decrease by $\frac{E}{|N|}$. If $c_i < E \leq c_j$, c_i is replaced by $\tilde{c}_i \equiv c_i - \frac{c_i}{|N|}$ and c_j by $\tilde{c}_j \equiv c_j - \frac{E}{|N|}$, then $\tilde{c}_i \leq \tilde{c}_j$. Thus, in the proof of Theorem 1, we could have chosen agent i to be the agent with the largest claim.

In the next paragraphs, we give an explicit construction of the recursive rule. In general, (for $|N| = 2$ and except if the larger claim is twice the smaller claim), the path of awards of the rule is the concatenation of an infinite number of segments. A graphical representation is possible for $|N| = 2$. The shape of the path of awards depends on the relative values of the claims. We distinguish two cases:

Case 1: $\boldsymbol{c_2 \leq 2c_1}$: If $E \leq c_1$, $\mu(c, E) = (\frac{E}{2}, \frac{E}{2})$, so $R(c, E) = (\frac{E}{2}, \frac{E}{2})$ (Figure 3a). If $c_1 < E \leq c_2$, $\mu(c, E) = (\frac{c_1}{2}, \frac{E}{2})$, and in $(c - \mu(c, E), E - \sum \mu_i(c, E))$, the amount to divide is no greater than the smaller claim, so equal division prevails. Thus, $x \equiv R(c, E) = (\frac{c_1}{4} + \frac{E}{4}, -\frac{c_1}{4} + 3\frac{E}{4})$. Note that as E increases, x moves up along a line of slope 3 (Figure 3a). The point reached when $E = c_2$ is $e^1 \equiv (\frac{c_1}{4} + \frac{c_2}{4}, -\frac{c_1}{4} + 3\frac{c_2}{4})$. If $c_2 < E$, $\mu(c, E) = \frac{c}{2}$. At first, equal division of any amount greater than c_2 prevails. The process described for $E \leq c_2$ is repeated for $E \leq \frac{c_2}{2}$ since this is the value of agent 2's revised claim (Figure 3b).

The path that results is as follows: divide box$[(0, 0), c]$ into four equal boxes by drawing a vertical line of abscissa $\frac{c_1}{2}$ and a horizontal line of ordinate $\frac{c_2}{2}$; divide the northeast box so defined into four equal boxes in a similar way; repeat. Let $d^1 \equiv (0, 0) + (\frac{c_1}{2}, \frac{c_1}{2})$, $d^2 \equiv \frac{c}{2} + (\frac{3c_1}{4}, \frac{3c_1}{4})$, $d^3 \equiv \frac{3c}{4} + (\frac{7c_1}{8}, \frac{7c_1}{8})$, and so on. Let $\sigma^1 \equiv$ seg$[(0, 0), d^1]$, $\sigma^2 \equiv$ seg$[\frac{c}{2}, d^2]$, $\sigma^3 \equiv$ seg$[\frac{3c}{4}, d^3]$, and so on. Let e^1 be the intersection of σ^2 with the line of slope 3 emanating from d^1, e^2 be the intersection of σ^3 with the line of slope 3 emanating from d^2, and so on. Now, the path for c is bro.seg$[(0, 0), d^1, e^1, d^2, e^2, \ldots]$ (Figure 3c).

The equality $c_2 = 2c_1$ identifies the boundary between Case 1 and Case 2 examined next. Then, $e^1 = d^2$, $e^2 = d^3$, and so on. The segments of slope 1 vanish and we are left with a concatenation of segments of slope 3 from $(\frac{c_1}{2}, \frac{c_1}{2})$ to c. Thus, the path for c is bro.seg$[(0, 0), (\frac{c_1}{2}, \frac{c_1}{2}), c]$ (Figure 4a).

Case 2: $\boldsymbol{c_2 > 2c_1}$: The description of the path for this case is more complex because the direction of the inequality between agent 2's claim and the amount to divide may not change until several iterations. The path begins as in Case 1 with a segment of slope 1, and it continues with segments of slope 3, slope 7,... $2^k - 1$, and so on, until $\frac{E}{2} = \frac{c_2}{2}$, and c_2 is revised down to $\frac{c_2}{2}$ instead of to $\frac{E}{2}$. We refer to this sequence of steps as Stage 1. The greater c_2 is in relation to c_1, the more steps in Stage 1. Stage 2 consists of a parallel sequence of steps, and the path continues with a sequence of segments of increasing slopes until once gain, the direction of the inequality between agent 2's claim and the amount to divide changes. Figure 4b illustrates the construction for (c, E) such that $c_2 = 2.5c_1$ and up to $E = 3c_1$.

5 Properties of the recursive rule

In this section, we undertake a systematic investigation of the properties of the recursive rule. The properties we consider are standard in the literature,

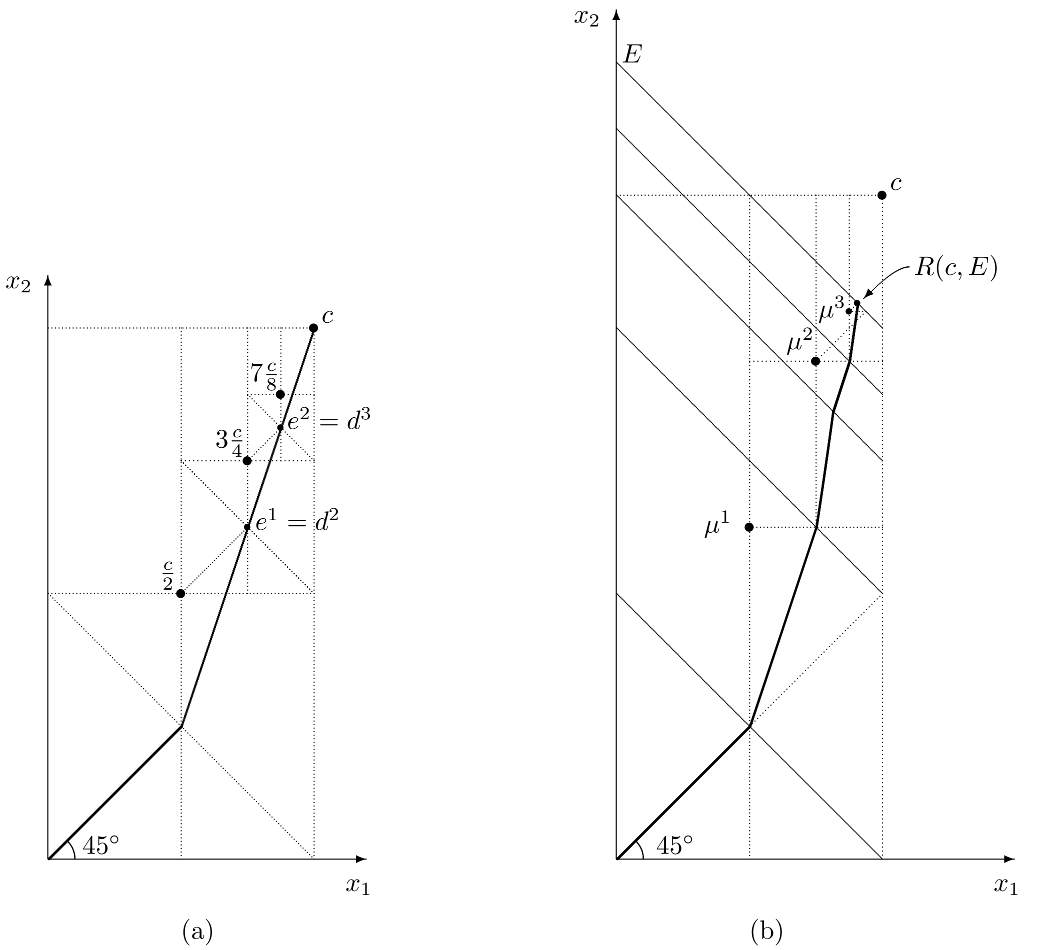

Figure 4: Path of awards for the recursive rule. (a) The case $c_1 = \frac{c_2}{2}$. (b) A configuration for which $c_1 < \frac{c_2}{2}$. The path consists of parts, each of which consists of sequences of increasing slopes, starting with a segment of slope 3. For short, we write μ^k for $\mu^k(c, E)$.

and we refer to Thomson (2003a, 2003b) for complete references.

By definition, the rule satisfies *reasonable lower bounds on awards.* It obviously satisfies *equal treatment of equals.* In situations in which some agents are deemed more deserving than others, this axiom is not desirable however, but if needed, the rule can be redefined so as to accommodate an asymmetric treatment of agents with equal claims. One introduces weights $\alpha \in \text{int}\Delta^N$ reflecting the extent to which certain agents are thought to be more deserving than others.[10] For each $i \in N$, let $\mu_i^\alpha(c, E) \equiv \alpha_i t_i(c, E)$. We now reformulate our lower bound as follows: the ***α-weighted reasonable lower bound*** is $S(c, E) \geqq (\alpha_i t_i(c, E))_{i \in N}$. The *$\alpha$-weighted bound* is satisfied by the weighted constrained equal awards rule with weights α and by the weighted versions of the Talmud rule with weights α (Hokari and Thomson, 2003a). However, the requirement $S(c, E) = \mu^\alpha(c, E) + S(c - \mu^\alpha(c, E), E - \sum \mu_i^\alpha(c, E))$ is met by only one rule, which is a weighted version of the recursive rule. We omit the proof, which follows that of Theorem 1.

The recursive rule satisfies ***order preservation*** (Aumann and Maschler, 1985), the requirement that awards should be ordered as claims are, and that so should losses. The proof relies on the fact that at each step of the recursion, awards are ordered as claims are, as shown above, and that, after revision, the order of claims is not reversed. Similarly, losses are ordered as claims are. Indeed, note that if $c_i \leq c_j$, then $c_i - \frac{1}{|N|} \min\{c_i, E\} \leq c_j - \frac{1}{|N|} \min\{c_i, E\}$, as can be seen by examining the three possible cases, $E \leq c_i$, $c_i < E \leq c_j$, and $c_j < E$.

The rule satisfies ***anonymity***, the requirement that the names of agents should not matter, and ***homogeneity***, the requirement that, starting from any problem, if the data of the problem are multiplied by some positive number, so should the recommended awards vector.

We now turn to two basic monotonicity properties. First is the requirement that when the amount available increases, each agent should receive at least as much as he did initially. The idea of monotonicity is central to the axiomatic literature on fair allocation (for a survey, see Thomson, 2003d).

Resource monotonicity: For each $(c, E) \in \mathcal{C}^N$ and each $E' > E$, if $\sum c_i \geq E'$, then $S(c, E') \geqq S(c, E)$.

The following lemma relates the sequences of revised problems obtained for two values of the amount available.

[10] The notation Δ^N designates the simplex in $\mathbb{R}^N$.

Lemma 4 *For each $(c, E) \in \mathcal{C}^N$ and each $E' > E$ such that $(c, E') \in \mathcal{C}^N$, let (c^k, E^k) and (c'^k, E'^k) be the revised problems of the k-th step, starting from (c, E) and (c, E') respectively. Then, for each $k > 1$, $c'^k \leqq c^k$ and $E'^k \geq E^k$.*

We relegate the proof to the appendix.

Proposition 1 *The recursive rule is resource monotonic.*

Proof: Let $(c, E) \in \mathcal{C}^N$, and $E' > E$ be such that $(c, E') \in \mathcal{C}^N$. Let $i \in N$. By Lemma 4, and using the notation of the lemma, for each $k > 1$, ${c'_i}^k \leq {c_i}^k$. Then, $\bar{c}_i \equiv \lim_{k\to\infty} {c_i}^k \geq \lim_{k\to\infty} {c'_i}^k \equiv \bar{c}'_i$. By definition of the recursive rule, $R_i(c, E') \equiv c_i - \bar{c}'_i \geq c_i - \bar{c}_i \equiv R_i(c, E)$. $\square$

Many models include the specification of individual parameters, representing initial ownership of assets, rights, obligations, and so on. Whenever these parameters are valuable resources—as in the case of assets or rights—it is natural to require that an increase in an individual's parameter should benefit him. If they are not valuable, what is natural to require is that an increase should penalize him. Here, the parameter falls in the first category and we require that if an agent's claim increases, he should receive at least as much as he did initially.

Claims monotonicity: For each $(c, E) \in \mathcal{C}^N$, each $i \in N$, and each $c'_i > c_i$, we have $S_i(c'_i, c_{-i}, E) \geq S_i(c, E)$.[11]

We may also be interested in how the other agents are affected by an increase in some agent's claim. We require that each of them should receive at most as much as he did initially.[12]

Others-oriented claims monotonicity: For each $(c, E) \in \mathcal{C}^N$, each $i \in N$, and each $c'_i > c_i$, we have $S_{N\setminus\{i\}}(c'_i, c_{-i}, E) \leqq S_{N\setminus\{i\}}(c, E)$.

Together with *efficiency*, (which is incorporated in the definition of a rule,) this property implies *claims monotonicity*. In the two-claimant case, the two properties are equivalent.

[11]The notation c_{-i} designates the vector c from which the i-th coordinate has been deleted, and the notation (c'_i, c_{-i}) the vector c in which the i-th coordinate has been replaced by c'_i.

[12]Thomson (1987) formulates a parallel for classical exchange economies.

It will be convenient to first show that the recursive rule satisfies *others-oriented claims monotonicity*, and deduce that it satisfies *claims monotonicity.* We first prove a lemma which relates the revised problems after each recursion when an agent's claim increases.

Lemma 5 *For each $(c, E) \in \mathcal{C}^N$, each $i \in N$, and each $c'_i > c_i$, let (c^k, E^k) and $({c'_i}^k, {c'}^k_{-i}, {E'}^k)$ be the revised problems of the k-th step, starting from (c, E) and (c'_i, c_{-i}, E) respectively. Then, for each $k > 1$, ${c'}^k \geqq c^k$ and ${E'}^k \leq E^k$.*

We relegate the proof to the appendix. It is by induction. We first show that for each $k > 1$, ${c'}^k \geqq c^k$. Then we show that at each step of the recursion, the set of agents whose claim is smaller than the amount available for the new problem is a subset of the corresponding set for the original problem. Using these two facts we conclude that for each $k > 1$, ${E'}^k \leq E^k$.

Our next result is an immediate consequence of Lemma 5 and the definition of the recursive rule.

Proposition 2 *The recursive rule is others-oriented claims monotonic.*

Proof: Let $(c, E) \in \mathcal{C}^N$, $i \in N$, and $c'_i > c_i$. By Lemma 5, for each $j \in N \backslash \{i\}$ and each $k > 1$, ${c'_j}^k \geq c^k_j$. Then, and using the notation of the lemma, $\bar{c}'_j \equiv \lim_{k\to\infty} {c'_j}^k \geq \lim_{k\to\infty} c^k_j \equiv \bar{c}_j$. By definition of the recursive rule, $R_j(c'_i, c_{-i}, E) \equiv c_j - \bar{c}'_j \leq c_j - \bar{c}_j \equiv R_j(c, E)$. □

The following proposition is a direct corollary of Proposition 2:

Proposition 3 *The recursive rule is claims monotonic.*

Next we turn to an important invariance property. Since the part of an agent's claim that exceeds the amount available cannot be recovered anyway, we might just as well ignore it: If an agent's claim is truncated at the amount available, the awards vector should not be affected. This property is satisfied by several important rules,[13] and as noted earlier, it is necessarily satisfied by a rule that has a counterpart in the theory of coalitional games.[14]

Invariance under claims truncation: For each $(c, E) \in \mathcal{C}^N$, we have $S(c, E) = S(t(c, E), E)$.

[13]See Aumann and Maschler (1985) and Dagan and Volij (1993).

[14]This is proved by Curiel, Maschler and Tijs (1987).

The proof of the next proposition is relegated to the appendix.

Proposition 4 *The recursive rule is claims truncation invariant.*

The recursive rule violates *reasonable lower bounds on losses*, ***self-duality*** (Aumann and Maschler, 1985; the requirement that the rule should coincide with its dual), and ***composition down*** (Moulin, 2000), which says that if the amount to divide decreases from some initial value, the awards vector should be obtainable directly, or by using as claims vector the awards vector calculated for the initial amount, and ***composition up*** (Young, 1987), which is an invariance property pertaining to the opposite possibility. It violates *minimal rights first*. To see this, let $N \equiv \{1, 2\}$ and $(c, E) \in \mathcal{C}^N$ be given by $(c, E) = (3, 6; 4)$. Since $c_2 = 2c_1$ and $E > c_1$, $R(c, E)$ is the point of intersection of $\text{seg}[(\frac{c_1}{2}, \frac{c_1}{2}), c] = \text{seg}[(1.5, 1.5), (3, 6)]$ with the budget line. Thus, $R_i(c, E) > 1.5$. On the other hand, $m(c, E) = (0, 1)$ and the revised problem is $(3, 5; 3)$. The amount to divide is equal to the smallest claim and we obtain equal division in the second step, namely $(1.5, 1.5)$. Thus, $R_1(c, E) > 1.5 = 0 + 1.5 = m_1(c, E) + R_1(c - m(c, E), E - \sum m_i(c, E))$.

We continue with variable-population properties. First, the recursive rule violates ***replication invariance***, the requirement that for each problem, the awards vector chosen for a replica of it should be the corresponding replica of the awards vector chosen for the initial problem. This can be seen as follows: let $N \equiv \{1, 2\}$ and $(c, E) \in \mathcal{C}^N$ be such $(c, E) = (2, 4; 2)$. Then, $\mu(c, E) = (1, 1)$ and $R(c, E) = (1, 1)$. We replicate this problem once, denoting by $2 * (c, E)$ the resulting problem and $2 * R(c, E)$ the corresponding replica of $R(c, E)$. We have $\mu(2 * (c, E)) = \mu(2, 4, 2, 4; 4) = (.5, 1, .5, 1)$, and $\mu(c - \mu(c, E)) = \mu(1.5, 3, 1.5, 3; 1) = (.25, .25, .25, .25)$, so that $R(2 * (c, E)) = (.75, 1.25, .75, 1.25) \neq 2 * R(c, E)$, in violation of *replication invariance*.

Let $r \in \mathbb{N}$ denote the order of replication. When a problem $(c, E) \in \mathcal{C}^N$ is replicated r times, let $r * N$ be the population in the replica problem, and $r * (c, E)$ the replica problem. In an r-replica of (c, E), we are led to calculating the minimum of c_i and rE, which for r large enough, is the former quantity. Then, proportional division is the outcome. Note that for each $c \in \mathbb{R}^N_+$ and each $r \in \mathbb{N}$, the path of the recursive rule for c^{r*N} starts with equal division.

Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$. When a rule satisfies *equal treatment of equals*, in an r-replica of (c, E), all clones of each member of N receive equal amounts, so the awards vector the rule selects is the r-replica of some awards vector x^r of (c, E). To say that the rule is *replication invariant* is to say that x^r is independent of r. If it is not *replication invariant*, it is natural to enquire whether the sequence $\{x^r\}$ of awards vectors so defined has a limit, and if yes, to identify the rule defined by associating to each problem this limit. Questions of this type are addressed by Chun and Thomson (2003) who obtain certain rules as limits of two rules that violate *replication invariance*.[15] Here, we have convergence too, and interestingly, the rule towards which convergence occurs is the proportional rule.

Theorem 2 *The awards vector selected by the recursive rule for a replica problem is the replica of an awards vector of the problem that is replicated that, as the order of replication increases, converges to the proportional awards vector of that problem.*

Proof: Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$ be given. If $E = 0$, the answer is straightforward, so let us assume that $E > 0$. Let $r * (c, E)$ be obtained by replicating r-times the problem (c, E). For r large enough, for each $i \in N$, $rE > c_i$, and at the first round, each agent $i \in r{*}N$ receives $\frac{1}{r|N|}c_i$. (The total distributed is $\frac{r}{|N|}\frac{1}{r}\sum_{j\in N} c_j = \frac{1}{|N|}\sum_{j\in N} c_j$.) Revised claims are proportional to original claims. At the second round, if the amount that remains to divide is still larger than the largest revised claim, proportional division to the revised claims prevails. Thus, total awards so far are still proportional to original claims. This goes on until a stage $k(r)$ at which the amount to divide is smaller than the largest claim revised $k(r)$ times. This remainder is divided among $r|N|$ claimants. Each copy of the original population receives at most $\frac{\max c_j}{r|N|}$, so the sum of the partial terms received by each member of each copy is a quantity that goes to zero as $r \to \infty$. $\square$

The recursive rule violates *consistency*. (The first application to claims problems of the idea of *consistency* is due to Young, 1987). We could give an example to make this point but we will instead derive it as a corollary of a proposition that addresses the more general question whether the two-claimant version of the rule has any *consistent* extension.[16]

[15] The random arrival and minimal overlap rules converge to the proportional and constrained equal losses rules respectively.

[16] In fact, the rule violates the weaker property of ***null claims consistency***, the requirement that if an agent's claim is 0, removing him should not affect the awards recom-

The answer is negative. The proof is based on a geometric technique developed in Thomson (2001). This technique exploits the following simple geometric implication of *consistency* of a rule: for each $N \in \mathcal{N}$, each $c \in \mathbb{R}^N_+$, and each $N' \subset N$, its path for c, when projected onto $\mathbb{R}^{N'}$, is a subset of its path for $c_{N'}$. Moreover, if the rule is *resource monotonic*, this projection actually coincides with the path for $c_{N'}$. *Resource monotonicity* holds here since the recursive rule satisfies this property in the two-claimant case, and this property is lifted (Dagan and Volij, 1997; Hokari and Thomson, 2003b). So, if the recursive rule had a *consistent* extension, this extension would be *resource monotonic*. The key to this sort of argument is to exploit the projection implication of *consistency* for entire paths of awards, not just point by point, and to understand which properties of paths are preserved by projections and which are not.

Proposition 5 *The two-claimant recursive rule has no bilaterally consistent extension to general populations.*

Proof: Let S be a *consistent* extension of the two-claimant recursive rule. Let $N \equiv \{1, 2, 3\}$ and $c \in \mathbb{R}^N_+$ be defined by $c \equiv (10, 14, 20)$. Let Π_3 be the path of S for $c_{\{1,2\}} = (10, 14) \in \mathbb{R}^{\{1,2\}}_+$. Since $c_2 < 2c_1$, Case 1 of the description of the rule given above applies. We will only need the first two segments of Π_3. Let Π_2 be the path for $c_{\{1,3\}} = (10, 20) \in \mathbb{R}^{\{1,3\}}_+$. Since $c_3 = 2c_1$, the boundary case covered under Case 1 applies, and $\Pi_2 = \text{bro.seg}[(0, 0), (\frac{c_1}{2}, \frac{c_1}{2}), c_{\{1,3\}}]$ (the case illustrated in Figure 4a). Let $k^1 \equiv (\frac{c_1}{2}, \frac{c_1}{2})$ and $k^2 \equiv k^1 + \frac{c_2-c_1}{4}(1, 3) = (-\frac{c_1}{4} + \frac{c_2}{4}, -\frac{c_1}{4} + 3\frac{c_2}{4})$ be the first two kinks of Π_3. Let $\ell^1 \equiv (\frac{c_1}{2}, \frac{c_1}{2})$ be the kink in Π_2, and ℓ^2 be the point of Π_2 whose first coordinate is equal to k^2_1. Since the first segment of Π_3 is $\text{seg}[(0, 0), (\frac{c_1}{2}, \frac{c_1}{2})] \subset \mathbb{R}^{\{1,2\}}$, and the first segment of Π_2 is $\text{seg}[(0, 0), (\frac{c_1}{2}, \frac{c_1}{2})] \subset \mathbb{R}^{\{1,3\}}$, the path for c begins with $\text{seg}[(0, 0, 0), (\frac{c_1}{2}, \frac{c_1}{2}, \frac{c_1}{2})] \subset \mathbb{R}^N$. This segment is contained in the plane of equation $x_2 = x_3$, and its projection onto $\mathbb{R}^{\{2,3\}}_+$

mended for the other claimants. To see this, let $N \equiv \{1, 2\}$ and $(c, E) \in \mathcal{C}^N$ be given by $(c, E) = (2, 4; 3)$. Then, $\mu(c, E) = (1, 1.5)$ and $R(c, E) = (1, 1.5) + (.25, .25) = (1.25, 1.75)$. Now, let $N' \equiv \{1, 2, 3\}$ and $(c', E') \in \mathcal{C}^{N'}$ be given by $(c', E') = (2, 4, 0; 3)$. We have $\mu(c', E') = (.666, 1, 0)$; revised claims are $(1.33, 3, 0)$ and the revised amount to divide 1.33. In the new problem (c'', E'') we have $\mu(c'', E'') = \frac{1}{3}(1.33, 1.33, 0)$. From that point on, the revised claims of agents 1 and 2, once truncated, are equal, so we obtain a sequence of equal division steps, which when taken to the limit, give us an equal division of 1.33. The final awards vector is $(.666, 1, 0) + (.666, .666, 0) \neq (1.25, 1.75, 0)$.

is $\mathrm{seg}[(0,0),(\frac{c_1}{2},\frac{c_1}{2})]$, (in the figure, $(\frac{c_1}{2},\frac{c_1}{2}) \in \mathbb{R}^{\{2,3\}}_+$ is the point m^1). Since the slope of the second segment of Π_3 is equal to that of the second segment of Π_2 (both slopes are equal to 3), $k^2_2 = \ell^2_3 = -\frac{c_1}{4} + 3\frac{c_2}{4} \equiv a$. A simple calculation shows that $a = 8$. Thus, the point in $\mathbb{R}^N$ whose projections onto $\mathbb{R}^{\{1,2\}}$ and $\mathbb{R}^{\{1,3\}}$ are k^2 and ℓ^2 has equal second and third coordinates, and its projection onto $\mathbb{R}^{\{2,3\}}$ belongs to the 45° line of that space. Thus, for S to be *consistent*, the path for $c_{\{2,3\}} = (14, 20) \in \mathbb{R}^{\{2,3\}}_+$ should also contain $\mathrm{seg}[m^1,(a,a)]$. However, since $c_3 < 2c_2$, Case 1 applies to $c_{\{2,3\}}$: the path of the recursive rule for $c_{\{2,3\}}$ is also piece-wise linear, its first two segments being $\mathrm{seg}[(0,0),(\frac{c_2}{2},\frac{c_2}{2})]$ and a segment of slope 3 with lower endpoint $(\frac{c_2}{2},\frac{c_2}{2})$: it has a kink at $(\frac{c_2}{2},\frac{c_2}{2})$. Since $2a = -\frac{c_1}{2} + 3\frac{c_2}{2} > c_2$, the point (a,a) lies above the line of equation $t_2 + t_3 = c_2$. We have obtained a contradiction. $\square$

6 A comparison with another lower bound

Other bounds have been proposed for rules than the one we studied here. A simple one is that each agent should receive the minimum of his claim and equal division (Moulin, 2002). This ***constrained equal division lower bound on awards*** is more restrictive than the *reasonable lower bounds on awards* (MTV, 2004a), and in fact, by itself, it characterizes the constrained equal awards rule in the two-claimant case.

Let us base on it an invariance axiom parallel to the one we based on reasonable awards: for each problem, the awards vector should be obtainable in two ways, (i) directly, or (ii) in two steps, by first assigning to each claimant his lower bound, and in a second step, assigning to him what the rule would in the appropriately revised problem.[17] It is straightforward to see that this property, independently of the number of claimants, is satisfied only by the constrained equal awards rule.

The *constrained equal division lower bound* is the largest *anonymous* bound that one can impose on an agent's award that depends only on the agent's own claim and the amount to divide. To see this, let $b(c_i, E)$ be a bound of this type imposed on claimant i's award in the problem (c, E). (*Anonymity* is reflected in the fact that the function b is independent of i.)

[17]Let $\nu_i(c,E) \equiv \min\{c_i, \frac{E}{|N|}\}$ and $\nu(c,E) \equiv (\nu_i(c,E))_{i\in N}$. Then, $x \geqq \nu(c,E)$ is the *constrained equal division lower bound* and the invariance property is $S(c,E) = \nu(c,E) + S(c - \nu(c,E), E - \sum_{i\in N} \nu_i(c,E))$.

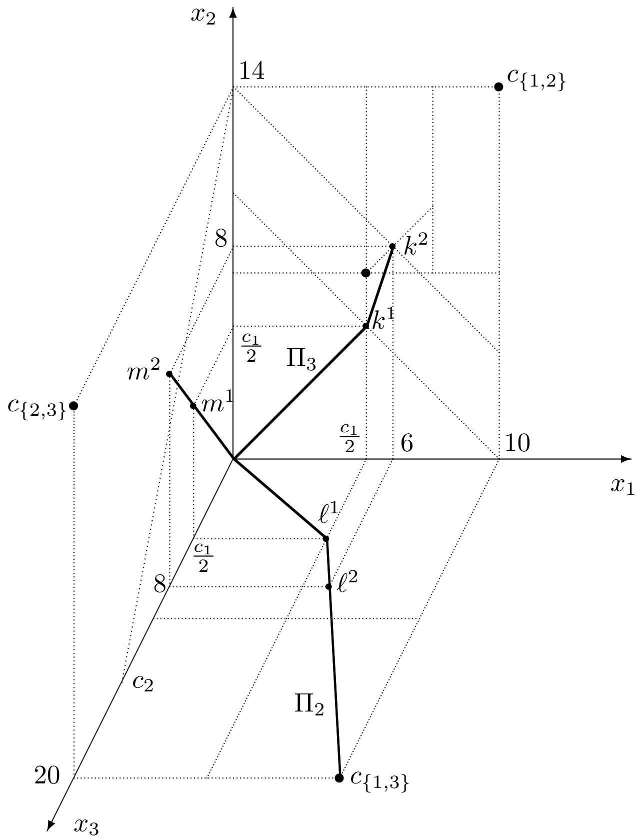

Figure 5: The two-claimant version of the recursive rule has no consistent extension to general populations. This figure pertains to the claims vector $c \equiv (10, 14, 20)$. It shows the first two segments of the path Π_3 of the recursive rule for $c_{\{1,2\}} = (10, 14) \in \mathbb{R}_+^{\{1,2\}}$, and its entire path Π_2 for $c_{\{1,3\}} = (10, 20) \in \mathbb{R}_+^{\{1,3\}}$. The path for c of a *consistent* extension of R (this path is not represented), if such an extension exists, can be constructed from these projections, Π_2 and Π_3. Its projection onto $\mathbb{R}^{\{2,3\}}$ contains seg$[(0,0), m^2]$.

By definition of a rule, we need $b(c_i, E) \leq c_i$. Also, for there to exist an awards vector meeting this bound for each agent, the profile $(b(c_i, E))_{i \in N}$ should be such that $\sum b(c_i, E) \leq E$. Fix $i \in N$. If $c_i > E$, in the well-defined problem $(\bar{c}, E) \in \mathcal{C}^N$ in which for each $j \in N$, $\bar{c}_j = c_i$, we obtain $|N| b(c_i, E) \leq E$. Altogether, $b(c_i, E) \leq \min\{c_i, \frac{E}{|N|}\}$, as claimed.

The property of a rule that it meets the *constrained equal division lower bound* is of course lifted since (i) in the two-claimant case, only one rule satisfies it, the two-claimant constrained equal awards rule, and (ii) the constrained equal awards rule, the only *consistent* rule that coincides with its two-claimant version,[18] also satisfies the bound. On the other hand, the property is not preserved under the attribution of minimal rights operator. Indeed, the rule obtained by subjecting the two-claimant constrained equal awards rule to this operator is the Talmud rule, which does not meet the bound.

The bound we have considered here is less demanding than the *constrained equal division lower bound*, but the invariance axiom based on it also leads to a unique rule.

7 Appendix

In this appendix we prove Lemmas 4 and 5.

Proof: (of Lemma 4) The proof is by induction.

Step 1 of the induction.

Part 1: $\boldsymbol{c'^2} \leqq \boldsymbol{c^2}$.

Let $i \in N$. Using the hypothesis $c_i^1 = c_i'^1$, we distinguish three cases:

Case 1: $c_i^1 = c_i'^1 > E'^1 > E^1$.

$$\begin{array}{rcll} c_i'^2 & \equiv & c_i'^1 - \frac{1}{n}\min\{c_i'^1, E'^1\} & \\ & = & c_i'^1 - \frac{1}{n}E'^1 & \text{(since } c_i'^1 > E'^1\text{)} \\ & \leq & c_i^1 - \frac{1}{n}E^1 & \text{(since } c_i^1 = c_i'^1 \text{ and } E'^1 > E^1\text{)} \\ & \leq & c_i^1 - \frac{1}{n}\min\{c_i^1, E^1\} & \\ & \equiv & c_i^2 & \end{array}$$

Case 2: $E'^1 \geq c_i^1 = c_i'^1 > E^1$.

[18] This is a consequence of the Elevator Lemma discussed in footnote 8.

$$\begin{array}{rcll} {c'_i}^2 & \equiv & {c'_i}^1 - \frac{1}{n}\min\{{c'_i}^1, {E'}^1\} & \\ & = & {c'_i}^1 - \frac{1}{n}{c'_i}^1 & (\text{since } {c'_i}^1 \le {E'}^1) \\ & \le & c_i^1 - \frac{1}{n}E^1 & (\text{since } c_i^1 = {c'_i}^1 \text{ and } {c'_i}^1 > E^1) \\ & \le & c_i^1 - \frac{1}{n}\min\{c_i^1, E^1\} & \\ & \equiv & c_i^2 & \end{array}$$

Case 3: ${E'}^1 > E^1 \ge c_i^1 = {c'_i}^1$.

$$\begin{array}{rcll} {c'_i}^2 & \equiv & {c'_i}^1 - \frac{1}{n}\min\{{c'_i}^1, {E'}^1\} & \\ & = & {c'_i}^1 - \frac{1}{n}{c'_i}^1 & (\text{since } {E'}^1 \ge {c'_i}^1) \\ & = & c_i^1 - \frac{1}{n}c_i^1 & (\text{since } {c'_i}^1 = c_i^1\ ) \\ & \le & c_i^1 - \frac{1}{n}\min\{c_i^1, E^1\} & \\ & \equiv & c_i^2 & \end{array}$$

Part 2: ${E'}^2 \ge E^2$.

Let $A \equiv \{i \in N : \ c_i^1 \le E^1\}$ and $A' \equiv \{i \in N : \ {c'_i}^1 \le {E'}^1\}$. We claim that that $A' \supseteq A$ (and therefore, ${A'}^C \subseteq A^C$). Indeed, let $i \in A$. Then, $c_i^1 \le E^1$. Since $E^1 < {E'}^1$, then ${c'_i}^1 = c_i^1 \le E^1 < {E'}^1$, which implies $i \in A'$.

Second, by the definitions of A and A',

$$E^2 = E^1 - \frac{1}{n}\sum_{i\in A} c_i^1 - \frac{1}{n}|A^C|E^1 = \frac{n-|A^C|}{n}E^1 - \frac{1}{n}\sum_{i\in A} c_i^1,$$

and

$$E'^2 = E'^1 - \frac{1}{n}\sum_{i\in A'} {c_i}'^1 - \frac{1}{n}|{A'}^C|E'^1.$$

Since $A' \supseteq A$, then $A' = A \cup (A' \setminus A)$. Thus,

$$E'^2 = E'^1 - \frac{1}{n}\sum_{i\in A} {c'_i}^1 - \frac{1}{n}\sum_{i\in A'\setminus A} {c'_i}^1 - \frac{1}{n}|{A'}^C|E'^1.$$

For each $i \in A' \setminus A$, we have ${c'_i}^1 \le {E'}^1$. Thus,

$$E'^2 \ge E'^1 - \frac{1}{n}\sum_{i\in A} {c_i}'^1 - \frac{1}{n}|A'\setminus A|E'^1 - \frac{1}{n}|{A'}^C|E'^1.$$

By the definitions of A and A', we have $(A' \setminus A) \cup {A'}^C = A^C$. Thus,

$$E'^2 \geq E'^1 - \frac{1}{n}\sum_{i\in A} {c_i}'^1 - \frac{1}{n}|A^C|E'^1 = \frac{n-|A^C|}{n}E'^1 - \frac{1}{n}\sum_{i\in A} {c_i}'^1.$$

Since $c_i^1 = {c_i}'^1$ and $E'^1 > E^1$,

$$E'^2 \geq \frac{n-|A^C|}{n}E^1 - \frac{1}{n}\sum_{i\in A} c_i^1 = E^2.$$

Step $\boldsymbol{k}$ of the induction. Let $k \geq 2$, and suppose that for each $\ell \in \{1,\ldots,k-1\}$, we have $c'^{\ell} \leqq c^{\ell}$ and $E'^{\ell} \geq E^{\ell}$.

Part 1: $\boldsymbol{c'^k} \leqq \boldsymbol{c^k}$.

Let $i \in N$. Using the induction hypothesis, we distinguish four cases:

Case 1: $c_i^{k-1} \geq c_i'^{k-1} \geq E'^{k-1} \geq E^{k-1}$.

$$\begin{array}{rcll} c_i'^k & \equiv & c_i'^{k-1} - \frac{1}{n}\min\{c_i'^{k-1}, E'^{k-1}\} & \\ & = & c_i'^{k-1} - \frac{1}{n}E'^{k-1} & (\text{since } c_i'^{k-1} \geq E'^{k-1}) \\ & \leq & c_i^{k-1} - \frac{1}{n}E^{k-1} & (\text{since } c_i^{k-1} \geq c_i'^{k-1} \text{ and } E^{k-1} \leq E'^{k-1}) \\ & \leq & c_i^{k-1} - \frac{1}{n}\min\{c_i^{k-1}, E^{k-1}\} & \\ & \equiv & {c_i}^k & \end{array}$$

Case 2: $c_i^{k-1} \geq E'^{k-1} \geq c_i'^{k-1} \geq E^{k-1}$, or $E'^{k-1} \geq c_i^{k-1} \geq c_i'^{k-1} \geq E^{k-1}$.

$$\begin{array}{rcll} c_i'^k & \equiv & c_i'^{k-1} - \frac{1}{n}\min\{c_i'^{k-1}, E'^{k-1}\} & \\ & = & c_i'^{k-1} - \frac{1}{n}c_i'^{k-1} & (\text{since } E'^{k-1} \geq c_i'^{k-1}) \\ & \leq & c_i^{k-1} - \frac{1}{n}E^{k-1} & (\text{since } c_i^{k-1} \geq c_i'^{k-1} \text{ and } c_i'^{k-1} \geq E^{k-1}) \\ & \leq & c_i^{k-1} - \frac{1}{n}\min\{{c_i}^{k-1}, E^{k-1}\} & \\ & \equiv & c_i^k & \end{array}$$

Case 3: $c_i^{k-1} \geq E'^{k-1} \geq E^{k-1} \geq c_i'^{k-1}$, or $E'^{k-1} \geq c_i^{k-1} \geq E^{k-1} \geq c_i'^{k-1}$.

$$\begin{array}{rcll} c_i'^k & \equiv & c_i'^{k-1} - \frac{1}{n}\min\{c_i'^{k-1}, E'^{k-1}\} & \\ & = & c_i'^{k-1} - \frac{1}{n}c_i'^{k-1} & (\text{since } E'^{k-1} \geq c_i'^{k-1}) \\ & \leq & c_i^{k-1} - \frac{1}{n}c_i^{k-1} & (\text{since } c_i^{k-1} \geq c_i'^{k-1}) \\ & \leq & c_i^{k-1} - \frac{1}{n}\min\{{c_i}^{k-1}, E^{k-1}\} & \\ & \equiv & c_i^k & \end{array}$$

Case 4: $E'^{k-1} \geq E^{k-1} \geq c_i^{k-1} \geq c_i'^{k-1}$.

$$
\begin{array}{rcll}
c_i'^k & \equiv & c_i'^{k-1} - \frac{1}{n}\min\{c_i'^{k-1}, E'^{k-1}\} & \\
& = & c_i'^{k-1} - \frac{1}{n}c_i'^{k-1} & (\text{since } E'^{k-1} \geq c_i'^{k-1}) \\
& \leq & c_i^{k-1} - \frac{1}{n}c_i^{k-1} & (\text{since } c_i^{k-1} \geq c_i'^{k-1}) \\
& \leq & c_i^{k-1} - \frac{1}{n}\min\{c_i^{k-1}, E^{k-1}\} & \\
& \equiv & c_i^k &
\end{array}
$$

Part 2: $\boldsymbol{E'^k \geq E^k}$.

Let $A \equiv \{i \in N : c_i^{k-1} \leq E^{k-1}\}$ and $A' \equiv \{i \in N : c_i'^{k-1} \leq E'^{k-1}\}$. We claim that $A' \supseteq A$ (and therefore $A'^C \subseteq A^C$). Indeed, let $i \in A$. Then, $c_i^{k-1} \leq E^{k-1}$. By the induction hypothesis, $c_i'^{k-1} \leq c_i^{k-1}$ and $E'^{k-1} \geq E^{k-1}$. Thus, $c_i'^{k-1} \leq E'^{k-1}$, which implies $i \in A'$.

Second, by the definitions of A and A',

$$E^k = E^{k-1} - \frac{1}{n}\sum_{i\in A} c_i^{k-1} - \frac{1}{n}|A^C|E^{k-1} = \frac{n-|A^C|}{n}E^{k-1} - \frac{1}{n}\sum_{i\in A} c_i^{k-1}$$

and

$$E'^k = E'^{k-1} - \frac{1}{n}\sum_{i\in A'} c_i'^{k-1} - \frac{1}{n}|A'^C|E'^{k-1}.$$

Since $A' \supseteq A$, then $A' = A \cup (A' \setminus A)$. Thus,

$$E'^k = E'^{k-1} - \frac{1}{n}\sum_{i\in A} c_i'^{k-1} - \frac{1}{n}\sum_{i\in A'\setminus A} c_i'^{k-1} - \frac{1}{n}|A'^C|E'^{k-1}.$$

For each $i \in A' \setminus A$, we have $c_i'^{k-1} \leq E'^{k-1}$. Thus,

$$E'^k \geq E'^{k-1} - \frac{1}{n}\sum_{i\in A} c_i'^{k-1} - \frac{1}{n}|A' \setminus A|E'^{k-1} - \frac{1}{n}|A'^C|E'^{k-1}.$$

By the definitions of A and A', we have $(A' \setminus A) \cup A'^C = A^C$. Thus,

$$E'^k \geq E'^{k-1} - \frac{1}{n}\sum_{i\in A} c_i'^{k-1} - \frac{1}{n}|A^C|E'^{k-1} = \frac{n-|A^C|}{n}E'^{k-1} - \frac{1}{n}\sum_{i\in A} c_i'^{k-1}.$$

By the induction hypothesis, $c_1'^{k-1} \leq c_i^{k-1}$ and $E'^{k-1} \geq E^{k-1}$. Thus, $E'^k \geq \frac{n-|A^C|}{n}E^{k-1} - \frac{1}{n}\sum_{i\in A} c_i^{k-1} = E^k$.

$\square$

The proof of Lemma 5 is parallel to that of Lemma 4.

Proof: (of Lemma 5) The proof is by induction. Without loss of generality suppose that the agent whose claim increases is claimant 1; that is, $c'_1 > c_1$.

Step 1 of the induction.

Part 1: $\boldsymbol{c'^2 \geqq c^2}$.

For each $i \in \{2, \ldots, n\}$, we have $c_i^1 = {c'_i}^1$, which implies ${c'_i}^2 \equiv {c'_i}^1 - \frac{1}{n}\min\{{c'_i}^1, E^1\} = c_i^1 - \frac{1}{n}\min\{c_i^1, E^1\} \equiv c_i^2$. Thus, for each $i \in \{2, \ldots, n\}$, ${c'_i}^2 \geq c_i^2$.

It remains to deal with claimant 1. Using the hypothesis ${c'_1}^1 > c_1^1$, we distinguish three cases.

Case 1: ${c'_1}^1 > c_1^1 \geq E^1 = E'^1$.

$$\begin{array}{rcll} {c'_1}^2 & \equiv & {c'_1}^1 - \frac{1}{n}\min\{{c'_1}^1, E'^1\} & \\ & \geq & {c'_1}^1 - \frac{1}{n}E'^1 & \\ & \geq & c_1^1 - \frac{1}{n}E^1 & \text{(since } {c'_1}^1 > c_1^1 \text{ and } E^1 = E'^1\text{)} \\ & = & c_1^1 - \frac{1}{n}\min\{c_1^1, E^1\} & \text{(since } c_1^1 \geq E^1\text{)} \\ & \equiv & c_1^2 & \end{array}$$

Case 2: ${c'_1}^1 > E^1 = E'^1 \geq c_1^1$.

$$\begin{array}{rcll} {c'_1}^2 & \equiv & {c'_1}^1 - \frac{1}{n}\min\{{c'_1}^1, E'^1\} & \\ & \geq & {c'_1}^1 - \frac{1}{n}{c'_1}^1 & \\ & \geq & c_1^1 - \frac{1}{n}c_1^1 & \text{(since } {c'_1}^1 > c_1^1\text{)} \\ & = & c_1^1 - \frac{1}{n}\min\{c_1^1, E^1\} & \text{(since } E^1 \geq c_1^1\text{)} \\ & \equiv & c_1^2 & \end{array}$$

Case 3: $E^1 = E'^1 \geq {c'_1}^1 > c_1^1$.

$$\begin{array}{rcll} {c'_1}^2 & \equiv & {c'_1}^1 - \frac{1}{n}\min\{{c'_1}^1, E'^1\} & \\ & \geq & {c'_1}^1 - \frac{1}{n}{c'_1}^1 & \\ & \geq & c_1^1 - \frac{1}{n}c_1^1 & \text{(since } {c'_1}^1 > c_1^1\text{)} \\ & = & c_1^1 - \frac{1}{n}\min\{c_1^1, E^1\} & \text{(since } E^1 \geq c_1^1\text{)} \\ & \equiv & c_1^2 & \end{array}$$

Part 2: $\boldsymbol{E'^2 \leq E^2}$.

Since $E^2 \equiv E^1 - \frac{1}{n}\sum_{i\in N}\min\{c_i^1, E^1\}$, $E'^2 \equiv E'^1 - \frac{1}{n}\sum_{i\in N}\min\{{c'_i}^1, E'^1\}$, $E'^1 = E^1$, and for each $i \in N$, ${c'_i}^1 \geq c_i^1$, then $E'^2 \leq E^2$.

Step $\boldsymbol{k}$ of the induction. Let $k \geq 2$ and suppose that for each $\ell \in \{1, \ldots, k-1\}$, we have $c'^\ell \geqq c^\ell$ and $E'^\ell \leq E^\ell$. We need to show that $c'^k \geqq c^k$ and $E'^k \leq E^k$.

Part 1: $\boldsymbol{c'^k} \geqq \boldsymbol{c^k}$.

Let $i \in N$. Using the induction hypothesis, we distinguish four cases:

Case 1: $c_i'^{k-1} \geq c_i^{k-1} \geq E^{k-1} \geq E'^{k-1}$.

$$\begin{array}{rcll} c_i'^k & \equiv & c_i'^{k-1} - \frac{1}{n}\min\{c_i'^{k-1}, E'^{k-1}\} & \\ & \geq & c_i'^{k-1} - \frac{1}{n}E'^{k-1} & \\ & \geq & c_i^{k-1} - \frac{1}{n}E^{k-1} & (\text{since } c_i'^{k-1} \geq c_i^{k-1} \text{ and } E^{k-1} \geq E'^{k-1}) \\ & = & c_i^{k-1} - \frac{1}{n}\min\{c_i^{k-1}, E^{k-1}\} & (\text{since } c_i^{k-1} \geq E^{k-1}) \\ & \equiv & c_i^k & \end{array}$$

Case 2: $c_i'^{k-1} \geq E^{k-1} \geq c_i^{k-1} \geq E'^{k-1}$, or $E^{k-1} \geq c_i'^{k-1} \geq c_i^{k-1} \geq E'^{k-1}$.

$$\begin{array}{rcll} c_i'^k & \equiv & c_i'^{k-1} - \frac{1}{n}\min\{c_i'^{k-1}, E'^{k-1}\} & \\ & \geq & c_i'^{k-1} - \frac{1}{n}E'^{k-1} & \\ & \geq & c_i^{k-1} - \frac{1}{n}c_i^{k-1} & (\text{since } c_i'^{k-1} \geq c_i^{k-1} \text{ and } c_i^{k-1} \geq E'^{k-1}) \\ & = & c_i^{k-1} - \frac{1}{n}\min\{c_i^{k-1}, E^{k-1}\} & (\text{since } E^{k-1} \geq c_i^{k-1}) \\ & \equiv & c_i^k & \end{array}$$

Case 3: $c_i'^{k-1} \geq E^{k-1} \geq E'^{k-1} \geq c_i^{k-1}$, or $E^{k-1} \geq c_i'^{k-1} \geq E'^{k-1} \geq c_i^{k-1}$.

$$\begin{array}{rcll} c_i'^k & \equiv & c_i'^{k-1} - \frac{1}{n}\min\{c_i'^{k-1}, E'^{k-1}\} & \\ & \geq & c_i'^{k-1} - \frac{1}{n}E'^{k-1} & \\ & \geq & E'^{k-1} - \frac{1}{n}E'^{k-1} & (\text{since } c_i'^{k-1} \geq E'^{k-1}) \\ & \geq & c_i^{k-1} - \frac{1}{n}c_i^{k-1} & (\text{since } E'^{k-1} \geq c_i^{k-1}) \\ & = & c_i^{k-1} - \frac{1}{n}\min\{c_i^{k-1}, E^{k-1}\} & (\text{since } E^{k-1} \geq c_i^{k-1}) \\ & \equiv & c_i^k & \end{array}$$

Case 4: $E^{k-1} \geq E'^{k-1} \geq c_i'^{k-1} \geq c_i^{k-1}$.

$$\begin{array}{rcll} c_i'^k & \equiv & c_i'^{k-1} - \frac{1}{n}\min\{c_i'^{k-1}, E'^{k-1}\} & \\ & \geq & c_i'^{k-1} - \frac{1}{n}c_i'^{k-1} & \\ & \geq & c_i^{k-1} - \frac{1}{n}c_i^{k-1} & (\text{since } c_i'^{k-1} \geq c_i^{k-1}) \\ & = & c_i^{k-1} - \frac{1}{n}\min\{c_i^{k-1}, E^{k-1}\} & (\text{since } E^{k-1} \geq c_i^{k-1}) \\ & \equiv & c_i^k & \end{array}$$

Part 2: $\boldsymbol{E'^k} \leq \boldsymbol{E^k}$. Let $A \equiv \{i \in N : c_i^{k-1} \leq E^{k-1}\}$ and $A' \equiv \{i \in N : c_i'^{k-1} \leq E'^{k-1}\}$.[19] We claim that $A' \subseteq A$ (and therefore $A'^C \supseteq A^C$). Indeed, let $i \in A'$. Then, $c_i'^{k-1} \leq E'^{k-1}$. By the induction hypothesis, $c_i^{k-1} \leq c_i'^{k-1}$ and $E'^{k-1} \leq E^{k-1}$. Thus, $c_i^{k-1} \leq E^{k-1}$, which implies $i \in A'$.

Next, by the definitions of A and A',

[19] The dependence of A and A' is not indicated, as this dependence is not relevant in the proof.

$$E'^k = E'^{k-1} - \frac{1}{n}\sum_{i\in A'} c_i'^{k-1} - \frac{1}{n}|A'^C|E'^{k-1} = \frac{n-|A'^C|}{n}E'^{k-1} - \frac{1}{n}\sum_{i\in A'} c_i'^{k-1}(*)$$

and

$$E^k = E^{k-1} - \frac{1}{n}\sum_{i\in A} c_i^{k-1} - \frac{1}{n}|A^C|E^{k-1}.$$

Since $A' \subseteq A$, then $A = A' \cup (A \setminus A')$. Thus,

$$E^k = E^{k-1} - \frac{1}{n}\sum_{i\in A'} c_i^{k-1} - \frac{1}{n}\sum_{i\in A\setminus A'} c_i^{k-1} - |A^C|E^{k-1}.$$

For each $i \in A \setminus A'$, we have $c_i^{k-1} \leq E^{k-1}$. Thus,

$$E^k \geq E^{k-1} - \frac{1}{n}\sum_{i\in A'} c_i^{k-1} - \frac{1}{n}|A \setminus A'|E^{k-1} - \frac{1}{n}|A^C|E^{k-1}.$$

By the definitions of A and A', we have $(A \setminus A') \cup A^C = A'^C$. Thus,

$$E^k \geq E^{k-1} - \frac{1}{n}\sum_{i\in A'} c_i^{k-1} - \frac{1}{n}|A'^C|E^{k-1} = \frac{n-|A'^C|}{n}E^{k-1} - \frac{1}{n}\sum_{i\in A'} c_i^{k-1}.$$

By the induction hypothesis, $c_i'^{k-1} \geq c_i^{k-1}$ and $E'^{k-1} \leq E^{k-1}$. Thus, by $(*)$,

$$E'^k = \frac{n-|A'^C|}{n}E'^{k-1} - \frac{1}{n}\sum_{i\in A'} c_i'^{k-1} \leq E^k.$$

□

To simplify notation, for each $k \geq 2$, let us denote the sum of the vectors of dividends up to Step k by

$$M^k(c,E) \equiv \sum_{l=1}^{k-1} \mu^l(c,E).$$

Then, for each $k \geq 2$, we can write the k-th revision of (c, E) as

$$(c^k, E^k) = \left(c - M^k(c, E), E - \sum_{i \in N} M_i^k(c, E) \right)$$

and the k-th vector of dividends as

$$\mu^k(c, E) = \mu \left(c - M^k(c, E), E - \sum_{i \in N} M_i^k(c, E) \right).$$

Proof: (of Proposition 4)

Let $(c, E) \in \mathcal{C}^N$ and consider $(t(c, E), E) \in \mathcal{C}^N$.

Let $i \in N$. By definition, $\mu_i(c, E) = \frac{1}{n} \min\{c_i, E\} = \frac{1}{n} t_i(c, E) = \frac{1}{n} \min\{t_i(c, E), E\} = \mu_i(t(c, E), E)$. Thus,

$$\mu(c, E) = \mu(t(c, E), E). \tag{1}$$

Claim: For each $k \in \mathbb{N}$, $\mu^k(c, E) = \mu^k(t(c, E), E)$.

The proof is by induction. The case $k = 1$ is covered by (1). Now, let $k \geq 2$ and suppose that, for each $l \in \{1, \ldots, k-1\}$, $\mu^l(c, E) = \mu^l(t(c, E), E)$. Then

$$M^k(c, E) = \sum_{l=1}^{k-1} \mu^l(c, E) = \sum_{l=1}^{k-1} \mu^l(t(c, E), E) = M^k(t(c, E), E).$$

By definition of the dividends,

$$\mu^k(c, E) = \mu \left(c - M^k(c, E), E - \sum_{i \in N} M_i^k(c, E) \right).$$

Since, as just proved, $M^k(c, E) = M^k(t(c, E), E)$, we have

$$\mu^k(c, E) = \mu \left(c - M^k(t(c, E), E), E - \sum_{i \in N} M_i^k(t(c, E), E) \right). \tag{2}$$

Let $i \in N$. There are two cases:

Case 1: $c_i \leq E$.

Then, $t_i(c,E) = c_i$ and from (2), we have

$$\mu_i^k(c,E) = \mu_i\left(t(c,E) - M^k(t(c,E),E), E - \sum_{j\in N} M_j^k(t(c,E),E)\right) = \mu_i^k(t(c,E),E),$$

where the second equality is by definition of the dividends. The claim is proved.

Case 2: $c_i > E$.

Then, $c_i > t_i(c,E) = E$, and

$$c_i - M_i^k(t(c,E),E) > t_i(c,E) - M_i^k(t(c,E),E) \geq E - \sum_{j\in N} M_j^k(t(c,E),E),$$

which implies:

$$t_i\left(c - M^k(t(c,E),E), E - \sum_{j\in N} M_j^k(t(c,E),E)\right) = E - \sum_{j\in N} M_j^k(t(c,E),E), \tag{3}$$

and

$$t_i\left(t(c,E) - M^k(t(c,E),E), E - \sum_{j\in N} M_j^k(t(c,E),E)\right) = E - \sum_{j\in N} M_j^k(t(c,E),E). \tag{4}$$

From (1), (2) and (3),

$$\mu_i^k(c,E) = \mu_i\left(E - \sum_{j\in N} M_j^k(t(c,E),E), E - \sum_{j\in N} M_j^k(t(c,E),E)\right),$$

and this equality, together with (1) and (4), yields $\mu_i^k(c,E) = \mu_i^k(t(c,E),E)$. The claim is proved.

Now, by definition of the recursive rule,

$$R(c,E) = \sum_{k=1}^{\infty} \mu^k(c,E) = \sum_{k=1}^{\infty} \mu^k(t(c,E),E) = R(t(c,E),E).$$

$\square$

8 References

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