Rochester Center for Economic Research

On properties of division rules lifted by bilateral consistency

Hokari, Toru and Thomson, William

Working Paper No. 536
December 2007

UNIVERSITY OF ROCHESTER

On properties of division rules lifted by bilateral consistency*

Toru Hokari and William Thomson

This version, October 8, 2007

*University of Tsukuba and University of Rochester, respectively. Hokari acknowledges support from the Ministry of Education, Culture, Sports, Science and Technology in Japan under grants No.15730089 and No.20344856. Thomson acknowledges support from NSF under grants SBR-9731431 and SES 0214691. We thank Youngsub Chun, Toyotaka Sakai, Rodrigo Velez, and an anonymous referee for their comments. Early versions of this paper were presented at the 2000 International Meeting of the Social Choice and Welfare Society, held in Alicante, and the 2004 Midwest Mathematical Economics Meeting, held in Bloomington. Liftingtheorems.tex

Abstract

We consider the problem of adjudicating conflicting claims in the context of a variable population. A property of rules is "lifted" if whenever a rule satisfies it in the two-claimant case, and the rule is bilaterally consistent, it satisfies it for any number of claimants. We identify a number of properties that are lifted, such as equal treatment of equals, resource monotonicity, composition down and composition up, and show that continuity, anonymity and self-duality are not lifted. However, each of these three properties is lifted if the rule is resource monotonic.

Keywords: claims problems; consistency; lifting; constrained equal awards rule; constrained equal losses rule.

JEL classification number: C79; D63; D74

1 Introduction

We consider the problem of allocating a resource among agents having conflicting claims over it. An example is when the liquidation value of a bankrupt firm has to be divided among its creditors. A "division rule" is a function that associates with each situation of this kind, or "claims problem", a division of the amount available. This division is interpreted as the choice that a judge or arbitrator could make. (The literature originates in O'Neill, 1982). Now, let us imagine that a rule has been applied to solve a certain problem and that some agents leave with their awards. If the situation is reevaluated at this point, the claims of the agents who stay are what they were initially and the endowment is the difference between the initial endowment and what the departing agents took with them. Alternatively, this difference can be calculated as the sum of what the remaining agents have been assigned. If, in this "reduced problem", the rule recommends for them the same awards as initially, and if this is true no matter what the initial problem is, and no matter who leaves, the rule is "consistent". An important special case of this requirement, known as "bilateral consistency", is when all but two claimants leave. It is the expression for the model at hand of a general principle that has been central in developments that have occurred in the last twenty years in game theory and the theory of resource allocation (see Young, 1994; Thomson, 2006a). Our purpose is to contribute to the understanding of its implications for division rules.

When studying any class of resource allocation problems or conflict situations, it is a standard research strategy to first deal with the two-agent case, and once a solution has been obtained for that case, to extend it to general populations by means of bilateral consistency. The two-agent case is conceptually and mathematically simpler. Indeed, the usually delicate issue of how to deal with coalitions does not arise then. Also, the analysis takes place in a space of lower dimension, often allowing the use of less sophisticated mathematical tools. In the context of the problem under investigation, this strategy has been followed by Dagan and Volij (1997), Moulin (2000), Hokari and Thomson (2003), and Thomson (2003b).

In the course of implementing this extension strategy, it has been observed in a number of contexts that if a two-agent solution is required to satisfy a certain property and it has a bilaterally consistent extension, then this extension satisfies the property in general. We will say then that the property is "lifted" from the two-agent case to the general case by bilateral consistency.

Our objective here is to identify which properties of division rules are lifted, and under what conditions.

We offer two theorems. The first one lists properties that are lifted "directly" so to speak, the only requirements imposed on a rule being that it satisfy the property in the two-claimant case and that it be bilaterally consistent. We then identify several properties that are not lifted. For each of them, we do so by constructing a bilaterally consistent rule that satisfies the property in the two-claimant case but not in the general case. The second theorem gives a list of properties that are lifted provided the rule also satisfies the requirement that, when the endowment increases, all agents should receive at least as much as they received initially. We can then say that this monotonicity property provides "assistance" in lifting other properties.

The properties of rules whose lifting we study are order properties, monotonicity properties, and independence properties, including all of the properties that have been most often discussed in the recent axiomatic study of the problem of adjudicating conflicting claims. They are mostly self-explanatory, but a reader interested in detailed motivation and discussion should consult the primary sources, which we cite when stating the properties, and the surveys by Herrero and Villar (2001), Moulin (2002), and Thomson (2003a, 2006b).

An early example of lifting is given by Dagan, Serrano and Volij (1997). We also note that Dagan and Volij (1997) have shown, and exploited the fact, that certain properties of rules are lifted by a version of consistency that they call "average consistency". A systematic investigation of lifting for other models also appears to be a fruitful endeavor.

2 The model

There is an infinite set of "potential" claimants, indexed by the natural numbers $\mathbb{N}$. In each instance however, only a finite number of them are present. Let $\mathcal{N}$ be the class of finite subsets of $\mathbb{N}$. A **claims problem with claimant set $\boldsymbol{N \in \mathcal{N}}$** is a pair $(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$ such that $\sum_N c_i \geq E$: $c \in \mathbb{R}^N_+$ is the **claims vector**—for each $i \in N$, c_i being the claim of agent i—and $E \in \mathbb{R}_+$ is the **endowment**. Let $\mathcal{C}^N$ be the class of these problems. A **rule** is a function defined on $\bigcup_{N\in\mathcal{N}} \mathcal{C}^N$, which associates with each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$ a vector $x \in \mathbb{R}^N_+$ such that $x \leqq c$ and whose coordinates add up to E, a property to which we refer to as efficiency.[1,2] Any vector x satisfying these two conditions is an **awards vector for (c, E)**. For each claims vector $c \in \mathbb{R}^N_+$, the locus of the awards vector chosen by the rule when the endowment varies from 0 to $\sum c_i$ is its **path of awards for c**.

The following property of a rule is the focus of our analysis. It says that, starting from any initial problem, if all but two claimants receive their awards as specified by the rule and leave, and the situation is reevaluated at that point, the rule should assign to each of the two remaining claimants the same amount as it did initially.[3] The problem involving the subgroup of remaining claimants is the **reduced problem associated with the initial recommendation and the subgroup**.[4]

Bilateral consistency: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $N' \subset N$ with $|N'| = 2$, if $x \equiv S(c, E)$, then $x_{N'} = S\big(c_{N'}, \sum_{N'} x_i\big)$.

The stronger version, *consistency*, is obtained by dropping the restriction $|N'| = 2$.

It will be convenient to have available the following concept (Aumann and Maschler, 1985): The **dual of a rule S** is the rule S^d defined by setting, for each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, $S^d(c, E) \equiv c - S(c, \sum c_i - E)$. **Two properties are dual of each other** if whenever a rule satisfies one of them, its dual satisfies the other. A property is **self-dual** if it coincides with its dual.

3 Lifting

Our first "lifting" theorem identifies a list of properties that, when satisfied in the two-claimant case by a *bilaterally consistent* rule, is satisfied for any number of claimants.

[1]Vector inequalities: $x \geqq y$, $x \geq y$, $x > y$.

[2]For surveys of the literature on division rules, see Thomson (2003a, 2006b).

[3]The many applications that have been made of the idea of consistency are surveyed by Thomson (2006a). Early applications to the problem at hand are due to Aumann and Maschler (1985) and Young (1987).

[4]Note that since we require rules to be such that for each $i \in N$, $x_i \in [0, c_i]$, then the sum of the claims of the agents who stay is still at least as large than the remaining endowment, so the reduced problem is indeed a well-defined claims problem.

The weaker version of a property obtained by restricting its applications to two-claimant problems is denoted with the prefix "2". For example, "2-resource monotonicity", defined shortly, means "resource-monotonicity in the two-claimant case". However, we state all of our fixed-population properties for rules as defined above.

• We start with the requirement that two agents with equal claims should receive equal amounts. This requirement is often imposed on rules. It is not always desirable, for instance if claimants represent agents with different characteristics—say a taxpayer may be single or may be a married couple—and the recent literature has considerably progressed so as to free us of it (Moulin, 2000; Hokari and Thomson, 2003; Thomson, 2003b), but in many applications, it is very natural.

Equal treatment of equals: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each pair $\{i, j\} \subseteq N$, if $c_i = c_j$, then $S_i(c, E) = S_j(c, E)$.

• If an agent's claim is at least as large as some other agent's claim, he should receive at least as much. Also, the loss he incurs should be at least as large as this other agent's loss. The requirement strengthens *equal treatment of equals.*[5]

Order preservation: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each pair $\{i, j\} \subseteq N$, if $c_i \geq c_j$, then $S_i(c, E) \geq S_j(c, E)$ and $c_i - S_i(c, E) \geq c_j - S_j(c, E)$.

• When the endowment increases, each claimant should receive at least as much as he did initially:[6]

Resource monotonicity: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $\bar{E} > E$, if $\sum c_j \geq \bar{E}$, then $S(c, \bar{E}) \geqq S(c, E)$.

• Suppose that a rule has been applied to some problem but that when agents show up to collect their awards, the endowment is found to be smaller than initially thought. In handling this new situation, two perspectives can be taken: (i) the initial division is ignored and the rule reapplied to the new

[5] This property is formulated by Aumann and Maschler (1985).

[6] Properties of this type are standard in all branches of game theory and economics. For a survey, see Thomson (2003c).

problem; (ii) the amounts initially assigned are used as claims in the division of the new endowment. We require that both perspectives should result in the same awards vector:[7]

Composition down: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $\bar{E} < E$, we have $S(c, \bar{E}) = S(S(c, E), \bar{E})$.

We will use the obvious fact that a rule satisfying *2-composition down* satisfies *2-resource monotonicity*.[8]

• The next requirement pertains to the opposite possibility, namely that when agents show up to collect their awards, the endowment is found to be greater than initially thought. Here too, two perspectives can be taken: (i) the initial division is ignored and the rule reapplied to the new problem; (ii) each agent's award is calculated in two installments; the first installment is his award for the division of the initial endowment; the second installment is what he gets when the rule is applied to divide the newly available resources, all claims being revised down by the first installments. We require that both perspectives should result in the same awards vector:[9]

Composition up: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $\bar{E} > E$ such that $\sum c_i \geq \bar{E}$, we have $S(c, \bar{E}) = S(c, E) + S(c - S(c, E), \bar{E})$.[10]

Composition down and *composition up* are dual properties (Moulin, 2000; Herrero and Villar, 2001).

It will help the proof of our lifting theorems to have available the following result, whose proof we omit:

Lemma 1 *If a property is lifted by* bilateral consistency *for any rule satisfying certain properties, the dual of the property is lifted for any rule satisfying the dual properties.*

[7]The property is formulated by Moulin (2000).

[8]This is because of the claims boundedness requirement $S(c, E) \leqq c$ imposed on rules. This implication holds for any number of claimants, but we need it only in the two-claimant case.

[9]The property is formulated by Young (1987).

[10]Note that this equality is well-defined since by definition, rules satisfy *claims boundedness*, that is, always select a vector that is dominated by the claims vector.

Our first lifting theorem states that each of the properties just defined is lifted by *bilateral consistency*.[11]

Theorem 1 *The following properties are lifted from the two-claimant case to the general case by* bilateral consistency*:*
(a) *equal treatment of equals;*
(b) *order preservation;*
(c) *resource monotonicity;*
(d) *composition down and composition up.*

Proof: We omit the proofs of (a) and (b), which are trivial.
(c) *Resource monotonicity.* The easy proof appears in Dagan, Serrano and Volij (1997), but we include it for completeness. Let $N \in \mathcal{N}$ with $|N| \geq 3$, $(c, E) \in \mathcal{C}^N$, and $\bar{E} > E$ be such that $\bar{E} \leq \sum c_i$. Let $x \equiv S(c, E)$ and $\bar{x} \equiv S(c, \bar{E})$. Suppose by contradiction that for some $i \in N$, $\bar{x}_i < x_i$. Then, there is $j \in N$ such that $\bar{x}_j > x_j$. Let $N' \equiv \{i, j\}$. By *bilateral consistency* applied twice, $x_{N'} = S(c_{N'}, \sum_{N'} x_k)$ and $\bar{x}_{N'} = S(c_{N'}, \sum_{N'} \bar{x}_k)$. The two problems $(c_{N'}, \sum_{N'} x_k)$ and $(c_{N'}, \sum_{N'} \bar{x}_k)$ (possibly) differ only in the endowment, and by *2-resource monotonicity*, we should have $\bar{x}_{N'} \geqq x_{N'}$ or $\bar{x}_{N'} \leqq x_{N'}$, which we know not to be true.

(d) *Composition down.* Let $N \in \mathcal{N}$ with $|N| \geq 3$, $(c, E) \in \mathcal{C}^N$, $\bar{E} < E$, $x \equiv S(c, E)$, $\bar{x} \equiv S(c, \bar{E})$, and $y \equiv S(x, \bar{E})$. We need to show that $\bar{x} = y$. We argue by contradiction.

Suppose that for some $i \in N$, $\bar{x}_i < y_i$. Then, there is $j \in N \backslash \{i\}$ such that $\bar{x}_j > y_j$. Let $N' \equiv \{i, j\}$. By *bilateral consistency*, $x_{N'} = S(c_{N'}, \sum_{N'} x_k)$, $\bar{x}_{N'} = S(c_{N'}, \sum_{N'} \bar{x}_k)$, and $y_{N'} = S(x_{N'}, \sum_{N'} y_k)$.

If $\sum_{N'} x_k \leq \sum_{N'} \bar{x}_k$, then by *2-resource monotonicity* (which, as we noted above, is implied by *2-composition down*) applied to $(c_{N'}, \sum_{N'} x_k)$ and $(c_{N'}, \sum_{N'} \bar{x}_k)$, we have $\bar{x}_i \geq x_i \geq y_i$, which contradicts $\bar{x}_i < y_i$.

If $\sum_{N'} x_k > \sum_{N'} \bar{x}_k$, then by *2-composition down*, $\bar{x}_{N'} = S(x_{N'}, \sum_{N'} \bar{x}_k)$. By *2-resource monotonicity* applied to $(x_{N'}, \sum_{N'} \bar{x}_k)$ and $(x_{N'}, \sum_{N'} y_k)$, we should have $\bar{x}_{N'} \geqq y_{N'}$ or $\bar{x}_{N'} \leqq y_{N'}$, which we know not to be true.

The proof for *composition up* follows from the statement just proved, the fact that this property is dual to *composition down*, and Lemma 1. □

[11] There is no logical relations between (a) and (b) of Theorem 1, even though the properties that are lifted are logically related. The reason is that both hypotheses and conclusions are strengthened as one passes from the statement pertaining to the first property to the statement pertaining to the second property.

Next, we identify several properties that are not lifted:

• The first one is another strengthening of *equal treatment of equals*. The rule should be invariant under permutations of agents.

Anonymity: For each pair $\{N, \bar{N}\}$ of elements of $\mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $\bar{c} \in \mathbb{R}^{\bar{N}}$, if there is a bijection $\pi\colon N \to \bar{N}$ such that for each $i \in N$, $c_i = \bar{c}_{\pi(i)}$, then for each $i \in N$, $S_i(c, E) = S_{\pi(i)}(\bar{c}, E)$.

Note that the idea can also be applied to the more limited case when $N = \bar{N}$.

• A small change in the endowment should not be accompanied by a large change in the chosen awards vector:

Continuity: For each $N \in \mathcal{N}$, each sequence $\{E^\nu\}_{\nu=1}^{\infty}$, and each $(c, E) \in \mathcal{C}^N$, if $E^\nu \le \sum_N c_i$ for each $\nu \in \mathbb{N}$, and $E^\nu \to E$, then $S(c, E^\nu) \to S(c, E)$.

• What is available should be divided symmetrically to "what is missing" (the difference between the sum of the claims and the endowment):[12]

Self-duality: For each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, $S(c, E) = c - S(c, \sum c_i - E)$.

We will show that none of the three properties just defined is lifted. We will actually provide a proof for these last two properties together, by means of a single example.[13] In defining the example, we use the following rule as an ingredient and the fact that it is *bilaterally consistent*.[14]

Reverse Talmud rule, $\boldsymbol{T^r}$: For each $(c, E) \in \mathcal{C}^N$ and each $i \in N$,

$$T_i^r(c, E) \equiv \begin{cases} \max\left\{0, \frac{c_i}{2} - \lambda\right\} & \text{if } \sum \frac{c_j}{2} \ge E, \\ \frac{c_i}{2} + \min\left\{\frac{c_i}{2}, \lambda\right\} & \text{otherwise,} \end{cases}$$

where in each case, $\lambda \ge 0$ is chosen so as to achieve efficiency.

[12]This property is formulated by Aumann and Maschler (1985).

[13]Thus, in the language of Section 4, *2-anonymity* provides no "assistance" in the lifting of *2-continuity*, nor does *2-continuity* provide "assistance" in the lifting of *2-anonymity*.

[14]The Talmud rule can be understood as a hybrid of the constrained equal awards rule defined later, and it dual, the so-called constrained equal losses rule. The former is applied if the endowment is at most as large as the half-sum of the claims, and the other otherwise. The reverse Talmud rule (Chun, Schummer, and Thomson, 2001) is defined in a similar way, but it starts with an application of the constrained equal losses rule, the constrained equal awards rule being used second.

Proposition 1 *Neither 2-continuity nor 2-self-duality is lifted from the two-claimant case to the general case by bilateral consistency.*

Proof: The proof is by means of an example of a rule S defined as follows.

Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$.

Case (i): $1 \notin N$. Then, $S(c, E) \equiv T^r(c, E)$.

Case (ii): $N = \{1, i\}$. Then, if $c_i \geq \frac{c_1}{2}$, the path of awards of S for c is the piecewise linear path connecting $(0, 0)$, $(0, \frac{c_i}{2} + \frac{c_1}{4})$, $(c_1, \frac{c_i}{2} - \frac{c_1}{4})$, and c (Figure 1a); otherwise it is the piecewise linear path connecting $(0, 0)$, $(0, c_i)$, $(\frac{c_1}{2} - c_i, c_i)$, $(\frac{c_1}{2} + c_i, 0)$, $(c_1, 0)$, and c (Figure 1b).

Case (iii): $1 \in N$, $|N| \geq 3$, and $E < \sum_{i \in N\setminus\{1\}} \min\left\{c_i, \frac{c_i}{2} + \frac{c_1}{4}\right\}$. Then, $S_1(c, E) \equiv 0$, and for each $i \in N \setminus \{1\}$, $S_i(c, E) \equiv T_i^r(c_{N\setminus\{1\}}, E)$.

Case (iv): $1 \in N$, $|N| \geq 3$,

$$\sum_{i \in N\setminus\{1\}} \min\left\{c_i, \frac{c_i}{2} + \frac{c_1}{4}\right\} \leq E < c_1 + \sum_{i \in N\setminus\{1\}} \max\left\{0, \frac{c_i}{2} - \frac{c_1}{4}\right\}.$$

Then, $S_1(c, E) \equiv t$, and for each $i \in N\setminus\{1\}$, $S_i(c, E) \equiv T_i^r(c_{N\setminus\{1\}}, E - t)$, where t is the largest $t' \in [0, c_1)$ such that

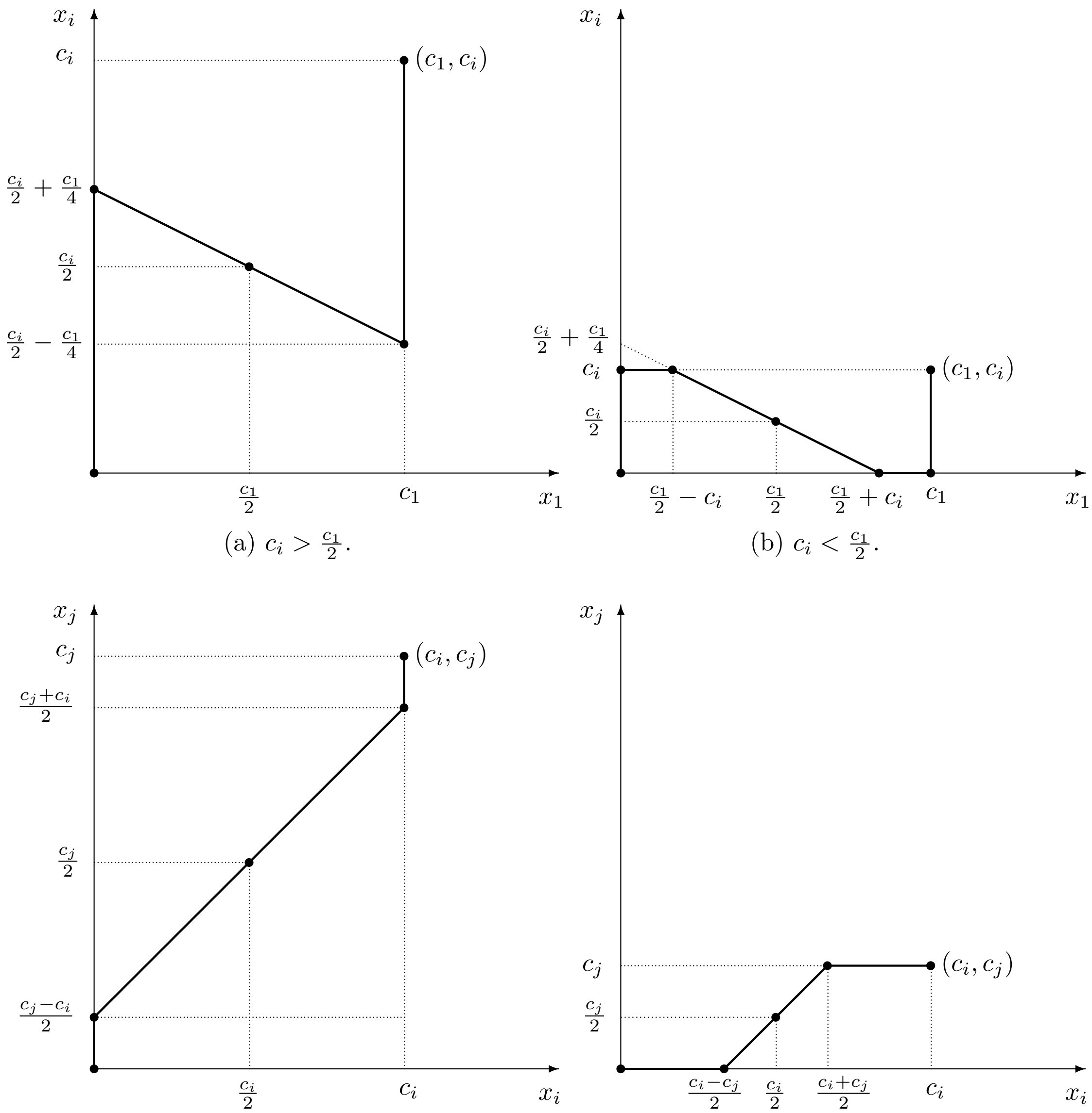
$$t' + \sum_{i \in N\setminus\{1\}} \max\left\{\min\left\{c_i, \frac{c_i}{2} + \frac{c_1}{4} - \frac{t'}{2}\right\},\ 0\right\} = E.$$

Case (v): $1 \in N$, $|N| \geq 3$, and

$$E \geq \max\left\{\sum_{i \in N\setminus\{1\}} \min\left\{c_i, \frac{c_i}{2} + \frac{c_1}{4}\right\},\ c_1 + \sum_{i \in N\setminus\{1\}} \max\left\{0, \frac{c_i}{2} - \frac{c_1}{4}\right\}\right\}.$$

Then, $S_1(c, E) \equiv c_1$, and for each $i \in N \setminus \{1\}$, $S_i(c, E) \equiv T_i^r(c_{N\setminus\{1\}}, E - c_1)$.

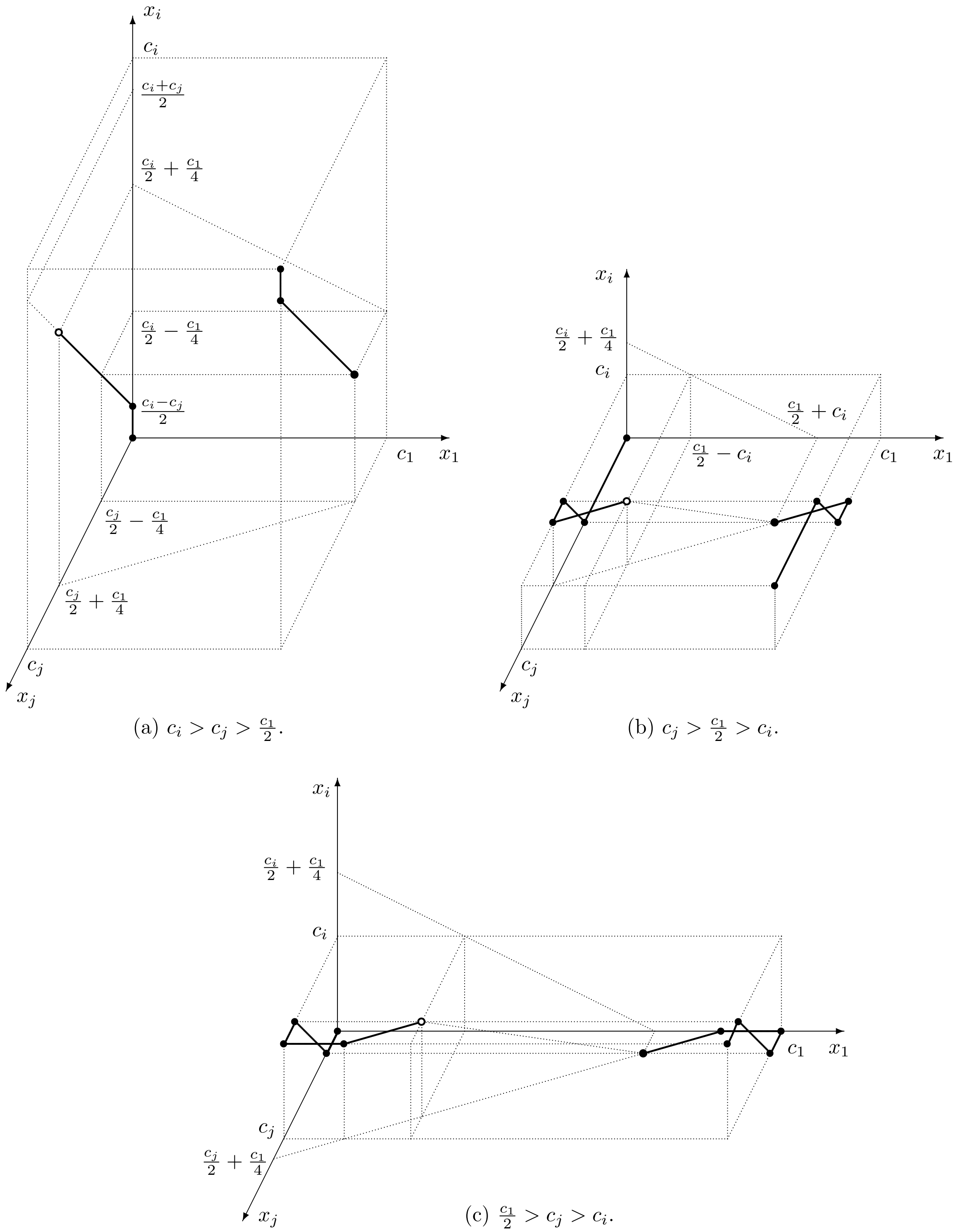
Note that the reverse Talmud rule is *continuous* and *self-dual*. From this fact and an inspection of Figure 1, we conclude that, in the two-claimant case, S is *continuous* and *self-dual*. As Figure 2 shows, S is neither *continuous* nor *self-dual* in general. *Self-duality* implies that when the endowment is equal to the half-sum of the claims, the half-claims vector is chosen, which is seen



(a) $c_i > \frac{c_1}{2}$. (b) $c_i < \frac{c_1}{2}$.

(c) $i, j \neq 1$ and $c_i < c_j$. (d) $i, j \neq 1$ and $c_i > c_j$.

Figure 1: The two-claimant case.



(a) $c_i > c_j > \frac{c_1}{2}$.

(b) $c_j > \frac{c_1}{2} > c_i$.

(c) $\frac{c_1}{2} > c_j > c_i$.

Figure 2: The three-claimant case. *Self-duality* and *continuity* are violated.

not to be the case on the figures;[15] it is also for an endowment equal to the half-sum of the claims that *continuity* is violated.

We assert that S is *bilaterally consistent.*

Let $N \in \mathcal{N}$ with $|N| \geq 3$, $(c, E) \in \mathcal{C}^N$, $x \equiv S(c, E)$, and $N' \subset N$ with $|N'| = 2$. We want to show that $x_{N'} = S(c_{N'}, \sum_{i \in N'} x_i)$. If $1 \notin N'$, then this equality holds since the reverse Talmud rule is *bilaterally consistent* and for each $i \in N \setminus \{1\}$, $x_i = T_i^r(c_{N \setminus \{1\}}, E - x_1)$.

Suppose that $N' = \{1, i\}$ for some $i \neq 1$. Three of the 5 cases listed in defining the rule may apply to (c, E).

If case (iii) applies, then $x_1 = 0$ and $x_i \leq \min\left\{c_i, \frac{c_i}{2} + \frac{c_1}{4}\right\}$.

If case (iv) applies, then $x_i = \max\left\{\min\left\{c_i, \frac{c_i}{2} + \frac{c_1}{4} - \frac{x_1}{2}\right\},\ 0\right\}$.

If case (v) applies, then $x_1 = c_1$ and $x_i \geq \max\left\{0, \frac{c_i}{2} - \frac{c_1}{4}\right\}$.

It can be seen from Figure 1 that, in each of these cases, (x_1, x_i) lies on the path of awards of S for (c_1, c_i). $\square$

It follows directly from the definition of *consistency* that if a rule satisfies this property, then for each population $N \in \mathcal{N}$ with three or more agents, each claims vector $c \in \mathbb{R}_+^N$, and for each subpopulation $N' \subset N$, the path of awards of the rule for c, when projected onto $\mathbb{R}^{N'}$, is a subset of its path of awards for the projection of c onto $\mathbb{R}^{N'}$, $c_{N'}$. If the rule is *resource monotonic*, the projection of the path for c onto $\mathbb{R}^{N'}$ actually coincides with the path for $c_{N'}$. For the example constructed here, the projection is a strict subset.

The proof of the next result is also by means of an example. We use the following rule as an ingredient and the fact that this rule is *bilaterally consistent.*[16]

Constrained equal awards rule, *CEA*: For each $(c, E) \in \mathcal{C}^N$ and each $i \in N$,

$$CEA_i(c, E) \equiv \min\{c_i, \lambda\},$$

where $\lambda \geq 0$ is chosen so as to achieve efficiency.

[15] The path for c consists of two connected components. One is a broken segment in five parts; the other is also a broken segment in five parts. The two components are symmetric of each other with respect to $\frac{c}{2}$ except that the fifth segment of the first component is open at its highest endpoint, whereas the fifth segment of the second component, counting down from c, is closed at its lowest endpoint. The two points belong to the plane of equation $\sum t_i = \sum \frac{c_i}{2}$.

[16] The constrained equal awards rule appears in Maimonides (Aumann and Maschler, 1985).

Proposition 2 *Anonymity is not lifted from the two-claimant case to the general case by bilateral consistency.*[17]

Proof: Let $\alpha > 0$ be given and consider the following rule: For each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, let $N_\alpha \equiv \{i \in N \mid c_i = \alpha\}$, $N_+ \equiv \{i \in N \mid c_i \geq 2\alpha\}$, and $N_- \equiv \{i \in N \mid \alpha \neq c_i < 2\alpha\}$.
Case (i): $E < \alpha|N_\alpha|$. Then,

$$S_i(c,E) \equiv \begin{cases} CEA_i(c_{N_\alpha}, E) \quad = \quad \frac{E}{|N_\alpha|} & \text{if } i \in N_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Case (ii): $\alpha|N_\alpha| \leq E < 2\alpha|N_+|$. Then,

$$S_i(c,E) \equiv \begin{cases} \alpha - \lambda & \text{if } i \in N_\alpha, \\ 2\lambda & \text{if } i \in N_+, \\ 0 & \text{if } i \in N_-, \end{cases}$$

where $\lambda \in [0, \alpha)$ is uniquely determined by $(\alpha - \lambda)|N_\alpha| + 2\lambda|N_+| = E$.
Case (iii): $\alpha|N_\alpha| < E$ and $2\alpha|N_+| \leq E < \sum_{i \in N_+} c_i$. Then,

$$S_i(c,E) \equiv \begin{cases} CEA_i(c_{N_+}, E) & \text{if } i \in N_+, \\ 0 & \text{otherwise.} \end{cases}$$

Case (iv): $N_+ \neq \emptyset$, $\alpha|N_\alpha| < E$, and $\sum_{i \in N_+} c_i \leq E < \sum_{i \in N_+} c_i + \alpha|N_\alpha|$. Then,

$$S_i(c,E) \equiv \begin{cases} CEA_i(c_{N_\alpha}, E - \sum_{j \in N_+} c_j) \quad = \quad \frac{E - \sum_{j \in N_+} c_j}{|N_\alpha|} & \text{if } i \in N_\alpha, \\ c_i & \text{if } i \in N_+, \\ 0 & \text{if } i \in N_-. \end{cases}$$

Case (v): $\sum_{i \in N_+} c_i + \alpha|N_\alpha| \leq E$. Then,

$$S_i(c,E) \equiv \begin{cases} \alpha & \text{if } i \in N_\alpha, \\ c_i & \text{if } i \in N_+, \\ CEA_i\big(c_{N_-}, E - \sum_{j \in N_+} c_j - \alpha|N_\alpha|\big) & \text{if } i \in N_-. \end{cases}$$

[17] Since the rule we construct to prove this result is *2-continuous*, the same example actually shows that, in the language of Section 4, *2-continuity* provide no assistance in lifting *2-anonymity*. Note that this rule could also be used to prove the point, already made in Proposition 1, that *continuity* is not lifted.

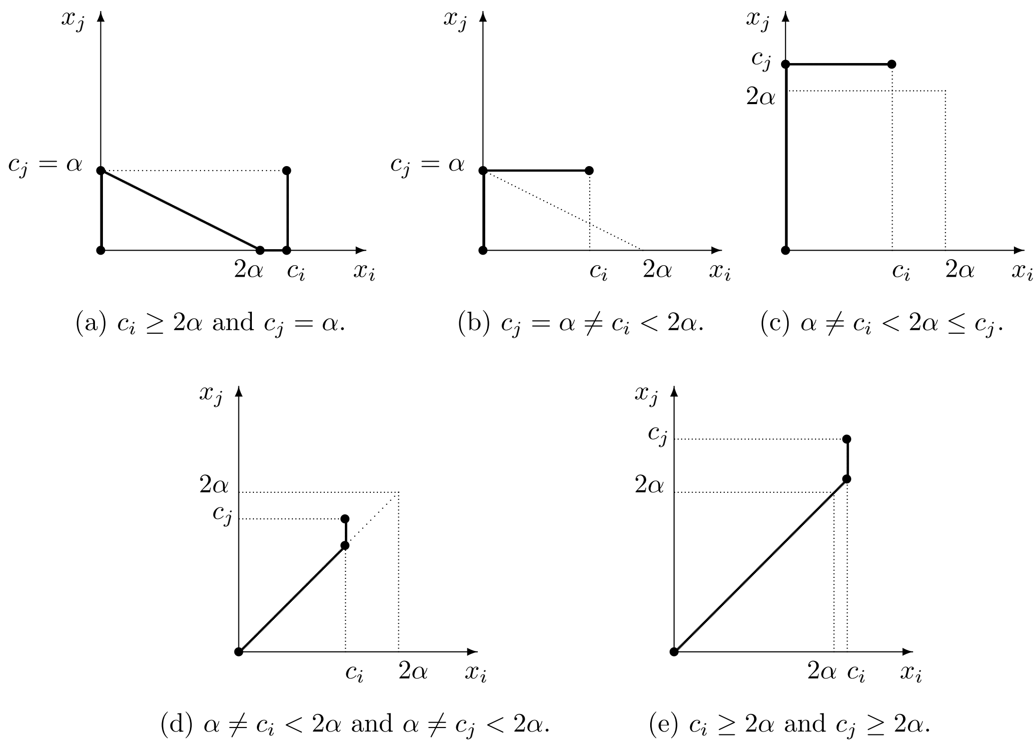


Figure 3: The two-claimant case.

Case (vi): $1 \in N_+$ and $2\alpha|N_+| \leq \alpha|N_\alpha| = E$. Then,

$$S_i(c,E) \equiv \begin{cases} \alpha & \text{if } i \in N_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

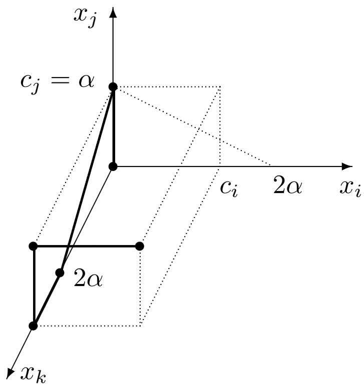
Case (vii): $1 \notin N_+ \neq \emptyset$ and $2\alpha|N_+| \leq \alpha|N_\alpha| = E < \sum_{i \in N_+} c_i$. Then,

$$S_i(c,E) \equiv \begin{cases} CEA_i(c_{N_+}, E) & \text{if } i \in N_+, \\ 0 & \text{otherwise.} \end{cases}$$

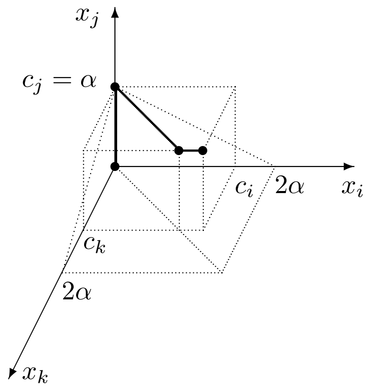
Case (viii): $1 \notin N_+ \neq \emptyset$, and $\sum_{i \in N_+} c_i \leq \alpha|N_\alpha| = E$. Then,

$$S_i(c,E) \equiv \begin{cases} CEA_i\big(c_{N_\alpha}, E - \sum_{j \in N_+} c_j\big) \quad = \quad \frac{E - \sum_{j \in N_+} c_j}{|N_\alpha|} & \text{if } i \in N_\alpha, \\ c_i & \text{if } i \in N_+, \\ 0 & \text{if } i \in N_-. \end{cases}$$

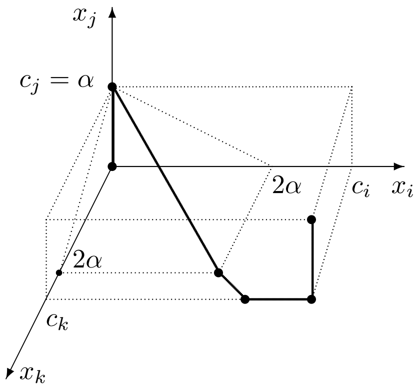
Note that cases (vi), (vii), and (viii) do not apply when there are only two claimants, and that the definition of S is anonymous except for these three cases. Thus, in the two-claimant case, S is *anonymous*. (See also Figure 3.) As panels (f) and (g) of Figure 4 show, it is not *anonymous* in general. We claim that it is *bilaterally consistent*.



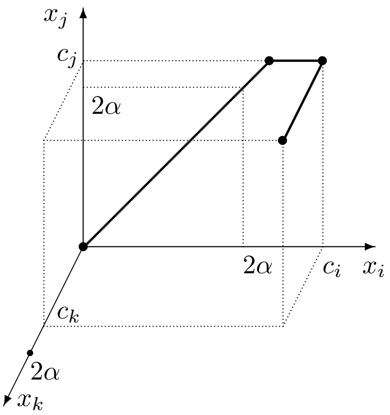
(a) $c_j = \alpha \neq c_i < 2\alpha$ and $c_k \geq 2\alpha$.



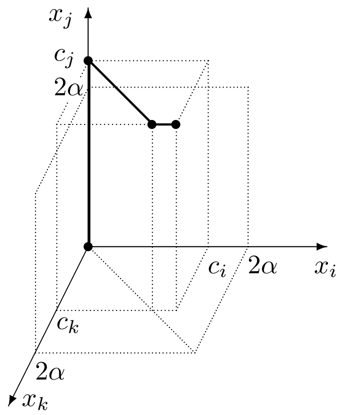
(b) $c_j = \alpha \neq c_i < 2\alpha$ and $\alpha \neq c_k < 2\alpha$.



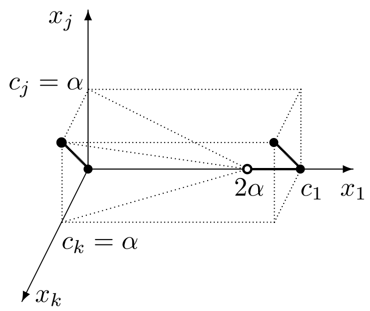
(c) $c_j = \alpha$, $c_i \geq 2\alpha$, and $c_k \geq 2\alpha$.



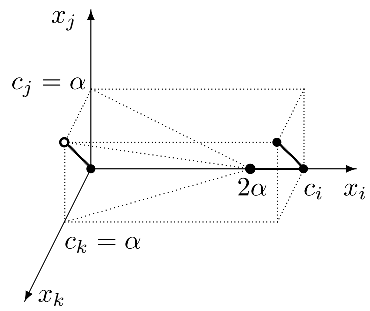
(d) $c_i \geq 2\alpha$, $c_j \geq 2\alpha$, and $\alpha \neq c_k < 2\alpha$.



(e) $c_j \geq 2\alpha$, $\alpha \neq c_i < 2\alpha$, and $\alpha \neq c_k < 2\alpha$.



(f) $c_j = c_k = \alpha$ and $c_1 \geq 2\alpha$.



(g) $c_j = c_k = \alpha$, $c_i \geq 2\alpha$, and $i \neq 1$

Figure 4: The three-claimant case. From panels (f) and (g), it can be seen that *anonymity* is violated when $c_i = c_1$.

Let $N \in \mathcal{N}$ with $|N| \geq 3$, $(c, E) \in \mathcal{C}^N$, $x \equiv S(c, E)$, and $N' \subset N$ with $|N'| = 2$. We want to show that $x_{N'} = S(c_{N'}, \sum_{i \in N'} x_i)$. Let $N' \equiv \{i, j\}$, $E' \equiv x_i + x_j$, and $(y_i, y_j) \equiv S(c_{N'}, E')$. Since $y_i + y_j = x_i + x_j$, it is enough to show that either $y_i = x_i$ or $y_j = x_j$.

First, suppose that $i, j \in N_\alpha$. If any of cases (i), (ii) with $\lambda > 0$, (iv), or (viii) applies to (c, E), then $E' < 2\alpha$ and case (i) applies to $(c_{N'}, E')$. Thus, $y_i = \frac{E'}{2} = x_i$. If any of cases (ii) with $\lambda = 0$, (v), or (vi) applies to (c, E), then $E' = 2\alpha$ and case (v) applies to $(c_{N'}, E')$. Thus, $y_i = \alpha = x$. If either of case (iii) or case (vii) applies to (c, E), then $E' = 0$ and hence $y_i = 0 = x_i$.

Suppose that $i, j \in N_+$. If either case (i) or case (vi) applies to (c, E), then $E' = 0$ and hence $y_i = 0 = x_i$. If case (ii) applies to (c, E), then $E' < 4\alpha$ and the same case applies to $(c_{N'}, E')$. Thus, $y_i = \frac{E'}{2} = x_i$. If either case (iii) or case (vii) applies to (c, E), then $4\alpha \leq E' \leq c_i + c_j$ and either case (iii) or case (iv) applies to $(c_{N'}, E')$. If case (iii) applies to (c, E), since CEA is *bilaterally consistent*, $y_i = x_i$. In case (iv), $y_i = c_i = x_i$. If any of cases (iv), (v), or (viii) applies to (c, E), then $E' = c_i + c_j$ and case (v) applies to $(c_{N'}, E')$. Thus, $y_i = c_i = x_i$.

Suppose that $i, j \in N_-$. If case (v) applies to (c, E), then the same case applies to $(c_{N'}, E')$. Since CEA is *bilaterally consistent*, $y_i = x_i$. If any other case applies to (c, E), then $E' = 0$ and hence $y_i = 0 = x_i$.

Suppose that $i \in N_\alpha$ and $j \in N_+$. If case (i) applies to (c, E), then $E' < \alpha$ and the same case applies to $(c_{N'}, E')$. Thus, $y_j = 0 = x_j$. If case (ii) applies to (c, E), then $E' = \alpha + \lambda < 2\alpha$ and the same case applies to $(c_{N'}, E')$. With the same λ, we have $\alpha - \lambda + 2\lambda = E'$. Thus, $y_i = \alpha - \lambda = x_i$. If either case (iii) or case (vii) applies to (c, E), then $2\alpha \leq E' = x_j \leq c_j$ and either case (iii) or case (iv) applies to $(c_{N'}, E')$. In both cases, $y_i = 0 = x_i$. If any of cases (iv), (v), or (viii) applies to (c, E), then $\alpha < E' \leq c_j + \alpha$ and either case (iv) or case (v) applies to $(c_{N'}, E')$. In both cases, $y_j = c_j = x_j$. If case (vi) applies to (c, E), then $E' = \alpha < 2\alpha$ and case (ii) applies to $(c_{N'}, E')$ with $\lambda = 0$. Thus, $y_i = \alpha = x_i$.

Suppose that $i \in N_\alpha$ and $j \in N_-$. If any of cases (ii) with $\lambda = 0$, (v), or (vi) applies to (c, E), then $\alpha \leq E'$ and case (v) applies to $(c_{N'}, E')$. Thus, $y_i = \alpha = x_i$. If any other case applies to (c, E), then $E' < \alpha$ and case (i) applies to $(c_{N'}, E')$. Thus, $y_j = 0 = x_j$.

Finally, suppose that $i \in N_+$ and $j \in N_-$. If any of cases (i), (ii) with $\lambda = 0$, or (vi) applies to (c, E), then $E' = 0$ and hence $y_i = 0 = x_i$. If any of cases (ii) with $\lambda > 0$, (iii), (iv), (vii), or (viii) applies to (c, E), then $E' = x_i \leq c_i$ and one of cases (ii), (iii), or (v) with $E' = c_i$ applies to $(c_{N'}, E')$.

In these cases, $y_j = 0 = x_j$. If case (v) applies to (c, E), then $c_i \leq E'$ and the same case applies to $(c_{N'}, E')$. Thus, $y_i = c_i = x_i$. $\square$

4 Assisted lifting

In our second lifting theorem, the hypothesis is made on the rule that it is *resource-monotonic*. Theorem 1c states that this property is lifted, so it suffices to impose it in the two-claimant case. The theorem pertains to the following properties:

• If the endowment increases, of two claimants, the award to the one with the larger claim should increase by at least as much as the award to the one with the smaller claim:[18]

Super-modularity: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, each $\bar{E} \in [0, E]$, and each pair $\{i, j\} \subseteq N$, if $c_i \leq c_j$, then $S_i(c, E) - S_i(c, \bar{E}) \leq S_j(c, E) - S_j(c, \bar{E})$.

• If an agent's claim is such that by substituting it to the claim of any other agent whose claim is higher, there is now enough to compensate everyone, then the agent should be fully compensated:[19]

Conditional full compensation: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$, if $\sum_{j \in N} \min\{c_j, c_i\} \leq E$, then $S_i(c, E) = c_i$.

• If an agent's claim satisfies the hypothesis of *conditional full compensation*, he should receive nothing in the problem in which the endowment is what it was initially minus the sum of the claims:

Conditional null compensation: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$, if $\sum_{j \in N} \min\{c_j, c_i\} \leq E$, then $S_i(c, \sum c_k - E) = 0$.

Conditional full compensation and *conditional null compensation* are dual properties (Herrero and Villar, 2002).

• If an agent's claim increases, he should receive at least as much as he did initially:

[18] This property is formulated by Dagan, Serrano, and Volij (1997).

[19] The property is formulated by Herrero and Villar (2002) under the name of "sustainability", and so is the next one, under the name of "independence of residual claims".

Claims monotonicity: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, each $i \in N$, and each $\bar{c}_i > c_i$, we have $S_i(\bar{c}_i, c_{-i}, E) \geq S_i(c, E)$.

• If an agent's claim and the endowment increase by the same amount, what he receives should increase by at most that amount.[20]

Linked claim-resource monotonicity: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, each $i \in N$, and each $\delta > 0$, we have $S_i(c_i + \delta, c_{-i}, E + \delta) - S_i(c, E) \leq \delta$.

Claims monotonicity and *linked claim-resource monotonicity* are dual properties (Thomson and Yeh, 2002):

Finally are two invariance requirements.

• One should be able to solve a problem in either one of the following two ways, either directly, or by first truncating claims at the endowment:[21]

Claims truncation invariance: For each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, $S(c, E) = S(t(c, E), E)$, where for each $i \in N$, $t_i(c, E) \equiv \min\{c_i, E\}$.

• One should be able to solve a problem in either one of the following two ways, either directly, or by first attributing to each claimant his "minimal right", defined to be the difference between the endowment and the sum of the claims of the other agents (or zero if this difference is negative).[22]

Minimal rights first: For each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, $S(c, E) = m(c, E) + S(c - m(c, E), E - \sum m_i(c, E))$.[23]

Claims truncation invariance and *minimal rights first* are dual properties (Herrero and Villar, 2001).

The proof of one of the statements of Theorem 2 below will also involve an additional lemma relating three properties, two of which we have already encountered and an additional one pertaining to a thought experiment that is the converse to that underlying *bilateral consistency*: it allows to deduce the desirability of a proposed awards vector x for some problem from the

[20]This property is formulated by Thomson and Yeh (2002).
[21]This property is formulated by Dagan and Volij (1993).
[22]This property is formulated by Curiel, Maschler and Tijs (1987).
[23]Note this is a well-defined problem.

desirability of the restriction of x to each two-claimant subgroup for the reduced problem obtained by imagining the departure of the members of the complementary group with their awards. If x is such that for each two-claimant subgroup, its restriction to that subgroup would be chosen by the rule for the associated reduced problem, then it should be chosen by the rule for the initial problem.[24]

Converse consistency: For each $N \in \mathcal{N}$ with $|N| \geq 3$, each $(c, E) \in \mathcal{C}^N$, and each $x \in \mathbb{R}^N$ such that $\sum x_i = E$, if for each $N' \subset N$ with $|N'| = 2$, we have $x_{N'} = S(c_{N'}, \sum_{N'} x_i)$, then $x = S(c, E)$.

The following Lemma is an easy consequence of the facts that *2-resource monotonicity* is lifted and that if a rule is *resource monotonic* and *bilaterally consistent*, it is *conversely consistent* (Chun, 1999).

Lemma 2 *If a rule is 2-resource monotonic and bilaterally consistent, it is conversely consistent.*

Theorem 2 *The following properties are lifted from the two-claimant case to the general case by bilateral consistency for any rule satisfying 2-resource monotonicity:*
(a) *anonymity;*
(b) *super-modularity;*
(c) *self-duality;*
(d) *conditional full compensation and conditional null compensation;*
(e) *claims monotonicity and linked claim-resource monotonicity;*
(f) *claims truncation invariance and minimal rights first.*

In the light of this theorem, it should be no surprise that the examples of rules used to prove Propositions 1 and 2 are not *2-resource monotonic.*

Proof: For each implication, let S be a rule satisfying the hypotheses.

(a) *Anonymity.* Let N, $\bar{N} \in \mathcal{N}$ with $|N| = |\bar{N}| \geq 3$, $(c, E) \in \mathcal{C}^N$ and $\bar{c} \in \mathbb{R}^{\bar{N}}$ be such that there is a bijection $\pi\colon N \to \bar{N}$ such that for each $i \in N$, $c_i = \bar{c}_{\pi(i)}$. We argue by contradiction. Suppose that there is $\ell \in N$ such that $x_\ell \equiv S_\ell(c, E) \neq S_{\pi(\ell)}(\bar{c}, E) \equiv \bar{x}_{\pi(\ell)}$. Then, there are $i, j \in N$ such that $x_i > \bar{x}_{\pi(i)}$ and $x_j < \bar{x}_{\pi(j)}$. By *bilateral consistency*, $(x_i, x_j) = S(c_i, c_j, x_i + x_j)$

[24]The literature concerning this property is surveyed by Thomson (2006a).

and $(\bar{x}_{\pi(i)}, \bar{x}_{\pi(j)}) = S(c_i, c_j, \bar{x}_{\pi(i)} + \bar{x}_{\pi(j)})$. The problems $(c_i, c_j, x_i + x_j) \in \mathcal{C}^{\{i,j\}}$ and $(c_i, c_j, \bar{x}_{\pi(i)} + \bar{x}_{\pi(j)}) \in \mathcal{C}^{\{\pi(i),\pi(j)\}}$ involve two different set of agents, and they (possibly) differ only in the endowment. By *2-anonymity* and *2-resource monotonicity*, we should have $(x_i, x_j) \geqq (\bar{x}_{\pi(i)}, \bar{x}_{\pi(j)})$ or $(x_i, x_j) \leqq (\bar{x}_{\pi(i)}, \bar{x}_{\pi(j)})$, which we know not to be true.

(b) *Super-modularity.* Let $N \in \mathcal{N}$ with $|N| \geq 3$, $(c, E) \in \mathcal{C}^N$, and $\bar{E} > E$ be such that $(c, \bar{E}) \in \mathcal{C}^N$. Let $i, j \in N$ be such that $c_i \leq c_j$ and $N' \equiv \{i, j\}$. Let $x \equiv S(c, E)$ and $\bar{x} \equiv S(c, \bar{E})$. We need to show that $\bar{x}_i - x_i \leq \bar{x}_j - x_j$. By *2-resource monotonicity* and Theorem 1c, for each $k \in N$, $x_k \leq \bar{x}_k$. By *bilateral consistency*, $x_{N'} = S(c_{N'}, \sum_{N'} x_k)$ and $\bar{x}_{N'} = S(c_{N'}, \sum_{N'} \bar{x}_k)$. Note that $\sum_{N'} x_k \leq \sum_{N'} \bar{x}_k$. By *2-super-modularity*, $\bar{x}_i - x_i \leq \bar{x}_j - x_j$.

(c) *Self-duality.* Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$. Let $x \equiv S(c, E)$ and $y \equiv S^d(c, E)$. We need to show that $x = y$. We argue by contradiction. Suppose that for some $i \in N$, $x_i < y_i$. Then, there is $j \in N \setminus \{i\}$ such that $x_j > y_j$. Let $N' \equiv \{i, j\}$. Since S is *bilaterally consistent*, so is S^d. Also, S and S^d coincide in the two-claimant case. Thus, $x_{N'} = S(c_{N'}, \sum_{N'} x_k)$ and $y_{N'} = S^d(c_{N'}, \sum_{N'} y_k) = S(c_{N'}, \sum_{N'} y_k)$. The problems $(c_{N'}, \sum_{N'} x_k)$ and $(c_{N'}, \sum_{N'} y_k)$ (possibly) differ only in the endowment, and by *2-resource monotonicity*, we should have $x_{N'} \geqq y_{N'}$ or $x_{N'} \leqq y_{N'}$, which we know not to be true.

(d) *Conditional full compensation.* Let $N \in \mathcal{N}$ with $|N| \geq 3$, $(c, E) \in \mathcal{C}^N$, and $x \equiv S(c, E)$. Let $i \in N$ be such that $\sum_{j \in N} \min\{c_j, c_i\} \leq E$. We need to show that $x_i = c_i$. We argue by contradiction. Suppose that $x_i < c_i$.

First, we claim that there is $j \in N \setminus \{i\}$ such that $x_j > c_i$. Indeed, if for each $k \in N \setminus \{i\}$, $x_k \leq c_i$, then $E = x_i + \sum_{k \in N \setminus \{i\}} x_k < \sum_{j \in N} \min\{c_j, c_i\} \leq E$, a contradiction.

Let $N' \equiv \{i, j\}$. By *bilateral consistency*, $x_{N'} = S(c_{N'}, \sum_{N'} x_k)$. Since $c_i < x_j \leq c_j$, then $(c_{N'}, 2c_i) \in \mathcal{C}^{\{i,j\}}$. Moreover, by *2-conditional full compensation*, $S(c_{N'}, 2c_i) = (c_i, c_i)$. The problems $(c_{N'}, \sum_{N'} x_k)$ and $(c_{N'}, 2c_i)$ (possibly) differ only in the endowment, and by *2-resource monotonicity*, we should have $x_{N'} \geqq c_{N'}$ or $x_{N'} \leqq c_{N'}$, which we know not to be true.

The proof for *conditional null compensation* follows from the statement just proved, the facts that this property is dual to *conditional full compensation*, that *resource monotonicity* is a *self-dual* property, and Lemma 1.

(e) *Claims monotonicity.* Let $N \in \mathcal{N}$ with $|N| \geq 3$ and $(c, E) \in \mathcal{C}^N$. Let $i \in N$ and $\bar{c} \in \mathbb{R}^N_+$ be such that $\bar{c}_i > c_i$ and for each $j \in N \setminus \{i\}$, $\bar{c}_j = c_j$. Let

$x \equiv S(c, E)$ and $\bar{x} \equiv S(\bar{c}, E)$. We need to show that $\bar{x}_i \geq x_i$. We argue by contradiction. Suppose that $\bar{x}_i < x_i$.

Let $j \in N \backslash \{i\}$ and $N' \equiv \{i, j\}$. We assert that $\sum_{N'} \bar{x}_k < \sum_{N'} x_k$. By *bilateral consistency*, $x_{N'} = S(c_{N'}, \sum_{N'} x_k)$ and $\bar{x}_{N'} = S(\bar{c}_i, c_j, \sum_{N'} \bar{x}_k)$. Suppose by contradiction that $\sum_{N'} \bar{x}_k \geq \sum_{N'} x_k$. By *2-resource monotonicity*, $S_i(\bar{c}_i, c_j, \sum_{N'} x_k) \leq S_i(\bar{c}_i, c_j, \sum_{N'} \bar{x}_k)$. By *2-claims monotonicity*, $S_i(c_i, c_j, \sum_{N'} x_k) \leq S_i(\bar{c}_i, c_j, \sum_{N'} x_k)$. Hence, $x_i = S(c_{N'}, \sum_{N'} x_k) \leq S_i(\bar{c}_i, c_j, \bar{x}_i + \bar{x}_j) = \bar{x}_i < x_i$, which is a contradiction.

Since $\bar{x}_i < x_i$, then for at least one $j \in N$, $\bar{x}_j > x_j$. Let $N' \equiv \{i, j\}$. By *bilateral consistency*, $x_{N'} = S(c_{N'}, \sum_{N'} x_k)$ and $\bar{x}_{N'} = S(\bar{c}_i, c_j, \sum_{N'} \bar{x}_k)$. By the previous paragraph, $\sum_{N'} \bar{x}_k < \sum_{N'} x_k$, so that by *2-resource-monotonicity*, $S_j(\bar{c}_i, c_j, \sum_{N'} x_k) \geq S_j(\bar{c}_i, c_j, \sum_{N'} \bar{x}_k) = \bar{x}_j > x_j$. By *2-claims monotonicity*, $S_i(\bar{c}_i, c_j, \sum_{N'} x_k) \geq S_i(c_{N'}, \sum_{N'} x_k) = x_i$. Thus, $\sum_{N'} S_k(\bar{c}_i, c_j, \sum_{N'} x_k) > x_i + x_j$, which contradicts the efficiency of S.

The proof for *linked claim-resource monotonicity* follows from the statement just proved, the facts that this property is dual to *claims monotonicity*, that *resource monotonicity* is a *self-dual* property, and Lemma 1.

(f) *Claims truncation invariance.* Let $N \in \mathcal{N}$ with $|N| \geq 3$ and $(c, E) \in \mathcal{C}^N$. Let $x \equiv S(t(c, E), E)$. We will show that $x = S(c, E)$. Let $N' \equiv \{i, j\} \subset N$. By *bilateral consistency*, $x_{N'} = S(t_{N'}(c, E), \sum_{N'} x_k)$. Let $E' \equiv \sum_{N'} x_k$. By *2-claims truncation invariance*, $S(t_{N'}(c, E), E') = S(t(t_{N'}(c, E), E'), E')$. Since $E' \leq E$, then $t(t_{N'}(c, E), E') = t_{N'}(c, E')$, so that $x_{N'} = S(t_{N'}(c, E'), E')$. Also, by *2-claims truncation invariance*, $S(t_{N'}(c, E'), E') = S(t(c_{N'}, E') = S(c_{N'}, E')$. Thus, recalling the definition of E', we have obtained $x_{N'} = S(c_{N'}, \sum_{N'} x_k)$. This conclusion can be reached for each $\{i, j\} \subset N$. Thus, by *converse consistency*, which holds by Lemma 2, $x = S(c, E)$, as asserted.

The proof for *minimal rights first* follows from the statement just proved, the facts that this property is dual to *claims truncation invariance*, that *resource monotonicity* is a *self-dual* property, and Lemma 1. □

Theorem 2a can also be proved as follows: (i) *2-anonymity* implies *2-equal treatment of equals*; (ii) *2-equal treatment of equals* and *2-resource monotonicity* are lifted by *bilateral consistency* (Theorem 1a,c); (iii) *resource monotonicity* and *bilateral consistency* together imply *consistency* (Chun, 1999); (iv) *equal treatment of equals* and *consistency* imply *anonymity* (Chambers and Thomson, 2002). The same argument can be used to prove that the fixed-population version of *2-anonymity* is lifted.

References

Aumann, R. and M. Maschler, 1985. Game theoretic analysis of a bankruptcy problem from the Talmud. Journal of Economic Theory 36, 195-213.

Chambers, C. and W. Thomson, 2002. Group order preservation and the proportional rule for bankruptcy problems. Mathematical Social Sciences 44, 235-252.

Chun, Y., 1999. Equivalence of axioms for bankruptcy problems. International Journal of Game Theory 28, 511-520.

Chun, Y., J. Schummer, and W. Thomson, 2001. Constrained egalitarianism: a new solution to bankruptcy problems. Seoul Journal of Economics 14, 269-297.

Curiel, I., M. Maschler and S. H. Tijs, 1987. Bankruptcy games. Zeitschrift für Operations Research 31,, A143-A159.

Dagan, N., R. Serrano, and O. Volij, 1997. A non-cooperative view of consistent bankruptcy rules. Games and Economic Behavior 18, 55-72.

Dagan, N. and O. Volij, 1993. The bankruptcy problem: a cooperative bargaining approach. Mathematical Social Sciences 26, 287-297.

Dagan, N. and O. Volij, 1997. Bilateral comparisons and consistent fair division rules in the context of bankruptcy problems. International Journal of Game Theory 26, 11-25.

Herrero, C. and A. Villar, 2001. The three musketeers: four classical solutions to bankruptcy problems. Mathematical Social Sciences 39, 307-328.

Herrero, C. and A. Villar, 2002. Sustainability in bankruptcy problems. TOP 10, 261-273.

Hokari, T. and W. Thomson, 2003. Claims problems and weighted generalizations of the Talmud rule. Economic Theory 21, 241-261.

Moulin, H., 2000. Priority rules and other asymmetric rationing methods. Econometrica 68, 643-684.

Moulin, H., 2002. Axiomatic cost and surplus sharing. In: Arrow, K and K. Suzumura, (eds) Handbook of Social Choice, North-Holland, Amsterdam. 290-357.

O'Neill, B., 1982. A problem of rights arbitration from the Talmud. Mathematical Social Sciences 2, 345-371.

Thomson, W., 2003a. Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey. Mathematical Social Sciences 45, 249-297.

Thomson, W., 2003b. On the existence of consistent rules to adjudicate conflicting claims: a geometric approach. Review of Economic Design. forthcoming.

Thomson, W., 2003c. Options monotonic allocation rules. Mimeo.

Thomson, W., 2006a. Consistent allocation rules. Mimeo.

Thomson, W., 2006b. How to Divide When there isn't Enough. Book manuscript.

Thomson, W. and C.-H. Yeh, 2006. Operators for the adjudication of conflicting claims. Mimeo. Journal of Economic Theory, forthcoming.

Young, P., 1987. On dividing an amount according to individual claims or liabilities. Mathematics of Operations Research 12, 398-414.

Young, P., 1994. Equity: In Theory and Practice. Princeton University Press, Princeton.