Contests with Rank-Order Spillovers

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January 2007
Current Draft : September 2008

Abstract

This paper presents a unified framework for characterizing symmetric equilibrium in simultaneous move, two-player, rank-order contests with complete information, in which each player's strategy generates direct or indirect affine "spillover" effects that depend on the rank-order of her decision variable. These effects arise in natural interpretations of a number of important economic environments, as well as in classic contests adapted to recent experimental and behavioral models where individuals exhibit inequality aversion or regret. We provide the closed-form solution for the symmetric Nash equilibria of this class of games, and show how it can be used to directly solve for equilibrium behavior in auctions, pricing games, tournaments, R&D races, models of litigation, and a host of other contests.

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1 Introduction

This paper presents a unified framework for analyzing equilibrium in simultaneous-move, two-player, rank-order contests with complete information, in which each player's strategy generates direct or indirect affine "spillover" effects that depend on the rank-order of her decision variable. We show that these effects arise in natural interpretations of a number of important economic environments, including contests adapted to recent experimental and behavioral models where individuals exhibit inequality aversion or regret. We provide a characterization of symmetric equilibria (both pure and mixed), closed-form expressions for these equilibria, and show how our results may be used to directly solve for equilibrium behavior in auctions, pricing games, tournaments, R&D races, models of litigation, and a host of other games.

Rank-order contests are ubiquitous. These take the form of environments in which players choose nonnegative bids (which may be interpreted as a proposed payment, effort, or the commitment of other scarce resources that are non-refundable) whose rank-order discontinuously influences the probability of winning some prize. Classic examples include homogeneous product Bertrand competition (Bertrand, 1883), in which the lowest price firm "wins" the profit from selling to demand at that price, as well as first and second-price auctions (Vickrey, 1961), where the player who submits the highest bid wins the item and pays either his own bid (in the first-price auction) or the bid of the second-highest bidder (in the second-price auction).

Many rank-order contests involve both winners and losers alike forfeiting payments. In a first-price all-pay auction, for instance, each player submits a non-refundable bid and only the higher bidder receives a prize. The war-of-attrition (Maynard Smith, 1974) is a second-price all-pay auction: The high bidder wins the prize and pays the amount bid by the second-highest bidder. These forms of competition have been widely used to model activities as diverse as patent and R&D races, lobbying and rent-seeking activities, litigation, advertising and political campaigns, tournaments as incentive devices in labor markets, competition for college admissions, sports competitions, urban architecture, and territorial contests among organisms.[1]

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[1] Applications in these areas include work by Dasgupta (1986), Kaplan, Luski and Wettstein (2003), Hillman and Riley (1989), Baye, Kovenock and de Vries (1993), Che and Gale (1998), Baye, Kovenock and de Vries (2005), Sahuguet and Persico (2006), Konrad (2004), Fu (2006), Groh, Moldovanu, Sela and Sunde

The principal motivation of this article is that *spillovers* are often important in rank-order contests; in many economic environments, one player's decision affects the other player's payoff, and the nature of this effect may depend on the rank-order of the players' choices. This is perhaps most obvious in second-price auctions where the high bidder pays the second highest bid, but spillovers also arise in a variety of economic contexts. For instance, an extensive literature starting with D'Aspremont and Jacquemin (1988) has examined the effects of positive spillovers in R&D competition that can arise when one player's R&D effort provides information that benefits its rival. Although D'Aspremont and Jacquemin (1988) does not involve rank-order effects, a growing literature, starting from an original observation by Dasgupta (1986), models the R&D process as a rank-order tournament (see also Che and Gale (2003) and Zhou (2006)). The results examined in this article apply to the positive spillovers arising in this context.

Rank-order spillovers also arise in models of litigation. Baye, Kovenock and de Vries (2005), for instance, examine equilibrium in a litigation game with incomplete information in which legal expenditures increase the quality of the case presented and where the "best case" wins. This turns the litigation process into a rank order contest in which the litigation incentives in legal systems, such as the American, British, Continental and "Quayle" systems, may be examined. Although the American system, where litigants pay their own legal costs, involves no spillovers, other fee-shifting rules, such as the British and Continental rules, which require that losers compensate winners for a portion of their legal costs, and the Quayle system, in which the loser reimburses the winner up to the amount actually spent by the loser, involve spillovers. Under the British and Continental rules there is an indirect spillover effect of the winner's expenditure on the loser that is negative. In the continuation we call this a *second-order negative spillover* effect. In the case of the Quayle system, there is an indirect spillover of the loser's expenditure on the winner that is positive. We call this a *first-order positive spillover* effect.

Our taxonomy of spillover effects may also be used to construct and analyze variants and extensions of the auction and contest literatures noted above. For instance, the classic partnership dissolution problem may be viewed as the auction of a business in which two partners simultaneously submit bids and the partner with the higher bid pays his bid to the partner with the lower bid in return for ownership of the business. In this case, the payment

(forthcoming), Helsley and Strange (2008), and Kura (1999).

of the winning partner is a *second-order positive spillover* effect on the loser. Similarly, both the first-price and second-price all-pay auctions, often used to model economic and biological contests, may be extended to include environments in which effort expended imposes both a rank-order contingent direct effect on the player expending the effort and a rank order spillover effect on the player's rival. For instance, if two organisms are engaged in a territorial fight, the effort of the winner may exact both a cost to the winner (a *first-order negative direct* effect) and a cost to the loser (a *second-order negative spillover* effect). The loser's effort may have a *second-order negative direct* effect on the loser's payoff and a *first-order negative spillover* effect on the winner.

An important class of economic environments where rank-order dependent spillovers arise naturally is the analysis of auctions adapted to recent experimental and behavioral models of individual choice. In Section 3 we show that our characterization may be used to examine behavioral models that include: (i) tournaments in which individuals exhibit inequality aversion in the spirit of Fehr-Schmidt (1999), (ii) first-price and all-pay auctions where players experience regret similar to that in the models of Engelbrecht-Wiggans (1989), Engelbrecht-Wiggans and Katok (2007), and Filiz-Ozbay and Ozbay (2007), and (iii) an all-pay auction in which players maximize relative fitness according to the finite agent Evolutionary Stable Strategies (ESS) equilibrium of Schaffer (1988)[2].

Section 3 also shows that many pricing games have rank-order dependent spillovers that may be analyzed within our framework. For instance, a variant of the classic Bertrand model, due to Varian (1980), has two sellers simultaneously setting a price and selling to three inelastic segments of demand with common choke price, r. One of these inelastic segments consists of price sensitive consumers who are aware of both prices in the market and who purchase from the lower-price seller, while the other two segments are attached to different firms and are each aware of only the price of that firm to which they are attached (as long as that price is at or below the choke price). Baye, Kovenock and de Vries (1992) have shown that this game has a structure similar to that of a first-price all-pay auction in which the bid is the difference between the choke price and a firm's price. In this context, the bid corresponds to the opportunity cost of the lost revenue from the seller's own uninformed segment that results from reducing price in order to attempt to capture the "prize" consisting

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[2]See also Hehenkamp, Leininger, and Possajennikov (2004) who, to the best of our knowledge, were the first to apply the ESS equilibrium concept in a (Tullock) contest.

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of the demand of the informed price-sensitive consumer segment.

Spillovers also arise naturally in the context of the Varian model when one examines popular price matching policies (see Lin (1988), Png and Hirshleifer (1987), Baye and Kovenock (1994)). If a high price seller institutes a price matching policy it will sell at its own price to consumers informed only of that price, but sell at its rival's price to a proportion of the informed customers who are willing to bear the cost of visiting the high price seller and taking it up on its offer to match the better price. In this case the rival's low price generates a spillover effect on the high price seller's payoff, but not vice-versa. Section 3 also includes additional applications of our results, including a "reference pricing" version of the Varian model that includes "relative bargain" seekers whose demand from the low-price firm depends on the ratio of the high price to the low price. With reference pricing, a rival's high price generates a spillover effect on low price seller's payoff, but not vice-versa.

All of these models have the property that they are special cases of the linear parameterized class of rank-order contests whose symmetric equilibria we characterize in this paper. In Section 2 we formally introduce this class of models and provide a general closed-form solution for the symmetric equilibria of the class. We characterize the symmetric equilibrium strategies as functions of "contest parameters," which when varied change the "rules" of the contest. In Section 3, we show how this characterization may be used to directly obtain closed-form solutions for symmetric equilibrium strategies in these and other economic environments. In Section 4 we conclude. The proofs are collected in the Appendix.

2   Model and Results

We study the symmetric Nash equilibria of the class of two-player games of complete information in which each player $i \in \{1, 2\}$ chooses an action (or bid) x_i from the strategy space $A = [0, \infty)$, and where payoffs are

$$u_i(x_i, x_j) = \begin{cases} W\left(x_i, x_j\right) \equiv v - \beta x_i - \delta x_j & \text{if } x_i > x_j \\ L\left(x_i, x_j\right) \equiv -\gamma - \alpha x_i - \theta x_j & \text{if } x_i < x_j \\ T\left(x_i, x_j\right) \equiv \frac{1}{2}W\left(x_i, x_j\right) + \frac{1}{2}L\left(x_i, x_j\right) & \text{if } x_i = x_j \end{cases} \quad (1)$$

We assume that $V \equiv v + \gamma > 0$ and $v \geq 0$; otherwise, one could simply redefine winning as losing and vice versa.[3] We also assume that at least one of the contest parameters β, δ, α, or θ is nonzero.[4] In the sequel, we let Γ denote an arbitrary game within this class.

The δ and θ parameters capture the externalities (negative or positive) that contestants may inflict on each other. We use the terms "first-order positive (negative) *spillover* effects" when $\delta < (>) 0$, and "second-order positive (negative) spillover effects" when $\theta < (>) 0$. This captures the fact that when x_i is the higher bid (the first in rank-order), the *spillover* effect of player j's bid, $j \neq i$, on player i's payoff is linear with coefficient $-\delta$. If $\delta > 0$, this effect is negative and if $\delta < 0$ this effect is positive. Likewise, when player i's bid is the lower bid (second in the rank-order), the spillover effect of player j's bid, $j \neq i$, is linear with coefficient $-\theta$. If $\theta > 0$, this effect is negative and if $\theta < 0$ this effect is positive. For similar reasons, we refer to β and α as the first- and second-order *direct* effects. If player i's bid x_i is the higher bid, or first, in the rank-order, the *direct* effect of player i's bid on player $i's$ payoff is linear with coefficient $-\beta$. If $\beta > 0$, the first order direct effect of an increase in player i's bid is negative and is positive if $\beta < 0$. Similar interpretations apply to the second-order direct effect, α.

Notice that, were the strategy space bounded and one is merely interested in establishing existence of a symmetric equilibrium, one could readily analyze this class of games using the Dasgupta and Maskin (1986) framework. For games of incomplete information, Lizzeri and Perisco (2000) examine existence and uniqueness of bidding strategies in auctions where $W(x_i, x_j)$ and $L(x_i, x_j)$ need not be linear, while Baye, Kovenock and de Vries (2005) provide closed-form expressions for equilibrium strategies in the linear case. In what follows, we provide closed-form solutions for symmetric pure- and mixed-strategy equilibria for the case of complete information, and allow for parameter configurations not accounted for in the Lizzeri-Perisco and Baye-Kovenock-deVries analysis.[5]

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[3] In particular, note that $V > 0$ implies $W(0, 0) > L(0, 0)$. Analysis of the case where $V \equiv v + \gamma = 0$ is available upon request from the authors.

[4] The case where $\beta = \delta = \alpha = \theta = 0$ corresponds to the game of "pick the greatest non-negative real number," which does not have an equilibrium when the strategy space is unbounded.

[5] Complete information analogues of the Lizzeri-Perisco axioms (labelled A1-A8) would require $\alpha \geq 0$ (A5, A7), $\beta > 0$ (A7), $\delta \geq 0$ (A7), and $\theta = 0$ (A7). Baye, Kovenock, and de Vries assume, in their incomplete information framework, that $(\beta, \alpha) > 0$, $\delta = (1 - \alpha)$, and $\theta = (1 - \beta)$.

2.1 Symmetric Pure Strategy Nash Equilibria

We first provide the conditions under which there exists a symmetric pure strategy equilibrium, x^*, such that each player earns the equilibrium payoff $U^* < \infty$. To this end, define $\eta \equiv \alpha + \theta - \beta - \delta$, so that η measures the change in the payoff per unit of expenditure at x^* derived solely from changes in the direct and spillover effects (and not from winning V) that result from the switch from being a loser to being a winner at x^*.

Proposition 1 Γ *has a symmetric pure strategy Nash equilibrium if and only if the following three conditions jointly hold: (i) $\beta \geq 0$, (ii) $\alpha \leq 0$ and (iii) $\eta < 0$. Furthermore, there is but one such equilibrium and it is given by*

$$x^* = -\frac{V}{\eta} \equiv \frac{v + \gamma}{\beta + \delta - \alpha - \theta}. \tag{2}$$

Examples of applications of Proposition 1 to games of complete information include the first price auction ($\gamma = 0$, $\beta = 1$, $\delta = \alpha = \theta = 0$) where $x^* = v$ and the second-price auction ($\gamma = 0$, $\delta = 1$, $\beta = \alpha = \theta = 0$) where $x^* = v$. Proposition 1 also implies that games such as the standard first-price all-pay auction ($\beta = \alpha = 1$, $\delta = \theta = 0$) and the second-price all-pay auction (also called the war of attrition, where $\delta = \alpha = 1$, $\beta = \theta = 0$) do not have symmetric pure-strategy equilibria. Since it is known that these special cases of Γ do have symmetric mixed-strategy equilibria, we provide a characterization of all such equilibria to Γ.

2.2 Symmetric mixed-strategy Equilibria

Let $F(x)$ be the cumulative distribution function describing a symmetric equilibrium mixed strategy of Γ, if one exists, and let $EU^* \in (-\infty, \infty)$ denote the associated equilibrium (expected) payoff.[6] If F has a density $f(x)$ on (m, u), then the expected payoff to a player that bids $w \in (m, u)$ against the rival's mixed-strategy F is given by

$$EU(w) = \int_0^w W(w, x) dF(x) + \int_w^\infty L(w, x) dF(x),$$

Since F is a symmetric equilibrium by hypothesis, $EU(w) = EU^*$ on $w \in (m, u)$. Hence,

$$\frac{dEU(w)}{dw} = [V + \eta w] f(w) - \alpha + (\alpha - \beta) F(w) = 0 \tag{3}$$

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[6]Note that the strategy space and payoffs in equation (1) are unbounded. Consequently, if a player's expected payoff against a rival's mixed-strategy does not exist (that is, the relevant integral diverges to $\pm\infty$), that strategy cannot comprise a symmetric mixed-strategy equilibrium.

on (m, u). The solution to this differential equation is given by

$$F(w) = \frac{\alpha}{\alpha - \beta}\left\{1 - \left[\frac{V + \eta m}{V + \eta w}\right]^{\frac{\alpha - \beta}{\eta}}\right\} + C\left[\frac{V + \eta m}{V + \eta w}\right]^{\frac{\alpha - \beta}{\eta}}, \qquad (4)$$

where $m \geq 0$ and $0 \leq C \leq 1$.

Notice that this derivation of the functional form for a symmetric equilibrium mixed-strategy is only heuristic, as it ignores mass points, the possibility of profitable deviations outside of (m, u), and furthermore, may not represent a well-defined distribution function for some parameter configurations. Our next proposition addresses these issues formally and characterizes the non-degenerate symmetric mixed-strategy equilibria to Γ.

Proposition 2 Γ *has a nondegenerate symmetric mixed-strategy equilibrium if and only if one of the following three sets of conditions holds: (i) $\beta > 0$ and $\alpha > 0$; (ii) $\beta = 0$, $\alpha > 0$ and either $\eta\theta = 0$ or $\eta < \alpha$; or (iii) $\beta = 0$, $\alpha < 0$ and either $\alpha < \eta < 0$ or $\eta < \theta = 0$. In cases (i) and (ii), the equilibrium is unique within the class of symmetric equilibria (pure or mixed). In case (iii) there exists a continuum of nondegenerate symmetric mixed-strategy equilibria, as well as a unique symmetric pure strategy equilibrium (given in Proposition 1). The nondegenerate symmetric mixed-strategy equilibria are atomless and described by the distribution function $F^*(w)$ on $[m^*, u^*)$, where*

$$F^*(w) = \begin{cases} \frac{\alpha}{\alpha - \beta}\left(1 - \left(\frac{V + \eta m^*}{V + \eta w}\right)^{\frac{\alpha - \beta}{\eta}}\right) & \text{if } \eta \neq 0; \alpha - \beta \neq 0 \\ \frac{\alpha}{\theta - \delta} \ln\left(\frac{V + (\theta - \delta)w}{V}\right) & \text{if } \eta \neq 0; \alpha - \beta = 0 \\ \frac{\alpha}{\alpha - \beta}\left(1 - \exp\left(-\frac{\alpha - \beta}{V}w\right)\right) & \text{if } \eta = 0; \alpha - \beta \neq 0 \\ \frac{\alpha}{V}w & \text{if } \eta = 0; \alpha - \beta = 0 \end{cases}, \qquad (5)$$

$$m^* = \begin{cases} 0 & \text{if } \alpha > 0 \\ m' \in \left(-\frac{V}{\eta}, \infty\right) & \text{if } \alpha < 0 \end{cases}, \quad \text{and} \qquad (6)$$

$$u^* = \begin{cases} -\frac{V}{\eta} & \text{if } \alpha > 0; \beta = 0; \eta < 0 \\ \frac{V}{\eta}\left[(\alpha/\beta)^{\frac{\eta}{\alpha - \beta}} - 1\right] & \text{if } \alpha > 0; \beta > 0; \alpha \neq \beta; \eta \neq 0 \\ \frac{V}{\eta}\left[\exp(\eta/\alpha) - 1\right] & \text{if } \alpha = \beta > 0; \eta \neq 0 \\ \frac{V}{\alpha - \beta} \ln\frac{\alpha}{\beta} & \text{if } \alpha > 0; \beta > 0; \alpha \neq \beta; \eta = 0 \\ V/\alpha & \text{if } \alpha > 0; \beta > 0; \alpha = \beta; \eta = 0 \\ \infty & \text{if } \text{otherwise} \end{cases}$$

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Furthermore, the corresponding equilibrium payoffs are given by

$$EU^* = \begin{cases} \frac{\theta v + \delta \gamma}{\theta - \delta} + \beta \frac{\theta}{\eta(\theta - \delta)} \left[1 - \left(\frac{\alpha}{\beta} \right)^{\frac{\eta}{\alpha - \beta}} \right] V & if \quad \eta \neq 0; \alpha \neq \beta; \theta \neq \delta; \beta \neq 0 \\[2mm] -\gamma + \frac{\theta V}{\eta} - \frac{\theta V}{\eta} \frac{\alpha}{\eta} \ln \frac{\alpha}{\beta} & if \quad \eta \neq 0; \alpha \neq \beta; \theta = \delta; \beta \neq 0 \\[2mm] \frac{\theta v + \delta \gamma}{\theta - \delta} + \frac{\alpha \delta}{\theta - \delta} m^* & if \quad \eta \neq 0; \alpha \neq \beta; \theta \neq 0; \beta = 0 \\[2mm] -\gamma - \alpha m^* & if \quad \eta \neq 0; \alpha \neq \beta; \theta = 0; \beta = 0 \\[2mm] \frac{\theta v + \delta \gamma}{\theta - \delta} + \beta \frac{\theta}{(\theta - \delta)^2} \left[1 - \exp \left(\frac{\theta - \delta}{\beta} \right) \right] V & if \quad \eta \neq 0; \alpha - \beta = 0 \\[2mm] \frac{\theta v + \delta \gamma}{\theta - \delta} + \beta \frac{\theta}{(\theta - \delta)^2} \left[\ln \left(\frac{\alpha}{\beta} \right) \right] V & if \quad \eta = 0; \alpha - \beta \neq 0; \beta \neq 0 \\[2mm] \frac{\theta v + \delta \gamma}{\theta - \delta} & if \quad \eta = 0; \alpha - \beta \neq 0; \beta = 0 \\[2mm] -\gamma - \frac{\theta}{2\alpha} V & if \quad \eta = 0; \alpha - \beta = 0 \end{cases} \tag{7}$$

Notice that all of the terms in equations (5), 6 and (7) are well-defined, since conditions (*i*) through (*iii*) which guarantee the existence of a non-degenerate symmetric mixed-strategy equilibrium imply: (a) $\alpha \neq 0$, (b) if $\alpha < 0$, then $\eta < 0$ and $\beta = 0$; (c) if $\alpha = \beta$, then $\alpha > 0$ and $\beta > 0$; (d) if $\alpha = \beta$ and $\eta \neq 0$, then $\theta \neq \delta$; (e) if $\alpha \neq \beta$ and $\eta = 0$, then $\theta \neq \delta$; and (f) if $\beta \neq 0$ then $\alpha > 0$ and $\beta > 0$.

The Appendix constructively derives all of the possible symmetric equilibria and the resulting payoffs in a series of lemmata, and also indicates when an equilibrium does not exist. The analysis in the Appendix implies that one may also obtain closed-form expressions for the equilibrium strategies by taking limits of equation (4). For instance, the functional form for the equilibrium distribution function in Proposition 2 when $\eta \neq 0$ *and* $\alpha = \beta$ obtains by taking the limit of equation (4) as $\alpha - \beta$ tends to zero.

Propositions 1 and 2, which characterize the parameter ranges where symmetric pure and nondegenerate mixed-strategy equilibria arise, together facilitate a complete partition of the parameter space into ranges of qualitatively different symmetric Nash equilibrium correspondences. We summarize this in

Proposition 3 *The symmetric equilibria to* Γ *are characterized as follows:*

(a) The unique symmetric equilibrium is in pure strategies if and only if one of the following three conditions holds: (i) $\beta > 0$, $\alpha \leq 0$, and $\eta < 0$; (ii) $\beta = 0$, $\alpha = 0$, and $\eta < 0$; or (iii) $\beta = 0$, $\eta \leq \alpha < 0$, and $\theta \neq 0$.

(b) The unique symmetric equilibrium is in nondegenerate mixed strategies if and only one of the following two conditions holds: (i) $\beta > 0$ and $\alpha > 0$; or (ii) $\beta = 0$, $\alpha > 0$ and either $\eta\theta = 0$ or $\eta < \alpha$.

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(c) *There is a unique symmetric pure strategy equilibrium and a continuum of nondegenerate symmetric mixed-strategy equilibria if and only if $\beta = 0$, $\alpha < 0$ and either $\alpha < \eta < 0$ or $\eta < \theta = 0$.*

Furthermore, if none of the conditions in (a) through (c) hold, Γ does not have a symmetric equilibrium (either pure or mixed).

It is important to emphasize that we have focused solely on symmetric equilibria. Asymmetric equilibria are known to exist, for instance, in the symmetric two player war of attrition ($\alpha = \delta = 1, \beta = \theta = 0$) and sad loser auction ($\alpha = 1, \beta = \theta = \delta = 0$). We also note that, were there a common finite upper bound on the strategy space for the players (such that the strategy space is compact), existence of a symmetric equilibrium is guaranteed by Lemma 7 of Dasgupta-Maskin (1986).

Before proceeding, we offer several observations about the four functional forms for the symmetric equilibrium mixed-strategies that can arise under different parameter configurations. First, note that the lower bound of the support, m^*, of every symmetric equilibrium mixed-strategy to Γ is zero when $\alpha > 0$, but an arbitrary positive number $m^* \in (-V/\eta, \infty)$ when $\alpha < 0$ (this accounts for the continuum of symmetric mixed-strategy equilibria that arise in this case). Second, notice that the equilibrium mixed-strategies take on different functional forms (including the uniform distribution, exponential distribution, as well as more exotic forms) depending on which of four regions (denoted R1-R4) the parameters lie:

R1: $\eta = 0$ and $\alpha = \beta$. In this case, conditions (i) through (iii) in Proposition 2 imply that only condition (i) can be satisfied, and therefore $\alpha = \beta > 0$ and $\theta = \delta$. The standard all-pay auction ($\alpha = \beta = 1$ and $\theta = \delta = 0$) is a special case of this configuration. More generally, this configuration is a modified first-price all-pay auction in which $\theta = \delta \neq 0$ is a "nuisance parameter" that does not influence the symmetric equilibrium mixed-strategy (which is a uniform distribution) but *does*, impact the players' equilibrium expected payoffs.

R2: $\eta \neq 0$ and $\alpha = \beta$. In this case, conditions (i) through (iii) in Proposition 2 imply that only condition (i) is satisfied, and therefore $\alpha = \beta > 0$ and $\theta \neq \delta$. In this case the asymmetric spillovers impact both the equilibrium payoffs and the symmetric mixed strategies (which take on a logarithmic form). To the best of our knowledge, contests or auctions with parameters in R2 have not here-to-for been examined in the literature.

Notice that a game with parameters in R1 is the limit of games with parameters in R2 as the spillovers become symmetric. Hence, games with parameters in R2 may be interpreted as a generalized first-price all-pay auction with asymmetric spillovers.

R3: $\eta = 0$ and $\alpha \neq \beta$. In this case, conditions (i) through (iii) in Proposition 2 imply that either (i) or (ii) applies, and therefore $\alpha > 0$, $\beta \geq 0$ and $\theta \neq \delta$. In this case the asymmetric spillovers impact the symmetric equilibrium payoffs but not the equilibrium mixed-strategies (which take the form of a truncated exponential distribution). The standard war of attrition ($\alpha = \delta = 1$, $\beta = \theta = 0$) is a special case of a Γ with parameters in R3. Notice that when $\beta > 0$, the symmetric mixed-strategy has a bounded upper support, whereas it is unbounded when $\beta = 0$ (and hence the symmetric mixed-strategies are a non-truncated exponential distribution).

R4: $\eta \neq 0$ and $\alpha \neq \beta$. In this case, any of the conditions (i), (ii), or (iii) may apply. This is, in a sense, the most general form of an all pay auction in which direct effects are asymmetric and spillovers may be either symmetric or asymmetric. When both α and β are positive, Proposition 2 implies that the support of the distribution is bounded. When $\beta = 0$, the support of the symmetric equilibrium mixed strategies is unbounded unless $\eta < 0$; the unbounded distribution is known as a Burr distribution with a Pareto type upper tail such that not all moments exist. Notice that part (ii) of Proposition 2 includes the case where $\beta = 0$, $\alpha > 0$, and $\theta = 0$. The Riley and Samuelson (1980) sad loser auction in which only the loser pays his bid ($\beta = \delta = \theta = 0$ and $\alpha = 1$) is a special case of this configuration. The case where $\beta = 0$, $\alpha > 0$ and $\alpha > \eta \neq 0$ may be viewed as a sad loser auction with spillovers ($\delta > \theta$). Finally, note that when $\alpha < \beta = 0$ and either $\alpha < \eta < 0$ or $\eta < \theta = 0$, there is a continuum of symmetric mixed-strategy equilibria which stem from the continuum of lower bounds for the support that arise when $\alpha < 0$. The Baye-Morgan (1999) folk-theorem for one-shot, homogeneous product Bertrand games, which entails both a continuum of symmetric equilibrium strategies and equilibrium payoffs, is an example of an economic environment that may be viewed as a Γ with parameters in R4.[7]

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[7]More technically, Example 1 in Baye-Morgan (1999) corresponds to a limit of this case.

3 Applications

We now are in a position to show how our characterization of symmetric equilibria may be used to obtain closed-form expressions for equilibrium strategies in economic environments that include auctions, contests, and pricing games. In so doing, we also show that our results may be used to generalize existing models to allow for spillover effects, and to examine the implications of alternative behavioral and evolutionary assumptions on equilibrium. Throughout this section, we suppress ties, which we assume are broken with the flip of a fair coin.

3.1 Auctions and Contests with Spillovers

In addition to standard auctions and contests (such as first- and second-price auctions, the war of attrition, and the all-pay auction), our framework may be used to characterize symmetric equilibria in more exotic economic applications.

Partnership Dissolution (The Self-Auction). Two partners wish to dissolve a partnership each values at v. They simultaneously submit bids; the high bidder wins the asset and pays his bid to the other partner. Here, the payoffs are given by

$$u_i\left(x_i, x_j\right) = \begin{cases} v - x_i & \text{if} \quad x_i > x_j \\ x_j & \text{if} \quad x_\iota < x_j \end{cases}$$

and thus the game may be viewed as a Γ with $V = v$, $\beta = -\theta = 1, \alpha = \delta = 0$, and $\eta = -2$. It follows from Proposition 1 that the only symmetric pure-strategy equilibrium is $x^* = v/2$. Furthermore, Proposition 3 reveals that the self-auction does not have a non-degenerate symmetric mixed-strategy equilibrium.

An Innovation Contest with Spillovers. One may also use our results to extend Dasgupta's (1986) all-pay auction formulation of an R&D race by allowing each firm's expenditure on R&D to induce an informational spillover that benefits the rival. In particular, suppose the winner receives a greater benefit per unit of expenditure from the loser's expenditure than the loser receives from the winner's expenditure. The corresponding payoffs

are

$$u_i(x_i, x_j) = \begin{cases} v - x_i - \delta x_j & if \quad x_i > x_j \\ -x_i - \theta x_j & if \quad x_\iota < x_j \end{cases}.$$

This game may be viewed as a Γ in which $V = v > 0$, $\alpha = \beta = 1$, and $\delta < \theta < 0$. Since $\alpha - \beta = 0$ and $\eta > 0$, Propositions 2 and 3 imply that the unique symmetric equilibrium is in nondegenerate mixed strategies and given by

$$F^*(x) = \frac{1}{\theta - \delta} \ln \left(1 + \frac{\theta - \delta}{v} x \right) \quad on \quad \left[0, \frac{v}{\theta - \delta} \left(\exp \left(\frac{\theta - \delta}{\alpha} \right) - 1 \right) \right].$$

Territorial Contests with Injuries. Next, consider a generalization of an all-pay auction formulation of a territorial contest in which the outcome of the battle depends on the intensity of effort put forth by the two players in the fight, where each player values the territory in dispute at v. Suppose the cost to a player per unit of intensity of effort is unity ($\alpha = \beta = 1$), and each unit of effort a player expends in the battle imposes a cost on its rival (through injury), so that $\delta, \theta > 0$. If the cost to the loser per unit of intensity of effort of the winner is greater than the cost to the winner per unit of intensity of effort of the loser ($\theta > \delta > 0$), then $\alpha - \beta = 0$ and $\eta = \theta - \delta > 0$. In this case, Proposition 2 reveals that the symmetric equilibrium of this game is identical to that in the above innovation contest with spillovers.

If, on the other hand, the efforts of the winner and loser entail symmetric injury ($\theta = \delta > 0$), Proposition 2 implies that the symmetric equilibrium mixed-strategy is given by

$$F^*(x) = x/v \quad on \quad [0, v],$$

which corresponds to the all-pay auction. However, expected payoffs are *not* zero as they are in the standard all-pay auction. Notice that, as a result of spillovers, both of these variants differ from the classic war of attrition ($\beta = \theta = 0$ and $\alpha = \delta = 1$), which lies in the region of R3 that entails an equilibrium distribution that is an exponential distribution.

Litigation Contests with Fee Shifting. Our framework may also be used to characterize symmetric equilibria for the complete information analogues of the Baye et al. (2005) model of litigation contests under incomplete information. In this application, players are litigants who compete by choosing (quality normalized) expenditures on legal services, with the player spending the higher amount winning the case valued at v. The fee shifting rules examined by Baye et al. may be examined under complete information using the tools developed in Section 2.

For instance, under the Continental rule, the loser pays his own legal expenditure and, additionally, reimburses the winner a fraction $(1-\beta) \in (0, 1)$ of the winner's legal expenditures. Thus, the Continental Rule is a Γ with $V = v$, $0 < \beta < \alpha = 1$, and $\delta = 0 < \theta = (1 - \beta)$. Since $\alpha > 0$, $\beta > 0$, $\alpha - \beta > 0$, and $\eta = 2(1 - \beta) > 0$, Proposition 2 implies that the corresponding equilibrium is

$$F^*(x) = \frac{1}{1-\beta} \left(1 - \left(\frac{v}{v + 2(1-\beta)x} \right)^{\frac{1}{2}} \right) \text{ on } \left[0, \frac{v}{2(1-\beta)} \left(\frac{1}{\beta^2} - 1 \right) \right]$$

and the equilibrium payoff is $EU^* = -v\left(1 - \beta\right) / \left(2\beta\right) < 0$.

In contrast, under the Quayle rule where the loser reimburses the winner up to the amount actually spent by the loser, $V = v$, $\alpha = 2$, $\beta = 1$, $\delta = -1$ and $\theta = 0$. Since $\alpha > 0$, $\beta > 0$, $\alpha - \beta > 0$ and $\eta = 2 > 0$, Proposition 2 implies that the corresponding equilibrium is

$$F^*(x) = 2 \left[1 - \left(\frac{v}{v + 2x} \right)^{\frac{1}{2}} \right] \text{ on } \left[0, \frac{3}{2}v \right]$$

and the equilibrium payoff is $EU^* = 0$. Furthermore, Proposition 1 reveals that symmetric pure-strategy equilibria do not arise in these litigation environments.

3.2 Price Competition

We mentioned earlier that our framework readily includes standard models of price competition under complete information that take the form of first-and second-price auctions (which have a unique symmetric equilibrium in pure strategies) as well variants such as Baye-Morgan (1999) (which have a continuum of symmetric mixed-strategy equilibria). We next show that several leading models of price competition in the industrial organization literature are also subsumed in our framework.

The Varian/Rosenthal Model. The price setting models of Varian (1980) and Rosenthal (1980) may be analyzed in our framework as follows. Here, two price-setting firms each service a fixed number, $L > 0$, of loyal (or uninformed) consumers who have unit demand up to a choke price , $r > 0$. Additionally, there are $S > 0$ price sensitive "shoppers" (or informed consumers) who always purchase from the firm charging the lowest price—provided it does not exceed r. Each firm produces at zero cost to earn a payoff of

$$\pi_i \left(p_i, p_j \right) = \begin{cases} \left(S + L \right) p_i & \text{if } p_i < p_j \\ L p_i & \text{if } p_i > p_j \end{cases}$$

To see that this model is a special case of our framework, define $x_i \equiv r - p_i \geq 0$ so that the above payoffs are equivalent to a game in which

$$u_i\left(x_i, x_j\right) = \begin{cases} \left(S + L\right) r - \left(S + L\right) x_i & \text{if} \quad x_i > x_j \\ rL - Lx_i & \text{if} \quad x_i < x_j \end{cases}$$

Thus, the Varian/Rosenthal models may be interpreted as a Γ with $v = \left(S + L\right) r$, $\gamma = -rL$, $V = rS > 0$, $\beta = \left(S + L\right) > 0$, $\delta = \theta = 0$, $\alpha = L > 0$, and $\eta = -S < 0$. Hence, by Proposition 2, the equilibrium (expressed in terms of the discount from the monopoly price, or $x \equiv r - p$) is

$$F^*\left(x\right) = \frac{L}{S}\left(\frac{r}{r - x} - 1\right) \quad \text{on} \quad \left[0, \frac{rS}{S + L}\right].$$

To write this expression in terms of the equilibrium distribution of prices, $G^*\left(p\right)$, use the fact that $x \equiv r - p$ and note that $G^*\left(p\right) = \Pr\left(P \leq p\right) = 1 - \Pr\left(x < r - p\right) = 1 - F^*\left(r - p\right)$, so that

$$G^*\left(p\right) = 1 - \frac{L}{S}\left(\frac{r - p}{p}\right) \quad \text{on} \quad \left[r\frac{L}{S + L}, r\right].$$

Price Matching Guarantees. One may also use our results to extend the Varian/Rosenthal models to allow for price matching policies, as in Png and Hirshleifer (1987) and Baye and Kovenock (1994). To see this, extend the Varian/Rosenthal models by assuming that the two firms not only list prices, but also promise to match a better price by the rival. Here, one interprets S as informed consumers who are aware of the firms' prices, L as uninformed consumers who are unaware of the firms' prices, and one assumes that only a proportion $\mu < 1/2$ of the informed customers are willing to bear the cost of visiting a firm charging the highest price to take it up on its offer to match a better price. Consequently, the firms' payoffs are

$$\pi_i\left(p_i, p_j\right) = \begin{cases} \left(L + \left(1 - \mu\right) S\right) p_i & \text{if} \quad p_i < p_j \\ Lp_i + \mu Sp_j & \text{if} \quad p_i > p_j \end{cases}$$

As before, let $x_i = r - p_i$, so that the payoffs may be rewritten as

$$u_i\left(x_i, x_j\right) = \begin{cases} r\left(\left(1 - \mu\right) S + L\right) - \left(\left(1 - \mu\right) S + L\right) x_i & \text{if} \quad x_i > x_j \\ r\left(L + \mu S\right) - Lx_i - \mu Sx_j & \text{if} \quad x_i < x_j \end{cases}$$

Note that $V = \left(1 - 2\mu\right) Sr > 0$, $\beta = \left(1 - \mu\right) S + L > 0$, $\alpha = L > 0$, $\theta = S\mu > 0$, and $\delta = 0$. In this case, $\alpha - \beta = -\left(1 - \mu\right) S < 0$ and $\eta = -\left(1 - 2\mu\right) S < 0$, so that Proposition

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2 implies

$$F^* (x) = \frac{L}{(1-\mu)S} \left(\left(\frac{r}{r-x} \right)^{\frac{(1-\mu)}{(1-2\mu)}} - 1 \right) \text{ on } \left[0, r \left(1 - \left(\frac{L}{((1-\mu)S+L)} \right)^{\frac{(1-2\mu)}{(1-\mu)}} \right) \right].$$

As before, one may easily re-write this distribution of discounts from the monopoly price purely in terms of the prices.

3.3   Behavioral and Evolutionary Extensions

The results in Section 2 may also be used to extend existing models to account for behavioral or evolutionary considerations that impact the payoffs in standard games. We discuss these applications next.

Reference Pricing.   One may use our results to analyze an extension of the Varian/Rosenthal models to account for reference pricing. To see this, suppose that in addition to shoppers and loyal consumers, there also exists a measure of "relative bargain seekers." As above, all consumer segments have a common choke price, $r > 0$. Relative bargain seekers, like shoppers, always purchase from the firm offering the lowest price. But unlike shoppers, the demand of relative bargain seekers depends on how low the best price is in comparison to the next best price: The lower the "best" price *relative* to that of the higher price, the greater their demand for the low priced good. To capture this behavior, assume that when firm i charges the lowest price, its demand from relative bargain seekers is $D_i \equiv \lambda p_j / p_i$ while firm j's demand from these consumers is zero. When $\lambda > 0$, this captures the behavioral phenomenon where the demand by one segment of consumers depends, in part, on their frame of reference.

With this extension, the payoffs are.

$$\pi_i (p_i, p_j) = \begin{cases} \left(S + L + \lambda \frac{p_j}{p_i} \right) p_i & \text{if } p_i < p_j \\ L p_i & \text{if } p_i > p_j \end{cases}.$$

As before, let $x_i \equiv r - p_i$, so that these payoffs are equivalent to

$$u_i (x_i, x_j) = \begin{cases} r (S + L + \lambda) - x_i (S + L) - \lambda x_j & \text{if } x_i > x_j \\ rL - L x_i & \text{if } x_i < x_j \end{cases}.$$

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Thus, this extension is a Γ with $V = r(S + \lambda) > 0$, $\alpha = L > 0$, $\beta = S + L > 0$, $\theta = 0$, $\delta = \lambda$, $\alpha - \beta = -S < 0$ and $\eta = -(S + \lambda) < 0$. Hence, Proposition 2 implies

$$F^*(x) = \frac{L}{S}\left(\left(\frac{r}{r-x}\right)^{\frac{S}{S+\lambda}} - 1\right) \text{ on } \left[0, r\left(1 - \left(\frac{L}{S+L}\right)^{\frac{(S+\lambda)}{S}}\right)\right],$$

or in terms of price,

$$G^*(p) = 1 - \frac{L}{S}\left(\left(\frac{r}{p}\right)^{\frac{S}{S+\lambda}} - 1\right) \text{ on } \left[r\left(\frac{L}{S+L}\right)^{\frac{(S+\lambda)}{S}}, r\right]$$

This distribution converges to that in the Varian/Rosenthal model as λ tends to zero.

Interestingly, when there are only loyal customers and relative bargain seekers ($S = 0$), then $\eta = -\lambda < 0$ and $\alpha - \beta = 0$. In this case, Proposition 2 implies

$$F^*(x) = \frac{L}{\lambda}\ln\frac{r}{r-x} \text{ on } \left[0, \left(1 - \exp\left(-\frac{\lambda}{L}\right)\right)r\right].$$

Inequality Aversion in a Job Tournament. Next, consider an environment where two workers compete in a job tournament but, in the spirit of Fehr-Schmidt (1999), exhibit a specialized form of inequality aversion such that they receive disutility from inequality of effort. In particular, suppose that the worker exerting the greater effort (x_i) receives a bonus valued at $\mu > 0$ and that payoffs are

$$u_i(x_i, x_j) = \begin{cases} \mu - x_i - b(x_i - x_j) & \text{if } x_i > x_j \\ -x_i - a(x_j - x_i) & \text{if } x_i < x_j \end{cases}$$

where $0 < a < 1$ and $0 < b$. This captures behavior where the winner experiences disutility for having "slaughtered" the loser, and the loser derived disutility from having been beaten badly. In this case, utilities may be written as

$$u_i(x_1, x_2) = \begin{cases} \mu - (1+b)x_i + bx_j & \text{if } x_i > x_j \\ -(1-a)x_i - ax_j & \text{if } x_i < x_j \end{cases}.$$

This behavioral environment may thus be viewed as a Γ in which $V = v = \mu > 0$, $\gamma = 0$, $\alpha = 1 - a > 0$, $\theta = a$, $\beta = 1 + b > 0$ and $\delta = -b < 0$. Note that $\alpha - \beta = -(a+b) < 0$ and $\eta = 0$.

Since $\alpha = 1 - a > 0$, it is immediate from Proposition 1 that there does not exist a symmetric pure strategy equilibrium. However, Propositions 2 and 3 imply that a unique

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nondegenerate symmetric mixed strategy equilibrium exists. The corresponding distribution of effort is

$$F^*(x) = \frac{1-a}{a+b} \left(\exp\left(\frac{a+b}{\mu}x\right) - 1 \right) \quad \text{on} \quad \left[0, \frac{\mu}{a+b} \ln\left(\frac{1+b}{1-a}\right)\right]$$

and each player earns an expected equilibrium payoff of

$$EU^* = \frac{a}{a+b} \left[1 + \frac{1+b}{a+b} \ln\left(\frac{1-a}{1+b}\right)\right] \mu.$$

It is interesting to note that if $a \in (0,1)$ but $b \in (-1,0)$ (so that the winner enjoys "slaughtering" the loser), the equilibrium strategies and payoffs have exactly the same form when $b \neq -a$. But when $b = -a$ (so that $\alpha - \beta = 0$ and $\eta = 0$), the equilibrium distribution effort is identical to that in the all-pay auction,

$$F^*(x) = \frac{1-a}{\mu}x \quad \text{on} \quad \left[0, \frac{\mu}{1-a}\right]$$

However, unlike the all-pay auction,

$$EU^* = -\frac{a}{2(1-a)}\mu.$$

Loss Aversion in a Job Tournament. Two workers compete in a job tournament and the worker exerting the greater effort (x_i) receives a bonus valued at $\mu > 0$. Thus, their income is

$$y_i = \begin{cases} \mu - x_i & \text{if } x_i > x_j \\ -x_i & \text{if } x_\iota < x_j \end{cases}$$

Suppose the workers' utility (over income) exhibits "loss aversion" in that $u_i = y_i$ if player i wins, and $u_i = \lambda y_i$ if player i loses, where $\lambda > 1$. In this case, utility (as a function of effort) is

$$u_i(x_i, x_j) = \begin{cases} \mu - x_i & \text{if } x_i > x_j \\ -\lambda x_i & \text{if } x_\iota < x_j \end{cases}$$

and this scenario may be analyzed as a Γ with $v = \mu$, $\gamma = 0$, $V = \mu > 0$, $\alpha = \lambda > 0$, $\beta = 1 > 0$, $\theta = \delta = 0$, and $\eta = \alpha - \beta = \lambda - 1 > 0$. Hence, Propositions 2 and 3 imply that the unique nondegenerate symmetric mixed-strategy equilibrium is given by

$$F^*(x) = \frac{\lambda x}{\mu + (\lambda - 1)x} \quad \text{on} \quad [0, \mu].$$

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Regret in Auctions. A growing literature has examined regret in auctions; see Engelbrecht-Wiggans (1989), Engelbrecht-Wiggans and Katok (2007), and Filiz-Ozbay and Ozbay (2007) and the references cited therein. To illustrate how our framework may be used to examine the implications of this behavioral assumption in complete information environments, consider a first-price auction where each player $i \in \{1,2\}$ values the item at v, but there is winner regret such that the payoffs are

$$u_i(x_1, x_2) = \begin{cases} v - x_i - \mu(x_i - x_j) & \text{if } x_i > x_j \\ 0 & \text{if } x_i < x_j \end{cases}.$$

Here, x_i is player i's bid and $v > 0$ is the value of the item; winner regret ($\mu > 0$) refers to the fact that the high bidder derives disutility from leaving money on the table (the difference between the winning and losing bid). The payoffs may be rewritten as

$$u_i(x_1, x_2) = \begin{cases} v - (\mu + 1)x_i + \mu x_j & \text{if } x_i > x_j \\ 0 & \text{if } x_i < x_j \end{cases}$$

which is a rank-order contest, Γ, with a *positive first-order spillover* effects. In particular, $V = v$, $\alpha = \theta = 0$, $\beta = (1 + \mu) > 0$, $\delta = -\mu$, and $\eta = -1$, so Proposition 1 implies that a symmetric pure strategy Nash equilibrium exists and is given by $x^* = v$. Furthermore, by Proposition 3, there are no symmetric mixed-strategy equilibria.

In a first-price auction with loser regret, payoffs are

$$u_i(x_1, x_2) = \begin{cases} v - x_i & \text{if } x_i > x_j \\ -\rho(v - x_j) & \text{if } x_i < x_j \end{cases}$$

where $\rho > 0$. Preferences with loser regret thus transform the standard auction into a rank-order contest with a *positive second-order spillover* effect, and one can use the results in Section 2 to conclude that the unique symmetric equilibrium (pure or mixed) is $x^* = v$.

The results of Section 2 may also be used to extend these behavioral models to include combined winner and loser regret in a first-price auction. In this case, both *first and second order spillover* effects arise, as the payoffs are

$$u_i(x_1, x_2) = \begin{cases} v - (\mu + 1)x_i + \mu x_j & \text{if } x_i > x_j \\ -\rho(v - x_j) & \text{if } x_i < x_j \end{cases}.$$

One can readily establish that the unique symmetric equilibrium is in pure strategies and given by $x^* = v$.

Furthermore, our results may be utilized to examine the implications of combined winner-loser regret in other auction environments. For instance, in an all-pay auction with winner-loser regret, payoffs are

$$u_i(x_1, x_2) = \begin{cases} v - (\mu + 1) x_i + \mu x_j & \text{if } x_i > x_j \\ -v\rho - (\rho + 1) x_i + \rho x_j & \text{if } x_i < x_j \end{cases}$$

This may be viewed as a Γ in which $V = (1 + \rho) v$, $\alpha = (1 + \rho) > 0$, $\beta = (1 + \mu) > 0$, $\theta = -\rho$, $\delta = -\mu$, and $\eta = 0$. When $\rho \neq \mu$, Propositions 2 and 3 imply that the unique symmetric equilibrium is

$$F^*(x) = \left(\frac{1 + \rho}{\rho - \mu}\right) \left(1 - \exp\left(-\frac{\rho - \mu}{1 + \rho}\frac{x}{v}\right)\right) \quad \text{on} \quad \left[0, \frac{1 - \delta}{\delta - \mu} \ln\left(\frac{1 + \rho}{1 + \mu}\right)\right]$$

and each player earns an expected payoff of

$$EU^* = v\rho + \rho (1 + \rho) v \frac{(1 + \mu) \ln \frac{1+\mu}{1+\rho} + \rho - \mu}{(\rho - \mu)^2}.$$

However, if $\rho = \mu$, so that $\alpha = \beta$, one obtains the standard all-pay auction form: $F^*(x) = x/v$. In this case, total expected effort is the same with symmetric winner loser regret as in the standard all-pay auction, but $EU^* = -\rho v/2$.

ESS in the All-Pay Auction. Finally, one may utilize our results to construct equilibrium strategies in certain evolutionary environments. To see this, consider a two player all-pay auction and note that the (finite agent) ESS equilibrium of Schaffer (1988) requires that each player maximizes the difference in payoffs. Thus

$$u_i(x_1, x_2) = \begin{cases} v - x_i - (-x_j) & \text{if } x_i > x_j \\ -x_i - (v - x_j) & \text{if } x_i < x_j \end{cases}$$

Hence, this may be viewed as a Γ where payoffs are

$$u_i(x_1, x_2) = \begin{cases} v - x_i + x_j & \text{if } x_i > x_j \\ -v - x_i + x_j & \text{if } x_i < x_j \end{cases}.$$

and $V = 2v > 0$, $\beta = \alpha = -\theta = -\delta = 1$, and consequently, $\alpha - \beta = 0$ and $\eta = 0$. One may therefore apply the results in Section 2 to conclude that the unique symmetric Nash

equilibrium to a game with these payoffs (which corresponds to the ESS equilibrium of a game with the original formulation of payoffs) is

$$F^*(x) = \frac{x}{2v} \text{ on } [0, 2v].$$

Among other things, this implies that there is overdissipation of rents in the ESS equilibrium. This is similar to the findings of Hehenkamp et al. (2004) for the case of a Tullock contest.

4   Conclusion

This paper has characterized symmetric equilibria (pure and mixed) in a parameterized class of two player complete information contests with rank-order spillovers. We derived explicit closed form solutions for the complete set of symmetric equilibrium strategies for this class of games, and showed that these strategies take on only a small number of functional forms that depend on the parameters in a systematic and easily verified way. We concluded by using this framework to formulate and solve several new contests. Not only are a plethora of existing models of auctions, contests, and price competition covered as special cases, but our results permit one to extend these models to allow for a broader array of preferences, spillover effects, and equilibrium concepts. The logarithmic equilibrium distribution that arises in the all-pay auction with asymmetric spillovers, for example, appears to be novel to the literature. We believe that Propositions 1 through 3 will provide positive spillovers for future applied work on auctions, contests, and pricing strategies, as well as behavioral economics and evolutionary game theory.

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Appendix

This Appendix provides the proofs of Propositions 1 through 3. Recall the definitions of W, L, and T are given in equation (1).

A1. Proof of Proposition 1

The following lemma is useful in proving Proposition 1.

Lemma 1 $x \in [0, \infty)$ *is a symmetric pure strategy Nash equilibrium of* Γ *if and only if the following two conditions hold:*

$$
\begin{aligned}
T(x, x) &\geq W(y, x) \text{ for all } y \geq x \\
T(x, x) &\geq L(y, x) \text{ for all } 0 \leq y \leq x
\end{aligned}
\tag{8}
$$

Proof. Note first that since

$$
T(x, x) = \frac{1}{2} W(x, x) + \frac{1}{2} L(x, x),
$$

the conditions in (8) imply

$$
T(x, x) = W(x, x) = L(x, x).
$$

$(\Longleftarrow)$ By hypothesis, $x \in [0, \infty)$ satisfies

$$
\begin{aligned}
T(x, x) &\geq W(y, x) \text{ for all } y \geq x \\
T(x, x) &\geq L(y, x) \text{ for all } y \leq x
\end{aligned}
$$

Hence, if player i plays the pure strategy $x_i = x$ when player j plays $x_j = x$, she earns a payoff of $U^* = T(x, x) = W(x, x) = L(x, x)$. The conditions in (8) imply that player i cannot gain by deviating from x, given that $x_j = x$.

$(\Longrightarrow)$ If (x, x) is a symmetric pure strategy Nash equilibrium, player i earns a payoff of $T(x, x)$ in this equilibrium. By way of contradiction, suppose there exists a $y \in [0, \infty)$ such that $y > x$ with $T(x, x) < W(y, x)$. Then player i could increase his payoff to $W(y, x) > T(x, x)$ by deviating from $x_i = x$ to $x_i = y$, a contradiction. Similarly, if there existed a $y \in [0, \infty)$ such that $y < x$ with $T(x, x) < L(y, x)$, player i could increase his payoff to $L(y, x) > T(x, x)$ by deviating from $x_i = x$ to $x_i = y$, a contradiction. $\blacksquare$

We conclude that the conditions in (8) are necessary and sufficient for the existence of a symmetric pure strategy Nash equilibrium. Note that the proof of Lemma 1 does not

rely on the linear structure for W and L in equation (1), and hence applies to more general formulations for payoffs.

We are now in a position to prove Proposition 1. We do so by exploiting the linear structure in equation (1) and applying Lemma 1.

Proof of Proposition 1. ($\Longrightarrow$) By way of contradiction, suppose $x \in [0, \infty)$ is a symmetric pure strategy Nash equilibrium so that player i earns his equilibrium payoff of $U^* = T(x, x) = W(x, x) = L(x, x)$ at (x, x). If condition (i) in Proposition 1 did not hold, then player i could deviate to earn $W(x + \varepsilon, x) > U^* = W(x, x)$, since $\beta < 0$ implies $W(x_i, x)$ is increasing in x_i, a contradiction. If condition (iii) did not hold, then $V + \eta x = W(x, x) - L(x, x) > 0$, which contradicts the conditions in (8). Finally, since $V > 0$, condition (iii) implies $x > 0$. Thus, if condition (ii) did not hold, then $x > 0$ and $\alpha > 0$, in which case player i could deviate to earn a payoff of $L(x - \varepsilon, x) > L(x, x) = W(x, x) = U^*$, since $\alpha > 0$ implies $L(x_i, x)$ is decreasing in x_i, a contradiction. ($\Longleftarrow$) Suppose conditions (i) through (iii) hold. It is immediate that condition (iii) implies that $x^* = -V/\eta$ is well-defined and $V + \eta x^* = W(x^*, x^*) - L(x^*, x^*) = 0$. Hence, $W(x^*, x^*) = L(x^*, x^*) = T(x^*, x^*)$. Next, note that there is no incentive to deviate from x^*, since (i) implies $T(x^*, x^*) \geq W(y, x^*)$ for all $y \geq x^*$, and (ii) implies $T(x^*, x^*) \geq L(y, x^*)$ for all $y \leq x^*$. By the Lemma 1, this implies that x^* is a symmetric pure strategy Nash equilibrium. Uniqueness follows from Lemma 1 and the fact that $x^* = -V/\eta$ is the unique solution to $W(x, x) - L(x, x) = 0$. ∎

A2. Proof of Proposition 2

Our second proposition is proved through a sequence of lemmas. We first demonstrate that if an atom exists at some point (x, x) in a nondegenerate symmetric mixed-strategy equilibrium of Γ, then (x, x) constitutes a symmetric pure strategy equilibrium as well. We then apply Proposition 1 to show that this can occur only over a restricted range of the parameter space and that any such atom is unique. Consequently, if the symmetric equilibrium is in nondegenerate mixed strategies, there must exist an absolutely continuous part of the mixed-strategy, and furthermore, it must satisfy the differential equation in equation (3). Given the linearity of differential equation (3), it readily follows that its solution over the interval (m, u) is unique (as it satisfies a Lipschitz condition). Lemma 4 provides an endpoint restriction on the lower bound of the distribution. We then examine (by exhaustion) a partition of the parameter space to show (i) when the differential equation can be solved in

a manner consistent with the corresponding restrictions on mass points and endpoints, (ii) which solutions indeed define equilibria in the sense that there is no incentive for a player to unilaterally deviate from his strategy, and (iii) whether the differential equation, mass point and endpoint restrictions are inconsistent, thereby implying nonexistence of a nondegenerate symmetric mixed-strategy. We also derive the corresponding equilibrium payoffs.

Lemma 2 *If there is an atom at some point $x \in [0, \infty)$ in a nondegenerate symmetric mixed-strategy equilibrium of Γ, then (x, x) is also a symmetric pure strategy equilibrium of Γ. Furthermore, there is no atom at $x = 0$.*

Proof. If there is an atom of size $q_x \in (0, 1)$ at some point x, it must be the case that $q_x [W (x + \varepsilon, x) - T (x, x)] \leq 0$ (there is no incentive to raise the bid above x) and, if in addition $x > 0$, $q_x [L (x - \varepsilon, x) - T (x, x)] \leq 0$ for small $\varepsilon > 0$ (there is no incentive to lower the bid below x). Furthermore, there can be no atom at $x = 0$, since $q_0 [W (0 + \varepsilon, 0) - T (0, 0)] \leq 0$ implies $W (0, 0) - T (0, 0) \leq 0$ from the linearity of W, contradicting $W (0, 0) - T (0, 0) = V > 0$. For $x > 0$, since $q_x > 0$ by hypothesis, $[W (x + \varepsilon, x) - T (x, x)] \leq 0$ and $[L (x - \varepsilon, x) - T (x, x)] \leq 0$. This implies $T (x, x) = W (x, x) = L (x, x)$, and furthermore, given the linearity of W and L,

$$T (x, x) \geq W (y, x) \text{ for all } y \geq x$$
$$T (x, x) \geq L (y, x) \text{ for all } y \leq x.$$

These are exactly the conditions (8) for a pure strategy solution from Theorem 1 and hence (x, x) must also be a pure strategy equilibrium point. ■

Lemma 3 *Suppose a symmetric equilibrium strategy of Γ has an atom of size $q_x \in (0, 1]$ at x. Then $\beta \geq 0$, $\alpha \leq 0$ and $\eta < 0$. Furthermore the atom is unique and located at $x = -V/\eta > 0$.*

Proof. Follows immediately from Lemma 2 and Proposition 1. ■

Importantly, Lemma 3 implies that if a nondegenerate symmetric mixed-strategy equilibrium exists, any atom (if one exists) associated with the strategy is necessarily unique (and given by $x = -V/\eta$). Consequently, the remaining absolutely continuous part is characterized by differential equation (3). We will use this fact to establish when nondegenerate

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symmetric mixed-strategy equilibria exist, their functional forms and the corresponding equilibrium payoffs. We also identify parameter configurations for which the set of nondegenerate symmetric mixed-strategy equilibria is empty. We also use this lemma to establish:

Lemma 4 *Suppose $\alpha > 0$. Then in any non-degenerate symmetric mixed-strategy equilibrium of Γ, the lower bound of the support is $m = 0$.*

Proof. The proof proceeds by way of contradiction. Suppose the lower bound of the support of the equilibrium mixed-strategy is $m > 0$, and let q_m be the size of an atom (possibly zero) at m : Then a player who bids m earns his equilibrium payoff of

$$
\begin{aligned}
U^* &= q_m T(m,m) + (1 - q_m)\left(-\gamma - \alpha m - \theta E_F\left[x|x > m\right]\right) \\
&= \frac{q_m}{2}V - \gamma + \frac{q_m}{2}\alpha m - \alpha m - \frac{q_m}{2}\left(\theta + \beta + \delta\right)m - (1 - q_m)\theta E_F\left[x|x > m\right].
\end{aligned}
$$

Deviating by bidding zero yields a payoff of

$$
U^{**} = -\gamma - \theta q_m m - \theta\left(1 - q_m\right) E_F\left[x|x > m\right].
$$

The difference in payoffs is thus

$$
U^{**} - U^* = \frac{q_m}{2}\left\{-V + \left(-\theta - \alpha + \beta + \delta\right)m\right\} + \alpha m = \alpha m > 0,
$$

since $q_m > 0$ implies $-V = \eta m$ by Lemma 3. Therefore it pays to deviate by bidding zero, a contradiction.

∎

We are now in a position to consider, case by case, the parameter configurations identified in Proposition 2. We do this through a sequence of lemmas that are collected according to the four parameter regions (R1 through R4) defining the different forms for the equilibrium mixed strategies in equation (5), and which establish existence, uniqueness, or non existence of equilibrium for parameter configurations within each case. We begin with

Case 1: $\alpha - \beta \neq 0; \eta \neq 0$

Lemma 5 *Suppose $\beta = 0$, $\alpha > 0$ and $\eta \neq 0$. Then there exists a symmetric equilibrium if and only if either $\theta = 0$ or $\eta < \alpha$. Furthermore, this equilibrium is unique and in nondegenerate mixed strategies as characterized in Proposition 2.*

Proof. Since $\alpha > 0$, Proposition 1 implies any symmetric equilibrium must be in nondegenerate mixed-strategies, and by Lemma 3, there are no atoms. By Lemma 4, $m = 0$. Hence, if a symmetric equilibrium exists, it necessarily has the form in equation (4) with $C = 0$:

$$F(x) = 1 - \left(\frac{V}{V + \eta x} \right)^{\alpha/\eta}. \tag{9}$$

This is a well-defined distribution function; when $\eta > 0$, the upper bound of its support is $u = \infty$; when $\eta < 0$, it is $u = -V/\eta < \infty$. Since $\beta = 0$, a player cannot gain by choosing an action $w > u$. Thus, for an equilibrium to exist, it is sufficient to show that $EU^* < \infty$ and a player's expected payoff against a rival who uses F is constant for any action in the support of F.

The expected payoff when a player chooses $x_i = w$ against such a strategy is

$$EU(w) = \int_0^w (v - \delta x) \, dF(x) + \int_w^u (-\gamma - \alpha w - \theta x) \, dF(x)$$

or, since this also holds at $w = 0$,

$$
\begin{aligned}
EU(0) &= \int_0^u (-\gamma - \theta x) \, dF(x) \\
&= -\gamma - \int_0^u \theta x \, dF(x)
\end{aligned}
$$

Evidently, when $\theta = 0$, $EU(0) = -\gamma$, so $EU(w) = -\gamma$ for all $w \in Support(F)$, and F is the unique symmetric equilibrium. Thus, suppose $\theta \neq 0$.

Consider first the case where $\eta > 0$. If $\alpha/\eta - 1 \leq 0$ (or equivalently, $\delta \leq \theta \neq 0$), the distribution in (9) has a "fat tail" and hence $\int_0^u x \, dF(x)$ does not exist. Thus, when $\beta = 0$, $\alpha > 0, \eta > 0$, and $\theta \neq 0$, we conclude there does not exist a symmetric equilibrium when $\alpha \leq \eta$. On the other hand, when $\alpha/\eta - 1 > 0$ (or equivalently, $\delta > \theta \neq 0$), the relevant integral does exist and the expected payoff is

$$EU^* = -\gamma - \theta \int_0^\infty x \alpha \left(\frac{V}{V + \eta x} \right)^{\alpha/\eta} \frac{1}{V + \eta x} \, dx = \frac{\theta v + \delta \gamma}{\theta - \delta}. \tag{10}$$

Since for all $w \in [0, \infty)$, $EU(w) = EU^*$, in this case it follows that F is the unique symmetric equilibrium.

Finally, if $\eta < 0$, then $u = -V/\eta$ and simple integration reveals

$$EU(0) = (\theta v + \delta \gamma) / (\theta - \delta) = EU(w)$$

for all $w \in [0, u]$, and hence F is the unique symmetric equilibrium. ∎

Lemma 6 *Suppose $\beta = 0$, $\alpha < 0$ and $\eta > 0$. Then there does not exist a symmetric mixed-strategy equilibrium.*

Proof. Since $\alpha < 0$ and $\eta > 0$, Lemma 3 implies that F contains no atoms. Hence, $C = 0$ in equation (4), and thus

$$F\left(w\right) = 1 - \left(\frac{V + \eta w}{V + \eta m}\right)^{-\alpha/\eta}.$$

But note that, since $-\alpha/\eta > 0$ and $\eta > 0$,

$$\left(\frac{V + \eta w}{V + \eta m}\right)^{-\alpha/\eta} > 1$$

for all $w > m$, which implies $F(w) \leq 0$, a contradiction. $\blacksquare$

Lemma 7 *Suppose $\beta = 0$, $\alpha < 0$ and either $\alpha < \eta < 0$ or $\eta < \theta = 0$. Then there exists a continuum of non-degenerate symmetric mixed-strategy equilibria, all of which are identified in Proposition 2. Furthermore, if $\beta = 0$, $\alpha < 0, \eta \leq \alpha$ and $\theta \neq 0$, there does not exist a nondegenerate symmetric mixed-strategy equilibrium.*

Proof. By Proposition 1, a unique symmetric pure-strategy equilibrium exists at $x^* = -V/\eta$. By Lemma 3, in any non-degenerate symmetric mixed-strategy equilibrium, there is at most a single mass point, and this atom is located at $-V/\eta$. Let $q \in [0,1)$ denote the size of any such mass point. By way of contradiction, suppose that the lower bound of the absolutely continuous part of F is $m > -V/\eta$ (that is, F contains a gap). Then the expected payoff to a player that bids $-V/\eta$ against F is

$$
\begin{aligned}
EU\left(\frac{V}{-\eta}\right) &= qT\left(\frac{V}{-\eta}, \frac{V}{-\eta}\right) + (1-q)\left(-\gamma - \alpha\left(\frac{V}{-\eta}\right) - \frac{1}{1-q}\int_m^\infty \theta x dF\left(x\right)\right) \\
&= \frac{q}{2}\left[v - \gamma + (\delta + \alpha + \theta)\frac{V}{\eta}\right] \\
&\quad + (1-q)\left(-\gamma + \alpha\frac{V}{\eta} - \frac{1}{1-q}\int_m^\infty \theta x dF\left(x\right)\right).
\end{aligned}
$$

A player that bids $m > -V/\eta$ against F earns an expected payoff of

$$EU(m) = q\left(v + \delta\frac{V}{\eta}\right) + (1-q)\left(-\gamma - \alpha m - \frac{1}{1-q}\int_m^\infty \theta x dF\left(x\right)\right).$$

Recall that $V = v + \gamma$ and that, under the conditions stated, $\eta = \alpha + \theta - \delta$. Provided $\int_m^\infty x dF\left(x\right)$ exists or $\theta = 0$, straightforward algebra reveals

$$EU(m) - EU\left(\frac{V}{-\eta}\right) = -\alpha\left(1-q\right)\left(m - \frac{V}{-\eta}\right) > 0,$$

a contradiction: there can be no atom below the lower bound of the absolutely continuous part of a symmetric mixed-strategy equilibrium.

We next show that under the conditions stated, $m > -V/\eta$ (which implies there are no mass points) and that there exists a continuum of symmetric equilibria of the form in Proposition 2. To see this, note that for $\alpha < 0$ and $\eta < 0$, equation (3) requires that $m > -V/\eta$ in order for $F(w)$ to be a well-defined (and non-degenerate) distribution function on an open interval above m. It follows that, when $\beta = 0$, $\alpha < 0$ and $\eta < 0$, the only candidate for a non-degenerate symmetric mixed-strategy equilibrium is

$$F(w) = 1 - \left(\frac{V + \eta m}{V + \eta w}\right)^{\alpha/\eta}, \tag{11}$$

on $[m, \infty)$, where $m \in (-V/\eta, \infty)$ is arbitrary. The expected payoff when a player chooses $x_i = w$ against F is

$$EU(w) = \int_m^w (v - \delta x)\, dF(x) + \int_w^\infty (-\gamma - \alpha w - \theta x)\, dF(x)$$

If $\alpha/\eta \leq 1$, the distribution has fat tails ($\int_m^\infty x\, dF(x)$ does not exist). In this case, when $\theta \neq 0$, $EU(w)$ is undefined and hence a non-degenerate symmetric mixed-strategy equilibrium does not exist. But if $\theta = 0$, the expected payoff to a player that bids $w = m$ is $EU(m) = -\gamma - \alpha m$, and hence for all $w \in [m, \infty)$, $EU(w) = EU(m)$. Since $\alpha < 0$, $EU(w) < EU(m)$ for $w < m$, and hence it does not pay to deviate by choosing an action below m.

When $\alpha/\eta > 1$ (which implies $\theta > \delta$) it follows that for all $w \in [m, \infty)$,

$$
\begin{aligned}
EU(w) &= EU(m) = -\gamma - \alpha m - \theta \int_m^\infty x \alpha \left(\frac{V + \eta m}{V + \eta x}\right)^{\alpha/\eta} \frac{1}{V + \eta x}\, dx \\
&= \frac{\theta v + \delta \gamma}{\theta - \delta} + \frac{\alpha \delta m}{\theta - \delta}.
\end{aligned}
$$

Again, since $\alpha < 0$, $EU(w) < EU(m)$ for $w < m$, it does not pay to deviate by choosing an action below m. ∎

Lemma 8 *Suppose $\beta < 0$, $\alpha \neq \beta$, $\alpha \neq 0$, and $\eta \neq 0$. Then there does not exist a symmetric equilibrium.*

Proof. Since $\beta < 0$, Proposition 1 implies there does not exist a symmetric pure strategy equilibrium, and by Lemma 3, there are no mass points in a nondegenerate symmetric

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mixed-strategy equilibrium. Since $F(m) = 0$, $C = 0$ in equation (4); hence, if a symmetric mixed-strategy exists, it must be of the form

$$F(w) = \frac{\alpha}{\alpha - \beta} \left(1 - \left(\frac{V + \eta m}{V + \eta w} \right)^{\frac{\alpha - \beta}{\eta}} \right). \tag{12}$$

We claim the distribution is unbounded. To see this, suppose to the contrary that $u < \infty$. Since F has no atoms, a player that bids u is certain to win and earn an equilibrium payoff of $EU(u) = v - \beta u - \delta E_F[x]$. But, since $\beta < 0$, a player who deviates by bidding $u' > u$ earns an expected payoff of $EU(u') = v - \beta u' - \delta E_F[x] > EU(u)$, a contradiction.

If $(\alpha - \beta)/\eta > 0$, equation (12) implies

$$\lim_{w \to \infty} F(w) = \frac{\alpha}{\alpha - \beta} \neq 1.$$

If $(\alpha - \beta)/\eta < 0$, then

$$\lim_{w \to \infty} F(w) = \pm \infty.$$

Hence, regardless of the sign of $(\alpha - \beta)/\eta$, F is not a well-defined distribution function, a contradiction. Thus there does not exist a symmetric mixed-strategy equilibrium in this case. ■

Lemma 9 *Suppose $\beta > 0$, $\alpha < 0$, $\eta \neq 0$. Then there does not exist a nondegenerate symmetric mixed-strategy equilibrium.*

Proof. If $\eta > 0$, then there are no mass points by Lemma 3. A symmetric equilibrium, if one exists, must therefore satisfy equation (4) with $C = 0$:

$$F(w) = \frac{\alpha}{\alpha - \beta} \left[1 - \left(\frac{V + \eta m}{V + \eta w} \right)^{\frac{\alpha - \beta}{\eta}} \right] \quad \text{for } w \in (m, u).$$

Since $(\alpha - \beta)/\eta < 0$ and $\eta w > \eta m \geq 0$ for $w > m$, the term is square brackets is negative. This and the fact that $\alpha/(\alpha - \beta) > 0$ implies $F(w) < 0$, which contradicts the assumption that F is a well-defined distribution function.

If $\eta < 0$, Lemma 3 implies that an equilibrium mixed-strategy may have a mass point at $w = -V/\eta$ (hence $C \geq 0$). Hence, equation (4) implies

$$F(w) = \frac{\alpha}{\alpha - \beta} + \left(C - \frac{\alpha}{\alpha - \beta} \right) \left[\frac{V + \eta m}{V + \eta w} \right]^{\frac{\alpha - \beta}{\eta}} \quad \text{for } w \in (m, u) \tag{13}$$

If the distribution is unbounded, $\lim_{w \to \infty} F(w) = \alpha/(\alpha - \beta) < 1$, a contradiction. Thus, suppose $u < \infty$.

Suppose first that the equilibrium distribution contains no mass point. Then the differential equation in equation (3) holds at u, and since $F(u) = 1$, we have

$$[V + \eta u] f(u) - \alpha + (\alpha - \beta) = 0.$$

Now, $f(u) \geq 0, \alpha < 0, \beta > 0$ and $\eta < 0$ implies $u < -V/\eta$. Hence, the derivative of equation (13) is

$$F'(w) = \alpha (V + \eta m)^{\frac{\alpha - \beta}{\eta}} (V + \eta w)^{-\frac{\alpha - \beta + \eta}{\eta}} < 0,$$

since $V + \eta w > 0$ for all $w \in [m, u]$, a contradiction.

Finally, suppose there is a mass point. We first show the mass point must be located at or below m. By way of contradiction, suppose there is a mass point at $-V/\eta > m$. In this case, the differential equation (3) holds at m, and $F(m) = 0$. Hence,

$$[V + \eta m] f(m) - \alpha = 0$$

Since $f(m) \geq 0, \alpha < 0$, and $0 \leq m < -V/\eta$, this is a contradiction.

Since the mass point must be at $-V/\eta$ and $m \geq -V/\eta$, the derivative of equation (4) is (for $w > m$)

$$F'(w) = \left(\frac{\alpha}{\alpha - \beta} - C \right) (\alpha - \beta) \left(\frac{V + \eta m}{V + \eta w} \right)^{\frac{\alpha - \beta}{\eta}} \frac{1}{V + \eta w}.$$

Since $F'(w) > 0$ for some $w > m$,

$$sgn(F'(w)) = sgn\left(\frac{\alpha}{\alpha - \beta} - C \right) > 0 \tag{14}$$

in order for F to be a well-defined distribution. Since the differential equation holds at $w = u$, setting $F(u) = 1$ implies

$$\frac{\beta}{\alpha} = \left[1 - \frac{\alpha - \beta}{\alpha} C \right] \left(\frac{V + \eta m}{V + \eta u} \right)^{\frac{\alpha - \beta}{\eta}}$$

The LHS is strictly negative by assumption, while the RHS is strictly positive by equation (14) and the fact that $m, u > -V/\eta$—a contradiction.

Hence, there does not exist a nondegenerate mixed-strategy equilibrium. ∎

Lemma 10 *Suppose $\beta > 0$, $\alpha > 0, \alpha \neq \beta$, and $\eta \neq 0$. Then there exists a unique symmetric equilibrium and it is in nondegenerate mixed-strategies as identified in Proposition 2.*

Proof. Since $\alpha > 0$, Lemma 3 implies there are no mass points, and Lemma 4 implies $m = 0$. Hence equation (4) implies

$$F(w) = \frac{\alpha}{\alpha - \beta}\left[1 - \left(\frac{V}{V + \eta w}\right)^{\frac{\alpha - \beta}{\eta}}\right]. \tag{15}$$

It is straightforward to show that, for all $\beta > 0$, $\alpha > 0, \alpha \neq \beta$,and $\eta \neq 0$, this a well-defined distribution function on $[0, u^*]$, where

$$u^* = \frac{V}{\eta}\left(\left(\frac{\alpha}{\beta}\right)^{\frac{\eta}{\alpha - \beta}} - 1\right) > 0.$$

Suppose first that $\theta \neq \delta$ (or equivalently, $\eta \neq \alpha - \beta$). The expected payoff to a player that bids $w = 0$ against a rival that employs F is

$$
\begin{aligned}
EU^* &= EU\,(w = 0) = -\gamma - \theta \int_0^{u^*} x dF\,(x)\\
&= \frac{\theta v + \delta\gamma}{\theta - \delta} + \beta\frac{\theta}{\eta\,(\theta - \delta)}\left[1 - \left(\frac{\alpha}{\beta}\right)^{\frac{\eta}{\alpha - \beta}}\right]V.
\end{aligned}
$$

Hence, $EU\,(w) = EU^*$ for all $w \in [0, u^*]$, and it does not pay to deviate to a $w > u^*$ since $\beta > 0$.

When $\theta = \delta$ (or equivalently, $\eta = \alpha - \beta$), the expected payoff to a player that bids $w = 0$ against a rival that employs F is

$$
\begin{aligned}
EU\,(w = 0) &= -\gamma - \theta \int_0^{u^*} x dF\,(x)\\
&= -\gamma + \frac{\theta V}{\eta} - \frac{\theta\alpha V}{\eta^2}\ln\frac{\alpha}{\beta}
\end{aligned}
$$

As above, since $\beta > 0$, a player cannot gain by deviating to a $w > u^*$. We conclude that F is the unique symmetric equilibrium in this case. ■

Lemma 11 *Suppose $\alpha - \beta \neq 0, \alpha = 0$ and $\eta \neq 0$. Then there does not exist a symmetric equilibrium in nondegenerate mixed strategies.*

Proof. Under the conditions stated, the solution to differential equation (3) is

$$F(w) = K \left[\frac{V + \eta w}{V + \eta m} \right]^{\frac{\beta}{\eta}} \tag{16}$$

for some $K \neq 0$. By hypothesis, $\beta \neq 0$. Suppose first that $\beta < 0$. Then there is no mass point by Lemma 3, and hence $F(u) = 1$ implies

$$u = \frac{K^{-\frac{1}{\beta}\eta}V + K^{-\frac{1}{\beta}\eta}\eta m - V}{\eta}$$

But since $\beta < 0$, this is a contradiction, since a player could improve his payoff by bidding above u.

Suppose next that $\beta > 0$. If $\eta > 0$ then once again there is no mass point by Lemma 3. Hence, $F(w) = 0$ implies $w = -V/\eta$. But since $-V/\eta < 0$, this is a contradiction.

Finally, suppose $\beta > 0$ and $\eta < 0$. Then

$$
\begin{aligned}
f(w) &= K(V + \eta w)^{\frac{\beta - \eta}{\eta}} (V + \eta m)^{-\frac{\beta}{\eta}} \beta \\
&= \beta \frac{F(w)}{V + \eta w}
\end{aligned}
$$

and hence $w < -V/\eta$. By Lemma 3, any mass point must be above the upper bound of the absolutely continuous part of F. Setting $F(w) = 0$ in equation (16) implies the lower bound of the distribution must be $-V/\eta$. But this is a contradiction, since by Lemma 3, the mass point must be located at this point. ∎

Case 2: $\alpha = \beta; \eta \neq 0$

Lemma 12 *Suppose $\alpha = \beta > 0$ and $\eta \neq 0$. Then there exists a unique symmetric equilibrium and it is in nondegenerate mixed strategies as identified in Proposition 2.*

Proof. Note first that the conditions of the Lemma imply $\theta \neq \delta$. Since $\alpha > 0$, Lemma 3 implies there are no mass points. Moreover, by Lemma 4, $\alpha > 0$ implies $m = 0$. Hence the differential equation in (3) implies

$$f(w) = \frac{\alpha}{V + \eta w}, \tag{17}$$

which together with $F(m) = 0$, implies that the unique F is

$$F(x) = \int_0^x \frac{\alpha}{V + \eta w} dw = \frac{\alpha}{\theta - \delta} \ln \left(\frac{V + (\theta - \delta)x}{V} \right) \tag{18}$$

on $[0, \frac{V}{\theta-\delta}(\exp(\frac{\theta-\delta}{\alpha}) - 1)]$, where we have used the fact $\eta = \theta - \delta$ under the conditions stated. The expected payoff of a player that bids $w = 0$ is

$$
\begin{aligned}
EU(0) &= \int_0^u (-\gamma - \theta x) \frac{\alpha}{V + \eta x} dx \\
&= \frac{\theta v + \delta \gamma}{\theta - \delta} + \frac{\alpha \theta}{(\theta - \delta)^2} \left(1 - e^{\frac{\theta-\delta}{\alpha}}\right) V
\end{aligned}
$$

and hence $EU(w) = EU(0)$ for all $w \in \left[0, \frac{V}{\theta-\delta}\left(\exp\left(\frac{\theta-\delta}{\alpha}\right) - 1\right)\right]$. Since $\beta > 0$, a player cannot gain by bidding above the upper bound of the support. ∎

Lemma 13 *Suppose $\alpha = \beta < 0$ and $\eta \neq 0$. Then there does not exist a symmetric mixed-strategy equilibrium.*

Proof. Lemma 3 implies there are no mass points, and hence equation (17) implies that, if an equilibrium exists, it must have the form

$$
\begin{aligned}
F(x) &= \int_m^x \frac{\alpha}{V + \eta w} dw \\
&= \frac{\alpha}{\theta - \delta} \ln\left(\frac{V + (\theta - \delta)x}{V + (\theta - \delta)m}\right)
\end{aligned}
$$

where we have used the fact that $\eta = \theta - \delta$ under the conditions stated. Since $F(u) = 1$ implies $u < \infty$, the support of F is bounded. But then F cannot be part of a Nash equilibrium since a player can increase his expected payoff by bidding above u, as $\beta < 0$. ∎

Lemma 14 *Suppose $\alpha = \beta = 0$ and $\eta \neq 0$. Then there does not exist a symmetric mixed-strategy equilibrium.*

Proof. Under the conditions stated, differential equation (3) implies

$$
(V + \eta w) f(w) = 0
$$

which contradicts the hypothesis that there is a nondegenerate mixed-strategy. ∎

Case 3: $\alpha - \beta \neq 0$; $\eta = 0$

Lemma 15 *Suppose $\eta = 0$, $\alpha - \beta \neq 0$, and $\alpha < 0$. Then there does not exist a symmetric equilibrium.*

Proof. First, note that since $\eta = 0$, there can be no mass point by Lemma 3. Under the conditions stated, differential equation (3) implies

$$V f(w) - \alpha + (\alpha - \beta) F(w) = 0$$

and hence the unique solution is

$$F(x) = \frac{\alpha}{\alpha - \beta} \left[1 - \exp\left(\frac{\beta - \alpha}{V} (x - m) \right) \right] \tag{19}$$

with density

$$f(x) = \frac{\alpha}{V} \exp\left(\frac{\beta - \alpha}{V} (x - m) \right).$$

If $\alpha < 0$, this is not a valid density and hence an equilibrium does not exist. ∎

Lemma 16 *Suppose $\eta = 0$, $\alpha - \beta \neq 0$, and $\alpha > 0$. Then if a symmetric equilibrium exists, $m = 0$ and the distribution function is of the form in equation (19) with $m = 0$.*

Proof. First, note that since $\eta = 0$, there can be no mass point by Lemma 3. Moreover, the solution to the differential equation takes on the form in equation (19). Since $\alpha > 0$, Lemma 4 implies $m = 0$. ∎

Lemma 17 *Suppose $\eta = 0$, $\alpha - \beta \neq 0$, $\alpha > 0$ and $\beta < 0$. Then there does not exist a symmetric mixed-strategy equilibrium.*

Proof. By Lemma 16, under this parameter configuration $F(x) \leq \alpha / (\alpha - \beta) < 1$ for all $x \geq 0$. Hence, F is not a valid distribution function. ∎

Lemma 18 *Suppose $\eta = 0$, $\alpha - \beta \neq 0$, $\alpha > 0$ and $\beta = 0$. Then there exists a unique symmetric equilibrium and it is in nondegenerate mixed strategies as characterized in Proposition 2.*

Proof. First, note that since $\eta = 0$, there can be no mass point by Lemma 3. Using Lemma 16 and setting $\beta = 0$, yields

$$F(x) = 1 - \exp\left(-\frac{\alpha}{V} x \right) \quad \text{on } [0, \infty).$$

Since this is an exponential distribution, with mean V/α, the expected payoff to a player that bids $w = 0$ against F is

$$
\begin{aligned}
EU(0) &= -\gamma - \theta \int_0^\infty x dF(x) \\
&= -\gamma - \theta \frac{V}{\alpha},
\end{aligned}
$$

and hence, $EU(w) = EU(0)$ for all $w \in [0, \infty)$. Thus, player cannot profitably deviate. ∎

Lemma 19 *Suppose $\eta = 0$, $\alpha - \beta \neq 0$, $\alpha > 0$ and $\beta > 0$. Then there exists a unique symmetric equilibrium and it is in nondegenerate mixed strategies as characterized in Proposition 2.*

Proof. First, note that since $\eta = 0$, there can be no mass point by Lemma 3. By Lemma 16, the distribution must have the form in equation (19) with $m = 0$. Since $\beta > 0$, this is a well-defined distribution function on $\left[0, \frac{V}{\alpha - \beta} \ln \frac{\alpha}{\beta}\right]$ regardless of the sign of $\alpha - \beta$. A player that bids $w = 0$ against F earns an expected payoff of

$$
\begin{aligned}
EU(0) &= -\gamma - \theta \int_0^{\frac{v+\gamma}{\alpha-\beta} \ln \frac{\alpha}{\beta}} w \frac{\alpha}{v+\gamma} e^{-\frac{\alpha-\beta}{v+\gamma} w} dw \\
&= \frac{\theta v + \delta \gamma}{\theta - \delta} + \beta \frac{\theta}{(\theta - \delta)^2} \left[\ln\left(\frac{\alpha}{\beta}\right)\right] V, \tag{20}
\end{aligned}
$$

where we have used the fact that $\eta = 0$ implies $\alpha - \beta = \delta - \theta$. Since $\beta > 0$, a player cannot gain by bidding above the upper bound of the support of F. ∎

Lemma 20 *Suppose $\eta = 0$, $\alpha - \beta \neq 0$, and $\alpha = 0$. Then there does not exist a symmetric equilibrium.*

Proof. Under the conditions stated, differential equation (3) implies

$$
V f(w) - \beta F(w) = 0.
$$

If $\beta < 0$, we have a contradiction, so suppose $\beta > 0$. The unique solution to this differential equation is

$$
F(w) = \exp\left(\frac{\beta}{V} w - Q\right).
$$

Since $\eta = 0$, Lemma 3 implies there are no mass points. This contradicts the fact that $F(w) > 0$ for all $w \in [0, \infty)$. Thus, if $\alpha = 0$ and $\eta = 0$, there does not exist a nondegenerate symmetric mixed-strategy equilibrium when $\beta > 0$ or $\beta < 0$. ∎

Case 4: $\alpha - \beta = 0$; $\eta = 0$

Lemma 21 *Suppose $\eta = 0$ and $\alpha = \beta < 0$. Then there does not exist a symmetric equilibrium.*

Proof. First, note that since $\eta = 0$, there can be no mass point by Lemma 3. Under the conditions stated, differential equation (3) implies

$$V f(w) - \alpha = 0$$

or

$$f(w) = \frac{\alpha}{V}.$$

But since $\alpha < 0$, this is not a well-defined density, a contradiction. ∎

Lemma 22 *Suppose $\eta = 0$ and $\alpha = \beta > 0$. Then there exists a unique symmetric equilibrium and it is in nondegenerate mixed-strategies as characterized in Proposition 2.*

Proof. First, note that since $\eta = 0$, there can be no mass point by Lemma 3. Under the conditions stated, differential equation (3) implies

$$V f(w) - \alpha = 0$$

or

$$f(w) = \frac{\alpha}{V}.$$

Furthermore, since $\alpha > 0$, Lemma 4 implies that $m = 0$. Hence, the unique solution to the differential equation is

$$F(x) = \int_0^x \frac{\alpha}{V} dw = \frac{\alpha}{V} x \text{ on } \left[0, \frac{V}{\alpha}\right]$$

The expected payoff to a player that bids $w = 0$ against F is

$$EU(0) = -\gamma - \theta \frac{V}{2\alpha},$$

and hence, $EU(w) = EU(0)$ for all $w \in \left[0, \frac{V}{\alpha}\right]$. Since $\beta > 0$, it does not pay to bid above the upper bound of the support, as doing so increases costs but not the probability of winning. ∎

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Lemma 23 *Suppose $\eta = 0$, $\alpha = \beta = 0$, and $\theta = \delta \neq 0$. Then there does not exist a symmetric equilibrium.*

Proof. First, note that since $\eta = 0$, there can be no mass point by Lemma 3. Under the conditions stated, differential equation (3) implies

$$V f(w) = 0,$$

or $f(w) = 0$ for all w. This is a contradiction. ■

Taken together, the above lemmas exhaustively describe all mixed-strategy equilibria (and nonexistence) as summarized in Proposition 2.

A3. Proof of Proposition 3

Follows directly from refining the partitions of the parameter space derived in Propositions 1 and 2.