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# When Does Extra Risk Strictly Increase an Option's Value?

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Abstract

It is well known that risk increases the value of options. This paper makes that precise in a new way. The conventional theorem says that the value of an option does not fall if the underlying option becomes riskier in the conventional sense of the mean-preserving spread. This paper uses two new definitions of "riskier" to show that the value of an option strictly increases (a) if the underlying asset becomes "pointwise riskier," and (b) only if the underlying asset becomes "extremum riskier."

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## 1. Introduction

A call option is the right to buy the asset at a strike price,  $P$ . It has been well known at least since Merton (1973) that the value of a call option increases with the riskiness of the underlying asset. If extra risk increases the probability that the market price exceeds  $P$ , then the value of the option increases. A standard finance text says

"The holder of a call option will prefer more variance in the price of the stock to less. The greater the variance, the greater the probability that the stock price will exceed the exercise price, and this is of value to the call holder." (Copeland & Weston, 3rd edition, p. 243)

But this is not quite correct, despite being the sort of thing that even experts say in conversation and in textbooks. As I am sure Thomas Copeland and Fred Weston knew as they wrote this passage, it is quite possible for the risk and variance of the underlying asset to increase while the value of the option remains and does not increase. The value will not fall, but it might remain unchanged. Suppose the strike price is \$50 and the current price of the asset is \$40. If the probability of the price being between \$10 and \$15 or \$45 and \$49 increases while the probability it is between \$38 and \$42 falls, the asset has become riskier, but there is no effect on the option value, because the probabilities of asset values above the strike price of \$50 have not changed.

This too, is well-known but it leaves open the question of what kind of risk does actually increase the value of options. It is false, strictly speaking, to say that additional risk increases the value. On the other hand, it is true but uninteresting to say that additional risk does not reduce the value. A great many variables do not reduce the value of an option, usually because they never affect the value either way. For introductory textbooks no great harm is done in stating a risk-value proposition loosely, but it is worth thinking about how we can come up with a proposition for this basic intuition that is both interesting and true.

One way out is to surrender generality in the kind of asset distribution that we describe. Bliss (2001), noting the problem of coming up with

a rigorous proposition points out that a sufficient condition for option value to increase with risk is that the underlying asset value have a two-parameter distribution such as the normal or lognormal. The relationships between option value and risk, however, clearly hold for much more general distributions (Bliss's attention in that article is about a different problem, also trivially solved by restricting attention to normal distributions (what to do when the underlying assets cannot be ordered using the standard definition of risk).)

The options literature has travelled down the route of studying particular stochastic processes for asset returns (diffusion or jump processes) rather than looking at general distributions for end-states as Merton (1973) did. This began with the lognormal diffusion processes of Fisher Black and Myron Scholes (1973) and continued with such generalizations as John Cox & Stephen Ross (1976) and Merton (1976). More recent entries in the literature include Yaacov Bergman, Bruce G. Huntly & Zvi Wiener (1996) and Masaki Kijima (2002). Other papers look at other considerations absent in the simplest model of one underlying asset, risk-neutral investors, and zero transaction costs. Ravi Jagannathan (1984), for example, looks at values when investors are not risk neutral, and value wealth more in particular states of the world. In such a situation, a riskier asset might not have a higher option value because the option might yield its highest returns in a state of the world when investors are wealthier anyway and hence value the return less. While the "extreme value theory" of, e.g., Chavez-Demoulin & Embrechts (2004) has turned to looking at the effects of unusual events on financial valuation, it is oriented toward estimation of the value of particular assets.

In this article I will return to the original problem of how risk affects option value, but from a different direction. First, we will see that if the underlying asset becomes riskier, then we can at least say that for some strike prices a call option will become more valuable (a very simple result, but worth noting (I will use call options rather than put options throughout, but the proof easily extends to puts)). Second, I will show that only if the underlying asset becomes riskier in the special way I call "extremum riskier"

will every call option will rise in value regardless of the strike price - a necessary condition for a rise in value. Third, I will show that if the underlying asset becomes riskier in the special way I call "pointwise riskier" then every call option will rise in value regardless of the strike price (a sufficient condition for a rise in value).

This paper's main contribution is a tidying up of one of the fundamental ideas in finance theory. In finance modelling it may be useful for those analysts who do not wish to assume normality of asset returns, particularly in real option theory, where option value enters only as part of a larger model of business decision making (see, e.g. Dixit & Pindyck [1994] or section 25.6 of Collier [2001]). The definitions here (and, in particular, the definition of "pointwise risk," may also be useful in other areas of economics. The paper's inspiration was an application in auction theory, to be able to derive a proposition that a bidder would be more willing to pay to acquire information if his initial uncertainty over the value of the object being auctioned was greater (see Rasmusen [2004]). Option value enters as a component with interesting comparative statics in other applications too. As long ago as Arrow & Fisher (1974) the idea was applied to cost-benefit analysis in environmental projects with irreversibility. Search theory is another application see Weitzman (1979) for a classic model in which the value of searches increases with uncertainty, or Varian (1999) for a more recent article. For such models it may be useful to identify assumptions on changes in distributions so that propositions can be found that say when a change in uncertainty strictly increases the payoff<sup>®</sup> from the option-creating action rather than just not reducing the payoff<sup>®</sup>.

## II. The Model

Let there be an asset with terminal value  $x_i$  with probability  $f(x_i)$ , where the values of  $x_i$  with positive probability are  $x_1 < x_2 < \dots < x_m$ . Denote by  $V_{\text{call}}(f; p)$  the current value of a call option on that asset with strike price  $p$  such that  $x_1 < p < x_m$ : This rules out strike prices of  $x_1$  or below and  $x_m$  and above, because they would lead to riskless options which would be exercised always or never. It does allow a strike price that does

not happen to equal any of the  $x_i$ . The call option entitles its owner to buy the asset at price  $p$  at the terminal time if he wishes. We will assume the discount rate is zero and use only two dates, the current date and the terminal date, to avoid distraction by the many issues that would otherwise arise (the date of exercise, diffusion versus jump processes, the time value of money, dividend payments, and so forth). Instead, our focus is on seeing how the option value would change if the underlying asset followed a different distribution  $g(x)$  which has the same mean as  $f(x)$ , so

$$E_x = \sum_{i=1}^n f(x_i)x_i = \sum_{i=1}^n g(x_i)x_i + \sum_{i=m+1}^n g(x_i)x_i; \quad (1)$$

where  $x_{m+1} < x_{m+2} < \dots < x_n$  are points in the support of  $g$  but not  $f$ . This allows for example,  $x_{m+1} < x_1$ , which says that  $g$  can have positive probability on  $x$  values less than or greater than the support of  $f(x)$ , or on values between  $x$ 's in  $f(x)$ 's support. Let us denote the cumulative distributions by  $F(x)$  and  $G(x)$ .

The value of a call option with strike price  $p$  is

$$\begin{aligned} V_{\text{call}}(f; p) &= \sum_{i=1}^n f(x_i) \max\{x_i - p, 0\} \\ &= \sum_{i=j}^n f(x_i)(x_i - p) \text{ where } j : x_{j-1} < p < x_j \end{aligned} \quad (2)$$

### Defining Risk

The standard definition of risk is based on the idea of the "mean preserving spread," which we can define as follows

Definition 1a: A mean-preserving spread consists of three numbers  $s(y_1)$ ,  $s(y_2)$ , and  $s(y_3)$  for  $y_1 < y_2 < y_3$  such that

$$s(y_1)y_1 + s(y_2)y_2 + s(y_3)y_3 = 0; \text{ (the mean is preserved)} \quad (3)$$

$$s(y_1) + s(y_2) + s(y_3) = 0; \text{ (the new probabilities sum to zero)} \quad (4)$$

and

$$s(y_1) > 0; s(y_2) < 0; s(y_3) > 0 \text{ (the probability is spread)} \quad (5)$$

Definition 1a is specialized to discrete probability distributions and it uses the idea of the "3-point mean preserving spread," developed in Petrakis & Rasmussen (1994) rather than the conventional "4-point mean preserving spread" of Rothschild & Stiglitz (1970), which has negative probability at two middle points rather than one. The two definitions of spread lead to equivalent definitions of risk. Definition 1b below orders distributions by risk identically whichever definition of spread is used. The 3-point definition is simpler and will lead to less clutter in proofs (as well as allowing an easy fix of the error in the main proof in Rothschild & Stiglitz [1970]). Note that Definition 1a does not require that the  $y_i$  equal any  $x_i$ : the spread can put positive probability on asset values which originally have zero probability. Formally, a spread added to  $f(x)$  also could result in probabilities that are negative or greater than one, but nobody would want to use such a spread.

Thus we arrive at Definition 1b, the definition of risk originated in Rothschild & Stiglitz (1970) (with earlier suggestions in Hadar & Russell [1969] and Haroch & Levy [1969]).

Definition 1b: Distribution  $g(x)$  is riskier than  $f(x)$  if  $g(x)$  can be reached from  $f(x)$  by a sequence of mean preserving spreads

This definition of risk has long been conventional, since it is equivalent to saying that the asset becomes less attractive to a risk-averse investor (one with a concave utility function) or that  $f$  is like  $g$  with noise added, although Definition 1b is only a partial ordering and many pairs of distributions cannot be ranked by it. In the option context, Bliss (2001) shows the importance of using Definition 1b instead of defining risk as simply higher variance, which is not an equivalent definition. Variance can increase without making an asset less attractive to a risk-averse investor, and option values

do not change in a uniform direction with changes in variance.<sup>1</sup>

It is perhaps worth reminding the reader of another statement of risk: in terms of stochastic dominance. Distribution  $F(x)$  "first-order stochastically dominates" distribution  $G(x)$  if  $F(t) \leq G(t)$  for all  $t$ , i.e., if  $G$  puts more probability on lower values of  $x$  than  $F$  does. Distribution  $F$  "second-order stochastically dominates" distribution  $G$  if  $\int_0^t F(x)dx \leq \int_0^t G(x)dx$  for all  $t$  with the inequality strict for at least one value of  $t$ .<sup>2</sup> A definition of risk that is equivalent to Definition 1b is that distribution  $G(x)$  is riskier than  $F(x)$  if  $F(x)$  second-order stochastically dominates  $G(x)$ . We will use densities rather than cumulative distributions in this article, however, because densities are easier to visualize and understand.

### Option Value Does Not Decline with Risk

The fundamental proposition in the theory of risk and options is the well-known Proposition 1: option value is weakly increasing in risk.

Proposition 1 (Merton [1970] Theorem 8, p. 149): If  $g$  is riskier than  $f$ , then  $V_{call}(f; p) \leq V_{call}(g; p)$  for any  $p$ .

<sup>1</sup> A nice example to show that increased variance can increase utility for a risk-averse person is the following. Let the utility be  $U = x$  for  $x < 10$ ,  $U = 10 + x - 2$  for  $x \geq 10$ , which is weakly concave. Suppose wealth is initially distributed as  $f = (.8 - .7, .2 + .12)$ , which has mean 8, variance 4 ( $= .8 \sigma^2 + .2 \sigma^2$ ), and utility 7.8 ( $= .8 \sigma^7 + .2 \sigma^11$ ). If the distribution is changed to  $g = (.2 - .0, .8 + .10)$ , the mean is still 8, the variance increases to 16 ( $= .2 \sigma^8^2 + .8 \sigma^2^2$ ), and utility rises to 8 ( $= .2 \sigma^0 + .8 \sigma^10$ ). Kurtosis, which increases when moving weight to the tails of the distribution, is equally unreliable for ranking the riskiness of distributions; it starts as 52 ( $= .8 \sigma^1^4 + .2 \sigma^4^4$ ) in this example and rises to 832 ( $= .2 \sigma^8^4 + .8 \sigma^2^4$ ). Note, too, that option value can rise with variance but does not necessarily do so: in this example,  $V_{call}(f; .11) = .2(12 - 11) = .2$  but  $V_{call}(g; .11) = 0$ .

<sup>2</sup> There is some scope for ambiguity here in whether the inequalities are weak or strong. Three levels that might be defined are "weak stochastic dominance," in which it is possible that the inequality is an equality for all  $t$ , so  $F$  and  $G$  are identical; "semiweak stochastic dominance," in which the inequality must be strict for at least one value of  $t$ ; and "strict stochastic dominance," in which the inequality must be strict for all values of  $t$ . Weak and semiweak stochastic dominance are what are standardly used in economic theorems. See, too, footnote 4 below.

Proof: From (2), the value of the call on the less risky asset,  $f$ , is

$$V_{\text{call}}(f;p) = \sum_{i=j}^{\infty} f(x_i)(x_i - p) \text{ where } j : x_{j-1} < p < x_j \quad (6)$$

and the value of the call on the riskier asset,  $g$ , is

$$V_{\text{call}}(g;p) = f(x_i)(x_i - p) + s(y_1)M \max(y_1 - p; 0) + s(y_2)M \max(y_2 - p; 0) + s(y_3)M \max(y_3 - p; 0); \quad (7)$$

If

$$0 < s(y_1)M \max(y_1 - p; 0) + s(y_2)M \max(y_2 - p; 0) + s(y_3)M \max(y_3 - p; 0); \quad (8)$$

then Proposition 1 is incorrect.

From definition equation (3), the spread is mean preserving so  $s(y_1)y_1 + s(y_2)y_2 + s(y_3)y_3 = 0$ , and by equation (4) the spread's probabilities add to zero, so  $[s(y_1) + s(y_2) + s(y_3)] = 0$ . Together, these imply that

$$s(y_1)(y_1 - p) + s(y_2)(y_2 - p) + s(y_3)(y_3 - p) = s(y_1)y_1 + s(y_2)y_2 + s(y_3)y_3 - p[s(y_1) + s(y_2) + s(y_3)] = 0; \quad (9)$$

a result that will be used below.

(i) Suppose  $p < y_1$ , so inequality (8) becomes

$$0 < s(y_1)(y_1 - p) + s(y_2)(y_2 - p) + s(y_3)(y_3 - p); \quad (10)$$

Equation (9) tells us that this is true as an equality.

(ii) Suppose  $p > y_3$ , so inequality (8) becomes

$$0 < s(y_1)(0) + s(y_2)(0) + s(y_3)(0); \quad (11)$$

This is obviously true as an equality.



(iii) Suppose that  $p \geq (y_1; y_3)$ . Then since  $M_{ax}(y_1; p; 0) = 0$  and  $M_{ax}(y_3; p; 0) = y_3 - p$ , we can rewrite expression (7) as

$$0 \cdot 0 + s(y_2)M_{ax}(y_2; p; 0) + s(y_3)(y_3 - p) \quad (12)$$

(a) If  $M_{ax}(y_2; p; 0) = 0$ , then inequality (12) is true as a strict inequality, since  $s(y_3) > 0$  and  $y_3 > p$ .

(b) If  $M_{ax}(y_2; p; 0) = y_2 - p$ , then inequality (12) is true if

$$s(y_2)(y_2 - p) + s(y_3)(y_3 - p) \geq 0 \quad (13)$$

Equation (9) tells us that  $s(y_1)(y_1 - p) + s(y_2)(y_2 - p) + s(y_3)(y_3 - p) = 0$ , so since  $s(y_1) > 0$  and, in case (iii),  $(y_1 - p) < 0$ , it follows that (12) is true as a strict inequality. Q. E. D.

Compare Proposition 1 with Proposition 1a, which differs only in the strength of the inequality.

Proposition 1a (false): If  $g$  is riskier than  $f$ , then  $V_{call}(f; p) < V_{call}(g; p)$  for any strike price  $p$ .

Disproof. Consider a call option with an exercise price of 4.5 and the asset price distribution shown in Figure 1.  $V_{call}(f; 4.5) = V_{call}(g; 4.5)$ , even though  $g$  is riskier than  $f$ . The increase in risk has no effect because only changes in the probabilities of terminal values greater than 4.5 would matter to the value of the call, and there are no such changes in the example.

### Figure 1: A Counterexample: Risk Does Not Increase Option Value

Propositions 1 and 1a differ only in the weakness of the inequality. That is enough, however, for  $\forall$  Proposition 1a: 'Option value increases with risk' to be false. Instead, we are left with  $\forall$  Proposition 1: 'Option value does not fall with risk,' which although true, is very weak. That kind of statement can be made of any variable outside the model: 'Option value does not fall with wealth,' or 'Option value does not fall with unemployment,' or 'Option value does not fall with the temperature in Bloomington'

The statement 'Option value does not fall with risk,' however, though it does not state the mathematical notation of Proposition 1, is unnecessarily weak. We can instead say that 'Option value does not fall with risk, and for at least one value of the strike price it increases' Proposition 1b expresses this in mathematical notation

Proposition 1b: If  $g$  is riskier than  $f$ , then there exists some exercise price  $p^0$  such that the associated call option is more valuable under  $g$  than under  $f$  but no exercise price  $p^0$  such that a call option is more valuable under  $f$ :

$$p^0: V_{\text{call}}(f; p^0) < V_{\text{call}}(g; p^0);$$

but

$$p^0: V_{\text{call}}(f; p^0) > V_{\text{call}}(g; p^0).$$

Proof:

The proof of Proposition 1 showed that if  $p \in (y_1; y_3)$ , then the value of the call strictly increases. Thus, simply pick  $p^0$  in  $(y_1; y_3)$  for one of the spreads that makes  $g$  riskier than  $f$ .

That there exists no value  $p^0$  for which option value declines is a direct corollary of Proposition 1. Q.E.D.

## 11b. New Definitions of Risk

Another approach is to find a definition of risk under which something like Proposition 1b is true, and the value of the option does increase with 'risk' regardless of the strike price.

Definition 2 (new): Distribution  $g(x)$  is pointwise riskier than  $f(x)$  if  $f$  and  $g$  have the same mean and there exist points  $\underline{x}$  and  $\bar{x}$  in  $(x_1; x_m)$  such that

(a) if  $x < \underline{x}$ , then  $g(x) \leq f(x)$  and if  $f(x) > 0$  then  $g(x) > f(x)$ ;

(b) if  $x \in [\underline{x}; \bar{x}]$ , then  $g(x) \leq f(x)$  and if  $f(x) > 0$  then  $g(x) < f(x)$ ;

(c) if  $x > \bar{x}$ , then  $g(x) \leq f(x)$  and if  $f(x) > 0$  then  $g(x) > f(x)$ .

Definition 2 says that  $g(x)$  is pointwise riskier than  $f(x)$  if it takes probability away from each point in the middle of the distribution and adds probability to each point at the two ends while preserving the mean. Distribution  $g_1(x)$  in Figure 2 is an example. Definition 2 also allows  $g(x)$  to add probability to points outside the interval  $[x_1; x_m]$  that is beyond the two extremes of the support of  $f(x)$ . Pointwise riskiness captures something of the same intuition as the idea of the mean-preserving spread: that probability is to be moved from the middle to the ends of the distribution. If  $g$  is pointwise riskier than  $f$ , it is also riskier in the conventional sense. Note

that the change from  $f$  to  $g$  need not be symmetric around the mean of the distribution, nor uniform even within the  $x_i$  each of the three regions  $[j-1; \bar{x}]$ ,  $[\bar{x}; \bar{x}]$ ,  $[\bar{x}; 1]$ . The pointwise riskier distribution might, for example, begin with a uniform  $f$  and then add .05 to  $f(x_1)$ , .01 to  $f(x_2)$ , and .20 to  $f(x_3)$ , reduce probability on  $f$  over  $[x_4; x_9]$ , and then increase probability again over  $[x_{10}; x_{20}]$ , so long as it preserves the mean of  $f$ . Every point on the "sides" must gain probability, but not necessarily the same amount of probability, nor must more extreme points gain more probability than less extreme ones<sup>3</sup>

Figure 2: Pointwise and Extremum Riskiness

Pointwise riskiness will be sufficient but not necessary for option value to increase with risk for all strike prices, as we will see in Proposition 5 once we have derived other results useful in proving sufficiency.<sup>4</sup> Distribution  $g_2(x)$

<sup>3</sup>The definition of pointwise riskiness can be modified easily for continuous densities  $f$  and  $g$  with convex and bounded support (that is, with an interval as a support with no gaps at which the definition would require  $g$  to have negative density).

<sup>4</sup>Since pointwise riskiness and second-order stochastic dominance both can be defined in terms of functions that cross a limited number of times, the reader may wonder if pointwise riskiness is the same as the strict second-order stochastic dominance of footnote

in Figure 2 is an example in which  $g$  is not pointwise riskier than  $f$ , but  $V_{\text{call}}(f; p) < V_{\text{call}}(g; p)$  nonetheless for all  $p$ .

If  $p$  is fixed,  $g(x)$  does not even have to be a mean preserving spread to increase the value of the call. But we are asking what changes to the asset distribution will increase the value of any call written on the asset.

Our other new definition of risk is one which is necessary for extra risk to increase option value: extremum risk. The definition of extremum riskiness would need modification for bounded continuous distributions. A definition in terms of cumulative distributions is then more convenient, if less intuitive:  $g$  is extremum-riskier than  $f$  if  $G(x_1 + \epsilon) > F(x_1 + \epsilon)$  and  $1 - G(x_m - \epsilon) < 1 - F(x_m - \epsilon)$  for arbitrarily small  $\epsilon$ . Cumulative distributions must be used because if  $f$  is a continuous density then each of the extrema has zero probability, even if positive density, and to change the value of an option it is necessary to change probabilities over an interval of  $f$ 's support, not just over one point. Thus  $g$  must put more probability on the intervals  $[x_1 - \epsilon; x_1 + \epsilon]$  and  $[x_m - \epsilon; x_m + \epsilon]$ .

Definition 3 (new): Distribution  $g(x)$  is extremum riskier than  $f(x)$  if

- (a) either  $f(x_1) < g(x_1)$ , or  $g(x) > 0$  for some  $x < x_1$ ;
- and
- (b) either  $f(x_m) < g(x_m)$ , or  $g(x) > 0$  for some  $x > x_m$ .

The distribution in part (c) of Figure 2 is extremum-riskier than the distributions in parts (a) and (b). The distribution in part (b) of Figure 3 is extremum-riskier than the distribution in part (a).

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2. It is not Distribution  $F$  strictly second-order stochastically dominates  $G$  if  $\int_0^t G(x) dx > \int_0^t F(x) dx$  for all values of  $t$  such that  $G(t) > 0$  and  $G(t) < 1$ . It could be, however, that  $G$  is pointwise riskier than  $F$  without  $F$  strictly second-order dominating  $G$ . Suppose, for example, that  $F$  is uniform, with  $F(1) = .25; F(2) = .5; F(3) = .75; F(4) = 1$  and  $G$  moves weight from the middle to the tails and is pointwise riskier so  $G(1) = .30; G(2) = .5; G(3) = .7; G(4) = 1$ . If we define  $D_F(t) = \int_0^t F(x) dx$  (and similarly for  $G$ ) then  $D_F(1) = .25; D_F(2) = .75; D_F(3) = 1.5$  and  $D_G(1) = .30; D_G(2) = .8; D_G(3) = 1.5$ . Since  $D_F(3) = D_G(3)$ ,  $F$  does not strictly dominate  $G$ .  $F$  does weakly dominate  $G$ , as we would expect since  $G$  is riskier in the conventional sense.

Figure 3: Extremum Risk versus Risk

Proposition 2: Consider two distributions  $f$  and  $g$ . A necessary condition for it to be true that  $V_{\text{call}}(g; p) > V_{\text{call}}(f; p)$  for any strike price  $p$  is that  $g$  be extremum-riskier than  $f$ .

Proof:

Hold  $p$  fixed. If  $f(x)$  and  $g(x)$  are identical for all  $x > p$  or for all  $x < p$ , then  $V_{\text{call}}(g; p) = V_{\text{call}}(f; p)$ .

(i) We will start with condition (b) in Definition 3. If  $f$  and  $g$  are identical for all  $x > p$ , then clearly the call value of equation (2) must be equal for  $f$  and  $g$ , since then  $f$  and  $g$  are identical for  $x_{j-1} < x_j$ , and only such values of  $x$  enter into equation (2), reproduced below.

$$V_{\text{call}}(f; p) = \sum_{i=j}^{\infty} f(x_i)(x_i - p) \text{ where } j : x_{j-1} < p < x_j \quad (14)$$

Thus for  $V_{\text{call}}(g; p) > V_{\text{call}}(f; p)$  to be true, it is necessary that  $g(x) > f(x)$  for some  $x > p$ .

Since this is true for every  $p$ , it must be true for  $p = x_{m+1} - \epsilon$ , for any small value  $\epsilon$ . Thus for some  $x > x_m$ ,  $g(x) > f(x)$ . But this can be true only if either  $g(x_m) > f(x_m)$ , or if  $g(x) > 0$  for some  $x > x_m$ . That is condition (b) in Definition 3.

(ii) We need

$$V_{\text{call}}(g; p) = \sum_{i=j}^{\infty} g(x_i)(x_i - p) + \sum_{i=k}^{\infty} g(x_i)(x_i - p) > \sum_{i=j}^{\infty} f(x_i)(x_i - p) = V_{\text{call}}(f; p) \quad (15)$$

where  $j : x_{j-1} < p < x_j$ ;  $k : x_{k-1} < p < x_k$

The distributions  $f$  and  $g$  have equal means so

$$E_g(x) - p = \sum_{i=1}^{\infty} g(x_i)(x_i - p) + \sum_{i=m+1}^{\infty} g(x_i)(x_i - p) = \sum_{i=1}^{\infty} f(x_i)(x_i - p) = E_f(x) - p \quad (16)$$

where  $j : x_{j-1} < p < x_j$ ;  $k : x_{k-1} < p < x_k$

Subtracting we know that

$$E_g(x) - p - V_{\text{call}}(g; p) < E_f(x) - p - V_{\text{call}}(f; p) \quad (17)$$

so

$$\sum_{i=1}^{\infty} g(x_i)(x_i - p) + \sum_{i=m+1}^{\infty} g(x_i)(x_i - p) < \sum_{i=1}^{\infty} f(x_i)(x_i - p) \quad (18)$$

where  $j : x_{j-1} < p < x_j$ ;  $k : x_{k-1} < p < x_k$

Now set  $p = x_1 + \epsilon^2$  for some small number  $\epsilon^2$ . Inequality (18) becomes

$$g(x_1)(x_{j-1} - p) + \sum_{i=m+1}^{x_1} g(x_i)(x_i - p) < f(x_1)(x_{j-1} - p) \quad (19)$$

where  $j : x_{j-1} < p < x_j$ ;  $k : x_{k-1} < p < x_k$ .

Both sides of this inequality are negative, since  $x < p$  over the range of  $x_i$  it includes so the inequality implies that either  $g(x_1) > f(x_1)$  or that  $g(x) > 0$  for some  $x < x_1$ . That is condition (a) in Definition 3.

Thus for the value of the call to be greater under  $g(x)$  for all  $p$ , it is necessary that  $g$  satisfy the conditions for being extremum riskier than  $f$ . Q.E.D.

We can now better discuss one part of the definition of extremum riskiness that may have puzzled the reader: why does it require that  $g$  add probability at both extremes not just the maximum? The answer is not that the definition has been crafted for application to puts as well as to calls; it turns out that even as a necessary condition for calls to become more valuable with extremum riskiness, we need probability to increase at both extremes. This was of course, needed in the proof of Proposition 2, but a numerical example will aid intuition.

The example, shown in Figure 4, in which  $g$  is made riskier than  $f$  by shifting probability away from  $x = 2$ , the mean to  $x = 1.33$  and  $x = 4.67$ . So doing makes  $g$  put more probability than  $f$  on the upper extremum of  $x = 4.67$ , but no more probability on the lower extremum of  $x = 0$ .

A call with a strike price above 1.33 will increase in value as a result of changing the distribution from  $f$  to  $g$ . But think about a call with a strike price of 1. It will have equal value under  $f$  and  $g$ , because the mean of the distribution conditional on  $x$  being greater than 1 has not changed. Or, looked at a bit differently, the probability of the state of the world ( $x = 0$ ) in which the call is not exercised has not changed at all.



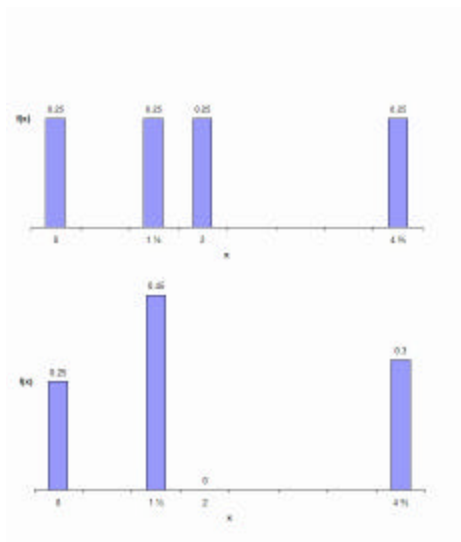


Figure 4: Why Extremum Risk Needs Spread at Both Extrema

The general problem is that unless both extrema are increasing, it is impossible to find a strike price such that the total amount of probability on prices above the strike price is unchanged. If the lower extremum does not increase, as in Figure 4's example, then choose the strike price to be very low, just above the extremum. The call is then like a bet that the price will exceed the lower extremum, and the probability of winning that bet is the same in  $f$  and  $g$ . If, on the other hand, the upper extremum does not increase, choose the strike price to be very high, just below the extremum.

Why is this just a necessary condition and not sufficient? Look back at Figure 3. In Figure 3,  $g(x)$  has more probability at the extremes than  $f(x)$  does (the probability of each extreme is .25 instead of .20) but it is not riskier in the conventional sense, because it cannot be reached from  $f(x)$  by a sequence of mean-preserving spreads. If the strike price is 4.5, then the call's value is higher under distribution  $g(x)$ , because the outcome  $x = 5$  occurs with probability .25 instead of  $f(x)$ 's .20.  $V_{\text{call}}(f; 4.5) = .20(5 - 4.5) = .10 < V_{\text{call}}(g; 4.5) = .25(5 - 4.5) = .125$ . If the strike price is 3.5, however,

the call's value is higher under distribution  $f(x)$ , because under  $g(x)$  the outcomes  $x = 4$  and  $x = 5$  together occur with probability .25 instead of .40 and  $V_{\text{call}}(f; 3.5) = .20(4 - 3.5) + .20(5 - 3.5) = .40 > V_{\text{call}}(g; 3.5) = .00(4 - 3.5) + .25(5 - 3.5) = .375$ .

Extremum riskiness already implies that  $g$  is not less risky than  $f$ , since more weight is in the far tail of the distribution, but it might be that  $f$  and  $g$  are not ordered by risk. Although neither conventional nor extremum riskiness is by itself sufficient to make calls more valuable, in combination they do yield a sufficient condition as stated in Proposition 3.

**Proposition 3:** Consider two distributions  $f$  and  $g$ . A sufficient condition for it to be true that  $V_{\text{call}}(g; p) > V_{\text{call}}(f; p)$  for any strike price  $p$  is that  
 (a)  $g$  is extremum-riskier than  $f$ ; and  
 (b)  $g$  is riskier than  $f$ .

**Proof:**

From Proposition 1 we know that if condition (b) is true, then  $V_{\text{call}}(g; p) > V_{\text{call}}(f; p)$ , that is Proposition 3's inequality is true at least weakly. Thus all that we need show is that condition (a) makes the inequality strict.

The proof of Proposition 1 showed that if a mean-preserving spread that made  $g$  riskier than  $f$  changed probability on three points  $y_1 < y_2 < y_3$ , then if the option's strike price were  $p < y_1$  or  $p > y_3$ , the option's value would be the same under  $f$  as under  $g$ .

Since  $g$  may be derived from  $f$  by a series of mean-preserving spreads, let  $y_1^m$  be the lowest  $x$  value that is changed, and  $y_3^m$  the highest. If the option's strike price were  $p < y_1^m$  or  $p > y_3^m$ , the option's value would be the same under  $f$  as under  $g$ . That is the possibility we are trying to rule out. But condition (a) says that  $g$  is extremum riskier. That implies that the probability of  $x_i$  less than or equal to  $x_1$  increases so  $y_1^m < x_1$ , and that the probability of  $x_i$  greater than or equal to  $x_m$  increases so  $y_3^m > x_m$ . Thus it is impossible (since we rule out the riskless options with  $p = x_1$  or  $p = x_m$ ) that  $p < y_1^m$  or  $p > y_3^m$ . As a result, the option values cannot be equal for any

and it must be that  $V_{\text{call}}(f;p) < V_{\text{call}}(g;p)$ . Q.E.D.

You might ask why I did not write Proposition 3 to say that conditions (a) and (b) are jointly necessary and sufficient, rather than just sufficient. If options on  $g$  are to be always more valuable than options on  $f$ , is it necessary that  $g$  be both riskier and extremum-riskier than  $f$ ? No.

Proposition 4: Consider two distributions  $f$  and  $g$ . The following two conditions are not necessary and sufficient for it to be true that  $V_{\text{call}}(f;p) < V_{\text{call}}(g;p)$  for any strike price  $p$ :

- (a)  $g$  is extremum-riskier than  $f$ ; and
- (b)  $g$  is riskier than  $f$ .

Proof: Conditions (a) and (b) are jointly sufficient, as Proposition 3 says Condition (a) by itself is necessary, as Proposition 2 says. Thus what we need to show to prove Proposition 4 is that there exist distributions  $f$  and  $g$  such that Condition (b) is violated but nonetheless  $V_{\text{call}}(f;p) < V_{\text{call}}(g;p)$  for any  $p$ . That is we must show that  $g$ 's options are always more valuable, but  $g$  is not riskier than  $f$ .

Consider the example in Figure 5. Distribution  $g$  is extremum-riskier than distribution  $f$ , but it is not riskier, because it has more probability at the mean  $x = 5$  (in fact,  $f(5) = 0$ ). The distributions  $f$  and  $g$  cannot be ordered by risk.

The value of a call option on an asset with density  $f$  and strike price  $p \in (2;8)$  is from equation (2),

$$V_{\text{call}}(f;p) = M \int_0^{2-p} f(x) dx + M \int_{2-p}^{4-p} (x-p) f(x) dx + M \int_{4-p}^{6-p} (x-p) f(x) dx + M \int_{6-p}^8 (x-p) f(x) dx \quad (20)$$

and the value of a call option on an asset with density  $g$  and strike price  $p$  is

$$V_{\text{call}}(g;p) = M \int_0^1 g(x) dx + M \int_1^{5-p} (x-p) g(x) dx + M \int_{5-p}^9 (x-p) g(x) dx \quad (21)$$

The possible values of  $p$  go from  $p = 2$  to  $p = 8$ , where the endpoints are not possible (as the option would then be always or never exercised). We will split this up into four intervals and examine each in turn.

Lemma 1:  $V_{\text{call}}(f; p) < V_{\text{call}}(g; p)$  for  $p \in (2; 4]$ :

Proof: Then  $V_{\text{call}}(f; p) = .25(4 - p) + .25(6 - p) + .25(8 - p) = .25(18 - 3p) = 4.5 - .75p$ . On the other hand,  $V_{\text{call}}(g; p) = .40(5 - p) + .30(9 - p) = 4.7 - .70p$ , which is greater than  $4.5 - .75p$ . Thus  $g$  has the more valuable options.

Lemma 2:  $V_{\text{call}}(f; p) < V_{\text{call}}(g; p)$  for  $p \in (4; 5]$ :

Proof: Then  $V_{\text{call}}(f; p) = .25(6 - p) + .25(8 - p) = 3.5 - .5p$ . On the other hand,  $V_{\text{call}}(g; p) = .40(5 - p) + .30(9 - p) = 4.7 - .70p$ . It is true that  $3.5 - .5p < 4.7 - .70p$  if  $.20p < 1.2$ , which is true if  $p < 6$ , and in particular if  $p \in (4; 5]$ . Thus  $g$  has the more valuable options.

Lemma 3:  $V_{\text{call}}(f; p) < V_{\text{call}}(g; p)$  for  $p \in (5; 6]$ :

Proof: Then  $V_{\text{call}}(f; p) = .25(6 - p) = 3.5 - .5p$ . On the other hand,  $V_{\text{call}}(g; p) = .30(9 - p) = 2.7 - .30p$ . It is true that  $3.5 - .5p < 2.7 - .30p$  if  $.8 < .2p$ , which is true if  $p > 4$ , and in particular if  $p \in (5; 6]$ . Thus  $g$  has the more valuable options.

Lemma 4:  $V_{\text{call}}(f; p) < V_{\text{call}}(g; p)$  for  $p \in (6; 8)$ :

Proof: Then  $V_{\text{call}}(f; p) = .25(8 - p) = 2 - .25p$ . On the other hand,  $V_{\text{call}}(g; p) = .30(9 - p) = 2.7 - .30p$ . It is true that  $2 - .25p < 2.7 - .30p$  if  $.7 < .05p$ , which is true if  $p > 14$ , and in particular is true if  $p \in (6; 8)$ . Thus  $g$  has the more valuable options.

Combining all four cases, we see that for any  $p \in (2; 8)$ ,  $g$  has more valuable options. Q. E. D.

To understand Proposition 4, start with the simpler idea that an option with price  $p$  can be more valuable under distribution  $g$  even if  $g$  is not riskier than  $f$ . That is true because for some particular  $p$ , the call's value is  $\sum_{i=j}^m f(x_i)(x_i - p)$  for  $j : x_{j-1} < p < x_j$ , which depends on all of the  $f$

distribution for every  $x_i > p$  but not on every  $x_i$  individually. Thus it is possible that  $g(x_k) < f(x_k)$  for some particular value of  $x_k > p$  in a way that makes it impossible to rank  $f$  and  $g$  by risk, but for that to be outweighed by  $g$ 's greater weight on most high values of  $x_i$ . Proposition 4 generalizes this to the idea that an option can be more valuable for any price  $p$  even though risk does not rise. The reason is that we can find a  $g$  that puts enough weight on its extrema compared to  $f$  that  $g$ 's expected values over  $x_i > p$  will be greater even if it puts more weight on the mean of  $x$  too.

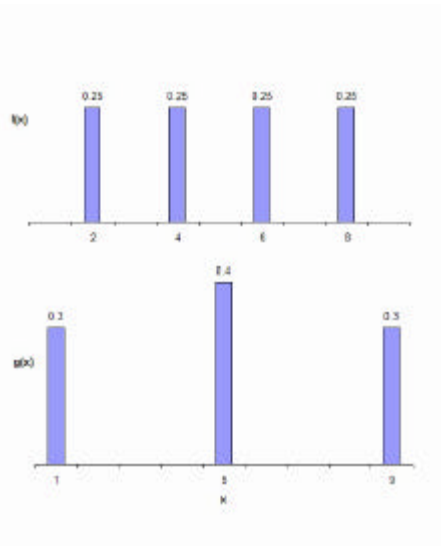


Figure 5: Why Riskiness and Extremum Riskiness Are Not Necessary for All Options To Increase in Value

Finally, let us leave extremum riskiness and look back to the second new definition of "riskier": pointwise riskiness. Pointwise riskiness

For many applications it is convenient to specify a simple sufficient condition for one option to be riskier than another. Indeed, my first motivation for this paper was to identify such a sufficient condition in the context of information acquisition during an auction (see Rasmussen [2004]). Pointwise riskiness is a sufficient condition that is simple and often plausible. Having already proved Proposition 3, it is easy to prove this

Proposition 5: If  $g$  is pointwise riskier than  $f$ , then for any  $p$ ,  $V_{\text{call}}(f; p) < V_{\text{call}}(g; p)$ .

Proof: If  $g$  is pointwise riskier than  $f$ , then it is also riskier and extremum riskier. It is riskier because we can move from  $f$  to  $g$  by a series of mean preserving spreads that take probability away from the middle interval  $[\underline{x}; \bar{x}]$  and move it to the extremes. It is extremum riskier because  $x_1 < \underline{x}$  and  $x_m > \bar{x}$ , so  $g$  puts more probability on  $x_1$  and  $x_m$  than  $f$  does. It follows from Proposition 3, proved above, that a call on  $g$  will be more valuable than a call on  $f$ . Q.E.D.

We have, of course, already found one sufficient condition for an option on  $g$  to be more valuable than an option on  $f$ . Proposition 3 said that riskiness plus extremum riskiness provides a sufficient condition. Proposition 5, in fact, is a tighter sufficient condition. If  $g$  is pointwise riskier than  $f$  it is always both riskier and extremum riskier, but  $g$  can be riskier and extremum riskier than  $f$  without being pointwise riskier. Nonetheless, pointwise riskiness is a useful concept, because it is simpler and more intuitive than standard plus extremum riskiness.

Propositions similar to Propositions 2, 3, 4, and 5 are easy to derive for put options as well as for call options. The propositions do not extend to exotic options that convey purchase or sale rights over ranges of prices that do not slice the real line into two (e.g., the right to buy if the price is either in the interval  $[3, 5.6]$  or in  $[7, 26]$ ). Neither the intuition nor the rigorous propositions extend to that kind of option, since an exotic option such as in my parenthetical example can increase in value when probability shifts from the extremes to the middle, a reduction in risk.

#### IV. Concluding Remarks

If distribution  $g$  is riskier than distribution  $f$ , then any call option on an asset whose value has distribution  $g$  will be at least as valuable as the equivalent option on an asset with distribution  $f$ . But the option on  $g$  might

not be more valuable, because the values might be equal. This paper has developed a necessary condition for all call options on an asset whose value has distribution  $g$  to be strictly more valuable than the equivalent option on an asset with distribution  $f$ , and two sufficient conditions for it, differing in strength and convenience. The necessary condition is that  $g$  be "extremum riskier": it must put more probability on the extreme values of the asset. One sufficient condition is that  $g$  be not only extremum riskier, but also riskier under the conventional definition of risk (that  $g$  can be reached from  $f$  by a series of mean-preserving spreads). A second sufficient condition, more restrictive but simpler, is that  $g$  be "pointwise riskier": asset values in the middle of  $g$  have higher probability than under  $f$ , and asset values outside the middle have lower probability.

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