Marginal Contributions and Externalities in the Value*

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Abstract

Our concern is the extension of the theory of the Shapley value to problems involving externalities. Using the standard axiom systems behind the Shapley value for an arbitrary exogenous coalition structure leads to the identification of bounds on players' payoffs around an "externality-free" value. In endogenizing the coalition structure, we analyze a two-stage process of coalition formation in whose second stage our axiomatic results are applied. We find reasons to explain inefficient coalition structures, and provide sufficient conditions for efficiency.

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1 Introduction

Since the path-breaking work of Shapley (1953), much effort has been devoted to the problem of "fair" distribution of the surplus generated by a collection of people that are willing to cooperate with one another. More recently, the same question has been posed in the realistic case where externalities across coalitions are present. This is the general problem to which this paper contributes. The presence of such externalities is an important feature in many applications. In an oligopolistic market, the profit of a cartel

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depends on the level of cooperation among the competing firms. The power of a political alliance depends on the level of coordination among competing parties. The benefit of an agent that refuses to participate in the production of a public good depends on the level of cooperation of the other agents (free-riding effect), and so on.

In the absence of externalities, Shapley (1953) obtained a remarkable *uniqueness* result. He characterized a unique solution using the axioms of efficiency, anonymity, additivity and null player. Today we refer to this solution as the Shapley value, which happens to be calculated as the average of marginal contributions of players to coalitions. This comes as a surprise at first glance: uniqueness is the consequence of four basic axioms, and nothing in those axioms hints at the marginality principle, of long tradition in economic theory. In the clarification of this puzzle, Young (1985) provides a key piece. He formulates the marginality principle as an axiom, i.e., that the solution should be a function of players' marginal contributions to coalitions. He drops additivity and null player as requirements. The result is that the only solution satisfying efficiency, anonymity and marginality is again the Shapley value.

In our extension of the theory of the Shapley value to settings with externalities, we shall pursue two approaches: axiomatic and strategic. In our axiomatic analysis, we will find that appealing systems of basic axioms that were used in problems with no externalities do not suffice to yield a unique solution. Thus, *multiplicity* of solutions seems essential to the problem at hand (this is confirmed by previous studies, in which authors must resort to new additional axioms to get uniqueness). Despite the multiple solutions, the novelty of our approach is to provide refined predictions based on payoff bounds implied by the axioms. In our strategic approach, we shall attempt to endogenize the coalition structure and turn to coalition formation questions.

In order to tackle the question in axiomatic terms, we sort out the effects of *intrinsic marginal contributions* of players to coalitions from those coming from *externalities*. The model we shall employ is that of partition functions, in which the worth of a coalition S may vary with how the players not in S cooperate. In the model, $v(S,\Pi)$ is the worth of S when the coalition structure is Π, S being an element of Π. To define player i's marginal contribution to coalition S –a trivial task in the absence of externalities–, it is now crucial to describe what happens after i leaves S. Suppose i plans to join T, another coalition in Π. The *total effect* on S of i's move is the difference $v(S,\Pi) - v(S \setminus \{i\}, \{S \setminus \{i\}, T \cup \{i\}\} \cup \Pi_{-S,-T})$. This effect can be decomposed into two. First, there is an *intrinsic marginal contribution effect* associated with i leaving S but before joining T, i.e., $v(S,\Pi) - v(S \setminus \{i\}, \{S \setminus \{i\}, \{i\}\} \cup \Pi_{-S})$. And second, there is an *externality effect*, which stems from the change in the worth of $S \setminus \{i\}$ when i, instead of remaining alone, joins T, i.e., the difference $v(S \setminus \{i\}, \{S \setminus \{i\}, \{i\}\} \cup \Pi_{-S}) - v(S \setminus \{i\}, \{S \setminus \{i\}, T \cup \{i\}\} \cup \Pi_{-S,-T})$. (Note how this latter difference is *not* a "partial derivative," a marginal contribution of player i to coalition S.) Our results follow from exploiting this decomposition.[1]

Assuming that the grand coalition forms, we investigate the implications of efficiency and anonymity, together with a weak version of marginality.[2] According to this last property, the solution may depend on all the total effects –the sum of the intrinsic marginal contribution and the externality effects. We find the first noteworthy difference with respect to the case of no-externalities, because in our larger domain these axioms are compatible with a wide class of linear and even non-linear solutions (Examples 1 and 2). However, despite such a large multiplicity of solutions, our first result shows that if the partition function can be written as the sum of a symmetric partition function and a characteristic function, a unique payoff profile is the consequence of the three axioms (Proposition 1). As will be formally defined in the sequel, this class amounts to having *symmetric externalities.*

But for general partition functions, as a result of our first finding, we seek the implications of strengthening the weak version of marginality, and we do so in two ways. First, we require monotonicity, i.e., a player's payoff should be increasing in all the total effects –the sum of his intrinsic marginal contribution and externality effects. Then, we are able to establish useful upper and lower bounds to each player's payoff (Proposition 2). And second, complementing this result, we require a marginality axiom, according to which a player's payoff should depend on the vector of intrinsic marginal contributions, not on the externality effect. The result is a characterization of an "externality-free" value on the basis of efficiency, anonymity and marginality (Proposition 3). In a second characterization result (Proposition 4), this solution is obtained using a system of axioms much like the original one due to Shapley (with a strong version of the dummy axiom that also disregards externalities).[3]

The externality-free value thus appears to be a natural reference point. Obviously, an analysis based solely on the externality-free value is not desirable in a model of external-

[1]In this decomposition, we are focusing on the "simplest path", i.e., that in which player i leaves S, and stays as a singleton before joining T. As we shall see, the singletons coalition structure acts as a useful origin of coordinates from which externalities are measured.

[2]Similar results can be established for any exogenous coalition structure, if the axioms are imposed atom by atom in the partition.

[3]Straightforward adaptations of the principle of balanced contributions (Myerson (1980)), and of some bargaining procedures (Hart and Mas-Colell (1996); Pérez-Castrillo and Wettstein (2001)), once applied to the larger domain of partition functions, lead to the externality-free value as well.

ities. This is why we do not insist on uniqueness, and accept the multiplicity of solutions inherent to the problem. The combination of both kinds of results –the externality-free value benchmark and the obtention of bounds around it– is a way to understand how externalities might benefit or punish a player in a context where normative principles are in place. In effect, the two results together provide a range for acceptable Pigouvian-like transfers (externality-driven taxes or subsidies among players) when efficiency is accompanied by our other normative desiderata.

To endogenize the coalition structure, in our strategic analysis we consider a two-stage cooperation process. In the first stage, players strategize over which coalition to join. In the second stage, after a coalition structure has emerged from the first, players commit to cooperate only with those players in the coalition they chose to join. What is critical to our process is the separation of group formation and payoff distribution in two sequential steps. Once the payoffs are determined for each coalition structure, one can resort to several techniques to solve the first stage of coalition formation. In our arguments, we use three. First, we focus on dominant partitions. Second, we consider the core of the corresponding "hedonic game" with externalities. And third, we propose a simple non-cooperative sequential game of coalition formation.

We find that the cooperative process may result in inefficient coalition structures. While these inefficiencies are sometimes due to the externalities, other causes are also identified (Examples 11, 12 and 13). The "Coase theorem logic" need not apply in our framework because the players' hands are tied by the normative principles behind the value once coalitions form. On the other hand, we provide sufficient conditions on the partition function under which the grand coalition is efficient and the unique dominant partition (Propositions 5, 6 and 7). These results (which rely on Propositions 1, 2 and 3, respectively), are to be compared with the existing literature, where often the formation of coalitions and the determination of payoffs is done simultaneously, resulting in games that are sometimes rather complex to solve. In contrast, one advantage of our two-stage process is its simplicity, which should make it useful in applications. Besides simplicity, the two-stage process is appropriate in scenarios where players' commitments to coalitions precede the determination of individual payoffs.

2 Definitions

Let N be the finite set of *players*. *Coalitions* are subsets of N. $P(N)$ denotes the set of all coalitions, and lower case letters denote the cardinality of coalitions ($s = \#S$, $n = \#N$, etc.). A *partition* of N is a set $\Pi = \{(S_k)_{k=1}^K\}$ $(1 \leq K \leq n)$ of disjoint coalitions that cover N, i.e. $S_i \cap S_j = \emptyset$, for each $1 \leq i < j \leq K$, and $N = \cup_{k=1}^K S_k$. By convention, $\{\emptyset\} \in \Pi$ for every partition Π. Elements of a partition are called *atoms*. A partition Π' is *finer* than a partition Π if each atom of Π' is included in an atom of Π: if $S' \in \Pi'$, then $S' \subseteq S$ for some $S \in \Pi$. We will say equivalently that Π is *coarser* than Π'. An *embedded coalition* is a pair (S, Π) where Π is a partition and S is an atom of Π. *EC* denotes the set of embedded coalitions. If S is a coalition and i is a member of S, then S_{-i} (resp. S_{+i}) denotes the set $S \setminus \{i\}$ (resp. $S \cup \{i\}$). Similarly, if Π is a partition and S is an atom of Π, then Π_{-S} denotes the partition $\Pi \setminus \{S\}$ of the set $N \setminus S$.

A *partition function* (Thrall and Lucas (1963)) is a function v that assigns to every embedded coalition (S, Π) a real number $v(S, \Pi)$, with the convention $v(\{\emptyset\}, \Pi) = 0$, for all Π. Externalities are *positive* (resp. negative) if $v(S, \Pi) \geq v(S, \Pi')$ for each embedded coalitions (S, Π) and (S, Π') such that Π is coarser (resp. finer) than Π'. There are *no externalities* if $v(S, \Pi) = v(S, \Pi')$ for all embedded coalitions (S, Π) and (S, Π'). In the latter case, a partition function is also called a *characteristic function*.

A partition function is *superadditive* if each coalition can achieve as much as the sum of what its parts can, i.e. $\sum_{k=1}^K v(S_k, \Pi') \leq v(S, \Pi)$, for every embedded coalition (S, Π), and every partition $\{(S_k)_{k=1}^K\}$ of S $(1 \leq K \leq s)$, with $\Pi' = \Pi_{-S} \cup \{(S_k)_{k=1}^K\}$.[4]

A *value* is a function σ that assigns to every partition function v a unique utility vector $\sigma(v) \in \mathbb{R}^N$. Shapley (1953) defined and axiomatized a value for characteristic functions:

$$\text{Sh}_i(v) := \sum_{S \subseteq N \text{ s.t. } i \in S} \frac{(s-1)!(n-s)!}{n!} [v(S) - v(S_{-i})]$$

for each player $i \in N$ and each characteristic function v. In the next sections, we are concerned with the extension of the theory of the Shapley value to partition functions.

3 Weak Marginality and Multiplicity of Solutions

Based on the marginality principle, Young (1985) proposes a beautiful axiomatization of the Shapley value for characteristic functions. We shall explore the implications of

[4]It is well-known that a characteristic function v is superadditive if and only if $v(R)+v(T) \leq v(R \cup T)$, for every *pair* (R, T) of disjoint coalitions. In other words, it is sufficient to check the case $K = 2$ in the absence of externalities. This observation does not extend to partition functions and one needs to consider all the possible K's (see Hafalir (2006, Example 1)). Notice that Hafalir (2006) uses the term "full cohesiveness" to describe superadditivity, while reserving the term "superadditivity" for the property with $K = 2$.

marginality, together with other basic axioms, on the class of partition functions. The first two axioms that we shall impose are hardly controversial.

Anonymity *Let π be a permutation of N and let v be a partition function. Then $\sigma(\pi(v)) = \pi(\sigma(v))$, where $\pi(v)(S,\Pi) = v(\pi(S), \{\pi(T)|T \in \Pi\})$ for each embedded coalition (S,Π) and $\pi(x)_i = x_{\pi(i)}$ for each $x \in \mathbb{R}^N$ and each $i \in N$.*

Efficiency $\sum_{i\in N} \sigma_i(v) = v(N)$*, for each partition function v.*

Anonymity means that players' payoffs do not depend on their names. We assume until Section 8 (included) that the grand coalition forms. Efficiency then simply means that the value must be feasible and exhaust all the benefits from cooperation, given that everyone cooperates. We will discuss the question of coalition formation in Sections 9 to 12. We will discuss in Section 11 the implications of the efficiency property once it is interpreted as overall efficiency on the class of superadditive partition functions.

Next, we turn to our discussion of marginality, central in our work. The marginal contribution of a player i within a coalition S is defined, for characteristic functions, as the loss incurred by the other members of S if i leaves the group. This number could depend on the organization of the players not in S when there are externalities. It is natural therefore to define the marginal contribution of a player within each embedded coalition.

To begin, one may consider the general case where a player may join another coalition after leaving S. Since many numbers qualify as marginal contributions, the resulting marginality axiom is rather weak. Still, it coincides with Young's concept of marginal contributions in the absence of externalities.

Weak Marginality *Let v and v' be two partition functions. If*

$$v(S,\Pi) - v(S_{-i}, \{S_{-i}, T_{+i}\} \cup \Pi_{-S,-T}) = v'(S,\Pi) - v'(S_{-i}, \{S_{-i}, T_{+i}\} \cup \Pi_{-S,-T})$$

for each embedded coalition (S,Π) such that $i \in S$ and each atom T of Π different from S, then $\sigma_i(v) = \sigma_i(v')$.

Suppose for instance that $N = \{i,j,k\}$. Then player i's payoff should depend only on the following seven real numbers:

$$A_i(v) = v(N,\{N\}) - v(\{j,k\}, \{\{i\},\{j,k\}\}),$$
$$B_i(v) = v(\{i,j\}, \{\{i,j\},\{k\}\}) - v(\{j\}, \{\{j\},\{i,k\}\}),$$

$$C_i(v) = v(\{i,j\},\{\{i,j\},\{k\}\}) - v(\{j\},\{\{i\},\{j\},\{k\}\}),$$
$$D_i(v) = v(\{i,k\},\{\{i,k\},\{j\}\}) - v(\{k\},\{\{k\},\{i,j\}\}),$$
$$E_i(v) = v(\{i,k\},\{\{i,k\},\{j\}\}) - v(\{k\},\{\{i\},\{j\},\{k\}\}),$$
$$F_i(v) = v(\{i\},\{\{i\},\{j,k\}\}),$$
$$G_i(v) = v(\{i\},\{\{i\},\{j\},\{k\}\}).$$

For general partition functions, there is no hope to get a characterization result of a value with Efficiency, Anonymity and this weak notion of marginality. To see this, consider the following examples:

Example 1 *The value σ^α, defined by $\sigma_i^\alpha(v) := \frac{1}{3}A_i(v) + \frac{1}{6}(\alpha B_i(v) + (1-\alpha)C_i(v)) + \frac{1}{6}(\alpha D_i(v) + (1-\alpha)E_i(v)) + \frac{1}{3}(\alpha F_i(v) + (1-\alpha)G_i(v))$, satisfies Anonymity, Efficiency, and Weak Marginality, for every $\alpha \in \mathbb{R}$. The values σ^α are instances of the average approach characterized by Macho-Stadler* et al *(2004), as they coincide with the Shapley value of a fictitious characteristic function v^α, where $v^\alpha(\{i\}) = \alpha v(\{i\},\{\{i\},\{j,k\}\}) + (1-\alpha)v(\{i\},\{\{i\},\{j\},\{k\}\})$.*

In addition, and what is perhaps more surprising, a large class of non-linear values satisfy the three axioms. (Recall that they imply linearity in the domain of characteristic functions.) In this sense, our approach differs substantially from Fujinaka's (2004). He proposes several versions of marginality, whereby a marginal contribution is constructed as a weighted linear average of the marginal contributions over different coalition structures. This assumption already builds linearity in most versions of Fujinaka's result.

Example 2 *Let $m : \mathbb{R} \to \mathbb{R}$ be any function. Then the value $\sigma^{\alpha,m}$, defined by $\sigma_i^{\alpha,m}(v) := \sigma_i^\alpha(v) + m(F_i(v) - G_i(v)) - \frac{m(C_i(v)-B_i(v))+m(E_i(v)-D_i(v))}{2}$, also satisfies the three axioms. Observe that the differences $C_i(v) - B_i(v)$, $E_i(v) - D_i(v)$, and $F_i(v) - G_i(v)$ measure the externality that the agents face. The function m transforms the externality that a player faces into a transfer paid equally by the two other players. The value $\sigma^{\alpha,m}$ is then obtained by adding to σ^α the net transfer that each player receives.*

We shall say that a partition function v is *symmetric* if $\pi(v)(S,\Pi) = v(S,\Pi)$, for each embedded coalition (S,Π) and each permutation π of the players. That is, the worth of an embedded coalition is a function only of its cardinality and of the cardinality of the other atoms of the partition.

The earlier examples illustrate the large class of linear and non-linear solutions compatible with the three axioms. Yet, one can obtain a surprisingly sharp prediction on an interesting subclass of partition functions. This is the content of our first result:

Proposition 1 *Let σ be a value that satisfies Anonymity, Efficiency and Weak Marginality. Let u be a symmetric partition function, and let v be a characteristic function. Then, $\sigma_i(u+v) = \frac{u(N)}{n} + Sh_i(v)$, for each $i \in N$.*

Proof: We proceed by induction, as in Young (1985), but starting the argument with the symmetric partition function u, instead of the null partition function. The set of characteristic functions is a vector subspace of the set of partition functions. Consider the following basis. For each nonempty coalition S, let e_S be the characteristic function defined as follows:

$$e_S(S') = \begin{cases} 1 & \text{if } S \subseteq S' \\ 0 & \text{otherwise.} \end{cases}$$

We prove that for all $i \in N$, $\sigma_i(u+v) = \frac{u(N)}{n} + \text{Sh}_i(v)$ by induction on the number of non-zero terms appearing in the basis decomposition of v.

Suppose first that all the terms are null. Then $v = 0$ and $\text{Sh}(v) = 0$. Hence $\sigma_i(u+v) = \sigma_i(u) = \frac{u(N)}{n} = \frac{u(N)}{n} + \text{Sh}_i(v)$, for each $i \in N$, where the second equality follows from the fact that σ satisfies Anonymity and Efficiency.

Suppose now that we have proved the result for all characteristic functions that have at most $k \geq 0$ non-zero terms when decomposed in the basis and suppose that

$$v = \sum_{S \in P(N) \text{ s.t. } S \neq \emptyset} \alpha(S) e_S$$

has exactly $k+1$ non-zero coefficients. Let S^* be the intersection of the nonempty coalitions S for which $\alpha(S) \neq 0$.

If $S^* = N$, the partition function $u+v$ is symmetric. Hence Anonymity and Efficiency for both σ and the Shapley value yield the result.

If $S^* \neq N$, let $i \in N \setminus S^*$, and let

$$v' = \sum_{S \in P(N) \text{ s.t. } i \in S} \alpha(S) e_S.$$

Player i has the same marginal contributions in $u+v$ and in $u+v'$. This follows from the fact that

$$v(R) - v(R_{-i}) = v'(R) - v'(R_{-i})$$

for each coalition R that contains i. Indeed, this equality can be rearranged as follows:

$$\sum_{S \in P(N) \text{ s.t. } i \notin S \text{ and } S \neq \emptyset} \alpha(S) e_S(R) = \sum_{S \in P(N) \text{ s.t. } i \notin S \text{ and } S \neq \emptyset} \alpha(S) e_S(R_{-i}).$$

Now, going term by term, observe first that if $S \subseteq R$ and $i \notin S$, then $S \subseteq R_{-i}$, and $e_S(R) = e_S(R_{-i}) = 1$. On the other hand, if $S \nsubseteq R$ and $i \notin S$, then $S \nsubseteq R_{-i}$ either, and $e_S(R) = e_S(R_{-i}) = 0$. Thus, our claim is proven.

Therefore, Weak Marginality implies that $\sigma_i(u + v) = \sigma_i(u + v')$. Note that v' has at most k non-zero terms in its basis decomposition. The induction hypothesis implies that $\sigma_i(u + v) = \frac{u(N)}{n} + \text{Sh}_i(v')$. Weak Marginality for the Shapley value implies that $Sh_i(v) = Sh_i(v')$. Hence $\sigma_i(u + v) = \frac{u(N)}{n} + \text{Sh}_i(v)$ for each $i \notin S^*$.

Now, for each pair of players $i, j \in S^*$, Anonymity for σ implies that $\sigma_i(u+v) = \sigma_j(u+v)$. Anonymity for the Shapley value implies that $\frac{u(N)}{n} + \text{Sh}_i(v) = \frac{u(N)}{n} + \text{Sh}_j(v)$. Efficiency implies that $\sum_{i \in N} \sigma_i(u+v) = u(N) + \sum_{i \in N} \text{Sh}_i(v)$. Hence $\sigma_i(u+v) = \frac{u(N)}{n} + \text{Sh}_i(v)$, for each $i \in S^*$, and the proof is complete. □

Proposition 1 is straightforward if v is the null partition function. It coincides with Young's (1985) result if u is the null partition function. The important conclusion we can draw from the proposition is that the three axioms together imply additivity on the class of partition functions that can be decomposed as the sum of a symmetric partition function and a characteristic function.[5] This result is not trivial, in view of Example 2. As an illustration of the power of Proposition 1, consider the following example:

Example 3 *This example features prominently in Maskin (2003). A similar example was first proposed by Ray and Vohra (1999, Example 1.2). It describes a simple "free rider" problem created by a public good that can be produced by each two-player coalition. The set of agents is $N = \{1, 2, 3\}$, and this is the partition function:*

$$v(N) = 24;$$

$$v(\{1,2\}) = 12; \quad v(\{1,3\}) = 13; \quad v(\{2,3\}) = 14;$$

$$v(\{i\}, \{\{i\}, \{j,k\}\}) = 9 \quad \textit{for all } i, j, k;$$

$$v(\{i\}, \{\{i\}, \{j\}, \{k\}\}) = 0 \quad \textit{for all } i, j, k.$$

[5] We argue at the end of Section 5 that a partition function v can be decomposed as the sum of a symmetric partition function and a characteristic function if and only if externalities are symmetric for v.

Observe that this partition function is not symmetric, and it is not a characteristic function. Yet, it can be decomposed as the sum of a characteristic function v' (where each coalition's worth is zero, except $v'(\{1,3\}) = 1$ and $v'(\{2,3\}) = 2$) and a symmetric partition function u. We conclude that any value σ that satisfies Anonymity, Efficiency and Weak Marginality (and we remark this is a large class) must be such that

$$\sigma(v) = \sigma(u + v') = (8, 8, 8) + (-0.5, 0, 0.5) = (7.5, 8, 8.5)$$

in this example.

Next, we wish to consider general partition functions again. Given the large class of solutions identified in Examples 1 and 2 above, we propose to follow two alternative paths. First, we shall strengthen Weak Marginality into a monotonicity property. Second, we shall look more closely at the notion of "marginal contributions" to propose an alternative marginality axiom. We undertake each of the alternatives in the next two sections.

4 Monotonicity and Bounds on Payoffs

This section investigates what happens when, in addition to requiring Efficiency and Anonymity, Weak Marginality is strengthened to the following monotonicity axiom. The result will be the derivation of useful bounds to the payoff of each player.

Monotonicity[6] *Let v and v' be two partition functions. If*

$$v(S, \Pi) - v(S_{-i}, \{S_{-i}, T_{+i}\} \cup \Pi_{-S,-T}) \geq v'(S, \Pi) - v'(S_{-i}, \{S_{-i}, T_{+i}\} \cup \Pi_{-S,-T})$$

for each embedded coalition (S, Π) such that $i \in S$ and each atom T of Π different from S, then $\sigma_i(v) \geq \sigma_i(v')$.

In words, if for a partition function the vector of marginal contributions of a player to the different coalitions, for any organization of the complement, dominates coordinate by coordinate that of a second partition function, the value must pay this player more in the first partition function. For instance, the value σ^{α} of Example 1 is monotonic, for each $\alpha \in [0, 1]$.

[6]Perhaps the term "Weak Monotonicity" would be more appropriate to emphasize the link with Weak Marginality, hence comparing all the total effects instead of limiting ourselves to the intrinsic marginal contributions (see Section 5). Yet, since weak monotonicity has other meanings, and since it will not be necessary to introduce other monotonicity properties, we opted for the term "monotonicity."

First, we point out in the following example that Monotonicity, combined with Anonymity and Efficiency, does not imply additivity either:

Example 4 *The value $\sigma^{\alpha,m}$ from Example 2 satisfies Monotonicity if $\alpha = 1/2$, $m(x) = x^2$ if $|x| \leq 1/12$, and $m(x) = (1/12)^2$ if $|x| \geq 1/12$.*

We may nevertheless bound each player's payoff from below and from above. The approach we follow seeks to obtain bounds that rely on decompositions of certain partition functions into the sum of a symmetric partition function and a characteristic function, the class of partition functions uncovered in Proposition 1.

Let $\mathcal{S}$ be the set of symmetric partition functions and let $\mathcal{C}$ be the set of characteristic functions. Let $i \in N$. For each partition function v, let $M_i(v)$ be the set of pairs $(u, v') \in \mathcal{S} \times \mathcal{C}$ such that

$$\begin{aligned} &v(S,\Pi) - v(S_{-i}, \{S_{-i}, T_{+i}\} \cup \Pi_{-S,-T}) \\ &\qquad \geq [u(S,\Pi) - u(S_{-i}, \{S_{-i}, T_{+i}\} \cup \Pi_{-S,-T})] + [v'(S) - v'(S_{-i})] \end{aligned}$$

for each embedded coalition (S,Π) such that $i \in S$ and each atom T of Π different from S. Monotonicity and Proposition 1 imply that $\sigma_i(v) \geq \frac{u(N)}{n} + \mathrm{Sh}_i(v')$, for each $(u, v') \in M_i(v)$. Therefore, the best lower bound following this approach is obtained by solving the following linear programming problem, which always has a unique optimal objective value:

$$\mu_i(v) = \max_{(u,v') \in M_i(v)} [\frac{u(N)}{n} + \mathrm{Sh}_i(v')].$$

Similarly, for each partition function v, let $N_i(v)$ be the set of pairs $(u, v') \in \mathcal{S} \times \mathcal{C}$ such that

$$\begin{aligned} &v(S,\Pi) - v(S_{-i}, \{S_{-i}, T_{+i}\} \cup \Pi_{-S,-T}) \\ &\qquad \leq [u(S,\Pi) - u(S_{-i}, \{S_{-i}, T_{+i}\} \cup \Pi_{-S,-T})] + [v'(S) - v'(S_{-i})] \end{aligned}$$

for each embedded coalition (S,Π) such that $i \in S$ and each atom T of Π different from S. Again, Monotonicity and Proposition 1 imply that $\sigma_i(v) \leq \frac{u(N)}{n} + \mathrm{Sh}_i(v')$, for each $(u, v') \in N_i(v)$. The best upper bound following this approach is obtained by solving the following linear programming problem:

$$\nu_i(v) = \min_{(u,v') \in N_i(v)} [\frac{u(N)}{n} + \mathrm{Sh}_i(v')].$$

That is, we have shown the following:

Proposition 2 *If σ is a value that satisfies Anonymity, Efficiency and Monotonicity, then*

$$\sigma_i(v) \in [\mu_i(v), \nu_i(v)],$$

for each $i \in N$.

These bounds in general improve upon those that one could obtain by using only symmetric partition functions or only characteristic functions (in particular, the "most pessimistic" and "most optimistic" characteristic functions that each player i can construct from the partition function. Player i's "most pessimistic" characteristic function minimizes over Π the worth $v(S, \Pi)$ of each coalition S where $i \in S$, and maximizes it for coalitions that do not include i. The opposite happens for his "most optimistic" characteristic function.)

One important question is whether the bounds provided by Proposition 2 are "tight," in the sense that for each partition function one can always find solutions to fill up the entire identified cube of payoffs. Although we do not know the answer to this question in general, at least for the case of three players the bounds are tight. We will elaborate on this point in Section 7.

5 Marginality and the Externality-Free Value

An alternative route to Monotonicity is to strengthen Weak Marginality into another marginality axiom. To do this, it will be instructive to look closer at the concept of marginal contribution in contexts with externalities.

Consider player i and an embedded coalition (S, Π) with $i \in S$. Suppose player i leaves coalition S and joins coalition $T \in \Pi$, $T \neq S$. One can view this as a two-step process. In the first instance, player i simply leaves S and, at least for a while, he is alone, which means that for the moment the coalition structure is $\{S_{-i}, \{i\}\} \cup \Pi_{-S}$. At this point, coalition S_{-i} feels the loss of player i's *marginal contribution*, i.e.,

$$v(S, \Pi) - v(S_{-i}, \{S_{-i}, \{i\}\} \cup \Pi_{-S}).$$

In the second step, player i joins coalition $T \in \Pi_{-S}$, and then S_{-i} is further affected, but not because of a marginal contribution from player i. Rather, it is affected because of the corresponding *externalities* created by this merger, i.e.,

$$v(S_{-i}, \{S_{-i}, \{i\}\} \cup \Pi_{-S}) - v(S_{-i}, \{S_{-i}, T_{+i}\} \cup \Pi_{-S,-T}).$$

If one views this as an important distinction, one should reserve the term (intrinsic) *marginal contribution* to the former difference. We shall do this in the sequel.

Formally, let i be a player and let (S, Π) be an embedded coalition such that $i \in S$. Then the (intrinsic) *marginal contribution of* i *to* (S, Π) is given by

$$mc_{(i,S,\Pi)}(v) = v(S, \Pi) - v(S_{-i}, \{S_{-i}, \{i\}\} \cup \Pi_{-S})$$

for each partition function v. Player i's *vector of intrinsic marginal contributions* is obtained by varying (S, Π): $mc_i(v) = (mc_{(i,S,\Pi)})_{(S,\Pi)\in EC \wedge i\in S}$. Here is the formal statement of the new marginality axiom for partition functions (note that it also reduces to Young's if applied to characteristic functions).

Marginality *Let i be a player and let v and v' be two partition functions. If $mc_i(v) = mc_i(v')$, then $\sigma_i(v) = \sigma_i(v')$.*

Consider now the following extension σ^* of the Shapley value to partition functions:

$$\sigma_i^*(v) := \mathrm{Sh}_i(v^*)$$

for each player $i \in N$ and each partition function v, where v^* is the fictitious characteristic function defined as follows:

$$v^*(S) := v(S, \{S, \{j\}_{j\in N\setminus S}\})$$

for each coalition S.

We call this the *externality-free value*, and we shall discuss it below. Our next result follows.

Proposition 3 *σ^* is the unique value satisfying Anonymity, Efficiency and Marginality.*

Proof: The set of partition functions form a vector space. We prove the proposition by defining an adequate basis. Let (S, Π) be an embedded coalition, where S is non-empty. Then $e_{(S,\Pi)}$ is the partition function defined as follows:

$$e_{(S,\Pi)}(S', \Pi') = \begin{cases} 1 & \text{if } S \subseteq S' \text{ and } (\forall T' \in \Pi' \setminus \{S'\})(\exists T \in \Pi) : T' \subseteq T, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 1 *The collection of vectors $(e_{(S,\Pi)})_{(S,\Pi)\in EC}$ constitutes a basis of the space of partition functions.*

Proof of Lemma 1: The number of vectors in the collection equals the dimension of the space. It remains to show that they are linearly independent, i.e.

$$\sum_{(S,\Pi)\in EC} \alpha(S,\Pi)e_{(S,\Pi)} = 0 \tag{1}$$

implies $\alpha(S,\Pi) = 0$ for each $(S,\Pi) \in EC$.

Suppose on the contrary that there exists a collection $(\alpha_{(S,\Pi)})_{(S,\Pi)\in EC}$ of real numbers satisfying (1) and such that $\alpha(S,\Pi) \neq 0$, for some $(S,\Pi) \in EC$. Let (S^*,Π^*) be an embedded coalition such that:

1. $\alpha(S^*,\Pi^*) \neq 0$;

2. $(\forall(S,\Pi) \in EC) : S \subsetneq S^* \Rightarrow \alpha(S,\Pi) = 0$;

3. $(\forall(S^*,\Pi) \in EC$ s.t. $\Pi \neq \Pi^*) : \Pi$ coarser than $\Pi^* \Rightarrow \alpha(S^*,\Pi) = 0$.

By definition, $e_{(S,\Pi)}(S^*,\Pi^*) = 0$ if S is not included in S^*. The second property of (S^*,Π^*) then implies that

$$[\sum_{(S,\Pi)\in EC} \alpha(S,\Pi)e_{(S,\Pi)}](S^*,\Pi^*) = [\sum_{(S^*,\Pi)\in EC} \alpha(S^*,\Pi)e_{(S^*,\Pi)}](S^*,\Pi^*).$$

By definition, $e_{(S^*,\Pi)}(S^*,\Pi^*) = 0$ if Π is not coarser than Π^*. The third property of (S^*,Π^*) then implies that

$$[\sum_{(S^*,\Pi)\in EC} \alpha(S^*,\Pi)e_{(S^*,\Pi)}](S^*,\Pi^*) = \alpha(S^*,\Pi^*).$$

Equation (1) thus implies that $\alpha(S^*,\Pi^*) = 0$, a contradiction. $\square$

Now we continue with the proof of Proposition 3. The properties of the Shapley value imply that σ^* satisfies the three axioms.

For uniqueness, let σ be a value satisfying the three axioms. We show that $\sigma(v) = \sigma^*(v)$ for each partition function v by induction on the number of non-zero terms appearing in the basis decomposition of v.

Suppose first that $v = \alpha e_{(S,\Pi)}$, for some $\alpha \in \mathbb{R}$ and some $(S,\Pi) \in EC$. Let $i \in N \setminus S$ and let (S',Π') be any embedded coalition. The two following statements are equivalent:

1. $S \subseteq S'$ and $(\forall T' \in \Pi'_{-S'})(\exists T \in \Pi) : T' \subseteq T$;

2. $S \subseteq S' \setminus \{i\}$ and $(\forall T' \in \{\{i\}, \Pi'_{-S'}\})(\exists T \in \Pi) : T' \subseteq T$.

Hence $e_{(S,\Pi)}(S', \Pi') = 1$ if and only if $e_{(S,\Pi)}(S'_{-i}, \{S'_{-i}, \{i\}, \Pi'_{-S'}\}) = 1$. So, $mc_i(v) = mc_i(v^0)$, where v^0 is the null partition function (i.e., $v^0(S, \Pi) = 0$ for each $(S, \Pi) \in EC$). Marginality implies that $\sigma_i(v) = \sigma_i(v^0)$. Anonymity and Efficiency imply that $\sigma_i(v^0) = 0$. Hence $\sigma_i(v) = 0 = \sigma^*_i(v)$. Anonymity implies in addition that $\sigma_j(v) = \sigma_k(v)$ and $\sigma^*_j(v) = \sigma^*_k(v)$, for all j, k in S. Hence $\sigma(v) = \sigma^*(v)$, since both σ and σ^* satisfy Efficiency.

Suppose now that we have proved the result for all the partition functions that have at most k non-zero terms when decomposed in the basis and let

$$v = \sum_{(S,\Pi) \in EC} \alpha(S, \Pi) e_{(S,\Pi)}$$

be a partition function with exactly $k+1$ non-zero coefficients. Let S^* be the intersection of the coalitions S for which there exists a partition Π such that $\alpha(S, \Pi)$ is different from zero. If $i \in N \setminus S^*$, then player i's marginal contribution vector in v coincides with his marginal contribution vector in the partition function

$$v' = \sum_{(S,\Pi) \in EC \text{ s.t. } i \in S} \alpha(S, \Pi) e_{(S,\Pi)}.$$

Marginality implies that $\sigma_i(v) = \sigma_i(v')$. Note that the number of non-zero terms in the basis decomposition of v' is at most k. Then, by the induction hypothesis, $\sigma_i(v') = \sigma^*_i(v')$. Since σ^* satisfies Marginality as well, we conclude that $\sigma_i(v) = \sigma^*_i(v)$. Anonymity implies in addition that $\sigma_j(v) = \sigma_k(v)$ and $\sigma^*_j(v) = \sigma^*_k(v)$, for all j, k in S^*. Hence $\sigma(v) = \sigma^*(v)$, since both σ and σ^* satisfy Efficiency. The proof of Proposition 3 is now complete. □

Marginality does not imply on its own that externalities play no role in the computation of payoffs. Indeed, a player's intrinsic marginal contribution to an embedded coalition (S, Π) depends on the composition of Π_{-S}. Instead, it is the combination of Marginality, Efficiency, and Anonymity that leads to the externality-free value. Hence, Proposition 3 is definitely not a trivial variation on Young's (1985) original theorem: some information present in the partition function has to be discarded, as a consequence of the combination of the three axioms. To gain an intuition for this, consider first a three-player partition function. The result tells us that player 1's payoff does not depend on $x = v(\{1\}, \{\{1\}, \{2, 3\}\})$. This does not follow from any of the three axioms taken separately (in particular, by Marginality it could depend on x, if $S = \{1\}$ and $\Pi_{-S} = \{\{2, 3\}\}$ in the definition of the axiom). Instead, the reasoning goes as follows.

Players 2 and 3's payoffs do not depend on x according to Marginality. Efficiency then implies that player 1's payoff cannot depend on x either.

To strengthen one's intuition, let us pursue this heuristic argument with four players. In principle, player 1's payoff could depend on fifteen numbers according to Marginality. Only eight of them are actually relevant to compute σ^*. Let us show for instance why player 1's payoff cannot depend on $y = v(\{1\}, \{\{1\}, \{2, 3\}, \{4\}\})$. Marginality implies that, apart from player 1's, only the payoff of player 4 could depend on y, or more precisely on $z - y$, where $z = v(\{1, 4\}, \{\{1, 4\}, \{2, 3\}\})$. Marginality implies also that the payoffs of players 2 and 3 do not depend on z. On the other hand, as we know from Proposition 3, the three axioms together imply that the solution must be an anonymous and additive function. Thus, the payoffs of players 1 and 4 depend identically on z (if z increases, both payoffs to players 1 and 4 also increase). Hence, Efficiency implies that player 4's payoff cannot depend on z, and therefore not on y either. Players 2 and 3's payoffs do not depend on y by Marginality. Hence player 1's payoff cannot depend on y, by Efficiency.

We regard the externality-free value σ^* as a fair compromise that takes into account the pure or intrinsic *marginal contributions* of players to coalitions, stripped down from externality components. Note also that σ^* satisfies Monotonicity. Then, the range of payoffs identified in Proposition 2 for each player captures how externalities affect his payoff, when one still requires Efficiency, Anonymity and Monotonicity. Thus, the size of the difference $\nu_i(v) - \sigma_i^*(v)$ expresses the maximum "subsidy" or benefit to player i, associated with externalities that favor him, and the difference $\sigma_i^*(v) - \mu_i(v)$ represents how much i can be "taxed" or suffer, due to harmful externalities, in a value that obeys these three axioms.

In the same way that σ^* serves as an interesting reference point in the space of payoffs, the characteristic function v^* can serve as an interesting reference point to measure the size of externalities. Let us define the *externality index* $\epsilon_v(S, \Pi)$ associated to (S, Π) as the difference $v(S, \Pi) - v^*(S)$ (same as in Example 2 and in Section 7).[7] We can now better understand the class of partition functions that was uncovered in Proposition 1 and that plays a key role in many parts of our paper. Externalities are *symmetric* for v if the associated externality indices $(\epsilon_v(S, \Pi))_{(S,\Pi)\in EC}$ form a symmetric partition function, i.e. the level of externality associated to any embedded coalition (S, Π) depends only on the cardinality of S and on the cardinality of the other atoms of Π. It is then not

[7]Any difference $v(S, \Pi) - v(S, \Pi')$ can be recovered from our externality indices, since $v(S, \Pi) - v(S, \Pi') = \epsilon_v(S, \Pi) - \epsilon_v(S, \Pi')$.

difficult to check that a partition function can be decomposed as the sum of a symmetric partition function and a characteristic function if and only if externalities are symmetric for v. Example 3 provides an illustration of this equivalence.

6 Alternative Approaches to the Externality-Free Value

A natural adaptation of Shapley's original axiomatic system also leads to the externality-free value σ^*. Let i be a player and let v be a partition function. We say that player i is *null* in v if his marginal contribution to any embedded coalition is nihil: $mc_i(v) = 0$. Note how this notion of a null player disregards externalities, as only intrinsic marginal contributions matter.

Null Player *Let v be a partition function. If player i is null in v, then $\sigma_i(v) = 0$.*

Additivity *Let v and w be two partition functions. Then, $\sigma(v + w) = \sigma(v) + \sigma(w)$.*

A null player must receive a zero payoff, according to the first axiom. Additivity essentially amounts to the linearity of the value.[8] It expresses a form of mathematical simplicity by requiring a strong specific functional form.

Proposition 4 *σ^* is the unique value satisfying Anonymity, Efficiency, Null Player and Additivity.*

Proof: The properties of the Shapley value imply that σ^* satisfies the four axioms.

Let σ be a value satisfying the four axioms, and let v be a partition function. Remember that v can be decomposed in the basis described in Lemma 1:

$$v = \sum_{(S,\Pi)\in EC} \alpha(S,\Pi)e_{(S,\Pi)}, \tag{2}$$

for some real numbers $\alpha(S,\Pi)$. Anonymity, Null Player, and Efficiency imply that $\sigma(\alpha(S,\Pi)e_{(S,\Pi)}) = \sigma^*(\alpha(S,\Pi)e_{(S,\Pi)})$, for each $(S,\Pi) \in EC$ (a similar argument was made in the proof of Proposition 3). Since both σ and σ^* are additive, we conclude that $\sigma(v) = \sigma^*(v)$. $\square$

Proposition 4 is equivalent to theorem 2 of Pham Do and Norde (2007). However, their proof relies on the canonical basis instead of using the basis identified in Lemma 1.

[8] To be precise, it implies linearity only with respect to linear combinations involving rational numbers.

Our basis has the advantage of allowing to prove a similar result on the smaller class of superadditive partition functions (see the discussion at the end of Section 11).

A player is null if his vector of marginal contributions is nihil. The other notion of marginal contributions, which contained the externality effects and was used to define Weak Marginality, leads in turn to a weaker version of the null player property. Player i is *null in the strong sense* if

$$v(S,\Pi) - v(S_{-i}, \{S_{-i}, T_{+i}\} \cup \Pi_{-S,-T}) = 0$$

for each embedded coalition (S,Π) such that $i \in S$ and each atom T of Π different from S. Clearly, if player i is null in the strong sense, then he is null. One can use this definition to propose a different null player axiom:

Weak Version of the Null Player Axiom *Let $i \in N$ and let v be a partition function. If player i is null in the strong sense, then $\sigma_i(v) = 0$.*

This is equivalent to the dummy player axiom of Bolger (1989) and Macho-Stadler *et al* (2004).[9] Macho-Stadler *et al* (2004, theorem 1) show that any solution that satisfies this version of the null player axiom, as well as the axioms of efficiency, linearity and (a strong version of) Anonymity is a Shapley value of a characteristic function that is obtained by performing averages of the partition function. Our externality-free value σ^* belongs to this class of solutions. Macho-Stadler *et al* also characterize a unique solution by adding an axiom of similar influence that σ^* does not satisfy.

In light of our first discussion concerning even non-linear solutions (recall Example 2), we prefer Proposition 3 to Proposition 4 (even though both lead to the same value), because we find Marginality more compelling than Additivity. It is easier to interpret and justify a restriction on the set of variables required to compute the players' payoffs, than to impose a specific functional form. It is interesting to note in that respect that the non-additive solution defined in Example 4 satisfies the strong symmetry and the similar influence axioms of Macho-Stadler *et al* (2004), in addition to satisfying Anonymity, Efficiency, and Weak Marginality. Once again, Additivity cannot be justified by the marginality principle that underlies their dummy player axiom (the weak version of mar-

[9] Bolger and Macho-Stadler *et al* say that player i is dummy if $v(S,\Pi) = v(S',\Pi')$ for each (S,Π) and each (S',Π') that can be deduced from (S,Π) by changing player i's affiliation. This clearly implies that player i is null in our strong sense. The converse is straightforward after proving that $v(S,\Pi) = v(S,\Pi')$ for each pair of embedded coalitions (S,Π) and (S,Π') such that $i \notin S$ and (S,Π') can be deduced from (S,Π) by changing only player i's affiliation. Indeed, if i is null in our strong sense, then $v(S,\Pi) = v(S_{+i}, \{S_{+i}\} \cup \{T_{-i} | T \in \Pi_{-S}\}) = v(S_{+i}, \{S_{+i}\} \cup \{T_{-i} | T \in \Pi'_{-S}\}) = v(S,\Pi')$.

ginality), even if one imposes their other requirements.

One can take a bargaining approach to understand the Shapley value. This was done for instance in Hart and Mas-Colell (1996) or Pérez-Castrillo and Wettstein (2001) for characteristic functions. It is not difficult to see that we obtain σ^* if we apply these procedures to superadditive partition functions. Indeed, it is assumed in these two papers that, when a proposal is rejected, the proposer goes off by himself and does not form a coalition with anyone else. We leave the details to the interested reader. Other new rules concerning the rejected proposers would lead to values that treat externalities differently (see Macho-Stadler *et al* (2005)).

Myerson's (1980) principle of balanced contributions (or the related concept of potential proposed by Hart and Mas-Colell (1989)) offers another elegant justification for the Shapley value. Instead of characterizing a value, i.e. a function that determines a payoff vector for each characteristic function, Myerson characterizes a payoff configuration, i.e. a vector that determines how the members of each coalition would share the surplus that is created should they cooperate. A payoff configuration satisfies the principle of balanced contributions if, for any two members i, j of any coalition S, the payoff loss that player i suffers if player j leaves S equals the payoff loss that player j suffers if player i leaves S. This principle, combined with efficiency within each coalition, implies that the players agree on the Shapley value within the grand coalition. This methodology can be easily extended to partition functions. As was the case with our analysis of Young's marginal contributions, the payoff loss that a player i suffers if a player j leaves S usually depends on what i does after leaving S (stay on his own, or join another atom and, if so, which one?). It may be worthwhile to study different scenarios. Here we simply observe that if a player is assumed to stay on his own after leaving a coalition (as was the case for our concept of intrinsic marginal contribution), then the extended principle of balanced contributions *à la* Myerson leads to a unique payoff configuration, and the resulting payoff for the grand coalition coincides once again with the payoff associated to the externality-free value. We again leave the details to the interested reader.

Section 5 and the current section together lead to an apparent conclusion. Suppose one simply wishes to find an extension of the Shapley value to environments with externalities by using any of the different approaches that have provided its foundations on the domain of characteristic functions. Then, the result is the externality-free value, using our *intrinsic* marginal contributions as the parallel concept to the standard marginal contributions of the reduced domain. But in dealing with externalities, any such approach must be complemented by a treatment of the externalities themselves, and that is what has led to the bounds around the externality-free value.

7 The Tightness of the Bounds on Payoffs

We now come back to the bounds obtained in Proposition 2, and how they are positioned relative to the externality-free value. In particular, we argue that for the class of three-player partition functions, the bounds obtained are tight. By solving the linear programs described to prove that proposition, one can show that:

$$\mu_i(v) = \sigma_i^*(v) + \frac{\min\{0, \epsilon_i(v) - \epsilon_j(v)\} + \min\{0, \epsilon_i(v) - \epsilon_k(v)\}}{6}$$

and

$$\nu_i(v) = \sigma_i^*(v) + \frac{\max\{0, \epsilon_i(v) - \epsilon_j(v)\} + \max\{0, \epsilon_i(v) - \epsilon_k(v)\}}{6},$$

where $\sigma_i^*(v)$ is player i's externality-free value payoff and $\epsilon_i(v) = v(\{i\}, \{\{i\}, \{j,k\}\}) - v(\{i\}, \{\{i\}, \{j\}, \{k\}\})$ is the externality index associated to player i. Notice that it is not the sign nor the magnitude of $\epsilon_i(v)$ that determine the bounds on player i's payoff, but instead how far $\epsilon_i(v)$ is from $\epsilon_j(v)$ and $\epsilon_k(v)$.

Thus, suppose there are three players, and let σ be a value that satisfies Anonymity, Efficiency, and Monotonicity. Let v be a partition function. Suppose without loss of generality that $\epsilon_1(v) \leq \epsilon_2(v) \leq \epsilon_3(v)$. Then

$$\sigma_1(v) \in [\sigma_1^1(v), \sigma_1^0(v)] \quad \text{and} \quad \sigma_3(v) \in [\sigma_3^0(v), \sigma_3^1(v)].$$

Here, σ^0 and σ^1 are the solutions σ^α of Example 1 for the cases of $\alpha = 0$ and $\alpha = 1$, respectively ($\sigma^0 = \sigma^*$). The bounds are intuitive. For the player who benefits the least from externalities (player 1), the lowest possible payoff compatible with the axioms happens at the Shapley value of the average characteristic function that puts all the weight on his worth when players 2 and 3 cooperate. His highest possible payoff is obtained at the Shapley value of the average characteristic function which his worth corresponds to the situation in which players 2 and 3 do not cooperate. Exactly the opposite happens for player 3, who benefits the most from externalities.

Therefore, the bounds we obtained are tight in the following sense. There exists a value (e.g. σ^0 or σ^1) that satisfies Anonymity, Efficiency, and Monotonicity, and such that, for each partition function v, some player i gets $\mu_i(v)$ and some player j gets $\nu_j(v)$, the bounds of Proposition 2. The next two examples further illustrate the tightness of the bounds.

Example 5 *Consider a variant of Example 3, in which player 1 is the only agent capable of free-riding from a two-player coalition, receiving a worth of 9, as before, when coalition* $\{2,3\}$ *gets together. However,*

$$v(\{2\},\{\{2\},\{1,3\}\}) = v(\{3\},\{\{3\},\{1,2\}\}) = 0.$$

One can easily check that $\sigma^0(v) = \sigma^*(v) = (7.5, 8, 8.5)$*,* $\sigma^1(v) = (10.5, 6.5, 7)$*, and that the bounds from Proposition 2 are:*

$$\mu_1(v) = 7.5, \qquad \nu_1(v) = 10.5;$$

$$\mu_2(v) = 6.5, \qquad \nu_2(v) = 8;$$

$$\mu_3(v) = 7, \qquad \nu_3(v) = 8.5.$$

That is, no monotonic solution σ *ever punishes player 1 or rewards 2 and 3 with respect to the payoffs in* σ^**. Given the nature of externalities in this example (player 1 is the only one who benefits from externalities), it is clear what direction externality-driven transfers should take for each player.*

In the next example, we show that the cube of payoffs identified by our bounds contains payoffs associated with non-linear solutions, which cannot be obtained through Monotonicity in the "average approach" of Macho-Stadler *et al* (2004). We also show that the axioms may imply restrictions on the players' payoffs that are not captured by the bounds.

Example 6 *Consider the following partition function:*

$$v(N) = 1;$$

$$v(\{i,j\}) = 0 \quad \textit{for all } i,j;$$

$$v(\{1\},\{\{1\},\{2,3\}\}) = 1/10;$$

$$v(\{2\},\{\{2\},\{1,3\}\}) = 1/20;$$

$$v(\{3\},\{\{3\},\{1,2\}\}) = 0;$$

$$v(\{i\},\{\{i\},\{j\},\{k\}\}) = 0 \quad \textit{for all } i,j,k.$$

The cube of payoffs associated with monotonic values is:

$$\mu_1(v) = \frac{1}{3}, \quad \nu_1(v) = \frac{1}{3} + \frac{1}{40};$$

$$\mu_2(v) = \frac{1}{3} - \frac{1}{120}, \quad \nu_2(v) = \frac{1}{3} + \frac{1}{120};$$

$$\mu_3(v) = \frac{1}{3} - \frac{1}{40}, \quad \nu_3(v) = \frac{1}{3}.$$

Using $\alpha = 0$ *in the average approach gives the externality-free value, which yields equal split of surplus:* $\sigma^*(v) = (1/3, 1/3, 1/3)$*. On the other hand, if* $\alpha = 1$*, the corresponding average approach value gives* $\sigma^1(v) = (\frac{1}{3} + \frac{1}{40}, \frac{1}{3}, \frac{1}{3} - \frac{1}{40})$*. Player* 2 *actually receives* 1/3 *according to every monotonic value obtained from the average approach.*

However, the non-linear monotonic value of Example 4 gives

$$\sigma^{\alpha,m}(v) = (\frac{1}{3} + \frac{131}{7200}, \frac{1}{3} - \frac{7}{7200}, \frac{1}{3} - \frac{124}{7200}).$$

Here, the transfers driven by externalities do not cancel out for player 2, because they enter non-linearly in the solution. Non-linear solutions are far more complex than their linear counterparts, as the next paragraph will suggest.

Let's rescale up the partition function of this example through multiplying it by 60. Call the resulting partition function v'*. The new bounds are:*

$$\mu_1(v') = 20, \quad \nu_1(v') = 21.5;$$

$$\mu_2(v') = 19.5, \quad \nu_2(v') = 20.5;$$

$$\mu_3(v') = 18.5, \quad \nu_3(v') = 20.$$

The average approach, under $\alpha = 0$*, yields* $\sigma^*(v') = (20, 20, 20)$*. With* $\alpha = 1$*, it yields* $\sigma^1(v') = (21.5, 20, 18.5)$*. Again, every monotonic average approach value will pay player 2 exactly 20. That is, so far everything has been rescaled up by the same factor of 60. However, consider our monotonic solution of Example 4. It yields*

$$\sigma^{\alpha,m}(v') = (20 + \frac{217}{288}, 20 + \frac{1}{288}, 20 - \frac{109}{144}),$$

which does not preserve the rescaling and which, more interestingly, changes the nature of transfers across agents vis-à-vis the externality-free payoff.

The partition function v *also illustrates another point. Let* v'' *be the partition function* *obtained from v by increasing the surplus of agent 2 to $1/10$ when he free-rides. If the value is monotonic, then player 2's payoff is larger in v'' than in v, while player 1's payoff is larger in v than in v''. On the other hand, if the value is anonymous, then players 1 and 2 must receive the same payoffs in v''. Hence player 1's payoff is larger than player 2's payoff in v. This conclusion cannot be reached by comparing the bounds, since $\nu_2(v) > \mu_1(v)$. Hence it is possible, for some partition functions, to reach conclusions regarding the players' payoffs that are not captured by the bounds.*

Another observation in support of the tightness of our bounds is that $\nu_i(v) \leq v(N) - \mu_j(v) - \mu_k(v)$ and that $\mu_i(v) \geq v(N) - \nu_j(v) - \nu_k(v)$ (for each i, j, k). The easy proof is left to the interested reader.

For more than three players, we do not know how tight the bounds of Proposition 2 are. However, in many examples, they provide an insightful refinement of the set of feasible payoffs.

8 Comparisons with the Axiomatic Literature

In this section we present some examples to draw comparisons with previous axiomatic approaches in the literature.

Myerson (1977) proposes the first value for partition functions. His key axiom is a version of the carrier axiom, used also in the domain of characteristic functions. This value violates Monotonicity and sometimes yields unintuitive predictions.

Example 7 *Consider the three-player partition function v where $v(\{1\}, \{\{1\}, \{2, 3\}\}) = 1 = v(N)$ and $v(S, \Pi) = 0$ otherwise. Myerson's value, $(1, 0, 0)$, falls outside of our cube of payoffs ($[1/3, 2/3]$ for player 1, and $[1/6, 1/3]$ for players 2 and 3). Hence Myerson's value is not monotonic.*

In a second three-player partition function, where $v(N) = v(\{1\}, \{\{1\}, \{2\}, \{3\}\}) = 1$ and $v(S, \Pi) = 0$ otherwise, Myerson's value assigns the payoffs $(0, 1/2, 1/2)$. Again, this falls outside of our range of payoffs, which is the same as before.

The cube of payoffs compatible with Monotonicity is the same for both partition functions. However, the externality-free value and the direction of the externality-based transfers do vary. The externality-free value yields $(1/3, 1/3, 1/3)$ in the first partition function, and taking into account externality-based transfers in monotonic values can only help player 1 at the expense of either player 2 or 3 – positive externalities on 1 when 2 and 3 cooperate. In contrast, σ^* yields $(2/3, 1/6, 1/6)$ in the second partition function, *and taking externalities into account can only hurt player 1 to favor 2 and 3 – negative externalities for 1 when 2 and 3 get together.*

Bolger's (1989) work is based on the additivity axiom, yet he characterizes a unique value, while using the weak dummy axiom as in Macho-Stadler *et al* (2004), by introducing an axiom of (expected) marginality. Bolger applies this axiom only on the class of "simple games." The expectation is computed by assuming that a player has an equal chance of joining any of the other atoms when he is leaving a group. Macho-Stadler *et al* (2004) show that Bolger's value cannot be obtained through the "average approach" and that it violates their axioms of strong symmetry and similar influence. Clearly, it also violates our null player axiom.

Example 8 *Consider the following three-player partition function, where*

$$v(N) = v(\{2,3\}) = 1;$$

$$v(\{1,2\}) = v(\{1,3\}) = 1/2;$$

$$v(\{2\},\{\{1\},\{2\},\{3\}\}) = a, \qquad v(\{3\},\{\{1\},\{2\},\{3\}\}) = a;$$

$$v(\{2\},\{\{2\},\{1,3\}\}) = 1-a, \qquad v(\{3\},\{\{3\},\{1,2\}\}) = 1-a;$$

$$v(\{1\},\{\{1\},\{2\},\{3\}\}) = v(\{1\},\{\{1\},\{2,3\}\}) = 0.$$

Because of the expected marginality axiom, player 1 is null "in average": for example, if he abandons coalition $\{1,2\}$ *and there is equal probability of him being alone or joining player 3, his expected marginal contribution (as defined by Bolger) to player 2 is 0. Bolger's value assigns to this partition function the payoffs* $(0,1/2,1/2)$*. This is true with independence of the size of the parameter* a*. In contrast,*

$$\sigma^*(v) = (\frac{0.5-a}{3}, \frac{2.5+a}{6}, \frac{2.5+a}{6}).$$

That is, as a *increases, the externality-free value transfers surplus from player 1 to players 2 and 3.*

The bounds depend on whether the externality is positive or negative. If $a < 1/2$*, then*

$$\mu_1(v) = \sigma_1^*(v) - \frac{1-2a}{3}, \qquad \nu_1(v) = \sigma_1^*(v),$$

$$\mu_2(v) = \sigma_2^*(v), \qquad \nu_2(v) = \sigma_2^*(v) + \frac{1-2a}{6},$$

$$\mu_3(v) = \sigma_3^*(v), \qquad \nu_3(v) = \sigma_3^*(v) + \frac{1-2a}{6}.$$

So, if externalities are to be taken into account in the determination of the solution, players 2 and 3 should expect positive transfers from player 1, with respect to the externality-free value.

However, the opposite happens if $a > 1/2$*:*

$$\mu_1(v) = \sigma_1^*(v), \qquad \nu_1(v) = \sigma_1^*(v) + \frac{2a-1}{3},$$

$$\mu_2(v) = \sigma_2^*(v) - \frac{2a-1}{6}, \qquad \nu_2(v) = \sigma_2^*(v),$$

$$\mu_3(v) = \sigma_3^*(v) - \frac{2a-1}{6}, \qquad \nu_3(v) = \sigma_3^*(v).$$

Macho-Stadler *et al* (2004) use a "similar influence" axiom to uniquely characterize a value within the class of solutions that they term the "average approach." In the previous example, the value identified by Macho-Stadler *et al* coincides with Bolger's and hence, it is insensitive to the parameter a.[10] To better understand the differences between Macho-Stadler *et al*'s and our approach, here is another example.

Example 9 *Suppose there are 101 players. Let* $v(N) = v(\{1\}, \{\{i\}_{i \in N}\}) = 1$ *and* $v(S, \Pi) = 0$ *otherwise. Macho-Stadler* et al*'s value coincides with the Shapley value of the average characteristic function* $\bar{v}$*, where* $\bar{v}(\{1\}) = 1/100!$*, a very small positive number. Their value thus prescribes something extremely close to equal split, paying about* $1/101$ *per player (player 1 getting a tiny bit more than the others). According to our approach, the externality-free value pays* $2/101$ *to player 1, and* $99/10100$ *to each of the others. Player 1 gets more than double of the share of the others if one ignores externalities. Our cube allows the range of payoffs for player 1 to vary between* $1/101$ *and* $2/101$*, underlining the fact that he will have to worry about making transfers to the others to bribe them not to cooperate. For the others, the range of payoffs places the externality-free value at the bottom, from which they can only improve through transfers from player 1. This seems to capture better what's going on in the problem. Indeed, the partition function is "strongly asymmetric" from player 1's point of view, and so it is counterintuitive to prescribe essentially the equal split, as the Macho-Stadler* et al *value does. Our cube of payoffs and the position of* σ^* *in it seem to better capture the "strong*

[10] The same payoff is prescribed for that partition function by the value proposed by Albizuri *et al* (2005), because of their embedded coalition anonymity axiom. This value is one of the "average approach" values of Macho-Stadler *et al* (2004).

asymmetry" of the partition function.

Fujinaka (2004) introduces different notions of marginalism, using exogenous weights to aggregate the different scenarios that could follow the departure of a coalitional member. Particularly, his axiom boils down to Marginality if one puts all the weight on the scenario where a player stays on his own after leaving a coalition. Hence it appears that Fujinaka obtained independently a result similar to our third proposition. His proof differs substancially from ours and does not make use of the basis we uncovered in Lemma 1. It is not clear whether his proof can be adapted to apply on the important subclass of superadditive partition functions, as ours does (see Section 11). Our decomposition of the total effect into the intrinsic marginal contribution and the externality effect at the beginning of Section 5 shows that, if one has to choose one specific scenario within the class of scenarios considered by Fujinaka, the one where a player stays on his own after leaving a coalition is well motivated and appealing. Actually, many of Fujinaka's values are not monotonic.

Example 10 *Consider the partition function from Example 5. Recall that Efficiency, Anonymity and Monotonicity of σ determine the following payoff intervals:*

$$\sigma_1(v) \in [7.5, 10.5]; \quad \sigma_2(v) \in [6.5, 8]; \quad \sigma_3(v) \in [7, 8.5].$$

Using Fujinaka's notation, consider the case where $\alpha_i(\{j\}, \{\{j\}, \{i\}, \{k\}\}) = -1$, and $\alpha_i(\{j\}, \{\{j\}, \{i, k\}\}) = 2$. The associated value yields the payoff $(13.5, 5, 5.5)$, which falls outside the cube.

Maskin (2003) also provides an axiomatic treatment of coalitional problems with externalities, although his axioms are best understood in the context of his specific strategic model. The best comparison between his work and ours is drawn from our coalition formation analysis, in the sequel.

9 Moving Away from the Grand Coalition: Other Exogenous Coalition Structures

Thus far we have been assuming that the grand coalition forms. We want to relax this assumption. We assume in this section that a specific coalition structure has formed, for some exogenous reasons, while leaving the endogenous formation of coalitions, and its implications in terms of efficiency, for the next section.

Suppose that a coalition structure Π, not necessarily the grand coalition, has materialized. All our previous results can now be applied to each atom S of Π in the following way. An (S,Π)-*value* is a function $\sigma^{(S,\Pi)}$ that assigns to every partition function v (defined for N, as before) a unique utility vector $\sigma^{(S,\Pi)}(v) \in \mathbb{R}^S$. It is then straightforward to phrase the properties of (S,Π)-Anonymity, (S,Π)-Efficiency, Weak (S,Π)-Marginality, (S,Π)-Monotonicity, and (S,Π)-Marginality, simply by thinking of S as a grand coalition on its own, whose members are taking Π_{-S} as given. For instance, (S,Π)-Efficiency would require that $\sum_{i\in S}\sigma_i^{(S,\Pi)}(v) = v(S,\Pi)$. The details are left to the interested reader.

Let Sh^S denote the Shapley value for characteristic functions defined over S:

$$\mathrm{Sh}_i^S(v) := \sum_{T\subseteq S \text{ s.t. } i\in T} \frac{(t-1)!(s-t)!}{s!}[v(T) - v(T_{-i})]$$

for each player $i \in S$, and each characteristic function v defined over S.

Proposition 1' Let $\sigma^{(S,\Pi)}$ be an (S,Π)-value that satisfies (S,Π)-Anonymity, (S,Π)-Efficiency, and Weak (S,Π)-Marginality. Let u be a symmetric partition function, and let v be a characteristic function. Then $\sigma_i^{(S,\Pi)}(u+v) = \frac{u(S,\Pi)}{s} + Sh_i^S(v_S)$, where v_S is the reduced characteristic function defined over S by $v_S(T) = v(T)$, for each $T \in P(S)$.

A slightly stronger result holds, as symmetry for u and the absence of externalities for v can be required over S only. We will not need such a stronger result in the remainder, and so we prefer instead to keep notation simple. Combining Proposition 1' with (S,Π)-Monotonicity, we will obtain bounds $\mu_i^{(S,\Pi)}$ and $\nu_i^{(S,\Pi)}$, computed as in Section 4.

Proposition 2' If $\sigma^{(S,\Pi)}$ is an (S,Π)-value that satisfies (S,Π)-Anonymity, (S,Π)-Efficiency, and (S,Π)-Monotonicity, then

$$\sigma_i^{(S,\Pi)}(v) \in [\mu_i^{(S,\Pi)}(v), \nu_i^{(S,\Pi)}(v)],$$

for each $i \in S$.

For each partition function v, let $v^*_{(S,\Pi)}$ be the fictitious characteristic function defined over S as follows:

$$v^*_{(S,\Pi)}(T) = v(T, \{T\} \cup \{\{j\}_{j\in S\setminus T}\} \cup \Pi_{-S})$$

for each $T \in P(S)$. Let then $(\sigma^{(S,\Pi)})^*$ be the (S,Π)-value defined as follows:

$$(\sigma^{(S,\Pi)})^*(v) = Sh^S(v^*_{(S,\Pi)}),$$

for each partition function v.

Proposition 3' $(\sigma^{(S,\Pi)})^*$ is the only (S,Π)-value satisfying (S,Π)-Anonymity, (S,Π)-Efficiency, and (S,Π)-Marginality.

10 Coalition Formation: Endogenous Coalition Structures

Our next task is to suggest how one could endogenize the coalition structure. We propose an explanation based on a two-stage process. The players first decide with whom they want to cooperate. The outcome of this first stage is a coalition structure Π. Payoffs are then determined within each atom of Π in the second stage. Different scenarios come to mind for determining the payoffs. One could think for instance of many explicit bargaining procedures, although here we will pursue a reduced form approach using the results listed in the previous section.[11] To summarize, players strategize across the board when forming coalitions, but are bound by the partition itself once the coalitions are formed. Apart from the simplicity of the analysis, this two-stage process may describe some relevant scenarios, in which players' commitments to coalitions precede the determination of individual payoffs.

Let x be the function that summarizes the outcome of the second stage of the game. It is thus a function that associates a vector in $\mathbb{R}^N$ to each coalition structure. With our approach, we fix a list $(\sigma^{(S,\Pi)})_{(S,\Pi)\in EC}$ of values, and

$$x_i(\Pi) = \sigma_i^{(S,\Pi)}(v),$$

for each coalition structure Π, and each player i, with S being the atom of Π to which i belongs.

We focus now on the question of coalition formation. One advantage of our approach is the relative simplicity of this problem, because the players' final payoffs are given as a function of the coalitions that form.[12] Still, different bargaining scenarios may lead to

[11] Yet, the methodology we develop is independent of the axioms. Interesting conclusions can be reached by applying alternative scenarios to solve the second stage. What is key to our reasoning is the two-stage cooperation process.

[12] In the absence of externalities, i.e. when $x_i(\Pi) = x_i(\Pi')$ for each i and each Π and Π' such that the atoms to which i belongs are the same in both partitions, these problems of coalition formation have been referred to as "hedonic games." If one likes this terminology, then our problem could be referred to as an hedonic game with externalities.

different outcomes. We articulate our arguments around three solutions.

(i) The first solution is extremely appealing, when it exists. Coalition S is *strictly dominant* if each member of S ranks S as the best coalition, whatever the other players do, i.e.

$$x_i(\Pi) > x_i(\Pi')$$

for each $i \in S$, each partition Π that contains S, and each partition Π' that does not contain S. If S is strictly dominant, we can be confident that it will form in frictionless bargaining scenarios.[13] Assuming that S forms, there may now be another coalition that is strictly dominant for the remaining players (i.e. in $N \setminus S$). Continuing in this fashion may lead to a unique partition. A partition Π is *strictly dominant* if it is obtained by the iterative formation of strictly dominant coalitions. Clearly, a strictly dominant partition often fails to exist, but if there is one such partition, then it is unique.

(ii) The core is our second solution to the coalition formation stage. A coalition S *blocks* a partition Π' if

$$x_i(\Pi) > x_i(\Pi')$$

for each $i \in S$ and each partition Π that contains S. The *core* is the set of partitions that are not blocked by any coalition. Instead of trying to determine the partition that will form, we eliminate those that are unstable. We want to eliminate with confidence and this is why we require the members of the objecting coalitions to be better off whatever the other players do after the deviation.[14]Any other expectation when considering the formation of an objecting coalition would lead to a smaller core. If there is a strictly dominant partition, then this is the only partition in the core. When there is no such partition, then the core may be empty (Shenoy (1979, example 7.5)) or may contain more than one partition.

(iii) The third solution we suggest is based on a specific non-cooperative bargaining scenario. Fix an order π for the players in N. Following the order π, each player $i \in N$ announces a coalition S that contains i. The outcome of this sequential move game of perfect information is a coalition structure, in which a coalition S forms if and only if each player in S has announced the coalition S. If there is a strictly dominant partition, then this is the equilibrium outcome for every order of the players. The proposed coalition formation game is a finite horizon extensive form of perfect information. It admits at

[13]This impression is confirmed by our two other solutions. If a coalition structure Π belongs to our second or third solution, and S is strictly dominant, then $S \in \Pi$.

[14]Such a construction is reminiscent of the maximin (or α-) representations of games in strategic form.

least one subgame perfect equilibrium[15] and it is "almost always" unique.[16] On the other hand, the equilibrium outcome may depend on the ordering of the players.

10.1 Inefficiency and its Causes

We are now ready to draw some qualitative conclusions about the outcomes of our two-stage cooperation process. Our first observation is that inefficiency may be entirely due to the presence of externalities in some situations.

Example 11 *Consider again the partition function of Example 3, which describes a basic free-rider problem:*

$$v(N) = 24;$$

$$v(\{1,2\}) = 12; \quad v(\{1,3\}) = 13; \quad v(\{2,3\}) = 14;$$

$$v(\{i\}, \{\{i\}, \{j,k\}\}) = 9 \quad \textit{for all } i,j,k;$$

$$v(\{i\}, \{\{i\}, \{j\}, \{k\}\}) = 0 \quad \textit{for all } i,j,k.$$

Suppose that the value $\sigma^{(S,\Pi)}$ satisfy (S,Π)-Anonymity, (S,Π)-Efficiency and Weak (S,Π)-Marginality, for each embedded coalition (S,Π). Our previous analysis implies that the payoffs in each coalition structure are:

$$x(\{N\}) = (7.5, 8, 8.5);$$

$$x(\{1\},\{2,3\}) = (9,7,7);$$

$$x(\{2\},\{1,3\}) = (6.5,9,6.5);$$

$$x(\{3\},\{1,2\}) = (6,6,9);$$

$$x(\{1\},\{2\},\{3\}) = (0,0,0).$$

There is no strictly dominant partition. The core of the coalition formation problem is

$$\{\{N\}, \{\{1\},\{2,3\}\}, \{\{2\},\{1,3\}\}, \{\{3\},\{1,2\}\}\}.$$

Apart from each of the three partitions containing a two-player coalition and the free

[15] Hence the subgame perfect equilibrium outcomes associated to this class of bargaining procedures do not necessarily belong to the core (as the core may be empty).

[16] If each player's payoffs at the terminal nodes are different, then there exists a unique subgame perfect equilibrium, which coincides with the backwards induction outcome.

rider as a singleton, the grand coalition is also stable because of our very conservative definition of blocking: an individual player does not block because he is afraid of the "incredible threat" posed by the two-player coalition, i.e., that they will not cooperate after he leaves. Our third solution takes care of this. Whatever the order π, *the coalition structure predicted by the unique subgame perfect equilibrium has the first mover alone in his singleton coalition, rationally anticipating that after he chooses to be alone, the other two will join together in the two-player coalition, as will surely happen. Of course, the specific coalition structure that emerges from the sequential game depends on the order, which assigns different bargaining power to players as a function of how early they speak in the game: each of the partitions containing a two-player coalition is the subgame perfect equilibrium outcome of the game for a given ordering. On the other hand,* v *is superadditive, and hence the efficient coalition structure is the grand coalition. The outcome of our two-stage cooperation process admits inefficient partitions (and in the case of the sequential game, it consists exclusively of inefficient partitions). This conclusion is robust, as it holds for different solution concepts in the coalition formation stage, and does not rely on strong assumptions concerning the list of appropriate values. Here, note how the "Coase theorem logic" does not work to solve inefficiencies by means of bargaining: the application of normative criteria within each coalition ties the players' hands while negotiating to form coalitions.*

Consider now the modified partition function v' *where the agents do not benefit from positive externalities:*

$$v'(N) = 24;$$

$$v'(\{1,2\}) = 12; \quad v'(\{1,3\}) = 13; \quad v'(\{2,3\}) = 14;$$

$$v'(\{i\}, \{\{i\}, \{j,k\}\}) = v'(\{i\}, \{\{i\}, \{j\}, \{k\}\}) = 0 \quad \textit{for all } i, j, k.$$

This is a characteristic function. Hence the outcomes obtained by applying the list of values $(\sigma^{(S,\Pi)})_{(S,\Pi)\in EC}$ *must coincide with the Shapley value of the reduced characteristic functions. We have:*

$$x'(\{N\}) = (7.5, 8, 8.5);$$

$$x'(\{1\}, \{2,3\}) = (0, 7, 7);$$

$$x'(\{2\}, \{1,3\}) = (6.5, 0, 6.5);$$

$$x'(\{3\}, \{1,2\}) = (6, 6, 0);$$

$$x'(\{1\}, \{2\}, \{3\}) = (0, 0, 0).$$

The efficient coalition structure, $\{N\}$, is now strictly dominant and is thus expected to be the outcome of most bargaining scenarios for the coalition formation stage. We conclude that the inefficiency may be entirely attributed to the presence of externalities in some situations.

Our second observation is that the presence of externalities is not necessarily the only factor responsible for inefficiency. We elaborate on this point in the next two examples.

Example 12 *Consider the partition function in Example 5. Suppose that the value $\sigma^{(S,\Pi)}$ satisfy (S,Π)-Anonymity, (S,Π)-Efficiency and (S,Π)-Monotonicity, for each embedded coalition (S,Π). Our previous analysis implies that the payoffs in the second stage are:*

$$x(\{N\}) = (x_1, x_2, x_3), \qquad 7.5 \leq x_1 \leq 10.5, \quad 6.5 \leq x_2 \leq 8, \quad 7 \leq x_3 \leq 8.5;$$

$$x(\{1\},\{2,3\}) = (9,7,7);$$

$$x(\{2\},\{1,3\}) = (6.5,0,6.5);$$

$$x(\{3\},\{1,2\}) = (6,6,0);$$

$$x(\{1\},\{2\},\{3\}) = (0,0,0).$$

Hence, one anticipates that different normative principles will lead to different coalition structures. To simplify the discussion, let us focus on the linear solutions introduced in Example 1. For instance, suppose that σ^0 (or σ^*) is the relevant solution to apply. The players' payoffs if the grand coalition forms are thus $(7.5, 8, 8.5)$. There is no strictly dominant partition. The core of the coalition formation game consists of the partitions $\{\{N\}, \{\{1\},\{2,3\}\}\}$. In the sequential game, the equilibrium is the latter partition if player 1 is the first mover, but it is the grand coalition otherwise. On the other hand, if $\sigma^{3/5}$ is the relevant solution to apply, then the players' payoffs if the grand coalition forms are $(9.3, 7.1, 7.6)$. The grand coalition is now strictly dominant. We conclude that, for a given solution of the second stage of the cooperation process, different bargaining scenarios to describe the first stage may lead to different predictions, some efficient, others not. Moreover, the application of different normative principles in the second stage may also change our predictions in terms of the overall efficiency of the cooperation process.

Example 13 *Let $N = \{1,2,3\}$ be the set of agents and let v be the following characteristic function:*

$$v(N) = 18;$$

$$v(\{1,2\}) = 16; \quad v(\{1,3\}) = 14; \quad v(\{2,3\}) = 12;$$

$$v(\{i\}) = 0 \quad \textit{for all } i.$$

The outcomes obtained by applying a list of values $(\sigma^{(S,\Pi)})_{(S,\Pi)\in EC}$ *that satisfy* (S,Π)*-Anonymity,* (S,Π)*-Efficiency and Weak* (S,Π)*-Marginality, must coincide with the Shapley value of the reduced characteristic functions. We have:*

$$x(\{N\}) = (7,6,5);$$

$$x(\{1\},\{2,3\}) = (0,6,6);$$

$$x(\{2\},\{1,3\}) = (7,0,7);$$

$$x(\{3\},\{1,2\}) = (8,8,0);$$

$$x(\{1\},\{2\},\{3\}) = (0,0,0).$$

The coalition structure $\{\{1,2\},\{3\}\}$ *is strictly dominant (first eliminate* $\{1,2\}$ *and then* $\{3\}$*). Hence it is also the unique core partition and it is the outcome of our third solution, for the six possible orderings of the players. Yet* v *is superadditive and hence the grand coalition is the only efficient outcome.*

10.2 Sufficient Conditions for Efficiency

So far we have emphasized the possibility of inefficient coalition structures as the prediction of the theory. The reason for these results is that inefficiencies are actually the result of insisting on normative principles of fairness to determine the payoffs once coalitions have formed. These inefficiency results yield coalition structures other than the grand coalition, even for superadditive partition functions. On the other hand, if the surplus to share when everyone cooperates is large enough, then one expects the grand coalition to be efficient, and actually to form. We are able to formalize this intuition in the context of our coalition formation theory, thanks to the normative principles introduced earlier.

Proposition 5 *Suppose that the value* $\sigma^{(S,\Pi)}$ *satisfies* (S,Π)*-Anonymity,* (S,Π)*-Efficiency and Weak* (S,Π)*-Marginality. Consider a partition function that can be decomposed into the sum of a symmetric partition function* u *and a characteristic function* v*. If*

1. $\frac{u(N,\{N\})}{n} \geq \frac{u(S,\Pi)}{s}$*, for each embedded coalition* (S,Π)*, and*

2. *v is convex: $v(S') - v(S' \setminus \{i\}) \geq v(S) - v(S \setminus \{i\})$ for each i, S, S' such that $i \in S \subsetneq S'$, and*

3. *either all the inequalities appearing in 1, or all the inequalities appearing in 2, are strict,*

then $\{N\}$ is the unique efficient coalition structure, it is strictly dominant, it is the unique element in the core, and it is the unique subgame perfect equilibrium outcome of the sequential game of coalition formation, for each ordering of the players.

Proof: Condition 1 implies that the grand coalition is efficient for u. Indeed, let $\Pi = (S_1, \ldots, S_K)$ be a partition of N. We have:

$$\sum_{k=1}^{K} u(S_k, \Pi) = \sum_{k=1}^{K} \sum_{i \in S_k} \frac{u(S_k, \Pi)}{s_k} \leq \sum_{i \in N} \frac{u(N, \{N\})}{n} = u(N, \{N\}).$$

It is not difficult to check that convexity implies superadditivity for v. Hence the grand coalition is efficient for v. Condition 3 then implies that the grand coalition is the only efficient coalition structure for $u + v$.

Proposition 1' implies that

$$x_i(\Pi) = \sigma_i^{(S,\Pi)}(u + v) = \frac{u(S, \Pi)}{s} + Sh_i^S(v_S),$$

for each $i \in S$, and each embedded coalition (S, Π) such that $i \in S$.

Observe that

$$Sh_i^S(v_S) = Sh_i^N(v'), \tag{3}$$

for each $i \in S$, and each $S \in P(N)$, where v' is the characteristic function defined by $v'(S') = v(S' \cap S)$, for each $S' \in P(N)$. Though a bit tedious, this property can be checked via the formula of the Shapley value. A more elegant way to proceed is to apply Young's (1985) result. Let S be a coalition, and let σ be the S-value defined as follows: $\sigma_i(\hat{v}) = Sh_i^N(\hat{v}')$, for each characteristic function $\hat{v}$ defined for S, with $\hat{v}'(S') = \hat{v}(S' \cap S)$ for each $S' \in P(N)$. It is not difficult to check that σ satisfies S-Anonymity, S-Efficiency,[17] and S-Marginality. Hence $\sigma = Sh^S$.

The convexity of v implies that

$$v'(S') - v'(S' \setminus \{i\}) \leq v(S') - v(S' \setminus \{i\})$$

[17]Notice indeed that players in $N \setminus S$ are null in $\hat{v}'$.

for each $S' \in P(N)$ that contains i. Monotonicity for the Shapley value implies that

$$Sh_i^S(v_S) = Sh_i^N(v') \leq Sh_i^N(v). \tag{4}$$

The inequality is strict, if v is strictly convex.

Combining equation (4) with conditions 1 and 3 from the statement,

we conclude that $\{N\}$ is strictly dominant. As already observed before, this also implies that $\{N\}$ is the only element in the core, and the unique subgame perfect equilibrium outcome of the sequential game of coalition formation, for each ordering of the players. □

Two particular cases of interest are obtained by taking either u or v as the null partition function. We also remark that condition 3 is important for Proposition 5 to hold. For example, suppose $v(S) = s$ for every $S \subseteq N$. For this characteristic function, every partition is in the core and is supported by a subgame perfect equilibrium of the coalition formation game.

If the partition function cannot be decomposed into the sum of a symmetric partition and a characteristic function, then the bounds can prove useful to obtain sufficient conditions for efficiency.

Proposition 6 *Suppose that the value $\sigma^{(S,\Pi)}$ satisfies (S,Π)-Anonymity, (S,Π)-Efficiency and (S,Π)-Monotonicity. If the partition function v is such that*

$$\nu_i^{(S,\Pi)}(v) < \mu_i^{(N,\{N\})}(v), \tag{5}$$

for each member i of S, and each embedded coalition (S,Π), then $\{N\}$ is the unique efficient coalition structure, it is strictly dominant, it is the unique element in the core, and it is the unique subgame perfect equilibrium outcome of the sequential game of coalition formation, for each ordering of the players.

Proof: Let $\Pi = (S_1, \ldots, S_K)$ be a partition of N. We have:

$$\begin{aligned}\sum_{k=1}^K v(S_k,\Pi) &= \sum_{k=1}^K \sum_{i\in S_k} \sigma_i^{(S_k,\Pi)}(v)\\ &\leq \sum_{k=1}^K \sum_{i\in S_k} \nu_i^{(S,\Pi)}(v)\\ &< \sum_{i\in N} \mu_i^{(N,\{N\})}(v)\\ &\leq \sum_{i\in N} \sigma_i^{(N,\{N\})}(v)\\ &= v(N,\{N\}).\end{aligned}$$

Hence the grand coalition is the unique efficient coalition structure. The rest of the theorem follows at once from Proposition 2'. □

Remember that the grand coalition was not always strictly dominant in Example 12. An inefficient coalition structure was a plausible outcome for some monotonic values. Proposition 6 implies that in that example the grand coalition is strictly dominant for any monotonic value if $v(N) > 27.5$, or if $v(\{2,3\}) < 9.5$. Even though it is easy to construct examples where Proposition 6 is applicable, it would be interesting to determine more primitive conditions on v that would ensure (5).

We have argued that the externality-free value, if taken alone, is not too interesting for determining the payoffs if the grand coalition forms, as part of the partition function is discarded in its computation. The profile of externality-free values (as characterized in Proposition 3'), on the other hand, generates a non-trivial problem of coalition formation, as the payoffs within S depends on the organization of $N \setminus S$.

We say that the externalities have a *negative impact* on player i's (intrinsic) marginal contributions if

$$mc_{(i,S,\Pi)} \leq mc_{(i,S,\Pi')}$$

for each embedded coalition (S,Π) that contains i, and each partition Π' that contains S and that is finer than Π.

Proposition 7 *Let $\sigma^{(S,\Pi)} = (\sigma^{(S,\Pi)})^*$, for each embedded coalition (S,Π). Let v be a partition function such that:*

1. *the associated characteristic function v^* defined in section 5 is strictly convex: $v^*(S') - v^*(S' \setminus \{i\}) > v^*(S) - v^*(S \setminus \{i\})$ for each i, S,S' such that $i \in S \subsetneq S'$, and*

2. *externalities have a negative impact on the players' marginal contributions.*

Then $\{N\}$ is the unique efficient coalition structure, it is strictly dominant, it is the unique element in the core, and it is the unique subgame perfect equilibrium outcome of the sequential game of coalition formation, for each ordering of the players.

Proof: We first prove that $\{N\}$ is the unique efficient coalition structure. Observe that externalities must be negative in v. Indeed, let (S,Π) and (S,Π') be two embedded coalitions such that Π is coarser than Π'. We may assume without loss of generality (up to a relabeling of the players) that $S = \{1,\ldots,s\}$. Let

$$\xi_i = v(\{1,\ldots,i\},\{\{1,\ldots,i\}\} \cup \{\{j\}|j = i+1,\ldots,s\} \cup \Pi_{-S})$$

$$-v(\{1,\ldots,i-1\},\{\{1,\ldots,i-1\}\}\cup\{\{j\}|j=i,\ldots,s\}\cup\Pi_{-S})$$

$$\xi'_i = v(\{1,\ldots,i\},\{\{1,\ldots,i\}\}\cup\{\{j\}|j=i+1,\ldots,s\}\cup\Pi'_{-S})$$
$$-v(\{1,\ldots,i-1\},\{\{1,\ldots,i-1\}\}\cup\{\{j\}|j=i,\ldots,s\}\cup\Pi'_{-S})$$

for each $i \in \{1,\ldots,s\}$. Observe that $v(S,\Pi) = \sum_{i=1}^{s}\xi_i$ and $v(S,\Pi') = \sum_{i=1}^{s}\xi'_i$. On the other hand, $\xi_i \leq \xi'_i$, for each $i \in \{1,\ldots,s\}$, because the externalities have a negative impact on the players' marginal contributions. Hence $v(S,\Pi) \leq v(S,\Pi')$. In particular, $v(S,\Pi) \leq v^*(S)$, for each embedded coalition (S,Π). Hence, for every partition $\Pi = \{S_1,\ldots,S_K\}$, we have:

$$\sum_{k=1}^{K} v(S_k,\Pi) \leq \sum_{k=1}^{K} v^*(S_k) < v^*(N) = v(N,\{N\}).$$

The strict inequality is a consequence of the strict superadditivity of v^* (implied by strict convexity, as already argued in the proof of Proposition 5).

Let (S,Π) be an embedded coalition and let i be a member of S. Consider the characteristic functions w and $(v^*_S)'$ defined on as follows:

$$w(S') = v(S'\cap S,\{S'\cap S\}\cup\Pi_{-S,+(S'\cap S)}\cup\{\{j\}|j\in S\setminus(S'\cap S)\})$$

$$(v^*_S)'(S') = v(S'\cap S,\{S'\cap S\}\cup\{\{j\}|j\in N\setminus(S'\cap S)\})$$

for $S' \subseteq N$. We have:

$$\begin{aligned}
x_i(\Pi) &= Sh_i^S(v^*_{(S,\Pi)}) \\
&= Sh_i^N(w) \\
&\leq Sh_i^N((v^*_S)') \\
&< Sh_i^N(v^*) \\
&= \sigma^*_i(v),
\end{aligned}$$

and hence $\{N\}$ is strictly dominant, which also implies that it is the unique element in the core, and that it is the unique subgame perfect equilibrium outcome of the sequential game of coalition formation, for each ordering of the players. The first equality follows from the definition of x and $(\sigma^{(S,\Pi)})^*$. The second equality is similar to (3) in the proof of Proposition 5. As for the weak inequality, observe that the marginal contributions of player i is not smaller in $(v^*_S)'$ than in w, because externalities have a negative impact on the players' marginal contributions. The inequality then follows, since the Shapley value is a weighted sum of these marginal contributions. The strict inequality follows from the strict convexity of v^* (similar to the argument developed in the penultimate paragraph in the proof of Proposition 5). The last equality follows from the definition of σ^*. □

As already observed in the proof of Proposition 7, if externalities have a negative impact on the players' marginal contributions, then externalities are negative. Negative externalities are not enough, though, to guarantee the formation of the grand coalition. Consider for instance the following four-player partition function: $v(N, \{N\}) = 1$, $v(\{2\}, \{\{1\}, \{2\}, \{3, 4\}\}) = v(\{2\}, \{\{2\}, \{1, 3, 4\}\}) = -10$, $v(S, \Pi) = 1/3$ for each embedded coalition (S, Π) such that $\{3, 4\} \subseteq S$ and $S \neq N$, and $v(S, \Pi) = 0$ for all the remaining embedded coalitions. Externalities are negative and v^* is convex (a slight variation would make it strictly convex). Yet $\{N\}$ is not strictly dominant, because $x_1(\{1, 2\}, \{3, 4\}\}) = 5 > 1/4 = x_1(\{N\})$. The coalition structure $\{\{1, 2\}, \{3, 4\}\}$ actually belongs to the core (as well as $\{N\}$). It is also the unique subgame perfect equilibrium outcome of the sequential solution associated to the ordering $(1, 3, 4, 2)$.

11 A Fully Normative Treatment of Superadditive Partition Functions

We have seen that our two-stage cooperation process may lead to inefficiencies, even for superadditive partition functions. An alternative route is to apply a fully normative analysis to solve simultaneously the problem of coalition formation and of payoff distribution. Indeed, the axiom of efficiency could be rewritten to express overall efficiency, instead of efficiency relative to a specific coalition that has cristalized. Which approach (the fully cooperative or the two-stage process) is more relevant depends on the context.

If the partition function is superadditive, then overall efficiency is equivalent to requiring that the sum of the payoffs exhaust the surplus associated with the grand coalition. It is thus important to observe that our axiomatic results remain valid on the class of superadditive partition functions.[18]

Young (1985) notice on page 71 that his proof may be adapted so that the argument involves only superadditive characteristic functions. A similar idea can be pursued to adapt the proofs of Propositions 1 and 3 once restricted to the class of superadditive

[18]If a value satisfy some properties on a large class of partition functions, then so does it on any smaller class. Nevertheless, there could be more values satisfying those properties on smaller classes, hence the necessity to adapt some arguments when it comes to prove uniqueness or the tightness of the bounds.

partition functions. We have: 1) If σ is a value that satisfies Anonymity, Efficiency and Weak Marginality on the class of superadditive partition functions, and if v is a superadditive partition function such that $v = u + v'$, for some $(u, v') \in \mathcal{S} \times \mathcal{C}$,[19] then $\sigma_i(u+v) = \frac{u(N)}{n} + Sh_i(v')$, for each $i \in N$; 2) σ^* is the unique value satisfying Anonymity, Efficiency and Marginality on the class of superadditive partition functions. Let's show how to adapt the proof of Proposition 3 to obtain this second statement. The details regarding the modified Proposition 1 are left to the interested reader. The problem is to prove uniqueness.[20] For this, we focus on the set $\mathcal{V}$ of superadditive partition functions that can be written as a difference $u - \sum_{(S,\Pi)\in EC} \beta(S,\Pi)e_{(S,\Pi)}$, for some symmetric partition function u, and some non-negative real numbers $\beta(S,\Pi)$. We can apply an inductive argument as before to prove that any value that satisfies Anonymity, Efficiency and Weak Marginality must coincide with σ^* on $\mathcal{V}$. Notice that superadditivity is preserved through summation, and that each partition function in the basis is superadditive. Deleting any of the terms of the form $\beta(S,\Pi)e_{(S,\Pi)}$ leads to another superadditive partition function, and hence one is sure to remain within $\mathcal{V}$ throughout the inductive argument. The last step of the proof is to show that $\mathcal{V}$ actually coincides with the entire set of superadditive partition functions. Indeed, let v be any superadditive partition function, let $\sum_{(S,\Pi)\in EC} \alpha(S,\Pi)e_{(S,\Pi)}$ be its decomposition in the basis, and let

$$\begin{cases} \bar{\alpha}(s) := \max_{(S,\Pi)\in EC \text{ s.t. } \#S=s} \alpha(S,\Pi), \text{ for each } 1 \leq s \leq n, \\ u := \sum_{(S,\Pi)\in EC} \bar{\alpha}(s)e_{(S,\Pi)}, \\ \beta(S,\Pi) := \bar{\alpha}(s) - \alpha(S,\Pi). \end{cases}$$

Then, $v = u - \sum_{(S,\Pi)\in EC} \beta(S,\Pi)e_{(S,\Pi)}$, where u is symmetric and $\beta(S,\Pi) \geq 0$ for all (S,Π), as desired.

The methodology leading to the bounds in Proposition 2 remains valid, but one has to restrict $M_i(v)$ and $N_i(v)$ to superadditive partition functions, hence possibly worsening the bounds.

Finally, one can also prove Proposition 4 on the class of superadditive partition functions, by placing all the negative terms appearing in the decomposition (2) on the left-hand side, so as to have only superadditive partition functions on both sides of the equality.

[19] u and v' need not be superadditive for the result to hold.

[20] The argument in the proof of Proposition 3 is not valid when restricting ourselves to superadditive partition functions, because the partition function v' defined in the induction is not necessarily superadditive.

12 Comparisons with the Coalition Formation Literature

The resolution of the second stage of our cooperation process (cf. Section 9) is inspired by Aumann and Drèze's (1974, Section 3) adaptation of the Shapley value to characteristic functions with an existing coalition structure. They reproduce in Section 12.6 an informal argument due to Michael Maschler to determine endogenously the coalition structure that forms. They observe that the final outcome may be inefficient for some superadditive characteristic functions. Example 13 formalizes and strengthens that observation, showing that an inefficient partition may be strictly dominant. In this context, Proposition 5 offers an interesting sufficient condition for efficiency. Our contribution is also to extend Aumann and Drèze's methodology to - and obtain some qualitative results for - problems with externalities.

Shenoy (1979) and Hart and Kurz (1983) study a similar two-stage cooperation process for characteristic functions. Shenoy introduces a dynamic solution to discuss situations where the core of the coalition formation game is empty. He studies the properties of - and relations that may exist between - the core and the dynamic solution, for different solutions of the second stage. His analysis of the Shapley value (see Section 7 of his paper) focuses mainly on "simple games." Hart and Kurz study yet another division rule once a coalition structure is formed. While Aumann and Drèze assume that a group cannot share more than the surplus it generates once it forms, Hart and Kurz apply the Owen value, which shares the surplus of the grand coalition whatever the coalition structure that forms. Outcomes are always efficient in the cooperation process studied by Hart and Kurz. Even though they focus only on characteristic functions, some externalities exist when the coalitions form, as a consequence of the definition of the Owen value. Their paper contains a discussion of the core, based on different possible reactions of the players left behind after the formation of a blocking coalition.

More recent papers focus on the problem of coalition formation, leaving the second stage of the cooperation process as a black box. Each player derives some utility from his membership to the different coalitions. Some authors say in that case that preferences are hedonic. Bogomolnaia and Jackson (2002) study the existence of core stable partitions and other weaker notions of stability. Banerjee *et al* (2001) introduce the concept of top coalition (similar to our concept of strict dominance when there are no externalities) to prove the non-emptiness of the core in some economic applications (see also Alcalde and Revilla (2004)). Proposition 5, applied to the case $u = 0$, could be added to that list of applications that admit a strictly dominant partition.

The problem of coalition formation for partition functions, or more generally for games with strategic externalities, has been the subject of a growing literature over the past few years; see Bloch (2002). Bloch (1996) for instance studies a specific bargaining procedure to solve the first stage of the cooperation process, while the second stage is solved by a fixed arbitrary division rule (hence adding the possibility of externalities to the hedonic games described in the previous paragraph). Beyond the fact that we study different solutions for determining the coalitions that form, we obtain interesting results by focusing on division rules that satisfy some of the normative principles introduced in the previous sections. Many other authors focus on specific bargaining procedures to simultaneously determine the coalitions that form and the distribution of the surplus (Ray and Vohra (1999), Montero (1999), Maskin (2003)). Ray and Vohra (2001) study a core-like solution, assuming that only subcoalitions can block. Our observation that inefficiency may be due entirely to the presence of externalities (see Example 11) confirms similar conclusions previously obtained in these different frameworks. Given the complexity of some of these models, attention is often restricted to symmetric partition functions. Proposition 5 is a result of that type when v equals 0.

Maskin (2003) shows that, for his cooperation process, the efficient partition forms whenever externalities are non-positive and the partition function is strictly superadditive (to be compared with the conclusion we reached in Example 13).[21] It is important to emphasize though that all these results on the efficiency of the "equilibrium" outcomes strongly depend on the type of cooperation process one considers. One can construct cooperation processes that lead to an efficient outcome with or without externalities (see Hafalir (2006) for an example related to the Shapley value). And one can also construct other bargaining processes that lead to an inefficient outcome only in the presence of externalities. In the end, the interest of these different results should be judged on their robustness and on the appeal of the cooperation process under study. In our process, the results are robust to different ways to model the first stage and to a wide class of payoff distribution rules in the second stage. Our two-stage cooperation process also describes relevant scenarios, in which players' commitments to coalitions precede the determination of individual payoffs.

[21] His result holds for three-player partition functions. Additional conditions are required for it to extend to partition functions with more than three players, as pointed out in de Clippel and Serrano (2005).

13 Conclusion

This paper has explored partition functions. Our basic approach is rooted in the concept of marginal contributions of players to coalitions. In problems involving externalities, we have argued how it is important to separate the concept of intrinsic marginal contributions from that of externalities.

The paper is divided into two parts, each corresponding to a different methodology. The first is axiomatic and presumes an exogenously given coalition structure. For instance, when the grand coalition is together, the implications of Anonymity, Monotonicity and Marginality are explored, leading to two main results. The first one establishes bounds to players' payoffs if they are to be derived from solutions that are monotonic with respect to the (weak version of) marginal contributions. The second result provides a sharp characterization of a solution that captures value-like principles, if one abstracts from the externalities. The combination of both results provides insights to the size of the Pigouvian-like transfers compatible with our normative principles. Based on this analysis, one can extend the results to arbitrary coalition structures.

In the second part of the paper, we make the coalition structure endogenous and ask questions of coalition formation. Our way of modeling coalition formation combines players' strategizing at the time of forming coalitions with the normative principles behind the value once a coalition structure is in place. We learn that those normative principles applied to every coalition are the source of inefficient coalition structures, but we are able to identify a handful of sufficient conditions on the partition function to obtain the grand coalition as the prediction of our theory.

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