Implementing the Mas-Colell Bargaining Set*

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Abstract. We provide a mechanism that approximately implements the Mas-Colell bargaining set in subgame perfect equilibrium. The mechanism is based on the definition of the Mas-Colell bargaining set, and respects feasibility in and out of equilibrium.

Resumen. Este trabajo propone un mecanismo que implementa aproximadamente el conjunto de negociación de Mas-Colell en equilibrio perfecto en subjuegos. El mecanismo está basado en la definición del conjunto de negociación de Mas-Colell, y respeta la factibilidad dentro y fuera del equilibrio.

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Andreu Mas-Colell remarked to one of us that the bargaining set does not deserve its name until its bargaining foundations are provided. Indeed, although the different versions of the bargaining set (for example, Aumann and Maschler (1964), Davis and Maschler (1963, 1967), Mas-Colell (1989), Dutta, Ray, Sengupta and Vohra (1989)) have been suggested to handle non-credible blocking by coalitions, a criticism often raised against the core, the non-cooperative implementation of bargaining sets has only recently been accomplished.

Serrano and Vohra (2001) show that several notions of bargaining sets for exchange economies cannot be Nash implemented since they do not satisfy Maskin monotonicity. The first attempts to investigate the bargaining foundations of bargaining sets are Einy and Wettstein (1999) and Perez-Castrillo and Wettstein (2000). While the former paper suggests mechanisms that violate feasibility out of equilibrium, the latter assumes the existence of extra players to overcome the same problem. However, Serrano and Vohra (2001) provide a feasible extensive form mechanism to implement the Aumann-Davis-Maschler (ADM) bargaining set in subgame perfect equilibrium. That mechanism is closely tied to the definition of the ADM bargaining set, and its subtleties make it clear that, not surprisingly, the precise details of the mechanism do matter for the result. The purpose of this note is to provide a modification of that mechanism to implement the Mas-Colell bargaining set in subgame perfect equilibrium.

Suppose there are l commodities and a set of consumers $N = \{1, \ldots, n\}$. A pure exchange economy $\mathbb{E}$ is defined as $\mathbb{E} = \{(X_i, u_i, \omega_i)_{i \in N}\}$, where $X_i \subseteq \mathbb{R}^l$, $u_i : X_i \mapsto \mathbb{R}$ and $\omega_i \in X_i$ refer to consumer i's consumption set, utility function and endowment respectively. We shall assume that for every $i \in N$, $\omega_i > 0$. We will use the convention $\gg, >, \geq$ to order vectors.

Let $\mathcal{N}$ denote the set of all non-empty subsets (coalitions) in N. For $S \in \mathcal{N}$, we use $-S$ to denote the complement of S. Given a collection of vectors or sets, one for each consumer, we will use subscripts to refer to their restrictions to a particular coalition. For example, $X_N = \prod_{i \in N} X_i$, $X_S = \prod_{i \in S} X_i$, and given $(x_i) \in X_N$, $x_S = (x_i)_{i \in S}$ and $x_{-S} = (x_i)_{i \notin S}$. We will denote by $u_S(x_S)$ the profile of utilities $(u_i(x_i))_{i \in S}$. For the grand coalition we will use $u(x)$ to denote $u_N(x_N)$.

We shall assume that the designer knows the endowments of the consumers but not their utility functions. Thus X_i and ω_i will remain fixed and economies will be distinguished simply by the utility functions of the consumers. Let $\mathcal{E}$ denote the class of economies in which for all $i \in N$, the following assumptions are made:

A1. $\forall i \in N, \quad u_i(.)$ is strictly monotonic, in the sense that $u_i(x_i) > u_i(x'_i)$ if $x_i > x'_i$.

A2. $\forall i \in N, \quad u_i(.)$ is continuous.

Each coalition S has a feasible set of consumption plans, $A_S = \{x \in X_S : \sum_{i \in S} x_i \leq \sum_{i \in S} \omega_i\}$.

An allocation $x \in A_N$ is *efficient* if there does not exist $x' \in A_N$ such that $u(x') > u(x)$. An allocation $x \in A_N$ is *individually rational* if $u(x) \geq u(\omega)$.

Given $x \in A_N$, an *objection* is a pair (S, y_S) satisfying:

(i) $y_S \in A_S$;

(ii) $u_S(y_S) > u_S(x_S)$.

Given $x \in A_N$ and an objection (S, y_S), a *counterobjection* is a pair (T, z_T) satisfying:

(i) $z_T \in A_T$;

(ii) $u_k(z_k) > u_k(y_k) \forall k \in S \cap T$

(iii) $u_k(z_k) > u_k(x_k) \forall k \in T \backslash S$.

We say that an objection is *justified* if there does not exist a counterobjection to it. The following definition was introduced in Mas-Colell (1989).[1] The (Mas-Colell) *bargaining set* $B(\mathbb{E})$ of an economy $\mathbb{E}$ is the set of efficient and individually rational allocations against which there does not exist a justified objection.

An *extensive game form or mechanism* is defined as a game tree with possibly simultaneous moves, i.e, as an array $\Gamma = (N, K, g)$, where N is the set of players, K a game tree and $g : Z \mapsto A_N$ is the outcome function, where Z denotes the set of terminal nodes of the tree K. We will use $g(z)_i$ to denote consumer i's commodity bundle corresponding to the allocation $g(z)$. The set of nodes of the tree K is denoted T. The initial node is t_0. Let M_i^t denote the set of choices available to player i at node t and let M_i denote the set of strategies of player i. Given an economy $\mathbb{E} = \{(X_i, u_i, \omega_i)\}$, the mechanism Γ defines an extensive form game $(\Gamma, \mathbb{E})$, where the payoff to the players corresponding to the strategy profile m is $u(g(m))$.

A *subgame perfect* equilibrium of a game $(\Gamma, \mathbb{E})$ is a strategy profile $\bar{m} \in M_N$ such that the restriction of the strategies to every subgame constitutes a Nash equilibrium in the subgame. Let $\mathrm{SPE}(\Gamma, \mathbb{E})$ denote the set of all allocations corresponding to subgame perfect equilibria of the game $(\Gamma, \mathbb{E})$.

A mechanism in extensive form Γ is said to *implement in subgame perfect equilibrium* the bargaining set in all economies over the class $\mathcal{E}$ if $\mathrm{SPE}(\Gamma, \mathbb{E}) = B(\mathbb{E})$ for all $\mathbb{E} \in \mathcal{E}$.

We shall construct a mechanism that is closely tied to the story underlying the Mas-Colell bargaining set. It can be seen as a multi-stage bargaining game with the following features. There is a pre-bargaining stage (the "conversation stage"), where a status quo is agreed upon before further negotiations. This stage also determines the protocol in further negotiations. In the next stage, an agent makes a proposal to a coalition (a potential 'objection'), to be sequentially ratified. A veto results in the status-quo while unanimous acceptance leads to the next stage in which potential 'counterobjections' are possible. A failed counterobjection imposes a penalty on the proposer. In equilibrium, all agents announce the same allocation, which is in the bargaining set, and the first proposal involves this status-quo allocation proposed to the grand coalition. This mechanism can be viewed as adding an additional stage to the extensive form mechanism constructed in Serrano and Vohra (1997) for implementing the core correspondence.

Our mechanism is related to the one proposed in Serrano and Vohra (2001) for implementing the Aumann-Davis-Maschler bargaining set. However, there are stubtle differences. The main difficulty involved in using the mechanism in Serrano and Vohra (2001), or any obvious modification thereof, to implement the Mas-Colell bargaining set stems from the way in which counterobjections are specified. Recall that a counterobjection used in defining $B(\mathbb{E})$ is required to make some member of the coalition strictly better-off (in contrast to counterobjections in the ADM sense, where it suffices for the members of the counter-objecting coalition to be exactly indifferent). For example, suppose $x \notin B(\mathbb{E})$ and (S, y) is a justified objection to x. While there does not exist a counterobjection (in the sense of $B(\mathbb{E})$) to (S, y), it is possible that there exists a coalition T and $z \in A_T$ such that $u_i(z_i) = u_i(y_i)$ for all $i \in S \cap T$ and $u_j(z_j) = u_j(x_j)$ for all $j \in T \setminus S$. In the kind of mechanism described in Serrano and Vohra (2001), x will be an equilibrium outcome, supported by strategies in which members of T accept the counterproposal (T, z) if the objection (S, y) is made, and the justified objection is, therefore, not made. It should be clear that imposing a small cost on the counter proposer can deter such frivolous counter proposals.[2] We shall formalize this idea by introducing an extensive form mechanism Γ'_δ with discounting, where $\delta \in (0, 1)$ denotes the common discount factor. We shall show that for δ close to 1, this mechanism approximately implements the Mas-Colell bargaining set in subgame perfect equilibrium. The set of limit points of SPE outcomes as δ converges to 1 yields the closure of $B(\mathbb{E})$. Indeed, one cannot do better. While the SPE payoff correspondence is upper hemicontinuous at $\delta = 1$, the Mas-Colell bargaining set is not closed in general (see Serrano and Vohra (2001) for an example).

We need some additional notation before defining our mechanism. For every i, pick $\epsilon_i \in \mathbb{R}^l_+$ such that $\omega_i - \epsilon_i \in \mathbb{R}^l_+$. This is possible since $\omega_i > 0$ for all i. Let Π denote the set of all permutations of N, i.e. one-to-one functions from N to N. Given $\pi = (\pi_i)$, where $\pi_i \in \Pi$ for every $i \in N$, define $p(\pi)$ to be the composition of the permutations (π_i), i.e., $p(\pi) = \pi_1(\pi_2(\ldots(\pi_i(\ldots\pi_n)\ldots)$. The i-th element of $p(\pi)$ will be denoted $p(\pi)_i$. Notice that for every $i \in N$, given π_{-i} and $\pi^* \in \Pi$, there exists $\pi'_i \in \Pi$ such that $p(\pi'_i, \pi_{-i}) = \pi^*$. In particular, any $i \in N$ can make a unilateral change in π_i to make him/herself the first player in the order p. We shall interpret $p(\pi)$ to be an endogenously determined protocol in our extensive form game.

The mechanism consists of four stages played in three periods. The pre-bargaining stage (stage 0) and stage 1 occur in period 1. Stage 2 occurs in period 2, and stage 3 occurs in period 3. Discounting takes place across periods. We describe the rules period by period.

Period $t = 1$.

Stage 0. Every player i chooses simultaneously from the choice set $M^0_i = A_N \times \Pi$. A typical choice of player i will be denoted $m^0_i = (x^i, \pi_i)$. Note that x^i refers to player i's announcement of an allocation, i.e., $x^i = (x^i_j)_{j \in N}$. We will generally use superscripts to denote an agent's announcement of a profile.

Let $m^0 = (m^0_i)$ represent the profile of stage 0 messages and let $1(m^0) = p(\pi)_1$ and $n(m^0) = p(\pi)_n$ denote the first and the last players according to the order $p(\pi)$.

If for any i and j, $x^i \neq x^j$, the outcome is that player $n(m^0)$ receives $\omega_{n(m^0)} - \epsilon_{n(m^0)}$, and all other players receive their initial endowments. If $x^i = x^j = x^*$ for all i and j in N, proceed to stage 1. In this case we will refer to x^* as the status-quo.

Stage 1. Player $i = 1(m^0)$ chooses a coalition S containing i, and $y \in A_S$. After player i's proposal, all players in $N \setminus \{i\}$ must respond. Responses occur sequentially. First, the members of S respond in the order induced by $p(\pi)$ and then the players in $N \setminus S$, using the same protocol.

(1.a) If the proposal is made to $S = N$, players in $S \setminus \{i\}$ can either accept it or default. If a player defaults in this case, the game ends and the status quo x^* is implemented immediately. If all accept the proposal, the game goes to stage 2.a.

(1.b) If the proposal is made to $S \neq N$, players in $S \setminus \{i\}$ can accept, reject or default. Players in $N \setminus S$ can either accept or reject.

If a player in $S \setminus \{i\}$ defaults, the game ends and the status quo x^* is implemented in period 2. If every player in $N \setminus \{i\}$ accepts the proposal, the game goes to stage 2.a. If a player rejects the proposal, the game moves to stage 2.b. In this case, let j be the first rejector.

Period $t = 2$.

Stage 2.a. Player i must either sign on to the proposal or default. If he signs on, the outcome is $(y, \omega_{N \setminus S})$. If he defaults, the outcome is x^*.

Stage 2.b. Player j names a coalition T that does not weakly contain S, but contains j and at least one other player, and a proposal $z \in A_T$.[3]Then, according to the protocol induced on $T \setminus \{j\}$ by the message profile of stage 0, the other players in T respond sequentially. They may accept or reject the proposal of player j. If any player in $T \setminus \{j\}$ rejects the proposal, player j receives 0. The outcome for the other players depends on the following two cases:

(i) If the first rejector belongs to $S \cap T$, the final outcome is $(y, 0_j, (\omega_k)_{k \notin S \cup \{j\}})$ if $j \notin S$, or $((y_k)_{k \in S \setminus \{j\}}, 0_j, (\omega_k)_{k \notin S})$ if $j \in S$.

(ii) If the first rejector belongs to $T \setminus S$, the final outcome is $(x^*_{-j}, 0_j)$.

If all players in $T \setminus \{j\}$ accept the proposal z, the game moves to Stage 3.

Period $t = 3$.

Stage 3. Player j makes the final choice whether to sign on to the counterproposal or to default. If he signs on, the final outcome is (z, ω_{-T}). If he defaults, the final outcome is x^*.

Payoffs from the mechanism are assigned as follows: if in the outcome that occurs in period $t = 1, 2, 3$ player $k \in N$ receives a bundle x_k, his payoff is $\delta^{t-1} u_k(x_k)$.

Thus, there is no discounting between the pre-bargaining stage and the time when a player would default and revert to the status quo. The novelty of our rules is that, if a proposal is accepted, it takes until the next period for the proposer to ratify it or not. In addition, although the first proposal may be made to subcoalitions, every player in N votes on it. While the players in the coalition have the additional option of reverting to the status quo, players outside can only resort to making a counteroffer if they do not like the proposal. Finally, the second proposal of stage 2.b is only voted on by the players in the coalition to which the counterproposal has been made. This, and the zero payoff to the proposer if the counterproposal is rejected, are the asymmetries that the counterproposal stage has with respect to the first proposal stage. These are related to the asymmetries of the bargaining set in treating objections and counterobjections.

Two remarks are in order: (1) all outcomes specified in the mechanism Γ'_δ are feasible, in and out of equilibrium; and (2) the wasteful features of the rules in stages 0 and 2 can be remedied if there are at least three agents, by allocating the wasted resources to a different agent.

Let $C \subseteq A_N$ be a subset of allocations. Denote by $\mathrm{cl}(C)$ the closure of C. We can now state and prove our result.

Theorem 1. *The mechanism Γ'_δ approximately implements in subgame perfect equilibrium the Mas-Colell bargaining set $B(\mathbb{E})$ as $\delta \to 1$ in the class of economies $\mathcal{E}$. That is, over the class of economies $\mathcal{E}$:*

(I) $B(\mathbb{E}) \subseteq \liminf_\delta SPE(\Gamma'_\delta, \mathbb{E})$.
(II) $\limsup_\delta SPE(\Gamma'_\delta, \mathbb{E}) \subseteq cl(B(\mathbb{E}))$.

Proof of Theorem 1: Part (I): We begin by showing that if $x^* \in B(\mathbb{E})$, then $x^* \in \mathrm{SPE}(\Gamma'_\delta, \mathbb{E})$ for δ close enough to 1, i.e., $x^* \in \liminf_\delta \mathrm{SPE}(\Gamma'_\delta, \mathbb{E})$. Consider a strategy profile $\bar{m}$ defined as follows:

(i) $\bar{m}_i^0 = (x^*, \pi^e)$ for all i, where π^e denotes the identity permutation.
(ii) Consider a subgame following a status quo y^* agreed upon in stage 0. If $y^* \in B(\mathbb{E})$, every player i proposes $(S, y) = (N, y^*)$ at every node of stage 1 where he has to make a proposal. If $y^* \notin B(\mathbb{E})$, every player i makes a proposal that, given the continuation strategies, maximizes his payoff.
(iii) Consider a subgame following a status quo y^* agreed upon in stage 0 and a proposal (S, y) made by player i. If $y^* \in B(\mathbb{E})$ and $(S, y) = (N, y^*)$, player $k \in N \setminus \{i\}$ defaults. Otherwise, he responds by backward induction, taking into account the continuation equilibrium. If (S, y) is accepted by $N \setminus \{i\}$, in stage 2.a player i signs on if and only if $u_i(y_i) \geq u_i(y_i^*)$.
(iv) Consider a subgame following a status quo y^* agreed upon in stage 0 and a stage 1 proposal (S, y) made by player i and rejected by player j. Then player j proposes one of the best (from his point of view) counterproposals (T, z), given the continuation strategies.
(v) In every subgame in stage 2.b following a status quo y^* agreed upon in stage 0, a proposal (S, y) rejected by j, and a counterproposal (T, z), each respondent k responds as follows.

Suppose all respondents following k, if any, accept the proposal. If $k \in T \cap S$, k accepts if and only if either one of the following conditions holds:

$$u_j(z_j) \geq u_j(y_j^*), \text{ and } \delta u_k(z_k) \geq u_k(y_k),$$

or

$$u_j(z_j) < u_j(y_j^*), \text{ and } \delta u_k(y_k^*) \geq u_k(y_k).$$

If $k \in T \setminus S$, then k accepts if and only if

$$u_j(z_j) \geq u_j(y_j^*), \text{ and } \delta u_k(z_k) \geq u_k(y_k^*).$$

Suppose k' is the first respondent following k who rejects the counterproposal. If $k \in T \cap S$, then k accepts if and only if $k' \in T \cap S$. If $k \in T \setminus S$, then k accepts if and only if $k' \in T \setminus S$.

This completes the description of all responses to a counterproposal in stage 2.b.

(vi) in stage 3, following a status quo y^* agreed upon in stage 0, a proposal (S, y), a counterproposal (T, z) made by j and accepted by $T \setminus \{j\}$, player j accepts if and only if $u_j(z_j) \geq u_j(y_j^*)$.

Obviously, (vi) is consistent with subgame perfection since stage 3 is a final stage with a one-person decision problem. By backward induction, it is also easy to see that (iii), (iv) and (v) are also consistent with subgame perfection. Here, continuity of utility functions and the fact that the acceptance thresholds are defined with weak inequalities ensure the existence of a strategy that maximizes the proposer's payoff in the subgame that starts in stage 2.b.

To see that (ii) corresponds to an equilibrium, by construction it follows that this is the case if $y^* \notin B(\mathbb{E})$. Suppose then that $y^* \in B(\mathbb{E})$ and consider a deviation by player i, who proposes (S, y). There are four potentially possible continuations after this deviation:

(a) If every player in $N \setminus \{i\}$ accepts (S, y), it must be an objection (this is without loss of generality: either (S, y) or a small perturbation thereof is an objection). Since $y^* \in B(\mathbb{E})$, there exists a counterobjection (T, z), where S is not a subset of T. Hence, for δ close enough to 1, any player in T has an incentive to reject (S, y) and propose (T, z), which will be accepted in stages 2.b and 3. But this contradicts the hypothesis that all players accept (S, y).

(b) A player $k \in S$ defaults. But in this case the status quo y^* is imposed in period 2 and the deviation is not profitable.

(c) A player j rejects the proposal (S, y), proposes (T, z) in stage 2.b, and $i \notin T$. In the continuation, j's counterproposal must be accepted by $T \setminus \{j\}$, as otherwise he would not be at a best response rejecting (S, y). But then, player i will receive either ω_i or y_i^*, so the deviation is not profitable.

(d) A player j rejects the proposal (S, y), proposes (T, z) in stage 2.b, and $i \in T$. But then there exists $k \in S, k \notin T$. By the same argument as in (c), the counterproposal (T, z) will be accepted by $T \setminus \{j\}$. Then, player k would receive in period 3 either ω_k or y_k^*. Therefore, he could deviate and default in stage 1, imposing the status quo in period 2.

We have shown that all four cases are either impossible or unprofitable continuations for player i. Hence, the strategies specified in (ii) are a best response in this subgame.

It is easy to see that (i) corresponds to best responses by all players in stage 0. It follows, therefore, that for sufficiently small discounting this profile constitutes a subgame perfect equilibrium whose outcome is $g(\bar{m}) = x^*$: hence, $x^* \in \liminf_\delta \mathrm{SPE}(\Gamma'_\delta, \mathbb{E})$.

Part (II): We now proceed to show that if $\bar{m}$ is the limit of a sequence of subgame perfect equilibria of $(\Gamma'_\delta, \mathbb{E})$ as $\delta \to 1$, then $g(\bar{m}) \in \mathrm{cl}(B(\mathbb{E}))$, i.e., $\limsup_\delta \mathrm{SPE}(\Gamma'_\delta, \mathbb{E}) \subseteq \mathrm{cl}(B(\mathbb{E}))$.

Consider a sequence of subgame perfect equilibria $\bar{m}(\delta)$ with $g(\bar{m}(\delta)) = \bar{z}(\delta)$ satisfying that $\lim_{\delta \to 1} \bar{z}(\delta) = g(\bar{m}) = \bar{z}$. Suppose $\bar{m}_i^0(\delta) = (x^i(\delta), \pi_i(\delta))$.

Claim 1. For all δ and for all $i, j \in N$, $x^i(\delta) = x^j(\delta)$.

Suppose not. Then player $j = n(\bar{m}^0(\delta))$ receives $\omega_j - \epsilon_j$. However, this player can gain by changing $\pi_j(\delta)$ to π'_j such that $j \neq p(\pi_{-j}(\delta), \pi'_j)_n$. This deviation from $\bar{m}_j(\delta)$ will result in j receiving ω_j instead of $\omega_j - \epsilon_j$, which contradicts the hypothesis that $\bar{m}(\delta)$ is a subgame perfect equilibrium.

In the sequel we refer to the status quo agreed in stage 0 as $x^*(\delta)$ and its limit as $\delta \to 1$ as x^*. A consequence of the proof of claim 1 is that, for any δ, any SPE outcome must be individually rational.

Claim 2. $u(\bar{z}) \geq u(x^*)$.

Suppose not. Then, there exists $j \in N$ and δ close enough to 1 such that $u_j(\bar{z}_j(\delta)) < u_j(x_j^*(\delta))$.

Notice that given $\pi_{-j}(\delta)$, by a suitable choice of π'_j, player j can make sure that $j = p(\pi_{-j}(\delta), \pi'_j)_1$. Suppose that by such a choice j becomes the first player in the order p and then proposes $(N, x^*(\delta))$ in stage 1. To avoid delay in going to stage 2.a, every player in $N \setminus \{j\}$ will want to default, resulting in the outcome $x^*(\delta)$. This contradicts that $\bar{z}(\delta)$ is a SPE outcome and the claim follows.

Claim 3. $u(\bar{z}) = u(x^*)$.

Suppose not. Then, by Claim 2, $u(\bar{z}) > u(x^*)$. Then, there exists δ close enough to 1 for which $u(\bar{z}(\delta)) > u(x^*(\delta))$. By monotonicity of preferences, there must exist $z' \in A_N$ and $i \in N$ such that $u(z') \gg u(x^*(\delta))$ and $u_i(z'_i) > u_i(\bar{z}_i(\delta))$. Suppose i changes her strategy to become the first player according to the protocol and proposes (N, z'). For δ close enough to 1, this proposal will be accepted by all the other players since $\delta u_j(z'_j) > u_j(x^*_j(\delta))$ for all $j \in N$ and is, therefore, a profitable deviation for player i. But this contradicts the hypothesis that $\bar{z}(\delta)$ is a SPE outcome.

A consequence of Claim 3 and its proof is that the limit of any sequence of SPE outcomes must be efficient.

The proof of the following claim is a simple application of backwards induction, and we omit it.

Claim 4. Let δ close enough to 1. Consider a subgame in stage 3 following a proposal $(S(\delta), y(\delta))$ and a counterproposal $(T(\delta), z(\delta))$, where both proposals have been accepted unanimously. If $u_j(x^*_j(\delta)) > u_j(z_j(\delta))$, then in stage 3, player j must reject, and the equilibrium outcome in this subgame must be $x^*(\delta)$.

To complete the proof, suppose $\bar{z} \notin \mathrm{cl}(B(\mathbb{E}))$. Consider a SPE outcome $\bar{z}(\delta)$ for δ arbitrarily close to 1. Of course, $\bar{z}(\delta) \notin \mathrm{cl}(B(\mathbb{E}))$. Let (S, y) be a justified objection against $\bar{z}(\delta)$ (or equivalently, against $x^*(\delta)$, the equilibrium outcome).

Let player i satisfy $u_i(y_i) > u_i(x^*_i(\delta))$. Let player i deviate from the equilibrium strategies by changing his permutation in stage 0 so as to become the proposer in stage 1. Let player i announce (S, y) in stage 1. We will show that this proposal is accepted by every $j \in N \setminus \{i\}$, contradicting that the strategy profile behind $\bar{z}(\delta)$ is a SPE. We show in the next claim that no player in $N \setminus \{i\}$ will reject this proposal made by player i.

Claim 5. Suppose (S, y) is a justified objection to $\bar{z}(\delta)$. Further, suppose $x^*(\delta)$ is the status quo agreed upon in stage 0, $u(\bar{z}(\delta)) = u(x^*(\delta))$, and that player i has made in stage 1 the proposal (S, y) in the game where δ is arbitrarily close to 1. Then no player $j \in N \setminus \{i\}$ rejects the proposal (S, y).

Suppose the Claim is false. Then $j \in N \setminus \{i\}$ counters with a proposal (T, z). We will now show that this counterproposal will not be accepted by all the responders. This will prove that player j must not reject (S, y) in a subgame perfect equilibrium, a contradiction to our first assumption. There are two cases to consider:

(a) there exists $k \neq j$, $k \in T \setminus S$ such that $u_k(x^*_k(\delta)) \geq u_k(z_k)$ or there exists $k \neq j$, $k \in S \cap T$ such that $u_k(y_k) \geq u_k(z_k)$.

Let k be the last responder for whom the above condition holds. Suppose the counterproposal is accepted by all the responders. Then the final outcome for player k is either z_k or $x^*_k(\delta)$ in stage 3. In either case, player k can do better by rejecting the proposal. This proves that the proposal z will not be accepted by all the players in $T \setminus \{j\}$.

(b) for all $k \neq j$, $k \in T \setminus S$, $u_k(x^*_k(\delta)) < u_k(z_k)$ and for all $k \neq j$, $k \in S \cap T$, $u_k(y_k) < u_k(z_k)$. Since $x^*(\delta) \notin \mathrm{cl}(B(\mathbb{E}))$ and (S, y) is a justified objection, this must mean that $u_j(z_j) < u_j(x^*_j(\delta))$. Then, we know from claim 4 that the final outcome must be $x^*(\delta)$ in stage 3. By rejecting the counterproposal, any $k \in T$, $k \neq j$ can obtain either y_k or $x^*_k(\delta)$, depending on whether $k \in S$ or $k \notin S$. In either case, rejecting the proposal is better than proceeding to stage 3 and receiving $x^*(\delta)$.

We have shown that according to the equilibrium strategies in $\bar{m}(\delta)$, a counterproposal by j will be rejected. And this will yield 0 to player j. On the other hand, accepting the proposal (S, y) made by player i and playing optimally in the continuation guarantee player j to received at least a utility of $\delta^2 u_j(\omega_j) > 0$. Clearly then (S, y) will not be rejected by j, and this completes the proof of Claim 5.

Clearly, to save discounting costs no player in $S \setminus \{i\}$ will default after (S, y). Therefore, the proposal (S, y) is accepted and player i's deviation is profitable, which is a contradiction to the fact that $\bar{z}(\delta)$ is a SPE outcome. $\square$

The implementation of the consistent bargaining set seems a more ambitious task, one which is left as an important open problem for future research. The reason is that the game called for should not fix a finite horizon. An extension of our mechanism along these lines seems to take us part of the way. In this mechanism, after the pre-bargaining stage, a sequence of proposals to coalitions is made. The members of the proposed coalition can choose first whether they default or not. To default terminates the game with either the status quo or with the implementation of a proposal made earlier to the defaulting player. If no player in the called coalition defaults, every player (including those outside of the coalition) votes on accepting or rejecting the proposal. If all accept, the proposer signs on to it in the next period or defaults to one of the previous rejected proposals where he was involved. If there is a rejection, the rejector must make a new proposal to a coalition the next period.

This mechanism yields individually rational and efficient allocations as SPE outcomes as $\delta \to 1$. The full analysis of this mechanism seems, however, far from being simple. Our conjecture is that the implementation of the consistent bargaining set can be obtained with a game along the lines suggested, where the option of reverting to previous status quos is introduced. Alternatively, one should explore other natural variants of coalitional bargaining procedures and compare their sets of equilibrium outcomes to the different versions of bargaining sets.

Endnotes

1. We abuse language slightly, as Mas-Colell (1989) did not require efficiency or individual rationality, which made his result of equivalence with Walrasian allocations the more surprising. Both requirements appear in Vohra (1991).

2. This problem can be avoided by modifying the definition of $B(\mathbb{E})$ to require counterobjections to hold with a weak inequality '$\geq$' instead of '$\gg$' (and imposing other restrictions on a counterobjection), as in Zhou (1994).

3. The restriction that S not be contained in T makes no difference for the definition of the Mas-Colell bargaining set. Specifically, as shown in Einy and Wettstein (1999), if $x \in B(\mathbb{E})$ and S is an objecting coalition, there always exists a counterobjecting coalition T such that S is not contained in T.

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