etadata, citation and similar papers at core.ac.uk
br
provided by Re

ELSEVIER

Journal of Mathematical Economics 38 (2002) 1–41

Mathematical ECONOMICS

www.elsevier.com/locate/jmateco

Incentives and the core of an exchange economy: a survey

Françoise Forges [a], Enrico Minelli [b], Rajiv Vohra [c,*]

[a] *THEMA, Université de Cergy-Pontoise, 33 Boulevard du Port, 95011 Cergy-Pontoise Cedex, France*
[b] *CORE, Université Catholique de Louvain, voie du Roman Pays 34, B-1348 Louvain la Neuve, Belgium*
[c] *Department of Economics, Brown University, Providence, RI 02912, USA*

Received 29 November 2000; received in revised form 31 October 2001; accepted 14 May 2002

Abstract

This paper provides a general overview of the literature on the core of an exchange economy with asymmetric information. Incentive compatibility is emphasized in studying core concepts at the ex ante and interim stage. The analysis includes issues of non-emptiness of the core as well as core convergence to price equilibrium allocations.
© 2002 Elsevier Science B.V. All rights reserved.

JEL classification: C71; D82; D51

Keywords: Core; Asymmetric information; Incentive compatibility; Exchange economy

1. Introduction

The core has proved to be a fruitful concept in analyzing cooperative outcomes in a general equilibrium framework with complete information. Its connections with Walrasian allocations also make it useful in understanding market economies. It is natural then to examine the core of an economy in a more realistic setting in which agents possess private information. While much has been done to understand the implications of incomplete information in non-cooperative games, as is well understood, outcomes generally depend crucially on the precise specification of the game. A cooperative approach based on the core may then be useful in so far as it abstracts from the details of the negotiation

* Tel.: +1-401-863-3836; fax: +1-401-863-1970.
E-mail addresses: francoise.forges@eco.u-cergy.fr (F. Forges), minelli@core.ucl.ac.be (E. Minelli), rajiv_vohra@brown.edu (R. Vohra).
URL: http://www.econ.brown.edu/ rvohra

0304-4068/02/$ – see front matter © 2002 Elsevier Science B.V. All rights reserved.
PII: S0304-4068(02)00073-3

procedure.[1] An additional motivation for the cooperative approach is that it may help in a better understanding of the various price equilibrium concepts that have been introduced in economies with incomplete information.

The notion of the core is based on the premise that any group of agents (a coalition) can cooperate and agree upon a coordinated set of actions which can then be enforced. A 'feasible' allocation of an economy belongs to the 'core' if no coalition can 'improve upon' it. At the outset, it should be recognized that this general description of the 'core' is ambiguous in the context of an economy with incomplete information.

First, it is necessary to be precise about the meaning of a 'feasible allocation'. In principle, an 'allocation' should now be seen as a state-contingent allocation satisfying physical resource constraints in each information state. But it is also important to distinguish between (i) the case in which private information eventually becomes publicly verifiable so that it is not necessary to impose incentive compatibility restrictions on allowable allocations, and (ii) the case in which private information is inherently unverifiable so that allowable contracts must be self-enforcing with respect to private information, in the sense of being incentive compatible.

Secondly, the meaning of 'improve upon' is not obvious. It depends on whether agents enter into coalitional contracts at the ex ante stage (before any agent receives private information) or at the interim stage (after each agent has received her private information). In the former case, expected utility (assuming von Neumann–Morgenstern utility representation) is the appropriate measure of an agent's well-being whereas in the latter case it is conditional expected utility (conditional on private information) which provides the appropriate measure.

In sum, an appropriate notion of the 'core', at the very least, must take account of whether the coalitional decision stage is ex ante or interim, and whether or not incentive constraints are relevant. This four-way taxonomy is inspired by the corresponding notions of efficiency identified in Holmström and Myerson (1983), and provides a useful perspective for viewing the literature.[2] The remainder of this section is structured with this taxonomy in mind. However, we will argue that incentive compatibility constraints are an important ingredient of a cooperative theory in asymmetric information economies, and for this reason our main focus will be on results which incorporate incentive constraints in describing the feasible allocations of each coalition.

1.1. *The ex ante stage*

Suppose cooperative agreements among agents are made at the ex ante stage for eventual consumption after the state of the world (the information state) is determined. The state of the world may include information about preferences as well as individual endowments.

[1] This is not to say that the cooperative theory should not be informed by developments in the non-cooperative theory. In fact, as we will argue, it is important even for cooperative theory, in environments with asymmetric information, to incorporate non-cooperative considerations as embodied in incentive compatibility constraints.

[2] The ex post stage, where decisions are made after the information state is known is no different from a model with complete information.

1.1.1. In the absence of incentive constraints

The simplest (classical) model to consider is one in which contracts are made ex ante but the true state of the world becomes known to all agents, and is publicly verifiable, before actual consumption takes place. In principle, therefore, any contract which satisfies the resource constraint can be enforced once it is agreed upon. (We will be assuming throughout that the number of consumers, commodities and states are all finite.) The corresponding model of an exchange economy is then the Arrow–Debreu model of contingent commodities (and symmetric uncertainty). With each commodity indexed by the state, a consumer's decision can be seen as one of choosing a state contingent commodity bundle. If these contingent commodity markets are complete, the notion of an Arrow–Debreu equilibrium is similar to that of a Walrasian equilibrium in an economy without uncertainty. Of course, one can apply to this economy the standard notion of the core. If preferences are represented by von Neumann–Morgenstern expected utility functions, an objection requires all members of a coalition to gain in terms of expected utility. The (ex ante) core is then the set of feasible state contingent allocations such that no coalition has a feasible state contingent allocation which increases the expected utility of each of its members. We shall refer to this notion of the core as the ex ante core of an Arrow–Debreu economy. It should be clear that this core bears the same relationship to the Arrow–Debreu equilibria as the core does to the Walrasian equilibria in an economy without uncertainty. In particular, an Arrow–Debreu equilibrium allocation belongs to the ex ante core of the Arrow–Debreu economy; both sets are non-empty under standard assumptions; as in Debreu and Scarf (1963), the set of allocations that remain in the core with replication converges to the set of Arrow–Debreu allocations.

It is also possible to consider a similar model but with asymmetric uncertainty in the sense that a coalition is restricted to using an allocation which is contingent only on the combined information of agents within the coalition. This translates into a corresponding measurability restriction on the allocations allowable to each coalition, as in Allen (1993) and Koutsougeras and Yannelis (1993) (discussed in Section 4). Note, however, that the resulting core contains the core of the Arrow–Debreu economy (since objections are made more difficult *only* for subcoalitions of the grand coalition) and is, therefore, non-empty.

1.1.2. Incentive compatibility

Suppose that agents make coalitional decisions at the ex ante stage but each agent receives (at the interim stage) private information which is not publicly verifiable before consumption takes place. Many interesting economic issues in the presence of incomplete information, such as adverse selection and moral hazard, pertain to such situations. In this case, the enforcement of a state contingent allocation relies on agents' claims regarding their private information. An agent who possesses information that is not available elsewhere in the economy may not have the incentive to truthfully reveal this information, i.e. a state contingent contract may be subject to strategic manipulation and, therefore, unenforceable. This is so even if there are no limits on communication among agents. More precisely, agents in a coalition may use any communication mechanism.[3] But the only allowable state

[3] In keeping with the usual story that when a coalition forms it cannot rely on the resources of the complement, it is natural to also insist that it cannot communicate with those in the complement (as in the last paragraph of Section 1.1.1). In other words, a coalition must rely on a communication mechanism defined with respect to the private information of agents within the coalition; see Section 4.1.

contingent contracts are those which are induced by Bayesian Nash equilibria of the corresponding communication game.[4] By the revelation principle (see, for example, Myerson, 1991), every such state contingent allocation is generated by a (Bayesian) incentive compatible direct mechanism. Hence, when incentive constraints are relevant, a state contingent allocation is more appropriately viewed as a direct mechanism. (We will sometimes use the term 'mechanism' even when incentive constraints are not imposed but this should cause no confusion.)

Define the set of feasible mechanisms for a coalition as those which satisfy incentive compatibility as well as the usual physical feasibility constraints. The corresponding characteristic function is now well-defined, and the *ex ante incentive compatible core* is simply the set of incentive compatible mechanisms such that no coalition can increase the expected utility of each member by choosing another incentive compatible mechanism using its own resources and information.

Non-emptiness of the ex ante incentive compatible core is not assured under the usual assumptions on the economy, as shown in Forges et al. (2002) and Vohra (1999); see Section 4.2. Positive results have, however, been obtained under a variety of additional assumptions[5] which we will discuss in Section 4.3. As shown by Forges et al. (2001), it is also possible that convergence to a corresponding notion of a market equilibrium does not obtain under the usual assumptions. As we explain in Section 6, convergence results do depend on the nature of the replication procedure.

1.1.3. (Private) measurability vs. incentive compatibility

To deal with the case in which private information does not become public, a different approach is often followed in the literature on market equilibria. If an agent trades with the anonymous market rather than with other agents directly, it is natural to require that an agent's trade be measurable with respect to her private information. The equilibrium notion introduced in Radner (1968) modifies the notion of an Arrow–Debreu equilibrium precisely by incorporating such measurability restrictions on agents' trades. An analogous ex ante core concept, the *private core*, is studied in Allen (1993) and Koutsougeras and Yannelis (1993), and a similar interim concept in Yannelis (1991).[6] The private core is related to equilibrium allocations in the sense of Radner (1968) in the same way as the core of the Arrow–Debreu economy is related to Arrow–Debreu equilibrium allocations. And non-emptiness and core convergence can be established under standard assumptions.[7]

However, the private measurability restriction, which is natural in the context of market equilibrium concepts, may not be appropriate in the context of the core. There are two reasons for this.

[4] Thus, in describing the cooperative possibilities available to a coalition it becomes necessary to rely on non-cooperative considerations in so far as contracts are contingent on private information.

[5] See Allen (1992), Forges and Minelli (2001), Forges et al. (2002), Ichiishi and Idzik (1996), McLean and Postlewaite (2000) and Vohra (1999).

[6] Measurability with respect to σ-algebras obtained from other forms of information sharing within a coalition lead to correspondingly different versions of the core; see Section 4.2 and the references cited therein.

[7] See Page (1997) and Yannelis (1991) for non-emptiness results in a model with a continuum of states, and Einy et al. (2001a) for convergence results.

First, the very notion of the core is based on agents making agreements to trade among themselves, not through an anonymous market. This clearly involves communication among agents, and it is then unreasonable to impose the restriction that an agent cannot entertain a contract which varies with information he does not possess. Of course, strategic considerations cannot be ignored in considering such a contract and incentive compatibility is therefore an important consideration. A possible rationale for requiring measurability with respect to private information is that, in an exchange economy, (under appropriate assumptions) it implies incentive compatibility; see Allen (1993), Koutsougeras and Yannelis (1993) and Section 4.1.1. But the converse is not true. There may exist many incentive compatible mechanisms only some of which (constant ones) satisfy private measurability, as in Example 2. In short, if incentive considerations are relevant they should be incorporated directly; measurability with respect to private information may be an unduly strong restriction.

Second, in a market equilibrium such as a fully revealing rational expectations equilibrium (Radner, 1979), communication through prices can make superfluous the a priori (strong) assumption of private measurability of trades. (There may exist a rational expectations equilibrium allocation which is incentive compatible but not measurable with respect to private information; see Example 2.) Of course, this form of communication is not available in the context of the core.

1.2. The interim stage

In many economic situations, agents already have private information when they contemplate engaging in state contingent trades with others. In other words, coalitions form at the interim stage rather than ex ante. As in the previous section, we begin by considering a model in which incentive constraints are not relevant and then turn to one incorporating incentive compatibility constraints. Our previous discussion on measurability in Section 1.1.3 continues to apply to the interim stage.

1.2.1. In the absence of incentive constraints

Suppose incentive constraints are not relevant. We place ourselves in the same model as in Section 1.1.1 except that agents already have their private information. In what follows, we rely crucially on the seminal contribution of Wilson (1978) on this subject. However, we shall find it convenient to formulate private information in terms of agents' types. This framework is equivalent to one in which private information is specified as a partition of the underlying set of states, as in Wilson (1978), but is especially useful in formulating incentive compatibility constraints; see Section 2 for further details. Let T_i denote the (finite) set of agent i's types. An information state then refers to a profile of types $(t_i) \in T \equiv \prod_i T_i$. The interpretation is that i knows her type, and for every $t_i \in T_i$ has a probability distribution on T_{-i} conditional on t_i. Of course, at the interim stage, the relevant utility function for an agent is then the conditional expected utility function—conditional on her type (private information). For the remainder of this section, the term 'better-off' for i of type t_i refers to an increase in the value of some conditional expected utility function, $U_i(\cdot|t_i)$.

It is not immediately obvious how the core ought to be defined for such an economy. More precisely, it is not obvious how the characteristic function should be constructed for the interim economy. What is the meaning of a coalitional improvement? Should it require *all* types of all agents to gain? As Holmström and Myerson (1983) argue, this is indeed the correct way to define an improvement for the grand coalition.[8] This may be the only way for an uninformed outsider to verify a Pareto improvement. However, this notion of domination should not be mimicked in defining objections in a coalition which is *not* the grand coalition. For example, consider the coalition consisting of agent i alone. Since i knows her type, say t_i, surely i will 'object' to a status-quo if she is better-off (an increase in $U_i(\cdot|t_i)$) with her own endowment. In other words, for an objection from a singleton coalition, $\{i\}$, it suffices that *some* type (not necessarily all types) of agent i can do better with her endowment. (The reader will notice that this is indeed consistent with the standard notion of interim individual rationality as, for example, in (10.7) and (10.8) on p. 485 of Myerson, 1991.) Fortunately, there is a formal way of defining objections for an arbitrary coalition which reconciles this seeming asymmetry in the way we have just defined objections for the grand coalition and for singleton coalitions.

The statement that all agents of *all types* can be made better-off turns out to be essentially equivalent to the statement that there is an informational event $E \subseteq T$ which is *common knowledge* to all i and all agents of *all types in* E can be made better-off (over E). And the statement that agent i of type t_i is better-off means that there is an informational event known to i (common knowledge to i) over which she is better-off. This idea, of an 'interim objection' by a coalition being common knowledge among members of the coalition, is the basis for the notion of the *coarse core* defined by Wilson (1978). A state contingent allocation belongs to the coarse core if there does not exist a coalition S, an event E which is common knowledge to all members of S, and a state contingent allocation feasible for S which makes all agents in S better-off over the event E. Wilson (1978) showed that the coarse core is non-empty under the standard assumptions on an economy. However, convergence of the coarse core to market equilibrium allocations does not generally hold, as shown by Serrano et al. (2001).

The restriction that objections be coordinated on a common knowledge event is motivated by the standard issues of adverse selection; see the examples in Wilson (1978) and Example 1.[9] While there is no doubt that coalitions should be *permitted* to object over a common knowledge event, there are situations in which it can be argued that coalitions can do more—they can share private information and thereby focus an 'objection' over an event which is not necessarily common knowledge. In the extreme case, one may allow agents in a coalition to choose how much of their private information they share among themselves, as in the *fine core* of Wilson (1978). But one can argue that this, too, is ad hoc. It is clearly desirable to develop a theory in which the amount of information shared by members of a coalition is endogenous. This issue, of information leakage, motivates the notion of a durable decision rule in Holmström and Myerson (1983). And similar ideas can be applied to develop alternative notions of the core for the interim stage, as we discuss in Section 5.3.[10]

---

[8] This is modulo the difference between an improvement and a strict improvement. See Section 3 for a formal definition, further justification, and examples.

[9] It is also possible to characterize the coarse core in terms of axioms, including appropriate notions of consistency and converse consistency, as shown by Lee and Volij (1996).

[10] While this is a conceptually difficult and as yet unsettled issue, there are several papers on the topic, including Dutta and Vohra (2001), Ichiishi and Sertel (1998), Lee and Volij (1996) and Volij (2000).

1.2.2. Incentive compatibility

If private information does not become publicly verifiable, for the reasons mentioned in Section 1.1.2, it is appropriate to introduce feasible mechanisms which satisfy incentive compatibility constraints in addition to the resource constraints.[11] It is now straightforward to define an analog of the coarse core in this setting—the *incentive compatible coarse core*—simply by restricting attention to incentive compatible and feasible allocations. Not surprisingly, incorporating incentive compatibility does make a significant difference. Note that an allocation in this core need not be first-best/classically efficient, i.e. it is possible that a mechanism in this core is interim Pareto dominated by one which is not incentive compatible. Under standard assumptions, there do exist mechanisms which are interim individually rational, incentive compatible and interim incentive efficient. Thus, the non-emptiness of the incentive compatible coarse core is not in doubt for a two-agent economy. There are other sufficient conditions, principally the case of non-exclusive information, discussed in Section 5, under which this core is non-empty. But, in general, the incentive compatible coarse core may be empty, as shown in Forges et al. (2002) and Vohra (1999). Identifying other sufficient conditions under which non-emptiness obtains remains an important issue for future work.

The rest of this paper is organized as follows. In Section 2, we introduce the basic notation and model. In Section 3, we review the Holmström and Myerson (1983) definitions of efficiency in incomplete information economies. Sections 4 and 5 discuss the core concepts and the issue of non-emptiness corresponding respectively to the ex ante and the interim stage. In Section 6, we turn to the question of core convergence.

2. The basic economy

We consider an exchange economy with n agents and l goods. The set of agents is denoted $N = \{1, \dots, n\}$. The private information of agent $i \in N$ is represented by i's type, $t_i \in T_i$, where T_i is a finite set. Let us set $T = \prod_{i=1}^{n} T_i$ and let us denote as $t = (t_i)_{i \in N}$ a typical element of T to represent the information state. Let q be a probability distribution over T. We assume, without loss of generality, that there are no redundant types, i.e. $q(t_i) > 0, \forall t_i \in T_i$. It should be stressed that $q(t) = 0$ for some $t \in T$ is allowed for. This is important since it permits the model to capture aspects of uncertainty which may be commonly known to all agents; see also footnote 24. With this in mind, it should be clear that this framework is essentially equivalent to one in which i's private information is formulated as a partition, P_i, of an underlying (finite) set of states of nature, Ω. (Given a partition P_i of Ω, let each element of P_i denote a particular type of agent i.) However, in order to define incentive compatibility it is essential to specify what the outcome is for *every* possible profile of claims regarding private information. And this, in effect, makes it necessary to consider outcomes over T.

We assume that each agent i has an initial endowment $e_i \in \mathbb{R}^l_+$, which does not depend on his type. Although this assumption can be relaxed (see Forges et al., 2002; Vohra, 1999),

[11] In another context, Demange and Guesnerie (2001) consider various concepts of interim cores with incentive compatibility in dominant strategies. A similar approach is followed by Hara (2000), who, in an economy with private values, proves the equivalence between the allocations in his notion of interim incentive compatible core and the (ex post) Walrasian allocations.

we refrain from doing so in the interests of simplicity. Agent i's preferences are represented by a (von Neumann–Morgenstern) utility function

$$u_i : T \times \mathbb{R}^l_+ \to \mathbb{R}, \quad i = 1, \dots, n$$

such that $\forall t \in T$, $u_i(t, \cdot)$ is increasing, continuous and concave. In particular, agent i's preferences can depend on the other agents' types (as in Akerlof, 1970, for instance). The basic economy is thus

$$E = \{N, (T_i, u_i, e_i)_{i \in N}, q\}.$$

The model is interpreted as follows: nature first chooses t in T according to q; every agent i is only informed of his own type t_i; consumption takes place afterwards. Three stages of information can be distinguished: ex ante, i.e. before the agents learn their types, *interim*, i.e. when every agent only knows his own type, and ex post, i.e. when all types are revealed publicly. Observe that in terms of negotiations over allocations, the ex ante stage and, even more so, the ex post stage may be fictitious; coalitional contracts may actually be negotiated at the interim stage.

Let

$$X = \left\{ x = (x_i)_{i \in N} \in (\mathbb{R}^l_+)^N \,\middle|\, \sum_{i \in N} x_i \leq \sum_{i \in N} e_i \right\}$$

denote the set of feasible allocations (in each state). A feasible (direct) *mechanism* is a function,

$$\mu : T \to X.$$

Note that a state contingent allocation is also a function from T to X. Conceptually, however, a state contingent allocation is different from a mechanism; a mechanism should be seen as a means to 'implement' a state contingent allocation.

If types are not verifiable, it becomes necessary to restrict attention to those mechanisms which are also informationally feasible. This is so even if there are no impediments to communication. Formally, agents may use any communication game in order to achieve a state contingent allocation. A communication game in our model starts with the move of nature choosing types in T according to q, specifies a set of messages M_i (or strategic choices) for each agent i and associates an outcome in X to each profile of messages, with resulting payoffs depending on types through the utility functions $u_i(\cdot)$. A (pure) strategy of agent i in this game is a mapping from T_i to M_i.[12] The informationally feasible allocations (from T to X) are those which correspond to a (Bayesian) Nash equilibrium of such a communication game. Fortunately, we do not need to consider the entire class of communication games. By the revelation principle (see, for example, Myerson, 1991), one can construct, for any Nash equilibrium of some communication game, an equivalent *truthful* Nash equilibrium of a *direct* communication game, which induces exactly the same allocation from T to X. The direct communication game, in which the set of messages

[12] In most of the paper, we focus on pure strategies and hence on deterministic mechanisms. We will turn to random mechanisms in Section 4.3.1.

of each agent i is canonically (a copy of) T_i is fully described by a mechanism μ, which should thus be viewed as defined over *reported* types. At the interim stage, every agent i must report a type $s_i \in T_i$ and receives thereafter the allocation $\mu_i(s)$, where $s = (s_i)_{i\in N}$. The conditions which express that telling the truth is a Nash equilibrium of the direct game are referred to as incentive compatibility constraints.[13]

The explicit incentive compatibility conditions are easily derived. By reporting s_i, agent i of type t_i will get expected utility

$$U_i(\mu|t_i, s_i) = \sum_{t_{-i}} q(t_{-i}|t_i)u_i[t_i, t_{-i}, \mu_i(s_i, t_{-i})]. \tag{1}$$

For $s_i = t_i$, let

$$U_i(\mu|t_i) = U_i(\mu|t_i, t_i)$$

denote the interim expected utility of agent i given his type t_i. His (ex ante) expected utility is

$$U_i(\mu) = \sum_{t_i} q(t_i)U_i(\mu|t_i).$$

Mechanism μ is *incentive compatible* if and only if

$$U_i(\mu|t_i) \geq U_i(\mu|t_i, s_i) \quad \forall i \in N, \quad \forall t_i, s_i \in T_i. \tag{2}$$

3. Efficiency

Holmström and Myerson (1983) distinguish six concepts of efficiency depending on the stage at which the agents' welfare is evaluated (ex ante, interim or ex post) and on whether incentive compatibility matters or not. They first introduce three different notions of domination for mechanisms.

Let μ and ν be feasible mechanisms:

ν *ex ante dominates* μ if and only if

$$U_i(\nu) > U_i(\mu) \quad \forall i \in N,$$

ν *interim dominates* μ if and only if

$$U_i(\nu|t_i) > U_i(\mu|t_i) \qquad \forall i \in N, \quad \forall t_i \in T_i,$$

ν *ex post dominates* μ if and only if

$$u_i(t, \nu(t)) > u_i(t, \mu(t)) \qquad \forall i \in N, \quad \forall t \in T.$$

We have departed slightly from the formal definition in Holmström and Myerson (1983) in using strict inequalities rather than weak inequalities and one strict inequality. This is

[13] We restrict ourselves to Bayesian incentive compatibility; Allen (1992, 1994) also considers an extremely strong version of incentive compatibility in dominant strategies.

simply to keep the notion of domination comparable to the way in which it is usually defined in the context of the core. It does not make any essential difference to the results we shall discuss.

Note in particular that the notion of interim domination requires that *all* types of all agents gain. This may be the only way in which an outsider can verify a Pareto improvement at the interim stage. It is also consistent with the phenomenon of adverse selection. Consider, for instance, a simple example of insurance across two states, s and t. If one agent knows the true state and the other does not, an interim Pareto improvement must ensure that the informed agent is better-off in both states; see Example 1 in Wilson (1978) and Example 1.

Let μ be a feasible mechanism; μ is ex ante (respectively, interim, ex post) *classically efficient* if and only if there is no feasible mechanism that ex ante (respectively, interim, ex post) dominates μ. Assume further that μ is incentive compatible; μ is ex ante (respectively, interim, ex post) *incentive efficient* if and only if there is no incentive compatible feasible mechanism that ex ante (respectively, interim, ex post) dominates μ.

Obviously, ex ante efficiency implies interim efficiency, which in turn implies ex post efficiency, and this holds for both the classical and incentive notions. Holmström and Myerson (1983), p. 1807, argue that only three concepts of efficiency are relevant: ex ante incentive efficiency, interim incentive efficiency and ex post classical efficiency. In particular, they define incentive ex post efficiency only for taxonomy purposes; we will therefore refer to ex post efficiency to denote the classical concept. If the agents must select a mechanism at the ex ante or the interim stage,[14] and cannot commit to report their types honestly, then incentive ex ante or interim efficiency are the appropriate efficiency concepts. It is well-known (and illustrated by Example 1) that incentive compatibility can be an important restriction in the sense that an incentive efficient mechanism need not be classically efficient.[15] In the next section, we shall extend these two notions of incentive efficiency in order to define the ex ante and the interim incentive compatible core.

We end this section with a couple of illustrative examples. Example 1 highlights the impact of incentive constraints; an interim incentive efficient mechanism need not be ex post efficient. Example 2 shows the difference between incentive compatibility and measurability restrictions on mechanisms; none of the mechanisms which are measurable with respect to private information may be interim incentive efficient.

Example 1 (Market for lemons). There are two consumers and two commodities. Suppose $T_1 = \{s, t\}$ while agent 2 is uninformed (and therefore has only one type). The information

[14] Observe that the agents can face the problem of choosing a *mechanism* only at the ex ante or the interim stage, and that such a decision problem only makes sense if they can communicate at the interim stage. We maintain these assumptions throughout the paper but mention alternative ones in Section 4.

[15] For sufficient conditions under which all incentive efficient mechanisms are first best (or classically) efficient, see Section 4.3.1. In an exchange economy with state independent endowments and monotonic preferences there always exists an ex post classically efficient mechanism which is incentive compatible—for example, a 'dictatorial' mechanism which assigns the aggregate endowment to a particular agent in all states. But, in other models, it is possible that *no* classically efficient mechanism is incentive compatible; see Holmström and Myerson (1983) for an example.

state can then be described by s or t. Suppose s and t are equally probable. Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

$$u_i(s, x^1, x^2) = x^2, \quad i = 1, 2.$$
$$u_1(t, x^1, x^2) = x^1 + x^2, \qquad u_2(t, x^1, x^2) = 1.5x^1 + x^2.$$

(Throughout, we will use superscripts to index commodities and subscripts to index consumers.) Let z_1 denote the net trade of consumer 1. The no-trade mechanism z^*, where $z_1^*(s) = z_1^*(t) = (0, 0)$, is not interim (or ex ante) efficient since the mechanism $z'_1(s) = (0, 0.1)$, $z'_1(t) = (-0.9, 1)$ interim Pareto dominates it. However, it is easy to check that z^* is interim *incentive* efficient. (An interim improvement for agent 1 in state t requires a trade at which the effective price of commodity 1 is greater than the price of commodity 2. However, incentive compatibility then implies the same trade in state s, which results in a lower expected utility for agent 2.) Note also that z^* is not ex ante incentive efficient since it is dominated (ex ante) by the (incentive compatible) trade z where $z_1(s) = z_1(t) = (-1, 0.6)$. Thus, an interim incentive efficient mechanism need not be ex post (classically) efficient nor ex ante incentive efficient. The fact that z does not interim dominate z^* points, again, to the importance of making *both types* of the informed agent better-off at the interim stage (adverse selection).

Example 2. The information structure is the same as in Example 1. The endowments (in both states) are $e_1 = e_2 = (1, 1)$, and the utility functions are

$$u_1(s, x^1, x^2) = x^1, \qquad u_1(t, x^1, x^2) = x^2,$$
$$u_2(s, x^1, x^2) = u_2(t, x^1, x^2) = x^1 + x^2.$$

Consider the mechanism with net-trades z, where $z_1(s) = (1, -1)$ and $z_1(t) = (-1, 1)$. This is incentive compatible as well as ex ante (and, therefore, also ex post) classically efficient. Clearly then, it is interim incentive efficient. Thus, incentive compatibility can be satisfied without sacrificing efficiency; the uninformed agent can safely delegate to the informed consumer the decision on how to trade.

The allocation corresponding to z is also the unique Arrow–Debreu equilibrium with the relative price equal to 1 in each state. (If in any state the relative price is not 1, the demand from agent 2 will violate the feasibility condition.) However, this allocation is not measurable with respect to the private information of the uninformed consumer, requiring her to trade contingent on information she does not possess. Since the relative prices are the same in both states, prices cannot reveal information and this allocation is, therefore, not a rational expectations equilibrium allocation. In fact there does not exist a rational expectations equilibrium. In this example, measurability is a very strong requirement while incentive compatibility is not. Notice also that the no-trade mechanism, z^*, cannot be interim dominated by *any* privately measurable mechanism. But it is interim dominated by the incentive compatible mechanism z', where $z'_1(s) = (0.9, -1)$ and $z'_1(t) = (-1, 0.9)$.

Modify the example so that $u_2(t, x^1, x^2) = \alpha x^1 + x^2$, where $\alpha \neq 1$. Then there exists a fully revealing rational expectations equilibrium but the corresponding allocation is not privately measurable.

4. The ex ante incentive compatible core

In this section, we extend the model developed in the previous two sections by allowing agents to form coalitions. We first define the ex ante incentive compatible core. We illustrate by a counter-example that the non-emptiness of this core cannot be guaranteed in general. We then identify special classes of economies in which the core is non-empty. In one of these classes, random mechanisms are crucial for the positive result.

4.1. Definition

Analogous to the definition of a feasible mechanism (for the grand coalition) $\mu : T \mapsto X$ as in Section 2, we can now define a feasible mechanism for a coalition, namely a subset of N. A mechanism μ satisfies the physical feasibility conditions for coalition S if

$$\sum_{i \in S} \mu_i(t) \leq \sum_{i \in S} e_i \qquad \forall t \in T. \tag{3}$$

Let the set of mechanisms satisfying (3) be denoted $\mathcal{F}_S$. Since a mechanism is usually interpreted as a communication device in a coalition, it is also appropriate to require it to depend only on information available within the coalition. A mechanism, μ, for S should be measurable with respect to the information available to S, i.e.

$$\mu_i(t) = \mu_i(t') \quad \forall i \in S, \quad \forall t, t' \in T : t_S = t'_S. \tag{4}$$

where $t_S = (t_i)_{i \in S}$. Let $\mathcal{F}_S^m$ denote the set of mechanisms satisfying both (3) and (4). We can now apply the definition of incentive compatibility (as in Section 2) to a feasible mechanism for coalition S. A mechanism $\mu \in \mathcal{F}_S^m$ is incentive compatible for S if it satisfies (2) for all $i \in S$. Let $\mathcal{F}_S^*$ denote the set of feasible and incentive compatible mechanisms for S (where "*" as in Holmström and Myerson, 1983 indicates incentive compatibility), i.e. $\mathcal{F}_S^*$ is the set of all mechanisms satisfying (2)–(4).

If we define $T_S = \prod_{i \in S} T_i$ and

$$X_S = \left\{ x = (x_i)_{i \in S} \in (\mathbb{R}_+^l)^S \,\middle|\, \sum_{i \in S} x_i \leq \sum_{i \in S} e_i \right\}, \tag{5}$$

then a mechanism in $\mathcal{F}_S^m$ can be seen as a mapping from T_S to X_S.[16] This formulation can sometimes be more convenient, as we will see in Section 4.3.2.

Let $\mu \in \mathcal{F}^*$ and let $\nu_S \in \mathcal{F}_S^*$ for some coalition S. In the same way as in Section 3, ν_S ex ante *dominates* μ for coalition S if and only if

$$U_i(\nu_S) > U_i(\mu) \quad \forall i \in S.$$

The ex ante *incentive compatible core* is the set of all mechanisms $\mu \in \mathcal{F}^*$ that are not ex ante dominated by any mechanism $\nu_S \in \mathcal{F}_S^*$ for any coalition S.[17]

---

[16] Using the notation of Section 2, $T \equiv T_N$, $X \equiv X_N$, etc. Note that $\mathcal{F}_N \equiv \mathcal{F} = \mathcal{F}^m$ and $\mathcal{F}_S^{m*} = \mathcal{F}_S^*$.

[17] Observe that the set of corresponding expected payoffs is just the *standard* core of the game defined by the *characteristic function*

$$V^*(S) = \{v \in \mathbb{R}^n \,|\, \exists \mu_S \in \mathcal{F}_S^* \text{ such that } v_i \leq U_i(\mu_S) \quad \forall i \in S\}.$$

We have focused on incentive compatible mechanisms that cannot be blocked by any coalition at the ex ante stage.In a similar way as in Section 3, one can also consider the "classical" ex ante core, which does not take account of incentive compatibility constraints. If incentive compatibility does not matter at all, it may be reasonable to allow coalitions to use allocations contingent on the entire type profile, i.e. to dispense with the measurability conditions (4) and consider any mechanism in $\mathcal{F}_S$. As we pointed out in Section 1.1.1, the corresponding core is then the core of an Arrow–Debreu economy with complete contingent markets, to which all classical (existence, convergence) results apply. Restricting coalition S's feasible allocations to $\mathcal{F}_S^m$ just reduces the set of objections (while $\mathcal{F}^m = \mathcal{F}$), so that the associated core is still non-empty (this core corresponds to the "fine core" in Allen, 1993 and to the "weak fine core" in Koutsougeras and Yannelis, 1993).[18]

Alternative measurability restrictions have been considered in the literature. For instance, agents in a coalition may be forbidden to communicate information in any way, leading to feasible sets for coalitions which are even more restricted than $\mathcal{F}_S^m$.[19]

More interestingly, every individual can be restricted to allocations that are measurable with respect to his own private information, which generates the "private core" (using the terminology of Yannelis, 1991; see also Allen, 1993; Ichiishi and Idzik, 1996; Hahn and Yannelis, 1997; Koutsougeras and Yannelis, 1993). A possible rationale for private measurability is that, under appropriate assumptions, it implies incentive compatibility (see, e.g. Allen, 1993; Koutsougeras and Yannelis, 1993). We turn to a clarification of this point in the next section. (The reader may move to Section 4.2 without any loss of continuity.)

4.1.1. Private measurability and incentive compatibility

In our model, a mechanism μ satisfies *private measurability* if for every i, $\mu_i(t)$ depends only on t_i. In other words:

$$\forall i \in N, \quad \mu_i(t) = \mu_i(t') \quad \forall t, t' \in T : t_i = t_i'.$$

The difference between incentive constraints and private measurability illustrated in Example 2 is relevant in comparing the corresponding core notions as well. For instance, in Example 2, the mechanism where consumer 1's net trade is given by $z_1(s) = (1, -1)$, $z_1(t) = (-1, 1)$ belongs to the ex ante incentive compatible core. However, this trade does not satisfy the requirement of private measurability with respect to consumer 2's information. The only privately measurable mechanisms are constant mechanisms, and the no-trade mechanism is the only one in the private core.

It has been noted that private measurability of μ implies incentive compatibility (see, for example, Allen, 1993; Koutsougeras and Yannelis, 1993; Vohra, 1999).[20] Since the

[18] Despite the terminology, these concepts should not be confused with the fine core introduced in Wilson, 1978 which is an interim concept. A recent paper which deals with Wilson's fine core is Einy et al., 2000; see Section 5.3.

[19] Allen (1993) and Koutsougeras and Yannelis (1993)'s "coarse core" is based on the latter assumption. Again, this notion should not be confused with the one introduced by Wilson (1978) (see Section 5).

[20] These papers assume that each consumer has an endowment which can vary with his own type. In that case private measurability of μ means that the corresponding net-trades depend only on i's types, and the conclusion of the following proposition should be read to say that the net-trades are constant with respect to the states.

definitions of private measurability as well as incentive compatibility are not the same in these papers, it is worthwhile to state a result explicitly in terms of our model and notation.

Proposition 1. *Suppose a mechanism μ satisfies private measurability and either*

(i) $\sum_i \mu_i(t) = \sum_i e_i$ *for all* t *(exact feasibility), or,*
(ii) *all utility functions are strongly monotonic and there does not exist another privately measurable mechanism* μ' *such that* $u_i(t, \mu'(t)) \geq u_i(t, \mu(t))$ *for all* i *and* t *with at least one strict inequality.*

Then μ *is constant with respect to the states, i.e.* $\mu(t) = \mu(t')$ *for all* $t, t' \in T$. *In particular,* μ *is incentive compatible.*

The first part of this proposition shows that if μ is privately measurable and satisfies exact feasibility, then μ is constant across states, and therefore incentive compatible. The second part shows that in so far as efficient allocations are concerned, this conclusions applies even if free disposal is permitted.

Proof of Proposition 1. Suppose (i) holds (exact feasibility). Let z denote the net-trades corresponding to μ. By private measurability, for each i, z_i depends only on t_i. By exact feasibility, $z_i(t_i) = -\sum_{j\neq i} z_j(t_j)$, from which it follows that $z_i(t) = z_i(t')$ for all i and for all $t, t' \in T$. □

Suppose all utility functions are strongly monotonic, and (i) does not hold. Now, z satisfies private measurability and $\sum_i z_i(t) \leq 0$ for all $t \in T$. By Lemma 6.3 in Ichiishi and Radner (1999), there exists a privately measurable mechanism μ' with associated net-trades z' such that $z'_i(t) \geq z_i(t)$ for all i and $t \in T$ and $\sum_i z'_i(t) = 0$ for all $t \in T$. Since μ does not satisfy condition (i), strong monotonicity implies that $u_i(t, \mu'(t)) \geq u_i(t, \mu(t))$ for all i and t with at least one strict inequality; a contradiction to (ii).

It should be emphasized that this result depends crucially on the exchange economy model. In a different model, this connection between private measurability and incentive compatibility may not hold.[21]

Finally, we check whether Proposition 1 still holds under other notions of private measurability and/or of incentive compatibility. Consider a model in which each agent has an information partition, P_i, defined on a set of states Ω. For $\omega \in \Omega$, let $P_i(\omega)$ denote the element of P_i which contains ω. In this context, a state-contingent allocation $x : \Omega \mapsto X$ is said to satisfy private measurability, as in Allen (1993) and Koutsougeras and Yannelis (1993), if

$$x_i(\omega) = x_i(\omega') \text{ for all } i, \text{ whenever } P_i(\omega) = P_i(\omega'). \tag{6}$$

As we have already mentioned, standard incentive compatibility requires extending the domain of x to $\prod_i P_i$, which can be identified with T. Suppose x satisfies (6) and exact feasibility for each $\omega \in \Omega$. If we extend the domain of x to T by specifying $x(t) = 0$ for

[21] For instance, Ichiishi and Idzik (1996) study 'Bayesian societies' in which agents use independent, measurable strategies (instead of mechanisms). They consider a case in which both private measurability and incentive compatibility are imposed.

all $t \in T$ such that $q(t) = 0$, it can be shown, by the same argument used in the proof of Proposition 1 (i), that x, so extended, is incentive compatible. But using free disposal in 0 probability states may be essential, as shown by the following example, which is similar to Example 2 in Krasa and Shafer (2001).

Example 3. Suppose there are two agents and one commodity. Let $\Omega = \{a, b\}$, $P_1 = P_2 = (\{a\}, \{b\})$ and $e_1 = e_2 = 1$. The utility functions are as follows:

$$u_1(a, x) = x, \qquad u_1(b, x) = 2x, \qquad u_2(a, x) = 2x, \qquad u_2(b, x) = x.$$

Consider the allocation x:

$$x(a) = (0, 2), \qquad x(b) = (2, 0).$$

This allocation does satisfy the present notion of private measurability, (6), because both agents know the true state. Let $T_i = \{a_i, b_i\}$ be the set of types for each i, so that (a_1, a_2) refers to state a and (b_1, b_2) refers to state b. For x' defined on T and agreeing with x on (a_1, a_2) and (b_1, b_2), incentive compatibility requires that $x'_1(b_1, a_2) \leq 0$ and $x'_2(b_1, a_2) \leq 0$. Thus, $x'_1(b_1, a_2) + x'_2(b_1, a_2) < e_1 + e_2$.

While Proposition 1 (i) applies to exactly feasible allocations satisfying (6) (by allocating 0 in 0 probability states), Proposition 1 (ii) cannot be similarly extended; the Ichiishi and Radner (1999) argument does rely on measurability with respect to types. Indeed, an allocation which satisfies (6) but not exact feasibility in all positive probability states may not be extendable to an incentive compatible allocation, as the next example shows. Of course, this weakens significantly the rationale for imposing (6) on the basis of incentive compatibility.

Example 4. As in the previous example, there are two consumers, one commodity and $e_1 = e_2 = 1$. Let $\Omega = \{a, b, c\}$, $P_1 = (\{a, b\}, \{c\})$ and $P_2 = (\{a, c\}, \{b\})$. Each of the three states is equally likely. The utility functions are $u_i(\omega, x) = x$ for $i = 1, 2$, $\omega \in \Omega$. Consider the following allocation, satisfying (6) but not exact feasibility:

$$x(a) = (0, 0), \qquad x(b) = (0, 2), \qquad x(c) = (2, 0).$$

Define the types as $T_1 = \{s_1, t_1\}$ and $T_2 = \{s_2, t_2\}$ so that (s_1, s_2) refers to a, (s_1, t_2) to b, (t_1, s_2) to c, and (t_1, t_2) is an incompatible report. Suppose x' is an extension of x to T. For x' to be measurable with respect to types, $x'(t_1, t_2) = (2, 2)$, which is infeasible, and so Proposition 1 (ii) cannot be applied. In fact, there is no way to make x' feasible as well as incentive compatible; even if $x'(t_1, t_2) = (0, 0)$, agent 1 gains by reporting t_1 when he is of type s_1, and agent 2 gains by reporting t_2 when he is of type s_2.

Koutsougeras and Yannelis (1993) also identify assumptions under which private measurability implies incentive compatibility (see Proposition 4.1and Theorem 4.1 in Koutsougeras and Yannelis, 1993). Private measurability in Koutsougeras and Yannelis, 1993 refers to ((6)) but more importantly, they introduce definitions of *coalitional Bayesian incentive compatibility* (Definitions 4.1 and 4.2), which, applied to individual agents, do *not* correspond to the standard concept of Bayesian incentive compatibility, as stated in condition (2).[22] For instance, consider in Example 4, the following allocation:

$$y(a) = (0, 0), \qquad y(b) = (1, 1), \qquad y(c) = (1, 1).$$

According to Definitions 4.1 or 4.2 in Koutsougeras and Yannelis (1993), this allocation is not incentive compatible, because agent 1 can 'gain' by pretending that the state is c when it is a. Notice though, that agent 1's partition does not allow him to recognize the difference between states a and b. If the state is actually b, his declaration that the state is c would be incompatible with agent 2's truthful report. As mentioned above, the standard notion of incentive compatibility makes it necessary to define outcomes for *all* possible declarations of types. For example, if we extend y to y' over T where $y'(t_1, t_2) = (0, 0)$, y' *is* incentive compatible according to condition (2).

4.2. Emptiness of the ex ante incentive compatible core

The question of the non-emptiness of the ex ante incentive compatible core in exchange economies has been recently settled negatively by Vohra (1999) and Forges et al. (2002). The second paper provides an example of a well-behaved economy with *quasi-linear* utility functions in which the ex ante incentive compatible core is empty, and this even if the grand coalition can enlarge its feasible set by relying on *random* mechanisms. By contrast, in Vohra (1999), the agents cannot make monetary transfers nor use lotteries. In the example constructed in Vohra (1999), the latter restriction is not innocuous: Forges and Minelli (2001) show that if random mechanisms are allowed in this economy, the corresponding ex ante incentive compatible core is non-empty (we shall return to this in Section 4.3.2). The negative result in Forges et al. (2002) is thus stronger than the one in Vohra (1999). Furthermore, the computations are simpler in Forges et al. (2002): given the transferable utility setting, it suffices to show that the game is not balanced. We briefly describe the example.

Example 5. The economy involves three agents and four goods (three consumption goods and money); agent 1 has two equiprobable types s and t, while agents 2 and 3 do not have private information ($T_1 = \{s, t\}$, $q(s) = q(t) = (1/2)$, T_2 and T_3 are singletons).

Let the endowments in consumption goods be

$$e_1 = (1, 0, 0), \quad e_2 = (0, 2, 0), \quad e_3 = (0, 0, 2).$$

Denote the consumption bundle as $x = (x^1, x^2, x^3)$ and the monetary transfer as m; let the (quasi-linear) utility functions be

$$u_i(r, x, m) = w_i(r, x) + m \quad i = 1, 2, 3, \quad r = s, t,$$

[22] The definition of coalitional Bayesian incentive compatibility in Koutsougeras and Yannelis (1993) is in terms of ex post utility. Allen and Yannelis (2001) use a notion of coalitional Bayesian incentive compatibility which is similar in spirit to Koutsougeras and Yannelis (1993) but is expressed in terms of interim expected utility (see the discussion in Hahn and Yannelis, 1997).

where

$$w_1(s,x) = 2x^1 + \min\{(x^2 + x^3), 2\},$$
$$w_2(s,x) = w_3(s,x) = 3x^1 + 2\min\{x^2, x^3\},$$
$$w_1(t,x) = x^1 + h(x^2) + h(x^3),$$
$$w_2(t,x) = w_3(t,x) = x^2 + x^3,$$

and

$$h(x) = \min\{0.9x, 0.05x + 0.85\}.$$

Let us first consider mechanisms which allow unlimited monetary transfers $m_i(s), m_i(t)$, $i = 1, 2, 3$ satisfying the feasibility constraints $\sum_i m_i(r) \leq 0$, $r = s, t$. It is easily checked that the ex ante incentive compatible core of this economy is just the core of a TU characteristic function game v^* (see Forges et al., 2002 for details). As a benchmark, we also consider the TU characteristic function v associated with the economy in the absence of incentive constraints.Obviously, $v^* \leq v$. We know that the core of v is not empty or, equivalently, by the Bondareva-Shapley theorem (see, e.g. Myerson, 1991), that v is balanced.

In the present example, v is in fact *exactly* balanced, in the sense that

$$v(N) = \tfrac{1}{2}[v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\})]. \tag{7}$$

To see this, notice that efficiency in coalition $\{2, 3\}$ is achieved by having each agent consume the same amount of each of the commodities 2 and 3 in each state. The corresponding aggregate utility is 4 in each state, and $v(\{2, 3\}) = 4$. In coalition $\{1, 2\}$ or $\{1, 3\}$, efficiency is achieved by having the agents swap their endowments in state s and no-trade in state t. This results in aggregate utility of 5 and 3 in states s and t respectively, and transfers (for example, of equal amounts in the two states) can then be made to distribute ex ante utility across the agents in any way, i.e. $v(\{1, 2\}) = v(\{1, 3\}) = 4$. In the grand coalition efficiency requires that in state s commodity 1 is given to agents 2 and 3 and agent 1 receives equal amounts of commodities 2 and 3, say b, where $b \leq 1$, and there is no trade in state t. The resulting aggregate utility is 7 in state s and 5 in state t. Again, transfers can be used to redistribute ex ante utility across the agents, and we have $v(N) = 6$ which implies (7).

A second important property of our example is that, in subcoalitions, the incentive conditions do not entail any loss of efficiency, i.e.

$$v^*(S) = v(S) \quad \forall S : |S| = 2 \tag{8}$$

This is obvious for coalition $\{2, 3\}$, which does not face any incentive compatibility constraint. Notice that the efficient mechanism we identified for coalition $\{1, 2\}$ is incentive compatible (without transfers) because $w_1(s, (0, 2, 0)) = w_1(s, (1, 0, 0)) = 2$ and $w_1(t, (1, 0, 0)) = 1$ while $w_1(t, (0, 2, 0)) = h(1) < 1$. And by making identical transfers in each state it is possible to satisfy incentive compatibility and achieve $v(\{1, 2\})$. Clearly, the same argument applies to $\{1, 3\}$.

By the Bondareva-Shapley theorem, the ex ante incentive compatible core is non-empty if and only if v^* is balanced, which, by (7) and (8) implies that $v^*(N) = v(N)$. To complete the proof of our assertion that the ex ante incentive compatible core is empty in this example, therefore, it only remains to be shown that $v^*(N) < v(N)$. Suppose not. Then there must exist a first best efficient mechanism μ, with a corresponding allocation of goods, x, and transfers, m, which is incentive compatible. As we have already observed, efficiency requires that $x_1(s) = (0, b, b)$, $0 \le b \le 1$ and $x_1(t) = (1, 0, 0)$. Incentive compatibility requires that

$$w_1(s, (0, b, b)) + m_1(s) \ge w_1(s, (1, 0, 0)) + m_1(t),$$
$$w_1(t, (1, 0, 0)) + m_1(t) \ge w_1(t, (0, b, b)) + m_1(s).$$

This implies (by summing the two inequalities) that

$$w_1(s, (0, b, b)) + w_1(t, (1, 0, 0)) \ge w_1(s, (1, 0, 0)) + w_1(t, (0, b, b)), \tag{9}$$

or, $1 + 2b \ge 2 + 2h(b)$, which obviously is not true (since $h(b) = 0.9b$ for $b \le 1$).

In Forges et al. (2002), it is proved that all the previous arguments go through when random mechanisms are allowed, and that the counter-example is fully robust with respect to the probability on states, the endowments, and the utility functions. In particular, the example can be modified so as to make the economy even more well-behaved. Forges et al. (2002) also uses the previous counter-example to demonstrate the emptiness of the core in a *Walrasian* economy, in which the agents are endowed with some limited amount of money.

4.3. *Sufficient conditions for non-emptiness*

4.3.1. *Cases in which the incentives problem disappears*

The literature on implementation and mechanism design has identified assumptions under which the incentives problem can be eliminated in the sense that for any reasonable (in particular, first-best) mechanism, an equivalent, incentive compatible mechanism can be constructed. This approach can be fruitfully applied to the problem at hand, since in the absence of incentive constraints, the ex ante core of the basic economy is non-empty.

Consider first the case of complete information, i.e. for all $t \in T$ such that $q(t) > 0$, $q(t|t_i) = 1$. Clearly, in this case, any inconsistency in the agents' declarations can be detected. By stipulating that no goods are allocated in case of an inconsistency, it is possible to make any mechanism, in particular any allocation in the ex ante core of the basic economy, equivalent to one which is incentive compatible.

A less trivial information structure with a similar feature is that of non-exclusive information, introduced by Postlewaite and Schmeidler (1986), under which the true state can be identified even if the type of any one individual is not known.

Information is said to be *non-exclusive* if

$$\forall t \in T : q(t) > 0, \quad \forall i \in N, \quad q(t_i|t_{-i}) = 1.$$

The next proposition is established in Vohra (1999).

Proposition 2. *If information is non-exclusive, the ex ante incentive compatible core is non-empty.*

As in the case of complete information, any unilateral lie gives rise to a reported state of null probability. If we modify any mechanism by requiring that it allocates no good at such states, expected utilities do not change but misreport of information is punished, so that the new mechanism is incentive compatible.[23]

Non-exclusivity of information can be interpreted as a notion of informational smallness. McLean and Postlewaite (1999) introduce a more refined concept. They parameterize information structures of pure common value economies (to be defined presently) by a measure that takes into account the relationship between the informational size of each individual, the level of aggregate uncertainty, and the extent to which individual signals influence the posterior distribution on the state; see McLean and Postlewaite (1999) for details. This leads to a precise measure of informational smallness under which it is possible to approximate almost any allocation of the underlying complete information economy by an incentive compatible mechanism. They use this characterization in McLean and Postlewaite (2000) to prove that the ex ante incentive compatible core is non-empty whenever individuals are informationally small in this sense. Their notion encompasses non-exclusive information, but also more general information structures.

Another special case of the McLean–Postlewaite model, analyzed independently by Krasa and Shafer (2001), is one in which information is, in some well defined sense, 'almost complete'. Consider a situation of *pure common values*, in which the utility of individuals depends only on the realization of a state of nature θ in some finite set Θ: $v_i(\theta, x)$. The elements of the set T_i can be viewed as signals, which do not enter the utility function directly.[24] For simplicity, let us set $T_i = \Theta$, and let $q \in \Delta(\Theta^{n+1})$ indicate the joint probability distribution on the state of nature and the signals. Each individual is informed on the realization of his own signal and cares about the signals received by other agents only if these signals contain some information concerning the realization of θ. If q is concentrated on the diagonal, d, of Θ^{n+1}, $q(\mathrm{d}(\Theta^{n+1})) = 1$, we are back to complete information: all individuals are perfectly informed about the realization of the state of nature. A situation of 'almost complete information' is captured by considering a sequence of pure common values economies $(E^k)_{k\geq 1}$ indexed by prior probabilities $q_k \in \Delta(\Theta^{n+1})$ converging to full information, $q_k \to q$ with $q(\mathrm{d}(\Theta^{n+1})) = 1$. Intuitively, when the economy is close to complete information we expect to be able to use simple punishment mechanisms, like those discussed for the case of complete and of non-exclusive information, to facilitate incentive compatibility.

[23] Observe that in the case of non-exclusive information, lies can be detected but not liars; hence all agents are punished in case of a misreport. As in Example 3, the punishment must be hard (all goods are confiscated) because the mechanisms in the ex ante core are not necessarily *interim* individually rational. Of course, in a two-agent economy, as in Example 3, non-exclusive information means complete information. If there are at least three agents, in the case of complete information it is possible to make the mechanism non-wasteful; see Krasa and Shafer (2001). However, if information is non-exclusive this is generally not possible even when there are at least three agents; see Example 6. Additional complications arise when endowments are allowed to be type-dependent. Indeed, for the class of mechanisms proposed in Forges et al. (2002) (see also the discussion before Proposition 4), the simple argument given in the text does not work.

[24] Notice that introducing the parameter θ in the model of Section 2 does not make it more general, since we can always define $u_i(t, x) = \sum_\theta v_i(\theta, t, x)q(\theta|t)$. Here, in the special case of common values, it is useful to keep track of θ explicitly.

The McLean and Postlewaite (2000) result makes use of the notion of the *strict* core of an economy, i.e. the set of core allocations such that, in every subcoalition, all members have a strictly higher utility than what they could get by deviating.[25]

Proposition 3. *Let E be a pure common value economy such that the strict core of the underlying Arrow–Debreu economy is non-empty. If all agents are informationally small, then the ex ante incentive compatible core of E is non-empty.*

The idea that correlation of beliefs facilitates the fulfillment of incentive compatibility conditions has also been used in the model of Section 2, under the additional assumption that utility functions $u_i(t, \cdot)$ are *quasi-linear*, i.e.

$$u_i(t, x, m) = w_i(t, x) + m \quad \forall i \in N, \ \ t \in T, \ \ x \in \mathbb{R}^l_+, m \in \mathbb{R}.$$

In this setting, it is understood that arbitrary monetary transfers are allowed. Feasibility requires the sum of transfers to be non-positive. The mechanism design literature (for example, Arrow, 1979; d'Aspremont and Gérard-Varet, 1979, 1982; Green and Laffont, 1979; Groves, 1973; Johnson et al., 1990) identifies conditions (on the beliefs and/or the utility functions) under which it is possible to construct money transfer schemes making (typically, first best) allocations incentive compatible. In our model, as shown in Forges et al. (2002), this implies that the incentive constraints do not affect the grand coalition, i.e. $v^*(N) = v(N)$. From this, one deduces immediately that the ex ante incentive compatible core is non-empty, since it is so even if subcoalitions can object with non-incentive compatible mechanisms.

More precisely, consider the following conditions on the beliefs q: even if agents get utility only from the monetary transfers (i.e. the utility functions w_i are identically 0), there exists a strictly incentive compatible and exactly feasible money transfer scheme. This is called *Condition B* in d'Aspremont and Gérard-Varet (1979, 1982) (see also Johnson et al., 1990). It can be interpreted as a form of correlation between the agents' beliefs. Under B, the ex ante incentive compatible core is non-empty; see Forges et al. (2002). d'Aspremont et al. (1990) show that this condition is *generic* provided that $n \geq 3$ and no player is fully informed.[26]

d'Aspremont and Gérard-Varet (1979, 1982) weaken the previous condition to one they refer to as *Condition C*, which is always satisfied by independent beliefs. They further assume that *values are private*, i.e.

$$w_i(t, x) = w_i(t_i, x) \quad \forall i \in N, \ \ t \in T, \ \ x \in \mathbb{R}^l_+.$$

With private values, first best efficient allocations can be implemented via Groves mechanisms, and correlation is not needed to elicit private information. As shown in Forges et al.

[25] See McLean and Postlewaite (2000) for conditions ensuring the non-emptiness of the strict core of complete information economies. Note also that they refer to the underlying Arrow–Debreu economy with contingent commodities (without incentive constraints) as the auxiliary economy.

[26] In view of the counter-example of Section 4.2, notice that the latter proposition typically relies on unlimited monetary transfers (see Forges et al., 2002 for further comments on the robustness of the counter-example in Walrasian economies).

(2002), Condition C is then sufficient for the non-emptiness of the ex ante incentive compatible core. Example 5 shows that the assumption of private values cannot be dropped in this statement. One can also deduce from d'Aspremont et al. (1990) that the ex ante incentive compatible core is non-empty if values are private and $|T_i| \leq 2$ for every $i \in N$.

Forges et al. (2002) further show that quasi-linearity is specially useful if the basic model of Section 2 is extended so as to allow for type-dependent initial endowments. Assume that each agent i of type t_i initially owns a bundle $e_i(t_i)$ (since agents know their initial endowments, these are privately measurable). In that case, it is natural to consider more general mechanisms than before, which ask every agent to show a bundle that is consistent with his reported type (see, e.g. Hurwicz et al., 1995). This considerably restricts the possibilities of lying: an agent can only pretend to be of a poorer type than he really is. In Forges et al. (2002) it is established that if the initial endowments are a one-to-one function of types (a condition which holds generically), feasible monetary transfers can be constructed so as to make the first best incentive compatible. Hence, as above, the ex ante incentive compatible core is non-empty.

Proposition 4. *Suppose each agent has a quasi linear utility function and endowments which are a one-to-one function of his types. Then the ex ante incentive compatible core is non-empty.*

4.3.2. Scarf's theorem and random mechanisms

We now turn to particular cases where, although incentives do matter, the ex ante incentive compatible core is non-empty. Two agents or linear utility functions are immediate examples[27] (see Ichiishi and Idzik, 1996; Vohra, 1999). Another class of examples has been identified in Forges and Minelli (2001). In order to get a positive result, they need to allow for random mechanisms.

In Holmström and Myerson (1983), the economy involves a set of feasible decisions (in our notation, X), but decision rules (i.e. mechanisms) are defined as mappings from the set of types T to the set of *probability distributions* over X. Random mechanisms are indeed necessary as soon as the revelation principle is applied in full generality and covers in particular the case where agents can use mixed strategies. Even if, as in our framework, the set X is convex and the utility functions $u_i(t, \cdot)$ are concave, allowing for random mechanisms can make a great difference, because of the incentive compatibility constraints. For instance, as already observed by Prescott and Townsend (1984a,b), a convex combination of deterministic incentive compatible feasible mechanisms need not be incentive compatible, so that the sets $V^*(S)$ which define the characteristic function corresponding to the ex ante incentive compatible core need not be convex either.

Let S be a coalition. Recall that X_S is the set of feasible commodity vectors for coalition S, at every state (see (5)). Let $\Delta(X_S)$ be the set of all probability distributions over the (Borel subsets of) X_S. A *feasible random mechanism* for coalition S is a transition probability from T_S to X_S, namely a mapping $\mu_S : T \to \Delta(X_S)$ such that

$$\mu_S(\cdot|t) = \mu_S(\cdot|t') \qquad \forall t, t' \in T : t_S = t'_S$$

[27] Building on Rosenmueller's model of fee games, Rosenmueller (1999), Haake (2001) also identified particular economies in which the ex ante incentive compatible core is not empty.

where $\mu_S(\cdot|t)$ denotes the image of t by μ_S. Feasibility requires that *every* allocation selected by the mechanism be in X_S. Allen (1992, 1994) and Prescott and Townsend (1984a,b) argue that in large economies, one may be satisfied with a weaker notion, *expected* feasibility (see also Section 6).[28] μ_S is *expected* feasible if

$$\int_{\mathbb{R}^l_+} \left(\sum_{i\in S} x_i\right) \mu_S(\mathrm{d}x|t) \leq \sum_{i\in S} e_i \quad \forall t \in T.$$

This property only depends of the marginal distributions of the mechanism over every agent's allocation.

The extension of the definitions of Section 2 (expected utility, Bayesian incentive compatibility, etc.) to random mechanisms is straightforward. For instance, (1) becomes

$$U_i(\mu_S|t_i, s_i) = \sum_{t_{-i}} q(t_{-i}|t_i) \int_{\mathbb{R}^l_+} u_i(t_i, t_{-i}, x_i)\mu_S(\mathrm{d}x_i|s_i, t_{-i}).$$

Similarly, all the notions introduced in Section 4.1, and in particular the ex ante incentive compatible core itself, apply to random mechanisms. The "modified (ex ante) incentive compatible core" of Allen (1992, 1994) is defined in a similar way by allowing coalitions to use expected feasible mechanisms. As shown in Allen (1992), it is always non-empty, as a consequence of Scarf (1967)'s theorem.

Proposition 5. *The "modified (ex ante) incentive compatible core", in which coalitions use expected feasible incentive compatible random mechanisms, is non-empty.*

To establish the previous result, let $\mathcal{S}$ be a balanced family of coalitions, with balancing weights λ_S, $S \in \mathcal{S}$: $\sum_{S\ni i} \lambda_S = 1$ for every $i \in N$ (it is understood that $\lambda_S = 0$ for $S \notin \mathcal{S}$). Let $V^*_{e,r}$ be the characteristic function defined as V^*, but allowing for expected feasible random mechanisms. The underlying game is balanced if $\bigcap_{S\in\mathcal{S}} V^*_{e,r}(S) \subseteq V^*_{e,r}(N)$.

Let $v \in \bigcap_{S\in\mathcal{S}} V^*_{e,r}(S)$. For every coalition $S \in \mathcal{S}$, there exists an expected feasible incentive compatible mechanism μ_S such that $v_i \leq U_i(\mu_S)$ $\forall i \in S$. In order to show that the game is balanced, one must construct, from the mechanisms μ_S, an expected feasible incentive compatible feasible mechanism μ_N for the grand coalition such that $v_i \leq U_i(\mu_N)$ $\forall i \in N$.

We will use $(\lambda_S)_{S\ni i}$ as a *lottery* over the coalitions containing agent i. Let, for every S, $\mu_{S,i}(\cdot|t)$ denote the marginal distribution of $\mu_S(\cdot|t)$ over agent i's bundles. Let us set

$$\mu_{N,i}(\cdot|t) = \sum_{S\ni i} \lambda_S \mu_{S,i}(\cdot|t) \qquad \forall i \in N, \quad t \in T \tag{10}$$

and let μ_N be the product probability distribution constructed from $\mu_{N,i}$, $i \in N$. μ_N is expected feasible and incentive compatible because both the expected feasibility constraints and the incentive constraints only depend on the marginal distributions of the mechanism. Furthermore, by the linearity of U_i, $v_i \leq U_i(\mu_N)$.

[28] The notion of feasibility in Prescott and Townsend (1984a,b) is even weaker than in this section; there, the average is taken over types as well.

In the absence of assumptions on the size of the economy, expected feasibility is not relevant, and one would like to apply Scarf's result to exactly feasible mechanisms. Even if, in the above proof, the mechanisms μ_S are exactly feasible, the mechanism μ_N constructed from (10) may not be. The difficulty typically comes from the fact that the agents are allocated goods independently of each other. It is thus natural to look for a mechanism μ_N whose marginal distributions $\mu_{N,i}$ satisfy (10), which is a key to incentive compatibility, but is not necessarily a product-mechanism. Unfortunately, it is possible that *no mechanism* with marginals as in (10) is feasible.

Let us go back to Example 5 and consider the balanced family $\mathcal{S} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ with the weights (1/2)($n = 3$). Let $x_{S,i}$ denote the state s allocations of the goods to agent i in coalition $S \in \mathcal{S}$ corresponding to the deterministic, first best efficient mechanism of S as identified in Example 5. Recall that

$$\begin{aligned} x_{\{1,2\},1} &= (0, 2, 0), \qquad & x_{\{1,2\},2} &= (1, 0, 0),\\ x_{\{1,3\},1} &= (0, 0, 2), \qquad & x_{\{1,3\},3} &= (1, 0, 0),\\ x_{\{2,3\},2} &= (0, 1, 1), \qquad & x_{\{2,3\},3} &= (0, 1, 1). \end{aligned}$$

Let μ_N be a feasible mechanism satisfying (10). In state s, $\mu_N(\cdot|s)$ must allocate either (0, 2, 0) or (0, 0, 2), with the same probability, to agent 1. And it must allocate either (1, 0, 0) or (0, 1, 1), with the same probability, to agent 2. But if $\mu_N(\cdot|s)$ satisfies this property for agent 1, feasibility precludes it from also having in its support the allocation of (0, 1, 1) to agent 2.

Forges and Minelli (2001) identify a class of economies where the construction of a feasible mechanism satisfying (10) is possible.

Proposition 6. *Assume that if agent i initially owns a positive quantity of some good, then no other agent initially owns this good (for every k, if $e_i^k > 0$, then $e_j^k = 0$ for every $j \not\equiv i$) and that in every state the utility functions are additively separable across goods, i.e.*

$$u_i(t, x) = \sum_{k=1}^{l} u_i^k(t, x^k) \quad \forall i \in N, \quad t \in T, \quad x \in \mathbb{R}_+^l.$$

Then the ex ante incentive compatible core associated with random mechanisms is non-empty.

The crucial assumption is that the utility function must be separable across goods. The assumption on endowments would have no bite if we did not require separability; one could always rename goods so as to satisfy it.

The main idea of the proof is to consider the marginal distributions that the mechanisms induce over *goods*, rather than over agents. The separability across goods makes it possible to allocate goods independently of each other. Eq. (10) can be deduced from the properties of balancing weights.

Forges and Minelli (2001) show that their proof can be modified so as to deal with economies that do not fully satisfy the above assumptions, like Vohra (1999)'s example, in which some agents have a non-separable utility function in one state. Recall that in this example, the ex ante incentive compatible core associated with deterministic mechanisms is empty. Hence, there are environments where random mechanisms are crucial for the non-emptiness of the ex ante incentive compatible core.

However, in Example 5, the ex ante incentive compatible core associated with random mechanisms is empty. This shows that the assumptions in the above proposition cannot be dropped. The same applies to the non-separability of the utility functions in Example 5.

5. The interim stage

At the ex ante stage there is no ambiguity about how a mechanism should be evaluated from the point of view of a particular agent (whether the evaluation is done by the agent or by an outsider)—expected utility provides the correct measure. (Whether or not incentive compatibility should be imposed is a separate issue.) In contrast, at the interim stage, account must clearly be taken of the fact that agents already possess their private information when they engage in coalitional negotiations. From the point of view of an agent, the appropriate utility measure is then conditional expected utility, conditional on the agent's type. An outsider evaluating a mechanism from the point of view of an agent should now consider the conditional expected utility of the agent for each of her types. Recall that this is the reason that the notion of an interim Pareto improvement as in Holmström and Myerson (1983) requires all agents of all types to gain. And this approach suggests that a notion of the core at the interim stage could be defined in a similar manner.[29] However, there is another subtlety involved in defining domination for a coalition which is a subset of the grand coalition. What is commonly known to agents within a coalition (even if they do not share their private information) may be more than what is known to an outsider. It is, therefore, appropriate to allow a coalition to concentrate a potential objection on an event which is common knowledge among members of the coalition. If agents within a coalition cannot pool their information it is then appropriate to assume that an objection can be concentrated only over an event which is common knowledge to agents in the objecting coalition. This is the basis of the coarse core developed in Wilson (1978), which we shall formally define in the next section. In order to clarify the issues involved, incentive compatibility will not be incorporated until Section 5.2.

5.1. *The coarse core*

Wilson (1978) provided a seminal analysis of the core at the interim stage (without imposing incentive constraints). He introduced two notions of the (interim) core: the coarse core and fine core. The former relates to the case in which the information possessed by agents cannot be pooled, and the latter to the case in which information can be shared in an arbitrary manner. We defer a discussion of the fine core until Section 5.3. In both cases, the fact that incentive constraints are ignored can be taken to imply that all information becomes publicly known at the (ex post) stage when actual trades are made. While Wilson formulates incompleteness of information by specifying partitions over a state space, we shall rely on the basic model of Section 2 in which private information is described by agents types. What follows is simply a re-formulation of Wilson's concept in the types framework.

[29] This is the approach followed in Ichiishi and Idzik (1996).

A key idea in defining the coarse core is the notion of a common knowledge event.[30] For an event $E \subseteq T$ let E_i denote the corresponding set of types for agent i, i.e. $E_i = \{t_i | t \in E\}$. An event $E \subseteq T$ is said to be *common knowledge* for S if

$$q(t'_{-i}|t_i) = 0 \quad \text{for all } i \in S, t_i \in E_i \quad \text{and} \quad (t'_{-i}, t_i) \notin E.$$

Based on the fact that all agents within a coalition can discern a common knowledge event, a coarse objection can be directed at any such event. An objection over event E requires all types in E_i to gain in interim utility. More precisely, let $\mu \in \mathcal{F}$ be a feasible mechanism. Coalition S has a *coarse objection* to μ if there exists an event E which is common knowledge for S and a mechanism $\nu_S \in \mathcal{F}_S$[31] such that

$$U_i(\nu_S|t_i) > U_i(\mu|t_i) \qquad \forall i \in S, \ \forall t_i \in E_i.$$

(Note that with this notion of dominance, there is no loss of generality in restricting attention to common knowledge events for S which are of the form $E = \prod_{i \in S} E_i \times \prod_{j \notin S} T_j$.)

The *coarse core* is the set of all feasible mechanisms to which no coalition has a coarse objection.

Observe that in identifying a coarse objection it is enough for the objecting coalition to be able to improve upon the status-quo over a common knowledge event. For the grand coalition, this turns out to be (essentially) equivalent to the requirement that the new allocation dominate the status-quo for all consumers of *all* types, as in the notion of interim efficiency. (The only reason this is not exactly so is that we have defined dominance in terms of strict inequalities, while in Holmström and Myerson, 1983 dominance is defined with weak inequalities and some strict inequality.) For if the grand coalition has a dominating mechanism over a common knowledge event E it can consider the same mechanism over E and the status-quo over the complement of E which ensures that no type of any agent loses and some types (those in E) gain strictly. It is worthwhile to stress that this argument does not necessarily apply to a coalition which is a strict subset of the grand coalition because such a coalition may have an objection over a common knowledge event but may not be able to assure itself of the status-quo utility over states not in E. This is most simply seen by considering an objection from a singleton coalition (recall the discussion of interim individual rationality in Section 1.2.1). For coalition $\{i\}$, the event $\{t \in T | t_i = t'_i\}$ is a common knowledge event, and i has a coarse objection if there is some $t_i \in T_i$ over which i can do better with his own endowment.[32] For this reason, the logical inclusion relationship between the set of ex ante efficient mechanism and interim efficient mechanisms does not extend to the ex ante core and the coarse core; the ex ante core is not necessarily a subset of

[30] The definitions appearing in Holmström and Myerson (1983) and Vohra (1999) are inaccurate, as pointed out to us by Claus–Jochen Haake, but the results in those papers hold for common knowledge as defined here (which corresponds to the definition in Chapter 10 in Myerson, 1991).

[31] As pointed out in Section 4.1, in the absence of incentive constraints it is appropriate to consider $\mathcal{F}_S$ rather than $\mathcal{F}_S^m$ as the set of feasible mechanisms for S. If coalition S is restricted to $\mathcal{F}_S^m$ the main result of this section remains unchanged since the core would then be larger than the one we will define presently.

[32] Notice that in Example 1, the mechanism where $z_1(s) = z_1(t) = (-1, 0.6)$ is ex ante individual rationality for consumer 1 but is not *not* interim individually rational because agent 1 in state t (knowing the state at the interim stage) is better-off not trading.

the coarse core.[33] And the non-emptiness of the coarse core cannot, therefore, be inferred simply from that of the ex ante core.

Wilson (1978) proved that, under standard assumptions (continuity and concavity of $u_i(t_i, \cdot)$ for all i and t_i, as imposed in Section 2), the coarse core is non-empty. His argument proceeds by constructing an appropriate NTU game as follows. For each agent i in the economy define a player corresponding to each of i's types and consider an NTU game in which a typical player is denoted (i, t_i); there are $\sum_i |T_i|$ players in this game. The utility function of player (i, t_i) is $U_i(\cdot|t_i)$. The 'grand coalition' is (N, T) and other allowable coalitions are restricted to be of the form $(S, E) = \{(i, t_i)|i \in S, t_i \in E_i\}$, where $S \subseteq N$ and E is a common knowledge event for S. The feasible utility set for coalition S is derived by applying the utility functions $U_i(\cdot|t_i)$ to the set of feasible allocations (deterministic mechanisms) from T to X_S. Note that mechanisms are deterministic and are not required to be incentive compatible. It can then be shown by a standard argument that for any balanced collection of coalitions and corresponding deterministic mechanisms, the mechanism constructed as in (10) is feasible for the grand coalition. The NTU game is, therefore, balanced and by Scarf (1967)'s theorem it has a non-empty core. The corresponding set of mechanisms is precisely the coarse core of the underlying economy.

Proposition 7. *The coarse core is non-empty.*

Wilson also pointed out (footnote 6 in Wilson, 1978) that, alternatively, non-emptiness of the coarse core follows from the observation that it contains equilibrium allocations of a constrained market process which we formally define in Section 6.2; see also Theorem 5.7 in Goenka and Shell (1997).

5.2. The incentive compatible coarse core

At this stage it should be clear how to incorporate incentive compatibility in the coarse core. The only change that needs to be made to the definition in the previous section is to require that the relevant mechanism satisfy incentive compatibility (as well as the measurability conditions (4)). This leads to the notion of the *incentive compatible coarse core* studied in Vohra (1999).

For an incentive compatible and feasible mechanism $\mu \in \mathcal{F}^*$, coalition S has an *incentive compatible coarse objection* if there exists an event E which is common knowledge for S and a mechanism $\nu_S \in \mathcal{F}_S^*$ such that

$$U_i(\nu_S|t_i) > U_i(\mu|t_i) \qquad \forall i \in S, \ \ \forall t_i \in E_i.$$

The *incentive compatible coarse core* consists of all incentive compatible and feasible mechanisms to which no coalition has an incentive compatible coarse objection.

In this case, an objection from the grand coalition corresponds to domination in the sense of interim incentive efficiency. While the argument is no longer as simple as the one concerning the coarse core and interim efficiency, it follows from Theorem 1 in Holmström

[33] If objections at the interim stage are required to make all types better-off, as in Ichiishi and Idzik (1996), then such an inclusion does indeed hold.

and Myerson (1983). Thus, the incentive compatible coarse core bears the same relationship to interim incentive efficiency as the coarse core does to interim efficiency.

Notice that in a two-agent economy, the incentive compatible coarse core is the set of interim individually rational and interim incentive efficient allocations.[34] Under the usual assumptions therefore, this set is non-empty.

As in Proposition 2, and by a similar argument (Proposition 3.1 in Vohra, 1999), non-emptiness holds if information is non-exclusive. Moreover, in this case, the mechanism used to ensure incentive compatibility can be chosen to be non-wasteful.

Proposition 8. *If information is non-exclusive, the incentive compatible coarse core is non-empty*.

Unfortunately, the positive result of Wilson, Proposition 7, does not extend to the case in which incentive constraints are imposed. An example of a three-consumer economy with an empty incentive compatible coarse core was provided in Vohra (1999). A much stronger negative result was established in Forges et al. (2002) pertaining to a quasi linear economy. In fact, Example 5 is one in which the incentive compatible coarse core is empty even if random mechanisms are allowed; see Forges et al. (2002) for details.

The results we have described so far for the coarse core are analogous to some of those on the ex ante core discussed in Section 4. It is natural then, to ask whether non-emptiness of the incentive compatible coarse core can be established under the other conditions discussed in Section 4.3.1. In particular, whether the notion of informational smallness of McLean and Postlewaite or the conditions in d'Aspremont and Gérard-Varet (1979, 1982) suffice to obtain a positive result for the incentive compatible coarse core. Unfortunately, the mechanism design approach of Section 4.3.1, which was so fruitful in the ex ante case, does not immediately extend to the interim case. While monetary transfers make it possible to transfer ex ante utility across consumers without affecting incentive constraints, the same need not be true in terms of transfers of interim utility (across types). Indeed, one ingredient of the approach in Section 4.3.1 was to construct, corresponding to a first-best outcome, a transfer scheme satisfying incentive compatibility. And an appropriate transfer scheme will typically affect interim utilities. Restrictions on interim utility (such as interim individual rationality) may be too demanding if one insists on first-best efficiency even in the case of independent, private values (Myerson and Satterthwaite, 1983).[35] While that approach may not longer be fruitful, the possibility remains that the incentive compatible coarse core is non-empty under the conditions of Propositions 3, 4 and 6. These are important open questions for future work in this area.

5.3. Information sharing

The coarse core is based on the assumption that a coalition can coordinate a potential objection only over an event which is common knowledge to members of the coalition. While

---

[34] In Example 1, the only mechanism in the incentive compatible coarse core is no-trade.

[35] While the Myerson-Satterthwaite result relies on a continuum of types, the same problem can arise with a finite number of types, as in Table 5.2 in Milgrom and Roberts (1992).

there is no doubt that a coalition must be permitted to direct an objection over a common knowledge event, it is worth considering alternative notions of the core (refinements of the coarse core) which allow a coalition to do more. In the extreme, suppose a coalition can choose any informational event which can be discerned by pooling the private information of its members. The corresponding notion of the core is the fine core of Wilson (1978), which we now describe in terms of our basic model.

Define an *admissible event* for coalition S to be an event of the form $E = \prod_{i\in S} E_i \times \prod_{j\notin S} T_j$, where $E_i \subseteq T_i$ for all $i \in S$ and $q(E) > 0$. A coalition may now rely on an admissible event to construct an objection. Let $q(t|E, t_i)$ denote the updated conditional probability of an agent whose type is $t_i \in E_i$ and who believes that the true state lies in E; set to 0 the probability of any state not in E and apply Bayes' rule. For an allocation x define $U_i(x|E, t_i)$ as the corresponding updated conditional expected utility. Note that if E is a common knowledge event, then $U_i(x|E, t_i) = U_i(x|t_i)$.

Coalition S is said to have a *fine objection* to $\mu \in \mathcal{F}$ if there exists an admissible event E for S and a mechanism $\nu_S \in \mathcal{F}_S$ such that

$$U_i(\nu_S|E, t_i) > U_i(\mu|E, t_i) \qquad \forall i \in S,\ \ \forall t_i \in E_i.$$

The *fine core* consists of all feasible mechanisms to which no coalition has a fine objection.

Clearly, the fine core is contained in the coarse core. The following example illustrates the differences, and shows that the fine core may be empty. This example is similar to Example 2 in Wilson (1978) except that it involves constant endowments, in keeping with our basic model.

Example 6. There are three consumers and one commodity. Each agent has an endowment of three units of the commodity in each state. For each agent i, $T_i = \{s_i, t_i\}$. However, only three states have positive probability. Let $a = (s_1, t_2, t_3)$, $b = (t_1, s_2, t_3)$ and $c = (t_1, t_2, s_3)$. Suppose $q(a) = q(b) = q(c) = 1/3$. Thus, agents 1, 2 and 3 can distinguish respectively states a, b and c, and any two agent coalition can identify the true state by pooling the information of its members. Since we are not concerned with incentive compatibility, it will suffice to consider state contingent allocations defined on the three positive probability states. The utility functions are as follows:

$$\begin{aligned}
&u_1(a, x) = \sqrt{x}, \quad &&u_1(b, x) = \sqrt{2x}, \quad &&u_1(c, x) = \sqrt{x}\\
&u_2(a, x) = \sqrt{x}, \quad &&u_2(b, x) = \sqrt{x}, \quad &&u_2(c, x) = \sqrt{2x}\\
&u_3(a, x) = \sqrt{2x}, \quad &&u_3(b, x) = \sqrt{x}, \quad &&u_3(c, x) = \sqrt{x}.
\end{aligned}$$

It is easy to see that the following allocation, y, belongs to the coarse core:

$$y(a) = (3, 2, 4), \qquad y(b) = (4, 3, 2), \qquad y(c) = (2, 4, 3).$$

However, coalition $\{1, 3\}$ has a fine objection to y over the state c, since $y_1(c) + y_3(c) < 6$; the event $\{t_1\} \times T_2 \times \{s_3\}$ is an admissible event for this coalition. In fact, the fine core in this example is empty. To see this, note that each agent must get at least three units in the state that he can discern with his own information. Moreover, every two agent coalition can identify each state with its pooled information, and must therefore get a total of at least six units in each state. This means that no-trade is the only possible allocation which can belong to the fine core. However, that has a (coarse) objection by the grand coalition using an allocation which is a perturbation of y, for example, y':

$$y'(a) = (3 + 2\epsilon, 2 - \epsilon, 4 - \epsilon), \qquad y'(b) = (4 - \epsilon, 3 + 2\epsilon, 2 - \epsilon),$$
$$y'(c) = (2 - \epsilon, 4 - \epsilon, 3 + 2\epsilon),$$

for ϵ small enough.

Incidentally, this example also shows the importance of free disposal in Proposition 2; recall footnote 23. The following allocation is in the ex ante core:

$$x(a) = (2.25, 2.25, 4.5), \quad x(b) = (4.5, 2.25, 2.25), \quad x(c) = (2.25, 4.5, 2.25).$$

Since information is non-exclusive, this allocation can be extended so as to make it incentive compatible. However, incentive compatibility will require that $x_i(t_1, t_2, t_3) \leq 2.25$ for $i = 1, 2, 3$, and thus $\sum_i x_i(t_1, t_2, t_3) < \sum_i e_i$.

Define the *ex post core* to be the set of all feasible mechanisms $\mu \in \mathcal{F}$ such that for all t such that $q(t) > 0$, $\mu(t)$ belongs to the core of the complete information economy in state t. Clearly, the ex post core is generally non-empty. Since Example 6 concerns a single good economy, the no-trade allocation is the only one in the ex post core. Example 6 also had the feature that any allocation not in the ex post core had an ex post objection by some two agent coalition. Since each two agent coalition can identify each state, such an objection is also a fine objection. This argument, showing that the fine core is a subset of the ex post core, can be applied to any economy in which the state can be identified by pooling the information of agents in *some* coalition with an ex post objection. Einy et al. (2000) show that this is generally the case in an atomless economy with a finite number of states. The proof is based on the argument that in an atomless economy, if there is an objection in a certain state, there exists an objection by an arbitrarily large coalition; see Vind (1972). And with a finite number of states it is then possible to construct such a coalition in which the state can be discerned by pooling the private information in the large coalition. The following result is proved in Einy et al. (2000).

Proposition 9. *In an atomless economy with a finite number of states such that pooled information of all the agents corresponds to full information, the fine core is contained in the ex post core.*

The coarse core and the fine core correspond to two polar extremes; the former ruling out pooling of information and the latter allowing for arbitrary forms of information pooling. It is natural then, to explore a theory which provides a basis for making endogenous the amount of private information that is shared within a coalition. There may be situations in which some members of a coalition could credibly convince others in the coalition of an event which is not common knowledge. Restricting attention to efficiency, Holmström and Myerson (1983) study this issue by considering a proposal for the grand coalition which is tested with a voting procedure and formalize the notion of durable decision rules. A durable decision rule is one to which there does not exist a threat from a proposal which is in some sense a credible objection to the status-quo even though it may not be an improvement over a common knowledge event.

Similar ideas can be applied to a notion of the core in which coalitions are permitted to carry out objections over events finer than a common knowledge event. Consider Example 6 again. Recall that coalition $\{1, 3\}$ has a fine objection to y when the state is c; for example, allocating $3-\epsilon$ to agent 1 and $3+\epsilon$ to agent 3. While this state is known to agent 3, should agent 1 believe agent 3's claim about the true state (even ignoring incentive compatibility)? Agent 1 may fear that the true state is b but this can be dispelled if agent 3 offers his entire endowment to agent 1 in case the state is a or b. In other words, when agent 1 knows that the true state is either b or c, he should be willing to accept from agent 3 the proposal allocating to him six units in b and $3-\epsilon$ in c. And agent 3 is better-off knowing the state to be c. Lee and Volij (1996) term such an objection a coarse $+$ objection. More generally, they define a coarse $+$ objection by coalition S as an objection over an event E which is common knowledge to a subset A of S such that all agents in A gain in terms of conditional expected utility over the event E, as in the definition of a coarse objection and all agents in $S\setminus A$ gain in (ex post utility) in *all* states. The corresponding core notion is defined as the *coarse* $+$ core. In Example 6, the coarse $+$core coincides with the fine core and is empty. In general, the coarse $+$core contains the fine core and is contained in the coarse core. Lee and Volij (1996) provide a characterization of both the coarse core and the coarse $+$ core in terms of consistency and converse consistency axioms (with appropriately defined reduced form games).

While the coarse $+$ core may require an uninformed agent to gain in all the states that he believes are possible, one could argue that it may be enough to convince such an agent of the true state. For instance, consider Example 6 and the allocation y' which differs from y only in state b:

$$y'(a) = (3, 2, 4), \quad y'(b) = (6, 3, 0), \quad y'(c) = (2, 4, 3).$$

As before, there is a fine objection from $\{1, 3\}$ in state c. But now this is not a coarse $+$ objection since agent 3 cannot offer more than six units to agent 1 in state b. Nevertheless, it seems reasonable to think that y' is a credible objection to y since it is not in the interest of agent 3 to propose this unless the state is actually c, in which case both 1 and 3 are better-off. The interests of 1 and 3 coincide and agent 1can safely delegate to agent 3 the decision about breaking away from the status-quo.[36] The idea here is that of self-selection. An uninformed agent can assume that if self-selection indicates that a certain proposal would be made by an agent of type t, then it is enough for an objection to be directed at the state t. This is the idea (along with appropriate incentive compatibility constraints) on which the *credible core* of Dutta and Vohra (2001) is based. It is not difficult to see that if incentive constraints are not imposed, then the credible core coincides with the fine core and can, therefore, be empty; see Dutta and Vohra (2001) for additional examples which take account of incentive constraints.

For a model in which actions of agents at the interim stage reveal additional information to others see Ichiishi and Sertel (1998). Abstracting from incentive constraints, Volij (2000) proposes a definition of the core that takes account of inferences drawn by agents based on the acceptance of a proposal by other members of the coalition. He constructs a sequence of

[36] In Example 6, this argument would not change the coarse $+$ core but other examples can be constructed where this would make a difference.

refinements of the information partition of each agent based on the types of others who would gain by accepting the new proposal. The limit of this procedure yields a new information partition for each player. An objection is required to make each player better-off at each step of the sequence, as well as in the limit.

5.4. The virtual utility approach

The basic model we presented in Section 2 incorporates incentive constraints in the definition of the feasible set for each coalition. Myerson (1984a), Myerson (1984b), Myerson (1995) develops an alternative approach in which incentive constraints are used to define the 'virtual utility' of agents corresponding to each state. One can thus consider (feasible) virtual utilities at any particular state, i.e. without having to directly consider the corresponding mechanism (across all states). This approach has been used by Myerson to develop a theory of cooperative games under incomplete information. In particular, it has been elegantly applied in Myerson (1984b) to extend the Shapley value to an environment with incomplete information. It has also been used in Myerson (1995) to generalize the notion of the inner core to games with incomplete information in the context of a dynamic matching process. Applying this approach to develop an interim core notion in an exchange economy will undoubtedly add to our understanding of the issues discussed throughout this section. In what follows we shall briefly describe the approach and illustrate in a simple example how the idea of virtual utility might be used to calculate a core-like solution.

The first step is similar to the one used by Shapley and Harsanyi in defining a value for NTU games (see Myerson, 1991, Chapter 9, and Myerson, 1992). One associates to any point on the Pareto efficient frontier of the grand coalition a vector of supporting weights and considers the fictitious game in which individuals are allowed to transfer utility at the rates specified by these weights. If the value of the modified transferable utility game is feasible in the original game, it is a Harsanyi-Shapley value. The difficulty in extending this approach to the case of incomplete information comes from the fact that allowing players to transfer utilities at the interim stage may alter the incentive structure of the game, so that the Pareto frontier of the modified game might be far removed from the one of the original game. Myerson's key insight is that there is an extension of the game, in which players are allowed to transfer appropriately defined virtual utilities in every state, which makes it possible to associate a supporting linear Pareto frontier to any given incentive efficient mechanism of the original game, exactly as in the case of complete information.

To illustrate this idea, refer to the basic economy introduced in Section 2; assuming for simplicity that the set of feasible allocation X is finite, the problem of finding incentive efficient mechanisms is to maximize the vector of interim expected utilities over the incentive constraints (2). With random mechanisms (see Section 4.3.2), these constraints define a convex set. The supporting hyperplane theorem then allows us to associate with any incentive efficient mechanism μ, vectors $\lambda \in \times_{i\in N}\mathbb{R}^{T_i}$ and $\alpha \in \times_{i\in N}\mathbb{R}^{T_i\times T_i}$ such that, in every state $t \in T$, $\mu(x|t) > 0$ only if x maximizes $\sum_i v_i(x, t, \lambda, \alpha)$, where the term $v_i(x, t, \lambda, \alpha)$ is player i's (λ, α)-virtual utility from bundle x_i in state t, defined as

$$\frac{1}{q(t)}\left[\left(\lambda_i(t_i)+\sum_{s_i}\alpha_i(s_i|t_i)\right)u_i(t, x_i)q(t_{-i}|t_i)-\sum_{s_i}\alpha_i(t_i|s_i)u_i(t_{-i}, s_i, x_i)q(t_{-i}|s_i)\right],$$

and $\alpha_i(s_i|t_i)$ is the multiplier associated with the constraint that individual i should not have an incentive to declare s_i when his true type is t_i (see Myerson, 1991, Theorem 10.1).

That is, any incentive efficient mechanism selects allocations that are ex post efficient in the virtual utility scales: in bargaining over mechanisms each player is forced, by incentive considerations, to act as if he was maximizing a distorted utility, which magnifies the differences between his true type and types that would be tempted to imitate him.

Example 7. There are two consumers and two commodities. Suppose $T_1 = \{s, t\}$ while agent 2 is uninformed. Suppose s and t are equally probable. Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

$$u_1(s, x^1, x^2) = 0.1x^1 + x^2 - 0.1, \qquad u_2(s, x^1, x^2) = 0.25x^1 + x^2 - 1.$$
$$u_1(t, x^1, x^2) = x^1 + x^2 - 1, \qquad u_2(t, x^1, x^2) = 1.5x^1 + x^2 - 1.$$

The main difference with Example 1 is that now there are gains from trade in both states. The constant terms allow us to identify no trade with the origin in the utility space. Utilities are linear, and we can restrict attention to deterministic mechanisms. Any incentive efficient mechanism gives all of the first good to player 2 in state s, and satisfies the incentive constraint for type s with equality. The set of interim expected utilities that can be achieved by means of incentive efficient and interim individually rational mechanisms is a triangle in $\mathbb{R}^3$ with vertices $(U_1(\cdot|s), U_1(\cdot|t), U_2(\cdot))$ equal to $(0.075, 0.075, 0)$, $(0.3375, 0, 0)$ and $(0, 0, 0.075)$. Using consumer 1's net trade vector to parameterize mechanisms, as in Example 1, the first vertex corresponds to the mechanism μ^1 with net trades $z_1(s) = (-1, 0.175)$, $z_1(t) = (0, 0.075)$, the second to the mechanism μ^2 with net trades $z_1(s) = (-1, 0.4375)$, $z_1(t) = (-0.375, 0.375)$, the third to the mechanism μ^3 with net trades $z_1(s) = (-1, 0.1)$, $z_1(t) = (0, 0)$.

The utility allocations in the triangle correspond to the mechanisms in the incentive compatible coarse core. Among these, μ^1 is the one preferred by player 1 when he is of type t, μ^2 when he is of type s, and μ^3 is the mechanism preferred by player 2.

The shape of the utility frontier determines the supporting weights, $\lambda_1(s) = 2/9$, $\lambda_1(t) = 7/9$, $\lambda_2 = 1$. Given these weights, the saddlepoint conditions for the Lagrangean give $\alpha_1(t|s) = 5/18$ as the only non-zero multiplier. Inserting these values in the expression above for virtual utilities, one can check that the only difference between virtual and real utilities is that in state t the virtual utility of player 1 is $v_1(x^1, x^2, t, \lambda, \alpha) = 1.5x^1 + x^2 - 1.5$. In an effort to separate himself from the bad quality seller (type s), the good quality seller (type t) acts as if he had an higher valuation of the good he owns.

The transferable virtual utility game in state s corresponds to the ex post economy in that state, and its core payoffs are vectors of the form $((y-0.1), (0.25-y))$, with $y \in [0.1, 0.25]$. In the transferable virtual utility game corresponding to state t there are no gains from trade, and the only payoff vector in the core is $(0, 0)$. These payoffs, when transformed in real expected utilities, corresponds to the segment in $\mathbb{R}^3$ between the point $(0, 0, 0.075)$ to the point $(0.15, 0.05357, 0)$, a line on the efficient frontier which connects the vertex of the triangle corresponding to player 2's most preferred mechanism to a point on the opposite side, where player 2 is down to his reservation utility.

Applying the virtual utility approach to this example thus allows us to identify a smaller set of outcomes than the incentive compatible coarse core. To understand why, let us first give an interpretation of the point $(0.15, 0.05357, 0)$. This utility allocation lies on the line connecting the vertex preferred by player 1 when he is of type s and the vertex preferred when he is of type t, and can thus be interpreted as the result of a compromise between the two types of player 1. The corresponding mechanism, μ^4, generates net trades $z_1(s) = (-1, 0.25)$, $z_1(t) = (-0.1071, 0.1607)$ and it has the property of maximizing the expected utility of player 1 under the constraint that the individual rationality for player 2 is guaranteed in *both* states. In this sense, μ^4 is 'safe', that is robust to the revelation of information, and among the safe mechanisms it is the best one for player 1. Myerson (1984a) argues that this mechanism is the natural outcome if player 1 has all the bargaining power but at the same time knows that his proposals, being made at the interim stage, may convey information to player 2.

The line connecting this point to the vertex preferred by player 2 can then be given a natural interpretation: it represents the efficient bargaining possibilities actually available to the two players. A point not on this line, even if incentive efficient and interim individually rational, is not a good candidate for being an outcome of bargaining. Consider, for example, mechanism μ^2, the one preferred by player 1 when his type is s. If he makes this proposal, player 2 should deduce that the state is s and refuse to participate, because type t would have done better with the 'safe' mechanism μ^4.

In the usual interpretation of the core, in which a status-quo is tested against deviations, this argument cannot be used to exclude μ^2. Indeed, if μ^2 is the status-quo, player 1 can credibly signal that the state is t by proposing μ^4, but he will not convince player 2 to deviate, because μ^2 gives to player 2 his best possible payoff in state t. Thus, the virtual utility approach suggests an argument for ruling out μ^2 which is different from those, based on refinements of the notion of domination, described in the previous section.

6. Replica economies and core convergence

The literature on the core of complete information economies offers two important insights on the role played by the number of agents. First, the core of replicated economies converges to the set of Walrasian allocations, and in an atomless economy, the core coincides (even in the presence of non-convexities) with the set of Walrasian allocations. Second, the non-emptiness of an approximate core is guaranteed in a large economy. Early papers on this topic are Aumman (1964) and Debreu and Scarf (1963); see Anderson (1994), Hildenbrand (1982) and Wooders (1994) for surveys. In this section, we investigate to what extent similar results hold for the two notions of the incentive compatible core discussed above.

A preliminary question concerns the definition of a large economy in the presence of asymmetric information. The standard notion, which requires that each individual owns only a negligible fraction of the aggregate endowment, is clearly not sufficient. It might well be the case that an individual, though negligible in terms of ownership of goods, still maintains market power as the only owner of some relevant piece of information. Informational smallness, as we saw in Section 4.3.1, is crucial in reducing the impact of incentive compatibility constraints.

Clearly, if we replicate the economy in such a way that two copies of the same individual have exactly the same information, the condition of non-exclusive information is immediately satisfied. With this type of ex post replication, incentive problems are absent in the replicated economy, the ex ante incentive compatible core coincides with the core of the complete information economy and classical results are restored. Interestingly, as we will see in Section 6.2, the convergence of the coarse core is problematic even in this simple case.

Gul and Postlewaite (1992) propose a different form of replication: types are independent across replicas and the utility of each individual depends only on the types of the replica to which he belongs. More precisely, given a basic economy $E = \{N, (T_i, u_i, e_i)_{i\in N}, q\}$, the m-times replicated economy, E^m, is defined as follows. Individuals are $(i, k) \in N \times M$, with $M = \{1, 2, \dots, m\}$. For every i, all the copies of i have the same set of types, $T_{i,k}$, a copy of T_i. The set of types for the economy is thus, the product of m copies of the set of states in the basic economy, $\bar{T}_m = \times_k T_k$, with T_k a copy of T, and we assume that the probability distribution over $\bar{T}_m$ is $\bar{q}_m = \times_k q_k$, the independent product of the probability distributions over types in every replica, $q_k = q$. For every i, all the copies of i have the same (state independent) initial endowment, $e_{i,k} = e_i$. The utility of each individual depends only on the types of other individuals in the same replica: for every (i, k), and every $\bar{t}_m \in \bar{T}_m$, the utility function of individual (i, k) over consumption in state $\bar{t}_m$ is

$$u_{i,k}(\bar{t}_m, \cdot) = u_i(t_k, \cdot).$$

This kind of replication process tries to capture a situation in which each individual maintains some truly private information even in the replicated economy, but his information has a direct impact only on a small number of other individuals. As we will see next, non-emptiness of an approximate ex ante incentive compatible core is indeed restored, but convergence to the appropriate notion of competitive equilibrium may fail.

6.1. The ex ante core

As we mentioned above, ex post replication leads to non-exclusive information and to the non-emptiness of the core. Given that incentive compatibility constraints are not binding, the corresponding equilibrium notion is the full information Arrow–Debreu equilibrium and convergence follows by standard arguments. Alternatively, one can follow Radner (1968) and impose the requirement that an agent's trades be measurable with respect to her private information. Equilibrium allocations so defined bear the standard relationship with the private core (see Section 1.1.3), which similarly imposes such measurability restrictions; see Einy et al. (2001a).[37]

When the economy is replicated in the way proposed by Gul and Postlewaite, on the other hand, each individual's information remains private even in the large economy and we need a notion of competitive equilibrium which takes incentives into account.

Such a notion was proposed by Prescott and Townsend (1984b), who consider a model in which the objects of trade are incentive compatible state contingent lotteries over consumption. A bundle for individual (i, k) in economy E^m is $\mu_{i,k} : \bar{T}_m \to \Delta(\mathbb{R}^l)$.

An allocation, $(\mu_{i,k})_{(i,k)\in N\times M}$, specifies an incentive compatible state-contingent lottery for every individual. It can be seen as a random mechanism for the grand coalition with

[37] Einy et al. (2001b) provide conditions for the convergence of the ex post core to the set of fully revealing rational expectations equilibrium allocations.

feasibility defined on average, across states and realizations of the lotteries: μ is feasible in E^m if

$$\sum_{i,k}\sum_{\bar{t}_m}\bar{q}_m(\bar{t}_m)\int x\mu_{i,k}(\mathrm{d}x|\bar{t}_m)\leq m\sum_i e^i. \tag{11}$$

Notice that this notion of average feasibility is even weaker than the one used in Section 4.3.2 to define the "modified incentive compatible core" of Allen (1992), because one averages also across realization of information.

Lotteries over consumption are priced by the average amount of resources they use. For a given vector $p\in\mathbb{R}^L_+$ of commodity prices, the price of a lottery μ is

$$\pi_p(\mu)=\sum_{\bar{t}_m}\bar{q}_m(\bar{t}_m)\int\sum_l p_l x_l\mu(\mathrm{d}x|\bar{t}_m).$$

With these definitions in place, a competitive equilibrium consists of a price and an allocation satisfying (11) such that each individual maximizes his expected utility over his budget set. Let the *average feasibility core* refer to the ex ante incentive compatible core with feasibility defined as (11). Its non-emptiness can be proved directly, as in Proposition 5, or by showing that a competitive equilibrium exists and the corresponding allocation is contained in this core. This notion of the core is useful in showing non-emptiness, in large economies, of the ex ante core in which feasibility holds approximately in *each* state.

Indeed, by appealing to the law of large numbers, Forges et al. (2001) show that, if we let m tend to infinity, the replication of an allocation which is feasible on average in the basic economy converges to an allocation which is feasible almost surely in the replicated economy, and use this property to prove a non-emptiness result.

Let $\epsilon>0$ and $\delta>0$. We say that x is (ϵ,δ)-feasible in E^m if the probability of violating feasibility by more than ϵ (in norm) is less than δ. For any given (ϵ,δ), the (ϵ,δ)-ex ante incentive compatible core defined using this notion of feasibility is non-empty when the number of replicas is large enough.

Proposition 10. *For all $\epsilon>0$, and all $\delta>0$ there exists M such that for all $m\geq M$, the (ϵ,δ)-ex ante incentive compatible core is non-empty.*

This proposition shows one way in which replicas help with the non-emptiness of a notion of the incentive compatible approximate core.[38] In particular, replicating sufficiently many times the economy in Example 4 of Section 4.2 would guarantee the existence of such a core. The special form of replication is crucial to be able to apply the law of large numbers. Does this type of replication allow us to extend to economies with asymmetric information the second classical result mentioned above, namely the core convergence theorem? The following example, taken from Forges et al. (2001), illustrates the type of problems that may arise: individuals who are ex ante identical are not treated equally in the core of the replicated economy.

[38] Non-emptiness results in large replica economies are also discussed, for the case of pure common values, in McLean and Postlewaite (2000).

Example 8. There are two consumers and two commodities. Suppose $T_1 = \{s, t\}$ while agent 2 is uninformed (and therefore has only one type). The information state can then be described by s or t. Suppose s and t are equally probable. Let $e_1 = e_2 = (1.5, 1)$.

$$u_1(s, x^1, x^2) = \log x^1 + x^2, \qquad u_2(s, x^1, x^2) = 2\log x^1 + x^2.$$

$$u_1(t, x^1, x^2) = 2\log x^1 + x^2, \qquad u_2(t, x^1, x^2) = \log x^1 + x^2.$$

The two individuals are thus ex ante identical, but the realized type of individual 1 determines ex post which of the two has a higher utility from consumption of the first good.

Consider the allocation $\hat{x}$ defined by $\hat{x}_1(s) = \hat{x}_2(t) = (1, 1.5)$ and $\hat{x}_1(t) = \hat{x}_2(s) = (2, 0.5)$. If we restrict attention to deterministic state - contingent allocations, $\hat{x}$ is (ex ante) Pareto-optimal. Furthermore, it is easy to check that it is also incentive compatible and individually rational. In particular, each individual obtains a gain from trade equal to $U_i(\hat{x}_i) - U_i(e_i) = 0.085$.

To show that $\hat{x}$ is in the (average feasibility) core of E we only have to check that $\hat{x}$ is Pareto-optimal even when we allow for state-contingent lotteries, but this follows easily from the concavity of the utility functions.

If we modify $\hat{x}$ by requiring an additional transfer of $\tau \leq 0.085$ units of good 2 from individual 2 to individual 1 in each state, we maintain incentive compatibility and individual rationality, and we obtain an allocation $\tilde{x}$ which also belong to the core of E.

We will show that $(\hat{x}, \tilde{x})$ belongs to the core of the two-fold replicated economy E^2, thereby violating the equal treatment property.

Consider coalition $\{11, 22\}$, formed by individual 1 in the first replica and individual 2 in the second replica.[39] In this coalition, if individual 11 must be guaranteed $U_1(\hat{x}_1)$, individual 22 cannot get more than his reservation utility $U_2(e_2)$, so that they do not have a profitable deviation.

To see this, consider the allocation $\bar{x}$ defined by $\bar{x}_{11}(s) = (1, 1.5)$, $\bar{x}_{11}(t) = (2, 0.5)$, $\bar{x}_{22}(s) = (1.5, 0.5)$, $\bar{x}_{22}(t) = (1.5, 1.5)$. Individual 11 obtains the same bundle as in $\hat{x}$, while individual 22, whose utility does not depend on the type of individual 11, obtains the same quantity of good 1 in both states. This allocation is incentive compatible and individually rational. Furthermore, $\bar{x}$ maximizes the sum of expected utilities in the coalition. By construction $U_1(\bar{x}_{11}) = U_1(\hat{x}_{11})$, hence individual 22's utility cannot exceed $U_2(\bar{x}_{22}) = U_2(e_2)$, as claimed.

As the example makes clear, the dependence of the utility of a given individual on the types of other individuals in his replica creates an 'informational externality' which breaks the usual argument for equal treatment (see e.g. Debreu and Scarf, 1963). In the special case of private values this externality is not present, and one may hope to get a positive convergence result.

The basic economy is said to satisfy the assumption of *independent private values* if, as in Section 4.3.1, $u_i(t, \cdot) = u_i(t_i, \cdot)$ and, moreover, $q = \times_i q_i$, $q_i \in \Delta(T_i)$. The replicated economy E^m is obtained from E exactly as above, but now the utility of each individual only depends on his own type and the fact of belonging to one replica or another is of no

[39] Deviations by other coalitions can be easily ruled out.

consequence. At an allocation in the core of the replicated economy, all the replicas of an individual must obtain the same level of utility. Private values are crucial to the result, as shown by the previous example. In Forges et al. (2001), this is used to prove.

Proposition 11. *With independent private values, if an allocation μ belongs to the (average feasibility) core of E^m for all $m \geq 1$, then μ is a competitive equilibrium allocation.*

6.2. The interim core

Serrano et al. (2001) study the relationship between the interim cores and corresponding price equilibrium notions for a replicated economy. They provide an example of a sunspot economy in which core convergence does not obtain. Since they consider ex post replicas (copies of the same agent have the same information), information is non-exclusive and the negative result is thus independent of incentive constraints.

The example of non-convergence in Serrano et al. (2001) is most simply described with respect to the coarse core and a price equilibrium concept which appears to have properties likely to yield convergence. The price equilibrium concept is adapted from Wilson (1978) and captures decision making at the interim stage. Let p denote a vector of state-contingent market prices where $p(t) \in \mathbb{R}^l$. For agent i of type t_i, let $X_i(t_i) = \{x_i(\cdot, t_i) \in \mathbb{R}^{l \times |T_{-i}|}\}$ denote the set of relevant state contingent commodity bundles. The budget set of agent i of type t_i at prices p is denoted

$$B_i(p|t_i) = \left\{ x_i(\cdot, t_i) \in X_i(t_i) \,\middle|\, \sum_{t_{-i}} p(t) \cdot x_i(t) \leq \sum_{t_{-i}} p(t) \cdot e_i \right\}.$$

A *constrained market equilibrium* consists of state contingent prices p and a feasible state contingent allocation x such that for all i and $t_i \in T_i$, $x_i(\cdot, t_i) \in \arg\max_{B_i(p|t_i)} U_i(\cdot|t_i)$.

It is easy to see that constrained market equilibria satisfy several properties that are analogous to those of Walrasian equilibria. In particular an equilibrium allocation belongs to the coarse core[40] and a replication of an equilibrium allocation is an equilibrium allocation of the corresponding replicated economy. This equilibrium notion thus provides a natural benchmark to which one might expect coarse core allocations of replicated economies to converge.

The result in Serrano et al. (2001) actually applies to any price equilibrium concept which has the property that an agent who can discern an information state, in equilibrium, maximizes his ex post utility given the prices corresponding to that state. More precisely, the critical property for their result (satisfied by many price equilibrium notions) is

Property P. Suppose (x, p) is an equilibrium and there exists $t \in T$ such that $q(t|t_i) = 1$. Then $x_i(t) \in \arg\max_{B_i(p|t)} u_i(t, \cdot)$.

Serrano et al. (2001) consider a particularly simple type of differential information economy, a sunspot economy consisting of two states and two kinds of consumers—those who

40 As pointed out in footnote 6 in Wilson (1978). See also Goenka and Shell (1997).

are fully informed and those who cannot distinguish between either state at the interim stage. The basic economy, and any ex post replication of it, can be seen as a restricted market participation economy of Cass and Shell (1983) in which informed consumers can participate only in spot markets. Moreover, a sunspot equilibrium is identical to a constrained market equilibrium. In equilibrium, informed agents maximize ex post utility subject to their ex post budget constraint, while uninformed consumers maximize expected utility subject to a single budget constraint (involving contingent commodities). The example is as follows.

Example 9. The basic economy, E, consists to two agents and two commodities. There are two equally probable (sunspot) states s and t. Agent 1 is uniformed while agent 2 is fully informed. Thus, we can identify the states with the types of agent 2, $T_2 = \{s, t\}$. The endowments and preferences are independent of the state and both agents have identical (ex post) utility functions:

$$e_1 = (0, 24), \qquad e_2 = (24, 0)$$
$$u_i(r, x^1, x^2) = (x^1 x^2)^{1/4} \text{ for } r = s, t \text{ and } i = 1, 2.$$

In this simple sunspot economy, with prices suitably normalized, there is a unique constrained market equilibrium which is also a sunspot equilibrium as well as rational expectations equilibrium; this equilibrium is $(\bar{x}, \bar{p})$, where

$$\bar{x}_i(r) = (12, 12), \quad \text{and} \quad \bar{p}(r) = (1/4, 1/4) \quad \text{for all } r = s, t \quad \text{and} \quad i = 1, 2.$$

Of course, $\bar{x}$ belongs the coarse core, and a replication of $\bar{x}$ along with $\bar{p}$ is the unique equilibrium in the replicated economy.

Coarse core allocations in this economy need not satisfy the equal treatment property, but for a completely different reason than the one explaining Example 8. Indeed, in a sunspot economy values are private, and there is no ‘informational externality’ of the kind described in the previous section. The problem results from the fact that a coarse improvement by a coalition containing both informed and uninformed consumers must make the informed better-off in *both* states. This corresponds to a restriction on the composition of an allowable coalition (as in Wilson’s proof of Proposition 7). Quite apart from the equal treatment property, this restriction on allowable coalitions can be enough for a failure of core convergence.

In the present example, there exists an allocation x such that the m-th replication of x belongs to the coarse core of E^m for all m but $x \neq \bar{x}$, i.e. x is not a sunspot equilibrium allocation.[41] There are several allocations with this feature, including the following:

$$x_1(s) = (9, 9), \qquad x_1(t) = (16, 16)$$
$$x_2(s) = (15, 15), \qquad x_2(t) = (8, 8).$$

Note that this allocation cannot be an equilibrium allocation for any equilibrium notion satisfying Property P; the commodity bundle (8, 8) is not on the offer curve of the informed consumer in state t.

[41] See Serrano et al. (2001) for details. Note that in our framework, the number of types increases with replication (even though there are only two underlying states of the world) but one can nevertheless concentrate on allocations for information states with positive probability.

It is instructive to see why x remains in the coarse for every replication. The standard Debreu-Scarf argument for ruling out x from the 'core' of a large economy would proceed by constructing a coalition containing a relatively small number of informed agents in state s and a relatively large number of informed agents in state t. But such a 'coalition' would have no meaning in the present context; a coalition must have the same number of informed consumers in each state.

As argued in Serrano et al. (2001), non-convergence pertains not only to the coarse core but to a variety of other interim core notions as well price equilibrium concepts.[42] In particular, it is possible to construct an example of an economy (not a sunspot economy) in which there is a constant (hence measurable) allocation which belongs to the core of each replicated economy but is not a price equilibrium allocation. Thus, non-convergence of the interim core to price equilibrium allocations is a robust phenomenon.

Acknowledgements

We thank a referee, Serkan Bahceci, Geoffroy de Clippel, Ali Khan, Herakles Polemarchakis, Roberto Serrano and Oscar Volij for many helpful comments. Vohra acknowledges support from NSF grant SES-0133113.

References

Akerlof, G., 1970. The market for lemons: quality uncertainty and the market mechanisms. Quarterly Journal of Economics 84, 488–500.

Allen, B., 1992. Incentives in Market Games with Asymmetric Information: The Core. CORE Discussion Paper 9221, Université Catholique de Louvain.

Allen, B., 1993. Market games with asymmetric information: verification and the publicly predictable core. Hitotsubashi Journal of Economics 32, 101–122.

Allen, B., 1994. Incentives in market games with asymmetric information: approximate NTU cores in large economies. In: Barnett, W., Moulin, H., Salles, M., Schofield, N. (Eds.), Social Choice, Welfare and Ethics. Cambridge University Press, Cambridge.

Allen, B., Yannelis, N., 2001. Differential information economies: introduction. Economic Theory 18, 263–273.

Anderson, R., 1994. Core convergence in perfectly competitive economies. In: Mertens, J.F., Sorin, S. (Eds.), Game Theoretic Approaches to General Equilibrium Theory. Kluwer Academic Publishers, Dordrecht.

Arrow, K., 1979. The property rights doctrine and demand revelation under incomplete information. In: Boskin, M. (Ed.), Economics and Public Welfare. Academic Press, New York.

d'Aspremont, C., Gérard-Varet, L.-A., 1979. Incentives and incomplete information. Journal of Public Economics 11, 25–45.

d'Aspremont, C., Gérard-Varet, L.-A., 1982. Bayesian incentive compatible beliefs. Journal of Mathematical Economics 10, 83–103.

d'Aspremont, C., Crémer, J., Gérard-Varet, L.-A., 1990. Incentives and the existence of Pareto-optimal revelation mechanisms. Journal of Economic Theory 51, 233–254.

Aumann, R., 1964. Markets with a continuum of traders. Econometrica 32, 39–50.

Cass, D., Shell, K., 1983. Do sunspots matter? Journal of Political Economy 91, 193–227.

Debreu, G., Scarf, H., 1963. A limit theorem on the core of an economy. International Economic Review 4, 235–246.

[42] For an interim core which may generally be empty, such as the fine core, non-convergence holds trivially.

Demange, G., Guesnerie, R., 2001. On coalitional stability of anonymous interim mechanisms. Economic Theory 18, 367–389.

Dutta, B., Vohra, R., 2001. Incomplete Information, Credibility and the Core. Working Paper No. 2001-02, Department of Economics, Brown University.

Einy, E., Moreno, D., Shitovitz, B., 2000. On the core of an economy with differential information. Journal of Economic Theory 94, 262–270.

Einy, E., Moreno, D., Shitovitz, B., 2001a. Competitive and core allocations in large economies with differential information. Economic Theory 18, 263–273.

Einy, E., Moreno, D., Shitovitz, B., 2001b. Rational expectations equilibria and the ex post core of an economy with asymmetric information. Journal of Mathematical Economics 34, 527–535.

Forges, F., Minelli, E., 2001. A note on the incentive compatible core. Journal of Economic Theory 98, 179–188.

Forges, F., Heifetz, A., Minelli, E., 2001. Incentive compatible core and competitive equilibria in differential information economies. Economic Theory 18, 349–365.

Forges, F., Mertens, J.-F., Vohra, R., 2002. The ex ante incentive compatible core in the absence of wealth effects. Econometrica 70, 1865–1892.

Goenka, A., Shell, K., 1997. Robustness of sunspot equilibria. Economic Theory 10, 79–98.

Green, J., Laffont, J.-J., 1979. Incentives in Public Decision Making. North-Holland, Amsterdam.

Groves, T., 1973. Incentives in teams. Econometrica 41, 617–631.

Gul, F., Postlewaite, A., 1992. Asymptotic efficiency in large economies with asymmetric information. Econometrica 60, 1273–1292.

Haake, C.J., 2001. Two-agent fee-economies. Mimeo, University of Bielefeld.

Hahn, G., Yannelis, N. 1997. Efficiency and incentive compatibility in differential information economies. Economic Theory 10, 383–411.

Hara, C., 2000. An equivalence theorem for the anonymous core. Mimeo, University of Cambridge.

Hildenbrand, W., 1982. Core of an economy. In: Arrow, K.J., Intrilligator, M.D. (Eds.), Handbook of Mathematical Economics, vol. II. North-Holland, Amsterdam.

Holmström, B., Myerson, R., 1983. Efficient and durable decision rules with incomplete information. Econometrica 51, 1799–1819.

Hurwicz, L., Maskin, E., Postlewaite, A., 1995. Feasible Nash implementation of social choice rules when the designer does not know endowments or production sets. In: Ledyard, J.O. (Ed.), The Economics of Informational Decentralization: Complexity, Efficiency and Stability. Kluwer Academic Publishers, Amsterdam, pp. 367–433.

Ichiishi, T., Idzik, A., 1996. Bayesian cooperative choice of strategies. International Journal of Game Theory 25, 455–473.

Ichiishi, T., Sertel, M., 1998. Cooperative interim contract and re-contract: Chandler's M-form firm. Economic Theory 11, 523–543.

Ichiishi, T., Radner, R., 1999. A profit-center game with incomplete information. Review of Economic Design 4, 307–343.

Johnson, S., Pratt, J., Zeckhauser, R., 1990. Efficiency despite mutually payoff-relevant private information: the finite case. Econometrica 58, 873–900.

Koutsougeras, L., Yannelis, N., 1993. Incentive compatibility and information superiority of the core of an economy with differential information. Economic Theory 3, 195–216.

Krasa, S., Shafer, W., 2001. Core concepts in economies where information is almost complete. Economic Theory 18, 451–471.

Lee, D., Volij, O., 1996. The core of economies with asymmetric information: an axiomatic approach. Journal of Mathematical Economics, in press.

McLean, R., Postlewaite, A., 1999. Informational size and incentive compatibility. Econometrica, in press.

McLean, R., Postlewaite, A., 2000. Informational size, incentive compatibility and the core of a game with incomplete information. Mimeo, University of Pennsylvania.

Milgrom, R., Roberts, J., 1992. Economics, Organization and Management. Prentice-Hall, New Jersey.

Myerson, R., 1984a. Two person bargaining problems with incomplete information. Econometrica 52, 461–487.

Myerson, R., 1984b. Cooperative games with incomplete information. International Journal of Game Theory 13, 69–96.

Myerson, R., 1991. Game Theory: Analysis of Conflict. Harvard University Press, Cambridge.

Myerson, R., 1992. Fictitious-tranfer solutions in cooperative game theory. In: Selten, R. (Ed.), Rational Interaction: Essay in Honor of John C. Harsanyi. Springer, Berlin, pp. 13–33.

Myerson, R., 1995. Sustainable matching plans with adverse selection. Games and Economic Behavior 9, 35–65.

Myerson, R., Satterthwaite, M., 1983. Efficient mechanisms for bilateral trading. Journal of Economic Theory 23, 265–281.

Page, F., 1997. Market games with differential information and infinite commodity spaces: the core. Economic Theory 9, 151–159.

Postlewaite, A., Schmeidler, D., 1986. Implementation in differential information economies. Journal of Economic Theory 39, 14–33.

Prescott, E., Townsend, R., 1984a. Pareto-optima and competitive equilibria with adverse selection and moral hazard. Econometrica 52, 21–45.

Prescott, E., Townsend, R., 1984b. General competitive analysis in an economy with private information. International Economic Review 25, 1–20.

Radner, R., 1968. Competitive equilibrium under uncertainty. Econometrica 36, 31–58.

Radner, R., 1979. Rational expectations equilibrium: generic existence and the information revealed by prices. Econometrica 47, 655–678.

Rosenmueller, J., 1999. Mechanisms in the core of a fee game. In: Gaul, W., Schader, M. (Eds.), Mathematische Methoden der Wirtschaftswissenschaften. Physica Verlag, Heidelberg.

Serrano, R., Vohra, R., Volij, O., 2001. On the failure of core convergence in economies with asymmetric information. Econometrica 69, 1685–1696.

Scarf, H., 1967. The core of an n-person game. Econometrica 35, 50–69.

Vind, K., 1972. A third remark on the core of an atomless economy. Econometrica 40, 585–586.

Vohra, R., 1999. Incomplete information, incentive compatibility and the core. Journal of Economic Theory 86, 123–147.

Volij, O., 2000. Communication, credible improvements and the core of an economy with asymmetric information. International Journal of Game Theory 29, 63–79.

Wilson, R., 1978. Information, efficiency and the core of an economy. Econometrica 46, 807–816.

Wooders M., 1994. Large games and economies with effective small groups. In: Mertens, J.F., Sorin, S. (Eds.), Game Theoretic Approaches to General Equilibrium Theory. Kluwer Academic Publishers, Dordrecht.

Yannelis, N., 1991. The core of an economy with differential information. Economic Theory 1, 183–198.