Incomplete Information, Credibility and the Core

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Abstract. An appropriate (interim) notion of the core for an economy with incomplete information depends on the amount of information that coalitions can share. The coarse and fine core, as originally defined by Wilson (1978), correspond to two polar cases, involving no information sharing and arbitrary information sharing, respectively. We propose a new core notion, the credible core, which incorporates incentive compatibility constraints, and is based on the idea that a coalition can coordinate its potential objection to a status-quo over an event that can be credibly inferred from the nature of the objection being contemplated.

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1 Introduction

Consider an exchange economy in which consumers have private information at the interim stage when state contingent contracts are made. Each agent knows her private information and has some probability assessment over the true information of others. We study an environment in which the only constraints on enforcing agreements are those arising from the incompleteness of information. A coalition can agree on a feasible state contingent contract (or net-trades) which is enforced by an agency using the private information reported by the agents. Naturally then, contracts need to be subjected to incentive compatibility constraints. An appropriate notion of the core provides a natural cooperative equilibrium concept for the problem of resource allocation in such an economy. One of the critical issues that arises in defining an appropriate core notion - and our central concern in the present paper - is the specification of the information that agents in a coalition are allowed to use in constructing an objection. In what way, if any, can members of a coalition share their private information? Put differently, over what kind of informational event is a coalition permitted to object? It should be borne in mind that this issue does not arise in defining the core at the ex ante stage; see Forges, Minelli and Vohra (2000) for additional discussion.

Wilson (1978), developed two distinct approaches that deal with this issue, and lead respectively to the notions of the coarse core and the fine core.[1] The coarse core is based on the assumption that a coalition can focus its potential objection on an event if and only if the event is commonly known to all members of the coalition. Thus the act of forming an objecting coalition does not change the private information of any agent. The fine core is based on the idea that the act of forming a coalition allows all members of the coalition to decide how much of their private information they wish to share with each other.

Thus, the coarse and the fine core correspond to two extreme informational assumptions on coalitional behavior - the former rules out information sharing or leakage while the latter permits arbitrary sharing of information. We argue that both of these polar cases are subject to criticism. In particular, we show by means of an example that there may be circumstances in which it is reasonable for coalitions to coordinate their actions on an event which is *not* a common knowledge event. Another example demonstrates that the fine core is also unreasonable since agents may not be able to pool their information in a credible manner.

In view of this discussion, it is natural to ask whether the theory can provide insights into the amount of private information that coalitions can be reasonably expected to pool. In other words, is it possible to make endogenous the amount of information that is pooled in a coalition? In this paper, we develop a notion of the core in which coalitions are allowed to coordinate their actions over an event that can be credibly inferred from the objection being contemplated. Our

[1] Wilson assumed that all information was publicly verifiable at the time of enforcement, and hence did not have to impose incentive compatibility. Our primary concern here will be with the incentive compatible versions of Wilson's core notions.

notion of credible objections is meant to capture the following idea. Suppose a contract is under consideration as an objection by a coalition, and agent i in the coalition claims that she is of type s_i. This claim is considered credible if agent i would prefer the new contract to the status-quo if and only if she were indeed of type s_i. Other agents should, therefore, be able to infer i's statement regarding the informational event.

The *credible core* is the set of allocations to which there is no credible objection.[2] Although the credible core is a fairly conservative way of allowing for endogenous information-sharing, we show, with an example, that it may be empty (even if utility functions are linear).[3] This reinforces Wilson's (1978) conclusion on the emptiness of the core. It appears that opening the door to information-sharing even in a very restricted way is enough to destroy non-emptiness of the core.

2 The Model

In this Section we describe the basic model of an exchange economy with incomplete information. Since our main interest lies in analyzing an environment in which private information cannot be verified at the enforcement stage, we shall impose incentive compatibility constraints on all contracts. For this reason, we find it convenient to formulate private information in terms of agents' types.

Let T_i denote the (finite) set of agent i's types. The interpretation is that $t_i \in T_i$ denotes the *private information* possessed by agent i. With $N = \{1, \ldots, n\}$ as the finite set of agents, let $T = \prod_{i \in N} T_i$. An information *state* for the economy refers to $t \in T$. We will use the notation t_{-i} to denote $(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots t_n\}$. Similarly $T_{-i} = \prod_{j \neq i} T_j$, and for any coalition S, a non-empty subset of N, $t_S = (t_i)_{i \in S}$ and $T_S = \prod_{i \in S} T_i$.

Each agent has a prior probability distribution q_i defined on T. We assume that none of the types is redundant in the sense that for every $i \in N$ and $t_i \in T_i$, there exists $t_{-i} \in T_{-i}$ such that $q_i(t) > 0$. For each $i \in N$ and $\bar{t}_i \in T_i$, the conditional probability of $t_{-i} \in T_{-i}$, given $\bar{t}_i$ is

$$q_i(t_{-i} \mid \bar{t}_i) = \frac{q_i(t_{-i}, \bar{t}_i)}{\sum_{t'_{-i} \in T_{-i}} q_i(t'_{-i}, \bar{t}_i)}.$$

We shall assume that all agents agree on zero probability events:

(A) If $q_i(t) > 0$ for some $i \in N$ and $t \in T$, then $q_j(t) > 0$ for all $j \in N$.

[2]The basic idea underlying the credible core has been used in a variety of different contexts. See, for instance, Cho and Kreps (1987), Holmström and Myerson (1983), Kahn and Mookherjee (1995). For related work, see also Forges (1994), Krasa (2000), Ichiishi and Sertel (1998) and Lee and Volij (1996).

[3]Wilson (1978) showed that the coarse core is non-empty in convex exchange economies but that the fine core may be empty. The incentive compatible coarse core is non-empty if preferences are linear but may be empty otherwise; see Vohra (1999) and Forges, Mertens and Vohra (2001).

Let $T^* = \{t \in T \mid q_i(t) > 0, \text{ for all } i \in N\}$. Agent i of type $t'_i \in T_i$ observes the event $P_i(t'_i) = \{t \in T^* \mid t_i = t'_i\}$. In this way we can define for each i a partition $P_i = \{P_i(t_i)\}_{t_i \in T_i}$ over T^* which represents the events that are discernible by i.

We assume that there are a finite number of commodities, so that the consumption set of any agent in any set is a subset of $\mathbb{R}^l_+$. The characteristics of an agent, namely, the consumption set, the endowments and utility function will depend in general on the state. We assume, for simplicity, that the endowment of agent i is defined by a function $\omega_i : T_i \mapsto \mathbb{R}^l$, so that $\omega_i(t_i) \in \mathbb{R}^l_+$ denotes i's endowment when her type is t_i. We also assume that $X_i(t) = \mathbb{R}^l_+$. Each consumer has a state dependent utility function $u_i : \mathbb{R}^l \times T \mapsto \mathbb{R}$. We will denote by $u_i(., t)$ the von Neumann-Morgenstern utility function of agent i in state t. We assume that for each $i \in N$ and $t \in T$, $u_i(., t)$ is continuous and concave.

We can now define an exchange economy as $\mathcal{E} = \{(u_i, X_i, \omega_i, T_i, q_i)_{i \in N}\}$. We shall assume:

(B) For all $i \in N, t \in T$ and $x \in \mathbb{R}^l_+$, $u_i(x, t) \geq u_i(0, t)$.

A state contingent contract is a function $x : T \mapsto \prod_i X_i$. The set of contracts *feasible* for the grand coalition is defined as:

$$A_N = \{x : T \mapsto \prod_i X_i \mid \sum_{i \in N} x_i(t) \leq \sum_{i \in N} \omega_i(t_i), \text{ for all } t \in T\}.$$

A contract x is *feasible for coalition S* if

(a) $\sum_{i \in S} x_i(t) \leq \sum_{i \in S} \omega_i(t_i)$ for all $t \in T$.

(b) $x_i(t) = x_i(t')$ for all $i \in S$, $t, t' \in T$ such that $t_S = t'_S$.

Requirement (b) reflects the idea that a coalition cannot rely on the participation of outsiders in choosing its mechanism. If information becomes publicly verifiable at the enforcement stage there would be no reason to insist on (b). The set of feasible contracts for coalition S is denoted A_S.

2.1 The Coarse Core

Suppose agents in a coalition do not (or cannot) share their private information. They can then coordinate their actions only over an event that is commonly known to them. A non-empty event $E \subseteq T$, is said to be *discernible without information pooling* by coalition S if $q_i(\hat{t}_{-i} \mid t_i) = 0$ (or, equivalently, if $q_i(\hat{t}_{-i}, t_i) = 0$) for all $i \in S$, $t \in E$ and $(\hat{t}_{-i}, t_i) \notin E$. Such an event is also termed a *common knowledge event* for coalition S.

For an event $E \subseteq T$, define for each $i \in N$, the set of types of i compatible with the event E as

$$E_i = \{t_i \in T_i \mid (t'_{-i}, t_i) \in E \text{ for some } t'_{-i} \in T_{-i}\}.$$

The conditional expected utility of consumer i corresponding to contract x, conditional on her being of type t_i, is

$$U_i(x_i \mid t_i) \equiv \sum_{t'_{-i} \in T_{-i}} q_i(t'_{-i}|t_i) u_i((x_i(t'_{-i}, t_i), (t'_{-i}, t_i)).$$

Accordingly, $y \in A_S$ *dominates* $x \in A_N$ for S over a common knowledge event E if

$$U_i(y_i \mid t_i) > U_i(x_i \mid t_i) \text{ for all } t_i \in E_i \text{ for all } i \in S,. \tag{D$'$}$$

In addition to physical feasibility and domination, we also need to impose incentive compatibility constraints to ensure that an objecting contract can be implemented by the coalition when private information cannot be verified.

Consider an allocation y. By pretending to be of type s_i, when her true type is t_i, agent i can obtain the net-trade corresponding to the state (t_{-i}, s_i) when the true state is t. Let the corresponding commodity bundle be denoted

$$y_i(t_{-i}, s_i \mid t_i) = y_i(t_{-i}, s_i) - \omega_i(t_{-i}, s_i) + \omega_i(t_i).$$

This deception yields conditional expected utility

$$U_i(y_i, s_i \mid t_i) \equiv \sum_{t'_{-i} \in T_{-i}} q_i(t'_{-i}|t_i) u_i(y_i(t'_{-i}, s_i \mid t_i), (t'_{-i}, t_i)).$$

We shall assume that a deception that leads to bankruptcy can never be profitable. This is equivalent to extending the domain of the utility function such that $u_i(y_i, t) = -\infty$ for all $y_i \notin \mathbb{R}^l_+$, for all $i \in N$ and $t \in T$.

A contract y is said to be *incentive compatible* for coalition S if

$$U_i(y_i \mid t_i) \geq U_i(y_i, s_i \mid t_i) \text{ for all } s_i, t_i \in T_i, \text{ for all } i \in S. \tag{IC$'$}$$

Coalition S has an *incentive compatible, coarse objection* to an incentive compatible contract $x \in A_N$ if there exists $y \in A_S$ and an event E that is common knowledge for S such that (D') and (IC') hold.[4] By the revelation principle, the set of incentive compatible contracts is identical to those which can be implemented as Bayesian Nash equilibria of a direct mechanism. A contract can therefore be viewed as a mechanism. It is also worth pointing out that in this context, the assumption of free disposal which we implicitly made in defining feasible allocations for a coalition is no longer innocuous. A coalition may be able to do better if free disposal is allowed in the presence of incentive constraints; see Forges, Mertens and Vohra (2001) and Forges, Minelli and Vohra (2000) for examples. The results we report here, however, do not rely on the free disposal assumption.

[4] One may argue that the incentive compatibility constraints as expressed in (IC') are too strong; it should be enough to require these constraints over the common knowledge event E. Fortunately, as we will show in Proposition 2.1 below, this would not alter the notion of an incentive compatible, coarse objection.

The *incentive compatible, coarse core* consists of all incentive compatible contracts $x \in A_N$ to which there exists no coarse objection.

2.2 The Fine Core

Suppose coalition S considers an event $E \subseteq T$ over which to coordinate its actions through a contract. The theory depends critically on the restrictions that are imposed on such an event. There are some basic restrictions which should always be imposed on such an event. Differences in various core notions will then depend on additional restrictions that might be imposed. It turns out that the basic restrictions we discuss below are already implicit in an event which is commonly known to a coalition.[5]

To consider the possibility that a coalition may be able to act over an event that is not necessarily commonly known to all members of the coalition, suppose all members of coalition S *believe* that the true state belongs to a non-empty set $E \subseteq T$. Clearly, there are some natural restrictions that ought to imposed on E (if E is not a common knowledge event) for such beliefs to be reasonable. First, it must be the case that E can be discerned without using the private information of those not in the coalition. So, if S considers an event E then a profile of types of agents outside S, $t'_{-S} \in T_{-S}$, can be excluded from E only if this is discernible with the private information of agents in S. Moreover, since all our domination notions will be based on evaluating conditional utilities, we can express this requirement as $E = E_S \times T_{-S}$. Second, E must reflect independent claims by members of a coalition in a mechanism, i.e., $E = [\prod_{i \in S} E_i] \times T_{-S}$, where $E_i \subseteq T_i$ for all i. Finally, E must be consistent with what each of the agents in the coalition know, given their private information. No agent, knowing her type, should rule out the possibility that the true state lies in E. In other words, for all $i \in S$ and $t_i \in E_i$, $q_i(E \mid t_i) > 0$.

Thus, a non-empty event $E \subseteq T$ is said to be *admissible* for coalition S if it is of the form:

$$E = \prod_{i \in S} E_i \times T_{-S}, \text{ where } E_i \subseteq T_i \text{ and } q_i(E \mid t_i) > 0 \text{ for all } t \in E, \text{ for all } i \in S.$$

Suppose a coalition can act over an admissible event E. The notion of domination used in (D') needs to be modified now taking account of the information that is contained in E.

The probability that agent i assigns to $t \in E$, conditional on her type being t_i, and the belief that the true state is in E, is given by

$$q_i(t_{-i} \mid t_i, E) = \frac{q_i(t_{-i} \mid t_i)}{q_i(E \mid t_i)} = \frac{q_i(t)}{\sum_{t'_{-i} \in E_{-i}} q_i(t'_{-i}, t_i)}.$$

Note that if E is an admissible event, this expression is well-defined since $q_i(E \mid t_i) > 0$ for $t \in E$. We can now define for each $i \in S$ and a type $t_i \in E_i$, the

[5] This will also make it clear that our definition of the incentive compatible, coarse core above is the same as the definition used in Vohra (1999).

conditional expected utility (conditional on E), for a contract x as

$$U_i(x_i \mid t_i, E) \equiv \sum_{t'_{-i} \in T_{-i} \mid (t'_{-i}, t_i) \in E} q_i(t'_{-i} \mid t_i, E) u_i(x_i(t'_{-i}, t_i), (t'_{-i}, t_i)).$$

For coalition S, $y \in A_S$ *dominates* $x \in A_N$ over an admissible event E if

$$U_i(y_i \mid t_i, E) > U_i(x_i \mid t_i, E) \text{ for all } t_i \in E_i \text{ for all } i \in S. \tag{D}$$

If coalition S uses the information corresponding to an admissible event E, we shall need to consider incentive compatibility with respect to E. Given a contract x and an admissible event E, the conditional expected utility (conditional on the information provided by E) to agent i of type t_i by pretending to be of type s_i is defined as

$$U_i(x_i, s_i \mid t_i, E) \equiv \sum_{t'_{-i} \in T_{-i} \mid (t'_{-i}, t_i) \in E} q_i(t'_{-i} \mid t_i, E) u_i(x(t'_{-i}, s_i \mid t_i), (t'_{-i}, t_i)).$$

A contract y is said to be *incentive compatible over an admissible event* E for coalition S if

$$U_i(y_i \mid t_i, E) \geq U_i(y_i, s_i \mid t_i, E) \text{ for all } s_i, t_i \in E_i, \text{ for all } i \in S. \tag{IC}$$

Coalition S has an *incentive compatible, fine objection* to an incentive compatible contract $x \in A_N$ if there exists $y \in A_S$ and an admissible event E for S such that (D) and (IC) hold.

The *incentive compatible, fine core* consists of all incentive compatible contracts $x \in A_N$ to which there exists no incentive compatible, fine objection.

To clarify the essential difference between the coarse core and the fine core it is important to check that the domination and incentive compatibility conditions are, in fact, the same in each case. Conditions (D) and (IC) in defining a fine objection reflect the fact that agents update their prior probability assessments based on the pooled information. No such updating is required in a coarse objection since no additional information becomes available to any agent through the process of constructing an objection over a common knowledge event. It can also be shown that imposing admissibility on a common knowledge event would imply no loss of generality.

Proposition 2.1 *Coalition S has an incentive compatible, coarse objection to x over an event E which is common knowledge for S, if and only if S has an incentive compatible, fine objection to x over an (admissible) event E' which is common knowledge for S.*

Proof: Suppose $y \in A_S$ is an incentive compatible, coarse objection of coalition S to x over a common knowledge event E. Let $E'_i = E_i = \{t_i \in T_i \mid (t'_{-i}, t_i) \in E$ for some $t'_{-i} \in T_{-i}\}$, and let $E' = \prod_{i \in S} E'_i \times T_{-S}$. Of course, E' is a common knowledge event for S since $E \subseteq E'$. Since no type is redundant, and E' is a common knowledge event, it follows that $q_i(E' \mid t_i) > 0$ for all $t \in E'$ and all $i \in S$. Thus E' is an admissible event. Since E' is a common knowledge event, it follows that for every $i \in S$ and $t \in E'$, $q_i(t_{-i} \mid t_i, E') = q_i(t_{-i} \mid t_i)$. Thus, $U_i(y_i \mid t_i, E') = U_i(y_i \mid t_i)$. Since (D') holds for all $t_i \in E_i = E'_i$ for all $i \in S$, this implies (D). Of course, (IC') implies (IC) over E'. Thus, y is a fine objection by S over a common knowledge (admissible) event E'.

To prove the converse, suppose y is an incentive compatible fine objection by S over E, a common knowledge, admissible event E. If $S = \{i\}$, we can assume without loss of generality, that $y_i(t) = \omega_i(t_i)$ for all $t \in T$. Of course, y then satisfies (IC') and since E is a common knowledge event, it follows that (S, y) is an incentive compatible coarse objection. Assume, therefore, that $|S| \geq 2$. Define $\tilde{y}$ such that for all $i \in S$,

$$\tilde{y}_i(t) = \begin{cases} y_i(t) & \text{for all } t \in E \\ 0 & \text{if } t_i \notin E_i \text{ and } t_j \in E_j \text{ for all } i \neq j \\ \omega_i(t_i) & \text{otherwise} \end{cases}$$

Note that since $|S| \geq 2$, it is possible to find $\tilde{y}$ of this form satisfying exact feasibility, i.e., free disposal is not needed. We now claim that $\tilde{y}$ is an incentive compatible coarse objection by S to x over the event E. Since E is a common knowledge event, for all $t \in E$ and $i \in S$,

$$U_i(\tilde{y}_i \mid t_i, E) = U_i(y_i \mid t_i, E) = U_i(y_i \mid t_i).$$

Thus (D) implies (D'). It remains to be shown that (IC) implies (IC'). Condition (IC) means that

$$U_i(\tilde{y}_i \mid t_i) \geq U_i(\tilde{y}_i, s_i \mid t_i) \text{ for all } s_i, t_i \in E_i \text{ for all } i \in S.$$

Given assumption (B), and the construction of $\tilde{y}$, this also holds for all $s_i, t_i \in T_i$. Thus $\tilde{y}$ is an incentive compatible coarse objection over E. ■

Proposition 2.1 shows that we can take admissibility, (D) and (IC) to be the necessary conditions in defining an objection. The coarse core adds to these conditions the requirement that objections are only permitted over common knowledge events. Clearly then, the incentive compatible, fine core is a subset of the incentive compatible, coarse core.

It is important to keep in mind that both (D) and (IC) are defined with respect to an admissible event E, reflecting the updated probability assessments inherent in E. In particular, a fine objection by S over a particular state, i.e., a fine objection by S over an event $E = \prod_{i \in S}\{t_i\} \times T_{-S}$, makes condition (IC) redundant. Thus a contract in the incentive compatible fine core must necessarily be ex-post efficient in the sense of Holmström and Myerson (1983).[6] Moreover, as Einy, Moreno and Shitovitz (2000) show, in an atomless economy, the fine core is a subset of the ex post core, i.e., a fine core allocation has

[6]The fact that an allocation in the fine core is ex-post efficient again points to the fact that an incentive compatible, fine objection may rely on an agent to believe unverifiable (and unreasonable) claims by another.

the property that in each state, the allocation is a core allocation of the full information economy for that state.

As is well known, even in the two-consumer case, there might not exist any incentive compatible allocation which is ex-post efficient (see, for example, Holmström and Myerson (1983) and Myerson and Satterthwaite (1983)). The incentive compatible, fine core may therefore be empty even in a two-consumer economy.[7] The incentive compatible, coarse core is non-empty in (well-behaved) two-consumer economies. It is also generally non-empty if preferences are linear; see Ichiishi and Idzik (1996) and Vohra (1999). However, Vohra (1999) shows that in well-behaved, three-consumer economies (with non-linear preferences) the incentive compatible, coarse may be empty.[8]

3 Credible Information Pooling

We shall argue that in some cases pooling of information is reasonable while in others it is not. Our aim is to formalize a notion of credible pooling of private information and a corresponding notion of a credible core. We begin with two simple, motivating examples. The first illustrates a situation in which information pooling seems reasonable, and provides a critique of the coarse core. The second illustrates a situation in which information pooling does not seem reasonable, and provides a critique of the fine core. These examples will also serve to introduce our notion of credibility.

Example 3.1

There are three consumers in an economy with two commodities. Each consumer i can be of two possible types. Let $T_i = \{a_i, b_i\}$. Of the eight information states, only three arise with positive probability. These states are denoted

$$t^1 = (a_1, b_2, b_3), \quad t^2 = (b_1, a_2, b_3) \quad t^3 = (b_1, b_2, a_3)$$

All consumers have identical priors q, where

$$q(t) = \begin{cases} 1/3 & \text{if } t \in T^* = \{t^1, t^2, t^3\} \\ 0 & \text{otherwise} \end{cases}$$

In each state with positive probability there is exactly one consumer who is fully informed; consumer i is the informed agent in state t^i.

The consumption set of each consumer is $\mathbb{R}^2_+$ in each state. The endowments are as follows:

$$\omega_i(t_i) = \begin{cases} (1, 0) & \text{if } t_i = b_i \\ (0.5, 0.5) & \text{if } t_i = a_i \end{cases}$$

[7] Wilson (1978) constructs a three-consumer example in which the fine core is empty. Recall that Wilson did not impose incentive compatibility.

[8] Forges and Minelli (2000) point out that the if random allocations are permitted, then the example in Vohra (1999) does have a non-empty ex ante core. However, Forges, Mertens and Vohra (2001) provide another example in which the incentive compatible coarse core is empty even if random allocations are allowed.

For $x = (x_1, x_2) \in \mathbb{R}^2_+$, the state-dependent utility functions are as follows.

$$u_1(x,t) = \begin{cases} 1.5(x_1 + x_2) & \text{if } t = t^3 \\ x_1 + x_2 & \text{otherwise} \end{cases}$$

$$u_2(x,t) = \begin{cases} 1.5(x_1 + x_2) & \text{if } t = t^1 \\ x_1 + x_2 & \text{otherwise} \end{cases}$$

$$u_3(x,t) = \begin{cases} 1.5(x_1 + x_2) & \text{if } t = t^2 \\ x_1 + x_2 & \text{otherwise} \end{cases}$$

Notice that both commodities are perfect substitutes. For a contract x let $\psi(x)$ denote the sum of the two commodities allocated to each consumer, i.e., $\psi_i(x,t) = x_{i1}(t) + x_{i2}(t)$. It can be shown that x belongs to the incentive compatible coarse core if and only if

$$\psi(x,t^1) = (1+\delta_1, 2-\delta_1, 0), \quad \psi(x,t^2) = (0, 1+\delta_2, 2-\delta_2), \quad \psi(x,t^3) = (2-\delta_3, 0, 1+\delta_3)$$

where $\delta_1, \delta_2, \delta_3 \in [0, 1/3]$.

The incentive compatible, coarse core contains in particular the allocation $\bar{x}(t)$, where

$$\begin{aligned} \bar{x}(t^1) &= ((0.5, 0.5), (2, 0), (0, 0)) \\ \bar{x}(t^2) &= ((0, 0), (0.5, 0.5), (2, 0)) \\ \bar{x}(t^3) &= ((2, 0), (0, 0), (0.5, 0.5)) \end{aligned}$$

This allocation is not in the incentive compatible, fine core.[9] Consumers 1 and 3 have a fine objection over the event $\{t^1\}$ since $\psi_1(\bar{x}, t^1) + \psi_3(\bar{x}, t^1) = 1$, while their aggregate endowment of the two commodities is 2. If private information can be shared, as is implicit in the notion of the fine core, then clearly $\bar{x}$ is not viable in state t^1. But, in the present example more can be said to justify a fine objection by agents 1 and 3. Suppose the state is t^1, which consumer 1 knows. Consumer 3 knows that the true state is either t^1 or t^2. Consider an offer from consumer 1 to consumer 3 of the full state contingent contract $\tilde{x}(t)$, where

$$(\tilde{x}_1(t), \tilde{x}_3(t)) = \begin{cases} ((1.1, 0), (0.4, 0.5)) & \text{if } t = t^1 \\ (\omega_1(t), \omega_3(t)) & \text{otherwise} \end{cases}$$

In state t^1, the corresponding net-trades are $z_1(t^1) = (0.6, -0.5)$, $z_3(t^1) = (-0.1, 0.5)$. In state t^1, the informed agent gives up 0.5 units of commodity 2 for 0.6 units of commodity 1. Note that t^1 is the only state in which her endowments permit her to make this trade. While 3 does not know whether the true state is t^1 or t^2, she does know that the informed agent would be better off with this contract only if the true state is t^1; if the state is actually t^2, the

[9]In fact, it can be shown, that in this example the fine core (with or without incentive constraints) is empty. The main difference between this example and Wilson's (1978) Example 2 is that in our example each agent's endowment depends on his own type.

net-trade $z_1(0.6, -0.5)$ is infeasible for agent 1. The informed agent's claim, that the state is t^1, is credible and should, therefore, be accepted by agent 3. Acceptance of this contract requires only that agent 3 infer (correctly) from the contract that the state is t^1, not that 1's private information becomes explicitly available to agent 3. In this respect this contract offers a sensible objection to the status-quo. Agents should be able to coordinate on an event that can be inferred simply by the fact that all members of the coalition are willing to sign a contract that is to their benefit only on the given event. In the present example, this makes it hard to justify the coarse core as the appropriate core notion.

We now give an example which shows that unlimited pooling of information, which is implicit in the definition of the fine core, may not be very appropriate under some circumstances.

Example 3.2

Consider a simpler version of Example 3.1 in which there is only one commodity. The information structure is the same as before, i.e., $T_i = \{a_i, b_i)$ for each i and $T^* = \{t^1, t^2, t^3\}$.

The consumption set of each consumer is $\mathbb{R}_+$ in each state and the endowment of each consumer is 1 in each state.

The state-dependent utility functions are as follows.

$$u_1(x,t) = \begin{cases} 1.5x & \text{if } t = t^3 \\ x & \text{otherwise} \end{cases}$$

$$u_2(x,t) = \begin{cases} 1.5x & \text{if } t = t^1 \\ x & \text{otherwise} \end{cases}$$

$$u_3(x,t) = \begin{cases} 1.5x & \text{if } t = t^2 \\ x & \text{otherwise} \end{cases}$$

The incentive compatible, coarse core consists of all allocations x of the form

$$x(t^1) = (1+\delta_1, 2-\delta_1, 0), x(t^2) = (0, 1+\delta_2, 2-\delta_2), \quad x(t^3) = (2+\delta_3, 0, 1+\delta_3)$$

where $\delta_1, \delta_2, \delta_3 \in [0, 1/3]$.

The contract $\bar{x}(t)$, where

$$\bar{x}(t^1) = (1,2,0), \quad \bar{x}(t^2) = (0,1,2), \quad \bar{x}(t^3) = (2,0,1)$$

is one such contract.

This allocation is not in the incentive compatible, fine core.[10] Consumers 1 and 3 have a fine objection over the event $\{t^1\}$ with a contract $\tilde{x}(t^1)$ such that $\tilde{x}_1(t^1) = 1+\epsilon$ and $\tilde{x}_3(t^1) = 1-\epsilon$ for $\epsilon \in (0,1)$. In fact, every fine objection must be of this form. But agent 3 cannot infer from this contract that consumer 1 is of type a_1 because consumer 1 would prefer the net trade ϵ in *both* states t^1 and t^2. Moreover, if the true state is t^2, consumer 3 by agreeing to the contract

[10] As in example 3.1, the incentive compatible, fine core is empty.

$\tilde{x}$, and accepting 1's claim that she is of type a_1, would be worse off compared to the status-quo $\bar{x}$. In this sense, the fine objection is not credible. The same argument holds for any fine objection to a contract that belongs to the coarse core. In this example, therefore, the coarse core seems more reasonable than the fine core.

The essential message from these examples is that the pooling of private information between members of a coalition should be permitted only if it can be justified as being credible. We now develop a notion of objections which incorporates this consideration.

Suppose each i in coalition S claims, independently, not to be of any type $\hat{t}_i \notin E_i$. This type, $\hat{t}_i$,cannot be ruled out by agent $j \in S$, with her private information, if

$$\text{for some } t \in E,\ \hat{t}_i \notin E_i,\ q_j(t_{-i}, \hat{t}_i) > 0 \tag{3.1}$$

By assumption (A), this must, of course, mean that $q_j(t_{-i}, \hat{t}_i) > 0$ for all $j \in N$. For each $i \in S$ let $V_i(E) \subseteq T_i \setminus E_i$ denote the set of all $\hat{t}_i$ satisfying (3.1). Of course, if the event E is not a common knowledge event, $V_i(E) \neq \emptyset$ for some $i \in S$.

Our credibility criterion imposes the restriction that none of the types in $V_i(E)$ should select (or pretend) to be some type in E_i.

Given an admissible event E for coalition S define for each $i \in S$ and $\hat{t}_i \in V_i(E)$,

$$q_i(t_{-i} \mid \hat{t}_i, E) = \frac{q_i(t_{-i}, \hat{t}_i)}{\sum_{t'_{-i} \in E_{-i}} q_i(t'_{-i}, \hat{t}_i)}.$$

Note that this expression is well-defined given the definition of $V_i(E)$.

For an event admissible for coalition S, we can now define for each $i \in S$ and a type $\hat{t}_i \in V_i(E)$, the conditional expected utility (conditional on E), of a contract x as

$$U_i(x \mid \hat{t}_i, E) = \sum_{t'_{-i} \in E_{-i}} q_i(t'_{-i} \mid \hat{t}_i, E) u_i(x(t'_{-i}, \hat{t}_i), (t'_{-i}, \hat{t}_i))$$

Similarly, define the conditional expected utility of x to $\hat{t}_i \in V_i(E)$ if $\hat{t}_i$ pretends to be of type $s_i \in T_i$ as

$$U_i(x, s_i \mid \hat{t}_i, E) \equiv \sum_{t'_{-i} \in T_{-i} \mid (t'_{-i}, \hat{t}_i) \in E} q_i(t'_{-i} \mid \hat{t}_i, E) u_i(x_i(t'_{-i}, s_i \mid \hat{t}_i), (t'_{-i}, \hat{t}_i))$$

Note that this is well-defined since $\hat{t}_i \in V_i(E)$.

Suppose $x \in A_N$, $y \in A_S$ and E is an admissible event for coalition S. A contract y is said to satisfy *self-selection* with respect to x over E if

$$U_i(y, s_i \mid \hat{t}_i, E) \leq U_i(x \mid \hat{t}_i, E) \text{ for all } \hat{t}_i \in V_i(E),\ s_i \in E_i \text{ for all } i \in S \qquad \text{(SS)}.$$

This constraint can be seen as an extension of (IC) to those types who are not supposed to be part of the objecting coalition. Notice that, as in (D) and

(IC), the probabilities used in computing conditional expected utility, are those corresponding to the event E over which the objection is supposed to take place. By the argument used in proving proposition 2.1, it can be shown that condition (SS) equivalent to one in which this inequality is required to hold for all $s_i \in T_i$ rather than for all $s_i \in E_i$.

Coalition S is said to have an *incentive compatible, credible objection* to an incentive compatible contract $x \in A_N$ if there exists $y \in A_S$ and an admissible event E such that (D), (IC) and (SS) are satisfied.

The *incentive compatible, credible core* consists of all incentive compatible allocations to which there does not exist an incentive compatible, credible objection.

The basic idea underlying our notion of the credible core is one that has been widely used in other, related contexts. In non-cooperative models with incomplete information it is an idea that appears in several equilibrium refinements; see, for example, the discussion of the intuitive criterion in Cho and Kreps (1987).[11] It is used by Kahn and Mookherjee (1995) in analyzing coalition proof Nash equilibrium under incomplete information. It is also related in spirit to the notion of *durability* studied by Holmström and Myerson (1983), which we will discuss in more detail in the next section. As in Forges (1994), we focus on coalitional moves in renegotiation and abstract away from the details of an explicit non-cooperative procedure of the kind considered in some of this related literature. While Forges (1994) considers durability of the decisions resulting from a mechanism, our analysis is concerned with durability, or credibility, of the mechanism itself.

Notice that if E is a common knowledge event for S then $V_i(E) = \emptyset$ for all $i \in S$, and (SS) is, therefore, trivially satisfied. Thus the credible core is a subset of the coarse core. On the other hand, the fine core is contained in the credible core since a credible objection is a particular kind of fine objection. In example 3.1, the credible core coincides with the fine core, and in example 3.2 it coincides with the coarse core. As shown by example 3.1, the credible core can be empty. Since utility functions in that example are linear, it follows that appealing to random allocations is not enough to establish non-emptiness.

Incentive constraints are important in the notion of the credible core. If incentive constraints are not imposed, it is easy to see that the credible core is identical to the fine core. This is so because a fine objection y over E by coalition S is then equivalent to one in which agent i is assigned 0 in every state t such that $t_{-i} \in E_i$ and $t_i \in V_i(E)$. This observation does not apply to a notion of credibility in which instead of requiring that the wrong types lose we require that the other (uninformed) agents gain regardless. This is the idea used by Lee and Volij (1996) in defining the coarse + core, without imposing incentive

[11] As in the intuitive criterion, the speech by an agent making a claim about her type is: 'If I am of the wrong type I would not gain over the status-quo, so you should believe me'. Think of status-quo as the equilibrium. It can be broken if there is a way to signal information in a credible way which would make them all better off. In the coarse core, breaking the equilibrium is difficult because it has to be common knowledge that all types are better off – akin to elimination by domination; the credible version is like the intuitive criterion.

constraints. More precisely, they define a coarse + objection by coalition S as an objection over an event E which is common knowledge to a subset A of S such that all agents in A gain in terms of conditional expected utility over the event E, as in the definition of a coarse objection and all agents in $S \setminus A$ gain in (ex post utility) in *all* states. In general, the coarse + core contains the fine core and is contained in the coarse core. While our notion of credibility reduces to the notion of fine objections in the present context, it is conceptually related to Volij (2000). Abstracting from incentive constraints, Volij (2000) proposes a definition of the core that takes account of inferences drawn by agents based on the acceptance of a proposal by other members of the coalition. He constructs a sequence of refinements of the information partition of each agent based on the types of others who would gain by accepting the new proposal. The limit of these procedure yields a new information partition for each player. An objection is required to make each player better-off at each step of the sequence, as well as in the limit. Our notion of credibility is clearly very similar to Volij's except that we impose a refinement of the information structure directly, without going through a sequence. The other difference is that we require improvements to be evaluated for the 'limit' information sets, and we impose incentive compatibility.

4 Credibility and Bayesian Nash Equilibria of Voting Games

As we have already pointed out, credibility is closely related in spirit to the Holmström-Myerson (1983) notion of durability. Holmström and Myerson construct a voting game for the grand coalition and define an incentive efficient decision rule to be durable if every alternative proposal is rejected in a sequential equilibrium of the voting game.[12] In this section we formally explore the connections between credible objections and Bayesian Nash equilibria of the voting game considered by Holmström and Myerson.

Consider an incentive compatible allocation $x \in A_N$ which is the status-quo. By the revelation principle, x can be seen as Bayesian Nash equilibrium outcome of the direct revelation game. Suppose coalition S is free to choose a new mechanism $y \in A_S$ to be played by members of S. The new mechanism y is to be interpreted as a competing proposal which is implemented if and only if all members of S vote to accept y instead of x. If all players in S accept y, they play a direct revelation game in which the strategy set for each $i \in S$ is T_i, and the outcome is $y(t)$ for $t \in T$. In case of a rejection, the outcome function used in the second stage is $x(t)$. Players vote confidentially, and only the outcome of the vote is revealed publicly. Thus a voting game is defined as $\gamma(S, y, x)$, in which the players are the members of S, $y \in A_S$ and $x \in A_N$. A *truthful* Bayesian Nash equilibrium of the voting game is one in which each

[12]They consider a fully non-cooperative framework in which coordination problems are not necessarily overcome in equilibrium. Since we are dealing with a cooperative solution, a coalition can coordinate on an allocation provided it is self-enforcing with respect to private information.

player reports truthfully in stage 2 (regardless of whether or not the proposal is accepted).

For our next result, we need one additional definition.

An allocation $x \in A_N$ is said to be *uniformly incentive compatible* if

$$u_i(x_i(t), t) \geq u_i(x_i(t_{-i}, s_i), t) \text{ for all } s_i, t_i \in T_i,\ t_{-i} \in T_{-i}, \text{ for all } i \in N.$$

Note that if there is only one informed agent, then this condition is equivalent to incentive compatibility.

Proposition 4.1 *Suppose $x \in A_N$ is incentive compatible and $\Gamma(S, y, x)$ is a voting game for coalition S. If there exists a truthful Bayesian Nash equilibrium, and an admissible event E, such that i accepts y if and only $t_i \in E_i$, and all players in S gain compared to the status-quo whenever y is accepted, then (S, y) is an incentive compatible credible objection to x over the event E. The converse also holds if x is uniformly incentive compatible.*

Proof: In the voting game, a strategy of player i is a mapping from T_i to $\{a, r\} \times T_i$, where a denotes acceptance and r denotes rejection of the proposal y. We will denote by $\alpha_i(t_i)$, the action chosen by player i of type t_i in stage 1, and by $\gamma(t_i)$, the reported type of player i in stage 2.

Suppose there exists a truthful Bayesian Nash equilibrium, σ, of the game (S, y) such that $\alpha_i(t_i) = a$ for all $t_i \in E_i$, and all players in S when y is accepted. We claim that $(S, \tilde{y}, E)$ is an incentive compatible, credible objection. The fact that all players in E gain implies that $\tilde{y}$ satisfies (DD) over E. Condition (SS) follows from the fact that all players outside E_S reject the proposal. Since σ is a Bayesian Nash equilibrium, it follows from an argument similar that used in proving the revelation principle that y satisfies (IC) over E.

To prove the converse, when x satisfies uniform incentive compatibility, let (S, y) be an incentive compatible, credible objection to x over the event E. Suppose x is uniformly incentive compatible. A strategy for player i of type i now consists of $\alpha_i(t_i) \in \{a, r\}$ and $\gamma_i(t_i, h) \in T_i$, where h denotes the observed history in stage 2. Let σ be a strategy profile in the game $\Gamma'(S, y)$ such that For each $i \in S$,

$$\alpha_i(t_i) = \begin{cases} a & \text{if } t_i \in E_i \\ r & \text{if } t_i \notin E_i \end{cases}$$

$$\gamma_i(t_i, h) = t_i \text{ for all } t_i \in T_i \text{ for all } h.$$

According to σ, a player accepts y if and only if she is of a type belonging to E_i, and all players report their types truthfully in stage 2. By following the strategy σ_i, the expected payoff to i is

$$\begin{aligned} U_i(\sigma \mid t_i) \quad &= \quad \textstyle\sum_{t'_{-i} \in E_{-i}} q_i(t'_{-i} \mid t_i) u_i(y_i(t'_{-i}, t_i), (t'_{-i}, t_i)) \\ &\quad\ + \textstyle\sum_{t'_{-i} \notin E_{-i}} q_i(t'_{-i} \mid t_i) u_i(x_i(t'_{-i}, t_i), (t'_{-i}, t_i)) \end{aligned}$$

Consider a deviation by player i of type $t_i \in E_i$ to a strategy σ_i' in which i accepts the proposal and then reports t_i' in stage 2 when y is played and s_i if x is played. The expected payoff is then

$$\begin{aligned} U_i(\sigma_{-i}, \sigma_i' \mid t_i) &= \textstyle\sum_{t'_{-i} \in E_{-i}} q_i(t'_{-i} \mid t_i) u_i(y_i(t'_{-i}, t'_i), (t'_{-i}, t_i)) \\ & \textstyle\sum_{t'_{-i} \notin E_{-i}} q_i(t'_{-i} \mid t_i) u_i(x_i(t'_{-i}, s_i), (t'_{-i}, t_i)) \end{aligned}$$

Suppose $t_i' \in E_i$. From (IC) it follows that

$$\begin{aligned} &\textstyle\sum_{t'_{-i} \in E_{-i}} q_i(t'_{-i} \mid t_i) u_i(y_i(t'_{-i}, t'_i), (t'_{-i}, t_i)) \leq \\ &\textstyle\sum_{t'_{-i} \in E_{-i}} q_i(t'_{-i} \mid t_i) u_i(y_i(t'_{-i}, t_i), (t'_{-i}, t_i)) \end{aligned} \tag{4.1}$$

If $t_i' \notin E_i$, then uniform incentive compatibility of x implies that

$$\begin{aligned} &\textstyle\sum_{t'_{-i} \in E_{-i}} q_i(t'_{-i} \mid t_i) u_i(x_i(t'_{-i}, t'_i), (t'_{-i}, t_i)) \leq \\ &\textstyle\sum_{t'_{-i} \in E_{-i}} q_i(t'_{-i} \mid t_i) u_i(x_i(t'_{-i}, t_i), (t'_{-i}, t_i)) \end{aligned} \tag{4.2}$$

Moreover, (DD) implies that

$$\begin{aligned} &\textstyle\sum_{t'_{-i} \in E_{-i}} q_i(t'_{-i} \mid t_i) u_i(x_i(t'_{-i}, t_i), (t'_{-i}, t_i)) \leq \\ &\textstyle\sum_{t'_{-i} \in E_{-i}} q_i(t'_{-i} \mid t_i) u_i(y_i(t'_{-i}, t_i), (t'_{-i}, t_i)) \end{aligned} \tag{4.3}$$

Thus (4.2) and (4.3) imply that (3.1) holds for all $t_i' \in T_i$. Using uniform incentive compatibility of x, we can assert that

$$\begin{aligned} &\textstyle\sum_{t'_{-i} \notin E_{-i}} q_i(t'_{-i} \mid t_i) u_i(x_i(t'_{-i}, s_i), (t'_{-i}, t_i)) \leq \\ &\textstyle\sum_{t'_{-i} \notin E_{-i}} q_i(t'_{-i} \mid t_i) u_i(x_i(t'_{-i}, t_i), (t'_{-i}, t_i)) \end{aligned} \tag{4.4}$$

Conditions (4.1) and (4.4) imply that

$$U_i(\sigma, \mid t_i) \geq U_i(\sigma_{-i}, \sigma_i' \mid t_i)$$

and the deviation cannot be profitable.

To complete the proof we need to show that a deviation by a player i of type $t_i \notin E_i$ cannot be profitable. The equilibrium strategy profile yields $U_i(x_i \mid t_i)$ to such a player. Since x is incentive compatible, it cannot be profitable to reject the proposal and misreport the type. Consider a deviation σ_i' in which this player accepts in stage 1, and reports t_i' if y is played in stage 2 and s_i if x_i is played in stage 2. This yields an expected payoff

$$\begin{aligned} U_i(\sigma_{-i}, \sigma_i' \mid t_i) &= \textstyle\sum_{t'_{-i} \in E_{-i}} q_i(t'_{-i} \mid t_i) u_i(y_i(t'_{-i}, t'_i), (t'_{-i}, t_i)) \\ & \textstyle\sum_{t'_{-i} \notin E_{-i}} q_i(t'_{-i} \mid t_i) u_i(x_i(t'_{-i}, s_i), (t'_{-i}, t_i)) \end{aligned}$$

From (SS) it follows that

$$\sum_{t'_{-i} \in E_{-i}} q_i(t'_{-i} \mid t_i) u_i(y_i(t'_{-i}, t'_i), (t'_{-i}, t_i)) \leq \sum_{t'_{-i} \in E_{-i}} q_i(t'_{-i} \mid t_i) u_i(x_i(t'_{-i}, t_i), (t'_{-i}, t_i))$$

From uniform incentive compatibility of x we obtain (4.4). Together, these conditions imply that

$$U_i(\sigma \mid t_i) = U_i(x_i \mid t_i) \geq U_i(\sigma_{-i}, \sigma_i' \mid t_i)$$

and a deviation is not profitable. This completes the proof that the credible deviation is sustained as a Bayesian Nash equilibrium of $\Gamma'(S, y)$. ■

The reason that uniform incentive compatibility is needed in proving the second part of the proposition is that if y is not accepted, players may gain information from this fact and not report truthfully in stage 2. If we consider a simpler voting game in which x is interpreted as the status-quo and assume that in case of a rejection, the outcome is x, then uniform incentive compatibility can be dispensed with. It is also worthwhile to note that if E is a common knowledge event, again uniform incentive compatibility is not needed in proving the second part of the proposition. In other words, an incentive compatible coarse objection can be seen as Bayesian Nash equilibrium of the voting game in which y is accepted whenever $t \in E$.

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