Money and indeterminacy over an infinite horizon [1]

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Abstract

Money provides liquidity services through a cash-in-advance constraint. The exchange of commodities and assets extends over an infinite horizon under uncertainty and a complete asset market. Monetary policy sets the path of rates of interest and accommodates the demand for balances. Competitive equilibria exist. But, for a fixed path of rates of interest, there is a non-trivial multiplicity of equilibrium paths of prices of commodities. Determinacy requires that, subject to no-arbitrage and in addition to rates of interest, the prices of state-contingent revenues be set.

KEYWORDS: Money, equilibrium, indeterminacy, monetary policy, fiscal policy.

JEL CLASSIFICATION NUMBERS: D50, E40, E50.

1 Introduction

1.1

An immediate extension of the model of general competitive equilibrium of Arrow, Debreu and McKenzie encompasses monetary economies with an operative transaction technology: a monetary authority supplies balances in exchange for assets, through open market operations, and by lump-sum transfers to individuals. Non-interest-bearing fiat money is dominated by interest-bearing nominal assets as store of value. The demand of money resulting from its role in facilitating transactions, here a factual starting point.

A general equilibrium formulation of a monetary economy emancipates monetary economics from the hypothesis of a representative individual, it allows for distributional effects and, possibly, for incomplete asset markets. This last extension is not treated here.

Monetary models typically involve economies that extend into the infinite future. The reason is commonly understood as a problem of consistency, emphasized by Hahn [21]. Suppose that a given supply of balances, $\bar{m}$, serves as a medium of exchange and is initially distributed to individuals, $\left(\ldots, \bar{m}^i, \ldots\right)$. In a one-period economy, each individual faces the budget constraint

$$\left(\frac{r}{1+r}\right) m^i + p \cdot z^i = \bar{m}^i,$$

where p are prices of commodities, z^i is the net trade, r is the nominal rate of interest and m^i is the demand for balances. Holding money has an opportunity cost measured by the nominal rate of interest. Equilibrium immediately requires that money be in zero net supply, that is, $\bar{m} = 0$. If not, a positive quantity of money,

$$\bar{m} - \left(\frac{r}{1+r}\right) \bar{m} = \left(\frac{1}{1+r}\right) \bar{m},$$

must be taxed away, as suggested by Lerner [24]. The amount that must be taxed away at the terminal date, $\bar{m}$, has discounted value at the initial date equal to

$$\bar{m} - \sum_{\tau=0}^{t} \left(\frac{r}{1+r}\right) \left(\frac{1}{1+r}\right)^{\tau} \bar{m} = \left(\frac{1}{1+r}\right)^{t+1} \bar{m}.$$

Over an infinite horizon, this converges to zero and initial outside claims are exhausted by transaction costs.

Recent formulations of a monetary economy have pointed out that an infinite horizon is not necessary in order for money to be valued at equilibrium. In Drèze and Polemarchakis [13], fiat money is created at no cost by banks that lend it to individuals and firms at nominal non-negative rates of interest. Banks keep balanced accounts, so that outstanding money is the counterpart of assets, claims on individuals and firms. Banks are owned and their profits, equal to interest earned on assets, accrue to shareholders. An alternative description is given by Dubey and Geanakoplos [14, 15], where profits of banks are not redistributed to individuals and the latter are endowed with initial nominal balances. Both institutional arrangements avoid Hahn's [21] paradox over a finite horizon, since balances used in transactions, m, do not coincide with nominal claims assigned to individuals, $\left(\ldots, \bar{m}^i, \ldots\right)$, so that Walras' Law,

$$\bar{m} - \left(\frac{r}{1+r}\right) m = 0,$$

can be satisfied by a non-vanishing supply of balances.[1]

Once proven to be consistent, finite time has the advantage of analytical tractability. Still, the extension to an infinite horizon, which motivates our research in this paper, is of interest. First, in order to show that, when moving from a finite to an infinite horizon, all qualitative conclusions are retained. Second, in order to allow for speculative trade in long-term assets. Third, in order to accommodate models commonly employed in contemporary monetary economics. Last, but not least, in order to contrast equilibrium properties of economies of finitely many infinitely-lived individuals with economies of overlapping generations.

1.2

We consider an economy that extends to an infinite future. Time is divided into an infinite number of dates. Uncertainty is described, following the canonical model of Debreu [8], by an event tree.

Policies are represented by contingent plans. At every date, after information is revealed, a public authority supplies balances at given nominal rates of interest (monetary policy), trades in assets (portfolio policy), which allows for open market operations, collects commodity taxes (fiscal policy) and makes transfers to individuals (transfer policy).

A finite number of individuals hold initial claims and trade sequentially in commodities, balances and assets. At every date, contingent on revealed information, individuals adjust their holdings of assets and balances subject to budget and solvency constraints; trade in commodities follows, subject to a cash-in-advance or Clower [7] constraint. The asset market is sequentially complete: all possible state-contingent revenues can be traded.

Our description encompasses the natural infinite-horizon extensions of the monetary economies of Drèze and Polemarchakis [13] and Dubey and Geanakoplos [14, 15].[2] Also, it reproduces, almost faithfully, the formulation of Woodford

[1] In Dubey and Geanakoplos [14, 15], $\ldots, \bar{m}^i, \ldots$ are initial balances held by individuals and $m - \bar{m}$ is the additional supply of balances. In Drèze and Polemarchakis [13], $\ldots, \bar{m}^i, \ldots$ are interpreted as transfers of bank profits to individuals and there are no initial balances. The distinction between the exogeneity of initial holdings of balances, unlike the endogeneity of transfers of bank profits, affects the determinacy of equilibria and differentiates the specifications.

[2] Drèze and Polemarchakis [13], however, consider a monetary economy over a finite horizon with production and general transaction technology.

[33] with a representative individual, which is, in turn, similar to the cash-in-advance economies studied in Wilson [32] and Lucas and Stokey [26].[3] All differences among these previous studies obtain here as differences in the way monetary and, possibly, fiscal policies are conducted.

Sequential completeness of the asset market serves to isolate the effects of liquidity constraints from those related to other obstacles to risk sharing and intertemporal consumption smoothing. Admitting that balances can be readjusted at every date, after information is revealed, eliminates any precautionary demand and distinguishes the transaction and the asset role of money.

1.3

If the public authority sets the one-period, risk-free nominal rates of interest at every date-event and it accommodates the demand for balances, suitable transfers guarantee that competitive equilibria exist for arbitrary fiscal and portfolio policies. In the terminology of Woodford [33, 34], the regime is Ricardian: as prices of commodities and assets vary, transfers adjust to satisfy intertemporal public budget constraints.

Alternatively, transfers are restricted: when negative transfers are ruled out, in the presence of a positive initial public liability, existence requires positive public revenues, coming from taxes and seignorage; without taxes, positive revenues from seignorage needs transactions, which parallels the analogous condition in Dubey and Geanakoplos [15]. In the terminology of Woodford [33, 34], the regime is non-Ricardian.

Interest rate policy does not suffice to determine the level and the path of nominal prices of commodities and assets at equilibrium: the prices of contingent revenues are unrestricted; so is the over-all price level; the latter, but not the former, have real allocative effects (in presence of initial nominal claims). Determinacy, up to the over-all price level, requires that, beyond risk-free rates of interest, also the prices of contingent revenues be somehow determined, subject to no-arbitrage constraints. Nominal determinacy results, as in the fiscal theory of the price level (Woodford [33, 34] and Cochrane [11]), when there are no transfers. Similarly, Dubey and Geanakoplos [14, 15], as well as Magill and Quinzii [27], obtain determinacy, for economies with a finite horizon, by adopting a non-Ricardian specification.

The indeterminacy associated with interest rate policy has real effects in economies with an incomplete asset market over an infinite horizon (Florenzano and Gourdel [16], Hernández and Santos [22], Levine and Zame [25] and Magill and Quinzii [28]) modified so as to accommodate cash-in-advance constraints.

Nakajina and Polemarchakis [29] work out an example that illustrates the indeterminacy of equilibrium with Ricardian policy and contrast it with the determinacy that results from a non-Ricardian specification. In addition, they demonstrate that a monetary authority may attain determinacy by setting the

[3] Grandmont and Younés [19, 20] study a temporary equilibrium of an economy under certainty with cash-in-advance constraints, an infinite horizon and heterogeneous individuals.

term structure of rates on interest, equivalent to setting the prices of contingent revenues.

2 The Economy

2.1

For a given countable set, $\mathcal{A}$, the space of all real valued maps on $\mathcal{A}$, $\ell(\mathcal{A})$, is an ordered vector space. Vector subspaces are the spaces of all bounded, $\ell_\infty(\mathcal{A})$, and summable, $\ell_1(\mathcal{A})$, real valued maps on $\mathcal{A}$, endowed with their respective norms. A vector, x, is positive (strictly positive, uniformly strictly positive) if, for every α in $\mathcal{A}$, $x_\alpha \geq 0$ ($x_\alpha > 0$, $x_\alpha \geq \epsilon > 0$). It is decomposed into a positive, $x^+ \geq 0$, and a negative, $x^- \geq 0$, part, so that $x = x^+ - x^-$. The positive cone of an ordered vector space consists of all its positive vectors. [4]

2.2

Time and the resolution of uncertainty are described by an event-tree, a countable set, $\mathcal{S}$, endowed with a (partial) order, $\succeq$. For every date-event, σ, an element of $\mathcal{S}$, t_σ denotes its date. The unique initial date-event is ϕ, with $t_\phi = 0$. For a given date-event, σ, $\sigma_+ = \{\tau \succ \sigma : t_\tau = t_\sigma + 1\}$ denotes the set of its immediate successors, a finite set; $\mathcal{S}_\sigma = \{\tau \in \mathcal{S} : \tau \succeq \sigma\}$ the set of all its (weak) successors, a subtree; $\mathcal{S}^t = \{\sigma \in \mathcal{S} : 0 \leq t_\sigma \leq t\}$ the set all date-events up to date t; $\mathcal{S}_t = \{\sigma \in \mathcal{S} : t_\sigma = t\}$ the set all date-events at date t. The construction is standard: see, for instance, Magill and Quinzii [28] and Santos and Woodford [30].

2.3

At every date-event, markets are open for commodities, assets and balances, which are numéraire. At every date-event, there is a finite set, $\mathcal{N}$, of tradable commodities, which are perfectly divisible and perishable. Consistently, the *commodity space* coincides with the space of all bounded real valued maps on $\mathcal{S} \times \mathcal{N}$ and prices of commodities, p, are a positive real valued map on $\mathcal{S} \times \mathcal{N}$.

The asset market is sequentially complete. A portfolio, holdings of assets, is described by its payoffs across date-events, v, a real valued map on $\mathcal{S}$. Prices of assets are *state prices*, the prices of revenues across date-events, a, a strictly positive real valued map on $\mathcal{S}$, normalized so that $a_\phi = 1$. At a date-event, a portfolio, with payoffs $(v_\tau : \tau \in \sigma_+)$ across its immediate successors, has market value

$$a_\sigma^{-1} \sum_{\tau \in \sigma_+} a_\tau v_\tau .$$

State prices are also commonly called prices of elementary Arrow securities.

[4] For details, see Aliprantis and Border [1]. Notice that, throughout the paper, we use the term 'positive' ('negative') to mean 'greater or equal than zero' ('less or equal than zero').

At given state prices, one-period, *nominal rates of interest*, r , a positive real valued map on $\mathcal{S}$, are defined, implicitly, by the equations

$$\sum_{\tau \in \sigma_+} a_\tau = \left(\frac{1}{1+r_\sigma}\right) a_\sigma \leq a_\sigma.$$

Indeed, since balances are storable, no-arbitrage requires that nominal rates of interest be positive.

2.4

There is a finite set of individuals. An individual, i, is described by preferences, $\succeq^i$, over the *consumption space*, the positive cone of the commodity space, and an endowment, e^i, of commodities, an element of the consumption space. We make two common assumptions on preferences and endowments of commodities.

Assumption 2.1 (Preferences) *The preferences of every individual are continuous (in the relative Mackey topology), convex and strictly monotone.*

Assumption 2.2 (Endowments) *The endowment of every individual is uniformly strictly positive.*

Continuity of preferences in the Mackey topology, introduced in Bewley [4], is a strong requirement.[5] In particular, it implies that the individual is impatient: sufficiently distant modifications, even unbounded ones, of consumption plans do not reverse the order of preference. Uniform impatience across individuals would be a stronger requirement. The much stronger assumption of a uniform rate of impatience across date-events, in the recent literature on incomplete asset markets over an infinite horizon (Hernández and Santos [22] and Magill and Quinzii [28]), is not needed here.[6]

[5] It encompasses, for example, preferences that are represented by an additively separable utility function,

$$\sum_{\sigma \in \mathcal{S}} \mu_\sigma \beta^{t_\sigma} u^i \quad x^i_\sigma \quad ,$$

where μ_σ is the probability of σ, $0 < \beta < 1$ is the discount factor, and u^i is a bounded, continuous, increasing, concave real valued map on $\ell(\mathcal{N})$ with $u^i(0) = 0$.

[6] Alternatively, one could include unbounded maps in the commodity space and preferences could be required to be continuous in the product topology. By rescaling units of measurement of different commodities, which does not affect continuity in the product topology, one can always suppose that the aggregate endowment is bounded. For the purposes of equilibrium theory, one need not consider individual consumptions that exceed this bound, though this may be contrary to the spirit of competitive equilibrium, since individuals might, indeed, contemplate unbounded consumption plans. Pursuing this direction, continuity in the product topology restricted to consumption plans uniformly bounded by the aggregate endowment is equivalent to continuity in the Mackey topology; this amounts to the impatience of individuals exceeding the rate of growth of the aggregate endowment. The stronger assumption of continuity in the product topology, as in Geanakoplos and Polemarchakis [18], keeps separate restrictions on preferences and restrictions on endowments.

2.5

A public authority (or a government, or a central bank) conducts monetary, fiscal, transfers and portfolio policies. The supply of balances, m, a positive real valued map on $\mathcal{S}$, is contingent on dates and information. *Monetary policy* consists of nominal rates of interest, r, that are set, while the supply of balances accommodates demand. Although our analysis could be adapted to cope with all arbitrarily set nominal rates of interest, we impose a restriction that facilitates presentation.

Assumption 2.3 (Monetary policy) *Nominal rates of interest, r, are (uniformly) bounded.*

Fiscal policy consists of taxes, $\left(\ldots, g^i, \ldots\right)$, bounded positive real valued maps on $\mathcal{S} \times \mathcal{N}$. In the aggregate, taxes are $g = \sum_i g^i$. We require consistency between monetary and fiscal policy, which is needed both to avoid problems of solvency, when individuals hold initial nominal debts, and to carry out a limit argument for the proof of existence of equilibria.[7]

Assumption 2.4 (Fiscal policy) *Given monetary policy, r, for every individual,*

$$\left(\frac{1}{1+r}\right) e^i - g^i$$

is uniformly strictly positive.

Transfer policy consists of lump-sum nominal transfers to individuals, $\left(\ldots, h^i, \ldots\right)$, real valued maps on $\mathcal{S}$. In the aggregate, transfers are $h = \sum_i h^i$. To avoid additional, redistributive indeterminacy, we assume that aggregate transfers are distributed to individuals according to fixed shares, $\left(\ldots, \zeta^i, \ldots\right)$. Moreover, we restrict transfers policies obtain well-defined present values at every date-event.

Assumption 2.5 (Transfer policy) *Transfers are distributed to individuals according to given shares, $\left(\ldots, \zeta^i, \ldots\right) \geq 0$, such that $\sum_i \zeta^i = 1$. Moreover, given state prices, a, at every date-event, the sum of present value transfers, $\sum_{\tau \in \mathcal{S}_\sigma} a_\tau h_\tau$, converges.*

Trade in assets by the public authority is described by public liabilities, w, a real valued map on $\mathcal{S}$, with a given initial value, $w_\phi = \delta$. The initial public

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[7]For a real valued map, x, on $\mathcal{S}$ and a real valued map, z, on $\mathcal{S} \times \mathcal{N}$, $xz = zx$ is the real valued map on $\mathcal{S} \times \mathcal{N}$ obtained by point-wise product, $(\ldots, x_\sigma z_\sigma, \ldots) = (\ldots, z_\sigma x_\sigma, \ldots)$. Moreover,

$$\frac{1}{1+r} \quad = \quad \ldots, \quad \frac{1}{1+r_\sigma} \quad , \ldots$$

and

$$\frac{r}{1+r} \quad = \quad \ldots, \quad \frac{r_\sigma}{1+r_\sigma} \quad , \ldots$$

are used for notational convenience.

liability, δ, corresponds to initial nominal claims, $\left(\ldots,\delta^i,\ldots\right)$, of individuals, that is, $\delta=\sum_i \delta^i$. To simplify the presentation, at no loss of realism, we assume that there is a strictly positive initial public liability.

Assumption 2.6 (Public liability) *The initial public liability, δ, is strictly positive.*

The public authority is subject, at every date-event, to a sequential budget constraint,

$$\left(\frac{r_\sigma}{1+r_\sigma}\right)m_\sigma + a_\sigma^{-1}\sum_{\tau\in\sigma_+} a_\tau w_\tau = w_\sigma + h_\sigma - p_\sigma\cdot g_\sigma.$$

Portfolio policy sets the composition of the public portfolio, but not its magnitude, at every date-event: it is represented by Θ, a real valued map on $\mathcal{S}$, and imposes the additional restriction that $(w_\tau : \tau\in\sigma_+)$ belong to the span of $(\Theta_\tau : \tau\in\sigma_+)$.[8]

Assumption 2.7 (Portfolio policy) *At every date-event, the vector $(\Theta_\tau : \tau\in\sigma_+)$ is positive and non-zero.*

Our representation of public policies incorporates a minimal requirement of consistency. Indeed, arbitrarily set policies determine, through the sequential public budget constraint, the evolution of public liabilities, for all given prices and demands of balances. For every date-event, σ , public liability at every immediately following date-event, τ, is given by

$$w_\tau=\left(\frac{\Theta_\tau}{\sum_{\xi\in\sigma_+} a_\xi\Theta_\xi}\right)\left(a_\sigma w_\sigma + a_\sigma h_\sigma - a_\sigma p_\sigma\cdot g_\sigma - \left(\frac{r_\sigma}{1+r_\sigma}\right)a_\sigma m_\sigma\right).$$

However, given policies might not satisfy an intertemporal public budget constraint at all prices and at all demands of balances.

Remark 2.1 Consolidated budget constraints are consistent with a specification of mutually independent monetary and fiscal authorities. Suppose that v and u represent, respectively, the liabilities of the monetary and fiscal authorities. The central bank is an institution which issues balances and runs balanced accounts: outstanding money, m, is matched by claims on individuals or the government, that is, $v=0$. The bank is owned by individuals or the government and distributes dividends, d, to share-holders: the imposition of balanced accounts implies, at every date-event, that

$$\left(\frac{r_\sigma}{1+r_\sigma}\right)m_\sigma = d_\sigma.$$

[8]In general, different constraints on portfolios (that is, for instance, $(w_\tau - m_\sigma : \tau\in\sigma_+)$ belongs to the span of $(\Theta_\tau : \tau\in\sigma_+)$) would lead to inconsistencies.

Consistently, the government is subject, at every date-event, to a sequential budget constraint,

$$\left(1-\sum_i \xi^i\right) d_\sigma + a_\sigma^{-1} \sum_{\tau\in\sigma_+} a_\tau u_\tau = u_\sigma + h_\sigma - p_\sigma \cdot g_\sigma,$$

where $\left(\ldots,\xi^i,\ldots\right) \geq 0$ and $\left(1-\sum_i \xi^i\right) \geq 0$ are, respectively, the shares of individuals and the government into the central bank. $\diamondsuit$

2.6

The constraints that an individual faces, at every date-event, are a budget constraint,

$$\left(\frac{r_\sigma}{1+r_\sigma}\right) m_\sigma^i + a_\sigma^{-1} \sum_{\tau\in\sigma^+} a_\tau w_\tau^i + p_\sigma \cdot \left(x_\sigma^i - e_\sigma^i\right) \leq w_\sigma^i + h_\sigma^i - p_\sigma \cdot g^i,$$

a liquidity constraint,

$$p_\sigma \cdot \left(x_\sigma^i - e_\sigma^i\right)^- - m_\sigma^i \leq 0,$$

and a solvency constraint

$$-a_\sigma^{-1} \sum_{\tau\in\mathcal{S}_\sigma} a_\tau \left(h_\tau^i - p_\sigma \cdot g^i + \left(\frac{1}{1+r_\tau}\right) p_\tau \cdot e_\tau^i\right) \leq w_\sigma^i,$$

where initial nominal claims are predetermined by the condition $w_\phi^i = \delta^i$. In the budget constraint, the nominal interest rate represents the opportunity cost of holding wealth in liquid form. Equivalently, here, of collecting proceeds of sales with a one-period lag.

Remark 2.2 Our constraints coincide with those of Woodford [33] and, in a finite horizon, with those of Dubey and Geanakoplos [15]. Liquidity constraints correspond to cash-in-advance. At every date, after information is acquired, σ, an individual has nominal claims w_σ^i, receives a transfer h_σ^i and pays taxes $p_\sigma \cdot g_\sigma^i$. He purchases a portfolio, with payoffs $\left(v_\tau^i : \tau \in \sigma_+\right)$, and balances n_σ^i so as to satisfy the constraint

$$n_\sigma^i + a_\sigma^{-1} \sum_{\tau\in\sigma_+} a_\tau v_\tau^i \leq w_\sigma^i + h_\sigma^i - p_\sigma \cdot g_\sigma^i.$$

He employs balances for the purchase of commodities, according to the constraint

$$p_\sigma \cdot \left(x_\sigma^i - e_\sigma^i\right)^+ - n_\sigma^i \leq 0,$$

and receives balances from the sale of goods. The end of period amount of balances is, therefore,

$$m_\sigma^i = n_\sigma^i - p_\sigma \cdot \left(x_\sigma^i - e_\sigma^i\right)^+ + p_\sigma \cdot \left(x_\sigma^i - e_\sigma^i\right)^- .$$

At the following date, after information is revealed, τ, nominal claims amount to $w_\tau^i = v_\tau^i + m_\sigma^i$. $\diamondsuit$

Remark 2.3 Solvency constraints serve to eliminate Ponzi schemes. They are equivalent to the restriction that an individual can incur any amount of nominal debt that can be repayed in finite time. The value of the endowment in commodities at a date-event is taxed at the nominal interest rate, since revenues from sales are carried over in the form of balances that do not earn interest. $\diamondsuit$

For an individual, a plan consists of a consumption plan, x^i, balances, m^i, and asset holdings, w^i. The sequential budget set is the set of all consumption plans which satisfy the sequential budget, liquidity and solvency constraints, for some balances and asset holdings, given initial nominal claims.

3 Equilibrium

For given monetary and fiscal policies, an *equilibrium* consists of portfolio and transfer policies, plans for individuals, prices of commodities and state prices (consistent with set nominal rates of interest) such that: (a) the plan of every individual is optimal subject to sequential budget, liquidity and solvency constraints, given initial nominal claims; (b) at every date-event, market clearing is achieved in markets of commodities,

$$\sum_i x^i_\sigma = \sum_i e^i_\sigma,$$

and assets,

$$\sum_i w^i_\sigma = w_\sigma,$$

where public liabilities, w, satisfy public sequential budget constraints at balances demanded by individuals, $m = \sum_i m^i$, given initial public liability.

4 Consolidation

Since the asset market is complete, the sequence of budget constraints faced by an individual reduces to a single constraint at the initial date-event.

Lemma 4.1 *At equilibrium, present value prices of commodities, ap, are a summable real valued map on $\mathcal{S} \times \mathcal{N}$.*

At equilibrium, therefore, the intertemporal budget constraint of an individual,

$$\sum_{\sigma \in \mathcal{S}} \left(\frac{r_\sigma}{1 + r_\sigma} \right) a_\sigma m^i_\sigma + \sum_{\sigma \in \mathcal{S}} a_\sigma p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma \right) \leq \delta^i + \sum_{\sigma \in \mathcal{S}} a_\sigma h^i_\sigma - \sum_{\sigma \in \mathcal{S}} a_\sigma p_\sigma \cdot g^i_\sigma,$$

is well-defined.

Lemma 4.2 *At equilibrium, a consumption plan is attainable under sequential budget, liquidity and solvency constraints if and only if it is attainable under the unique intertemporal budget constraint and sequential liquidity constraints. Optimality of a consumption plan requires that the intertemporal budget constraint be satisfied with equality and, at every date-event,*

$$\left(\frac{r_\sigma}{1+r_\sigma}\right)\left(p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right)^- - m^i_\sigma\right) = 0.$$

The transversality condition takes the form

$$\lim_{t\to\infty} \sum_{\sigma\in\mathcal{S}_t} a_\sigma w^i_\sigma = 0.$$

As the liquidity constraint is binding whenever the nominal rate of interest is positive, the intertemporal budget constraint of an individual reduces to

$$\textstyle\sum_{\sigma\in\mathcal{S}} \left(\frac{r_\sigma}{1+r_\sigma}\right) \pi_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right)^- + \sum_{\sigma\in\mathcal{S}} \pi_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right) \leq$$

$$\textstyle\delta^i + \zeta^i \sum_{\sigma\in\mathcal{S}} a_\sigma h_\sigma - \sum_{\sigma\in\mathcal{S}} \pi_\sigma \cdot g^i,$$

where $\pi = ap$ are present value prices of commodities, a summable positive real valued map on $\mathcal{S}\times\mathcal{N}$. At equilibrium, aggregation across individuals yields

$$\sum_{\sigma\in\mathcal{S}} \left(\frac{r_\sigma}{1+r_\sigma}\right) \pi \cdot \sum_i \left(x^i_\sigma - e^i_\sigma\right)^- + \sum_{\sigma\in\mathcal{S}} \pi_\sigma \cdot g_\sigma = \delta + \sum_{\sigma\in\mathcal{S}} a_\sigma h_\sigma,$$

which is the intertemporal public budget 'constraint'. It is clear that state prices are of no allocative relevance, unless transfer policy is arbitrarily set.

5 Existence and Determinacy

We successively address the issues of existence of an equilibrium, determinacy of the overall price level and determinacy of rates of inflation. In proving existence of an equilibrium, monetary, fiscal and, possibly, portfolio policies are given exogenously, whereas transfer policies are endogenous and, possibly, subject to additional restrictions. The determinacy of rates of inflation is studied for given portfolio policies, since, otherwise, a full indeterminacy, up to consistency with set nominal rates of interest, always obtains.

In addition to all already stated hypotheses, conditions for existence of an equilibrium are distinguished according to their specific role. Those for public solvency ensure that intertemporal public budget can be balanced, at the outset, at some overall price level, when one restricts transfers to be positive or one does not allow for transfers at all. Those for private solvency guarantee that every individual is able to honor the initial debt when the overall price level is varied subject to public solvency. Finally, those for public debt consistency ensures that public solvency obtains sequentially, at every date-event, under given portfolio policy.

Existence requires that initial public liability, which is nominal, be covered by public revenues, consisting of seignorage and taxes, which are real, and, possibly, negative transfers, which are nominal,

$$\sum_{\sigma \in \mathcal{S}} \left(\frac{r_\sigma}{1 + r_\sigma} \right) \pi \cdot \sum_i \left(x_\sigma^i - e_\sigma^i \right)^- + \sum_{\sigma \in \mathcal{S}} \pi_\sigma \cdot g_\sigma = \delta + \sum_{\sigma \in \mathcal{S}} a_\sigma h_\sigma .$$

If negative transfers are allowed, this restriction is vacuous: negative transfers can always be issued so as to guarantee a balanced intertemporal budget. When negative transfers are not allowed, under strictly positive, but otherwise arbitrary, real revenues, a sufficiently high price level permits covering an arbitrary initial liability. However, if there are no taxes, public solvency requires seignorage, which might fail to be strictly positive: consistently, conditions for individuals to be willing to hold balances, at set strictly positive nominal rates of interest, are to be explicitly considered.

Assumption 5.1 (Public solvency) *One of the following conditions holds:*

1. *Negative transfers are allowed.*
2. *Fiscal policy is non-zero.*
3. *Monetary policy is strictly positive and trade occurs at equilibrium.*

Trade at equilibrium occurs when gains to trade are higher than transaction costs related to nominal rates of interest, as pointed out by Dubey and Geanakoplos [14, 15]. The issue is further investigated below when we carry out a duality analysis.

Assumption 5.2 (Trade at equilibrium) *There exists an allocation, $\left(\ldots, x^i, \ldots \right)$, that Pareto dominates the initial allocation, $\left(\ldots, e^i, \ldots \right)$, and satisfies, at every date-event,*

$$\sum_i x_\sigma^i + \left(\frac{r_\sigma}{1 + r_\sigma} \right) \sum_i \left(x_\sigma^i - e_\sigma^i \right)^- \leq \sum_i e_\sigma^i .$$

Since individuals may hold initial nominal debts, private solvency,

$$\delta^i \geq \sum_{\sigma \in \mathcal{S}} \pi_\sigma \cdot g_\sigma^i - \sum_{\sigma \in \mathcal{S}} \left(\frac{1}{1 + r_\sigma} \right) \pi_\sigma \cdot e_\sigma^i = \sum_{\sigma \in \mathcal{S}} \pi_\sigma \cdot \left(g_\sigma^i - \left(\frac{1}{1 + r_\sigma} \right) e^i \right),$$

is to be ensured by a high enough overall price level. When transfers are issued, intertemporal public budget can be balanced at all high enough price levels and, therefore, private solvency can be guaranteed. If transfers are not issued, however, the overall price level serves to balance intertemporal public budget and, consequently, private solvency might fail when there are initial nominal debts.

Assumption 5.3 (Private solvency) *One of the following conditions holds:*

1. *Positive transfers are allowed.*
2. *Initial nominal claims of individuals are positive.*

As time evolves, public liabilities are determined by portfolio policy. They are to be balanced, at every date-event, by public revenues,

$$\sum_{\tau \in \mathcal{S}_\sigma} \left(\frac{r_\tau}{1 + r_\tau} \right) \pi_\tau \cdot \sum_i \left(x_\tau^i - e_\tau^i \right)^- + \sum_{\tau \in \mathcal{S}_\sigma} \pi_\tau \cdot g_\tau = a_\sigma w_\sigma + \sum_{\tau \in \mathcal{S}_\sigma} a_\tau h_\tau .$$

If negative transfers can be issued, this restriction is vacuous. When transfers can only be positive, situations in which public real revenues vanish must be ruled out. If transfers are not allowed, in addition, situations in which public liabilities vanish must be ruled out as well.

Assumption 5.4 (Public debt consistency) *One of the following conditions holds:*

1. *Negative transfers are allowed.*
2. *Positive transfers are allowed and fiscal policy is strictly positive.*
3. *Portfolio and fiscal policies are strictly positive.*

Equilibrium exists for given monetary and fiscal policies when public and private solvency is guaranteed. With unrestricted transfer policy, solvency requires no additional conditions. With positive transfer policy and without transfer policy, public solvency requires positive public revenue, either taxes or seignorage. In addition, without transfer policy, private solvency is only guaranteed if there are no initial private nominal debts. For given portfolio policy, with positive transfers policy and without transfer policy, the existence of an equilibrium might fail when public revenue is not positive at every date-event.

Proposition 5.1 (Existence) *For given monetary and fiscal policies, under solvency conditions, an equilibrium exists with unrestricted transfer policy, with positive transfer policy or without transfer policy. For given monetary, fiscal and portfolio policies, under solvency and public debt consistency conditions, an equilibrium exists with unrestricted transfer policy, with positive transfers policy or without transfer policy.*

With unrestricted transfer policy and positive transfer policy, there is in fact a multiplicity of equilibria, corresponding to the overall price level, up to a lower bound accounting for solvency of individuals and, possibly, positivity of transfers. One may also argue that, generically, if trade occurs, such a multiplicity is of real allocative relevance.

Proposition 5.2 (Overall Price Level) *For given monetary, fiscal and portfolio policies, under solvency and public debt consistency conditions, a multiplicity of equilibria exists with unrestricted transfer policy or with positive transfers policy. Such a multiplicity can be indexed by the overall price level, up to a lower bound which guarantees solvency of every individual and, if required, positivity of transfers.*

Up to consistency with nominal rates of interest, state prices, of no allocative relevance, are indeterminate, unless transfers are ruled out and, in addition, the portfolio policy is given. Without transfers, at every date-event, portfolio policy imposes

$$\sum_{\tau \in \mathcal{S}_\sigma} \left(\frac{r_\tau}{1 + r_\tau} \right) \pi_\tau \cdot \sum_i \left(x^i_\tau - e^i_\tau \right)^- + \sum_{\tau \in \mathcal{S}_\sigma} \pi_\tau \cdot g_\tau = a_\sigma w_\sigma,$$

where all terms, but state price, are predetermined, thus adding additional restrictions. If positive transfers are allowed, it only requires

$$\sum_{\tau \in \mathcal{S}_\sigma} \left(\frac{r_\tau}{1 + r_\tau} \right) \pi_\tau \cdot \sum_i \left(x^i_\tau - e^i_\tau \right)^- + \sum_{\tau \in \mathcal{S}_\sigma} \pi_\tau \cdot g_\tau \geq a_\sigma w_\sigma,$$

leaving, consequently, state prices partly indeterminate.

The indeterminacy of state prices, subject to consistency with nominal rates of interest, translates into an indeterminacy of rates of inflation, subject to Fisher Equations. Indeed, given (strictly positive) present value prices of commodities, π, one obtains, at every date-event,

$$\left(\frac{\|p_\tau\|}{\|p_\sigma\|} : \tau \in \sigma_+ \right) = \left(\frac{a_\sigma}{a_\tau} \frac{\|\pi_\tau\|}{\|\pi_\sigma\|} : \tau \in \sigma_+ \right).$$

Thus, state prices account for the variability of rates of inflation, subject to Fisher Equations,

$$\sum_{\tau \in \sigma_+} \frac{\|p_\sigma\|}{\|p_\tau\|} \frac{\|\pi_\tau\|}{\|\pi_\sigma\|} = \frac{1}{1 + r_\sigma}.$$

Proposition 5.3 (Rates of Inflation) *For given monetary, fiscal and portfolio policies, with unrestricted transfer policy or with positive transfer policy, state prices are indeterminate, up to consistency with nominal rates of interest and, if required, positivity of transfers. These are (countably)infinitely many degrees of purely nominal multiplicity, corresponding to the variability of inflation rates, up to consistency with Fisher Equations and, if required, positivity of transfers. For given monetary, fiscal and portfolio policies, without transfer policy, state prices are determinate, subject to strict positivity of public liabilities.*

Determinacy of the overall price level and of state prices corresponds to the fiscal theory of price determination (for example, Cochrane [11] and Woodford [33, 34]). It bears similarities, furthermore, with the determinacy of monetary equilibria obtained by Dubey and Geanakoplos [15] under given money supply, since it is there equivalent to a control on the nominal rate of interest.

6 Efficiency

Neither of the Welfare Theorems holds in a monetary economy under strictly positive nominal rates of interest: (a) equilibrium allocations, in general, fail to be Pareto efficient; (b) Pareto efficient allocations cannot, in general, be sustained as equilibrium allocations (though they can under suitable redistributions of endowments of commodities). More importantly, one can construct robust examples of economies exhibiting *Pareto-ranked equilibria* at given nominal rates of interest.[9] The concept of constrained efficiency suitable for monetary economies is not evident. The next section, which may be seen as a digression, is devoted to a related duality property.

7 Duality

We carry out a duality analysis using an auxiliary notion of supportability. One may be interested in such an analysis for two reasons: (a) it allows us to provide a characterization of equilibria without any explicit reference to prices; (b) it gives a better understanding of the displacement from Pareto efficiency caused by liquidity constraints. Throughout, nominal rates of interest are considered as given and are bounded.

An allocation, $\left(\ldots, x^i, \ldots\right)$, is *feasible* if, for every date-event,

$$\sum_i x^i_\sigma - \sum_i e^i_\sigma \leq 0.$$

It is said to be *supportable* (respectively, *weakly supportable*) if there is no Pareto dominating (respectively, weakly Pareto dominating) allocation, $\left(\ldots, z^i, \ldots\right)$, which satisfies, at every date-event,

$$\sum_i z^i_\sigma + \sum_i \left(\frac{r_\sigma}{1 + r_\sigma}\right)\left(z^i_\sigma - e^i_\sigma\right)^- \leq \sum_i e^i_\sigma + \sum_i \left(\frac{r_\sigma}{1 + r_\sigma}\right)\left(x^i_\sigma - e^i_\sigma\right)^- .$$

Notice that supportability is defined for given nominal rates of interest *and* endowments of commodities. The latter seems unavoidable if the notion is to be suitable for a duality analysis.

Clearly, every supportable feasible allocation is also weakly supportable. The converse is true as well since preferences are continuous and strictly monotone.

[9] There are two individuals and two commodities. Let $r > 0$ be given. Individual 1's preferences are represented by $x^1_1 + (1+r)^{-1} x^1_2$ and endowments are $(0,1)$. Individual 2's preferences are represented by $(1+r)^{-1} x^2_1 + x^2_2$ and endowments are $(1,0)$. A symmetric allocation is represented by $0 \leq \theta \leq 1$, with consumptions $x^1_\theta = (\theta, 1-\theta)$ and $x^2_\theta = (1-\theta, \theta)$. It is simple to verify that, for every $0 \leq \theta \leq 1$, x^1_θ, x^2_θ is an equilibrium with prices $\pi_\theta = (1,1)$. There is actually a continuum of equilibria, for a symmetric distribution of initial nominal claims, ranking from the no-trade to the symmetric Pareto-efficient allocation. All of them, but no trade, can be sustained as equilibria with symmetric strictly positive initial claims.

Lemma 7.1 *A feasible allocation is supportable if and only if it is weakly supportable. Moreover, it is supportable only if, at every date-event,*

$$\sum_i x_\sigma^i - \sum_i e_\sigma^i = 0.$$

We now establish a variation on the First Welfare Theorem.

Proposition 7.1 (First 'Welfare' Theorem) *Every equilibrium allocation is supportable.*

We say that present value prices of commodities, π, *support* a feasible allocation, $\left(\ldots, x^i, \ldots\right)$, whenever, for every individual, $z^i \succeq^i x^i$ implies

$$\sum_{\sigma \in \mathcal{S}} \pi_\sigma \cdot \left(z_\sigma^i - x_\sigma^i\right) \geq \sum_{\sigma \in \mathcal{S}} \left(\frac{r_\sigma}{1 + r_\sigma}\right) \pi_\sigma \cdot \left(\left(x_\sigma^i - e_\sigma^i\right)^- - \left(z_\sigma^i - e_\sigma^i\right)^-\right).$$

If an allocation is supported by some present value prices of commodities, consumption plans of individuals are (weakly) optimal at those prices.

We can now present a formulation of the Second Welfare Theorem.

Proposition 7.2 (Second 'Welfare' Theorem) *Every supportable feasible allocation is supported by some (non-zero) present value prices of commodities.*

Supportability can be interpreted as the absence of retrading benefits if trade were physically costly, as pointed out by Dubey and Geanakoplos (2001).

Proposition 7.3 (Gains to Trade) *Let $\left(\ldots, x^i, \ldots\right)$ be a supportable feasible allocation. Then there is no Pareto dominating allocation, $\left(\ldots, z^i, \ldots\right)$, which satisfies, at every date-event,*

$$\sum_i z_\sigma^i + \sum_i \left(\frac{r_\sigma}{1 + r_\sigma}\right) \left(z_\sigma^i - x_\sigma^i\right)^- \leq \sum_i x_\sigma^i.$$

A supportable feasible allocation coincides with a Pareto-efficient no-trade allocation of an economy with redistributed initial endowments and a costly trading, or marketing, technology, as in Foley [17]. In such an economy, competitive firms, or intermediaries, produce (net) trades across individuals using a linear technology, which involves some destruction of resources (in our cases, such costs corresponds to liquidity costs). It should be clear, however, that an equilibrium allocation does not, in general, correspond to a Pareto efficient allocation of an economy where transactions involve real costs, since liquidity costs have in fact no real counterpart.

8 Comments

8.1

Our approach generalizes the representative-individual cash-in-advance economy to a true equilibrium setting with heterogeneous individuals and sequential trade. It provides a promising alternative to overlapping generations economies and the self-insurance economy of Bewley [5] for addressing issues of monetary theory and macroeconomics.

A simple Clower constraint captures the transaction role of money.[10] We show the existence of a competitive equilibrium when a monetary authority sets nominal rates of interest under the assumption of a sequentially complete asset market. Further research is to be devoted to the issue of existence under the control of monetary quantities. Furthermore, asset market incompleteness and incomplete information are relevant issues to be explored. A major difficulty emerges, it should be noticed, when one introduces markets for contingent commodities, real assets and, in general, any contractual arrangement involving the delivery of commodities.

8.2

Our Propositions 5.2 and 5.3 point to limited relevance of the fiscal theory of the price level (Woodford [33, 34] and Cochrane [11]), given that surplus disposability (that is, public budget surplus can be distributed to individuals through transfers) is an innocuous assumption.

8.3

In an economy with heterogeneous individuals, the occurrence of sunspot fluctuations need not be related to indeterminacy. Under interest rate pegging, extrinsic uncertainty might still affect the real allocation of resources at equilibrium even though nominal rates of interest are not contingent at all.[11] Nominal rates of interest put bounds on the variability of consumptions across sunspot states.

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[11] Our duality analysis points out where the usual argument for the ineffectiveness of sunspots breaks down: sunspots could be effective at a supportable allocation since averaging, while making all individuals better off by convexity of preferences, would violate adapted feasibility condition.

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Proof of Lemma 4.1

Solvency constrains imply that

$$\sum_{\tau \in \mathcal{S}_\sigma} a_\tau \left(h^i_\tau + \left(\frac{1}{1+r_\tau} \right) p_\tau \cdot e^i_\tau - p_\tau \cdot g^i_\tau \right)$$

takes finite value at every non-initial date-event. The regularity condition on transfers ensures that

$$\sum_{\sigma \in \mathcal{S}} a_\sigma p_\sigma \cdot \left(\left(\frac{1}{1+r_\sigma} \right) e^i_\sigma - g^i_\sigma \right)$$

is finite as well, which, using interiority assumptions, proves the claim.

Proof of Lemma 4.2

Suppose that a plan $\left(x^i, m^i, w^i\right)$ satisfies sequential budget, liquidty and solvency constraints, given initial nominal claims. Multiplication of the sequential budget constraints by a_σ and summation over $\mathcal{S}^t$ yield

$$\textstyle\sum_{\sigma \in \mathcal{S}_{t+1}} a_\sigma w^i_\sigma + \sum_{\sigma \in \mathcal{S}^t} \left(\frac{r_\sigma}{1+r_\sigma} \right) a_\sigma m^i_\sigma + \sum_{\sigma \in \mathcal{S}^t} a_\sigma p_\sigma \cdot x^i_\sigma \leq$$

$$\textstyle\delta^i + \sum_{\sigma \in \mathcal{S}^t} a_\sigma h^i_\sigma + \sum_{\sigma \in \mathcal{S}^t} a_\sigma p_\sigma \cdot e^i_\sigma - \sum_{\sigma \in \mathcal{S}^t} a_\sigma p_\sigma \cdot g^i_\sigma.$$

The solvency constraint at every date-event, then, implies

$$\textstyle\sum_{\sigma \in \mathcal{S}^t} \left(\frac{r_\sigma}{1+r_\sigma} \right) a_\sigma m^i_\sigma + \sum_{\sigma \in \mathcal{S}^t} a_\sigma p_\sigma \cdot x^i_\sigma - \sum_{\sigma \in \mathcal{S}^t} \left(\frac{r_\sigma}{1+r_\sigma} \right) a_\sigma p_\sigma \cdot e^i_\sigma \leq$$

$$\textstyle\delta^i + \sum_{\sigma \in \mathcal{S}} a_\sigma h^i_\sigma + \sum_{\sigma \in \mathcal{S}} \left(\frac{1}{1+r_\sigma} \right) a_\sigma p_\sigma \cdot e^i_\sigma - \sum_{\sigma \in \mathcal{S}} a_\sigma p_\sigma \cdot g^i_\sigma.$$

Since the left-hand side is bounded, the first term is non-decreasing and the other two terms converge, taking the limit as $t \to \infty$ implies

$$\sum_{\sigma \in \mathcal{S}} \left(\frac{r_\sigma}{1+r_\sigma} \right) a_\sigma m^i_\sigma + \sum_{\sigma \in \mathcal{S}} a_\sigma p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma \right) \leq \delta^i + \sum_{\sigma \in \mathcal{S}} a_\sigma h^i_\sigma - \sum_{\sigma \in \mathcal{S}} a_\sigma p_\sigma \cdot g^i_\sigma.$$

Therefore, $\left(x^i, m^i\right)$ satisfy the intertemporal budget constraint and sequential liquidity constraints.

Conversely, suppose that a plan $\left(x^i, m^i\right)$ satisfies the intertemporal budget constraint and sequential liquidty constraints and define w^i, at all non-initial date-events, by

$$a_\sigma w^i_\sigma = \sum_{\tau \in \mathcal{S}_\sigma} \left(\frac{r_\tau}{1+r_\tau}\right) a_\tau m^i_\tau + \sum_{\tau \in \mathcal{S}_\sigma} a_\tau p_\tau \cdot \left(x^i_\tau - e^i_\tau\right) + \sum_{\tau \in \mathcal{S}_\sigma} a_\tau p_\tau \cdot g^i_\tau - \sum_{\tau \in \mathcal{S}_\sigma} a_\tau h^i_\tau.$$

Solvency constraints are satisfied, since liquidity constraints imply that

$$-a_\sigma^{-1} \textstyle\sum_{\tau \in \mathcal{S}} \left(\frac{1}{1+r_\tau}\right) a_\tau p_\tau \cdot e^i_\tau \leq$$

$$-a_\sigma^{-1} \textstyle\sum_{\tau \in \mathcal{S}_\sigma} \left(\frac{1}{1+r_\tau}\right) a_\tau p_\tau \cdot \left(x^i_\tau - e^i_\tau\right)^- + a_\sigma^{-1} \sum_{\tau \in \mathcal{S}_\sigma} a_\tau p_\tau \cdot \left(x^i_\tau - e^i_\tau\right)^+ \leq$$

$$w^i_\sigma + a_\sigma^{-1} \textstyle\sum_{\tau \in \mathcal{S}_\sigma} a_\tau h^i_\tau - a_\sigma^{-1} \sum_{\tau \in \mathcal{S}_\sigma} a_\tau p^i_\tau \cdot g^i_\tau.$$

To see that sequential budget constraints are satisfied as well, observe that, at every non-initial date-event, the definition of w^i implies that

$$\left(\frac{r_\sigma}{1+r_\sigma}\right) m^i_\sigma + a_\sigma^{-1} \sum_{\tau \in \sigma_+} a_\tau w^i_\tau + p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right) = w^i_\sigma + h^i_\sigma - p_\sigma \cdot g^i_\sigma.$$

At the initial date-event, the intertemporal budget constraint and the definition of w^i imply that

$$\left(\frac{r_\phi}{1+r_\phi}\right) m^i_\phi + \sum_{\sigma \in \phi^+} a_\sigma w^i_\sigma + p_\phi \cdot \left(x^i_\phi - e^i_\phi\right) \leq \delta^i + h^i_\phi - p_\phi \cdot g^i_\phi.$$

At an optimal plan, the intertemporal budget constraint must hold with equality since preferences are strictly monotone. Moreover, it is clear that the liquidity constraint is non-binding only if the nominal rate of interest is zero.

Concerning transversality, a plan satisfies solvency constraints only if

$$\liminf \sum_{\sigma \in \mathcal{S}_t} a_\sigma w^i_\sigma \geq 0.$$

It, then, suffices to show that a plan is maximal only if

$$\limsup \sum_{\sigma \in \mathcal{S}_t} a_\sigma w^i_\sigma \leq 0.$$

If not, then, for infinitely many dates, n, and some $\epsilon > 0$,

$$\epsilon + \textstyle\sum_{\sigma \in \mathcal{S}^n} \left(\frac{r_\sigma}{1+r_\sigma}\right) a_\sigma m^i_\sigma + \sum_{\sigma \in \mathcal{S}^n} a_\sigma p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right) \leq$$

$$\delta^i + \textstyle\sum_{\sigma \in \mathcal{S}^n} a_\sigma h^i_\sigma - \sum_{\sigma \in \mathcal{S}^n} a_\sigma p_\sigma \cdot g^i_\sigma.$$

From the limit, since all series must converge, it follows that

$$\sum_{\sigma\in\mathcal{S}}\left(\frac{r_\sigma}{1+r_\sigma}\right)a_\sigma m^i_\sigma+\sum_{\sigma\in\mathcal{S}}a_\sigma p_\sigma\cdot\left(x^i_\sigma-e^i_\sigma\right)<\delta^i+\sum_{\sigma\in\mathcal{S}}a_\sigma h^i_\sigma-\sum_{\sigma\in\mathcal{S}}a_\sigma p_\sigma\cdot g^i_\sigma,$$

which violates optimality.

Proof of Propositions 5.1-5.3

Outline

The proof is organized as follows. First, we show that, at equilibrium, present value prices of commodities can be determined independently of state prices. Second, we show that there is a multiplicity of state prices compatible with a given equilibrium allocation provided that transfers are allowed.

Abstract Equilibrium

Let X^i be the consumption space of individual i, the positive cone of $\ell_\infty\left(\mathcal{S}\times\mathcal{N}\right)$, and Π the space of normalized present value prices of commodities, the subset of the positive cone of $\ell_1\left(\mathcal{S}\times\mathcal{N}\right)$ satisfying the normalization $\|\pi\|_1=1$. For (π,x) in $\ell_1\left(\mathcal{S}\times\mathcal{N}\right)\times\ell_\infty\left(\mathcal{S}\times\mathcal{N}\right)$, $\pi\cdot x=\sum_{\sigma\in\mathcal{S}}\pi_\sigma\cdot x_\sigma$ denotes the duality operation.

We consider the following abstract notion of equilibrium: it consists of present value prices of commodities, π, an allocation, $\left(\ldots,x^i,\ldots\right)$, an aggregate transfer, β, and a positive index for the overall price level, μ, such that: (a) market clearing is achieved,

$$\sum_i x^i-\sum_i e^i=0;$$

(b) for every individual,

$$z^i\succ^i x^i \text{ implies } \pi\cdot\left(z^i-x^i\right)+\left(\frac{r}{1+r}\right)\pi\cdot\left(\left(z^i-e^i\right)^- -\left(x^i-e^i\right)^-\right)>0$$

and

$$\pi\cdot\left(x^i-e^i\right)+\left(\frac{r}{1+r}\right)\pi\cdot\left(x^i-e^i\right)^-=\mu\left(\delta^i+\zeta^i\beta\right)-\pi\cdot g^i.$$

To offset the redundancy stemming from the choice of the unit of account, we add the normalization $\pi\in\Pi$. Notice that, in an abstract equilibrium, $\mu=0$ is allowed.

Truncation

Suppose that all vector spaces are finite-dimensional, which corresponds to a truncated economy. We show that an abstract equilibrium exists for some transfer β when individuals possibly hold initial debt positions. Choose any positive

μ small enough for nonemptyness of the budget constraint of every individual evaluated at $\beta = -\delta$ and at all normalized present value prices of commodities. Consider the space of all

$$f = \left(\left(\ldots, x^i, \ldots\right), \pi, \beta\right) \in \cdots \times X^i \times \cdots \times \Pi \times B = F,$$

where X^i is consumption space of individual i, Π is the space of normalized present value prices and $B = \{\beta \geq -\delta\}$. And the correspondence $\hat{f} \twoheadrightarrow \bar{f}$ defined by: (a) $\bar{x}^i$ is an optimal choice subject to

$$\hat{\pi} \cdot \left(x^i - e^i\right) + \left(\frac{r}{1+r}\right) \hat{\pi} \cdot \left(x^i - e^i\right)^- \leq \mu \left(\delta^i + \zeta^i \hat{\beta}\right) - \hat{\pi} \cdot g^i;$$

(b) $\bar{\beta}$ solves

$$\left(\frac{r}{1+r}\right) \hat{\pi} \cdot \sum_i \left(\hat{x}^i - e^i\right)^- + \hat{\pi} \cdot g = \mu \left(\delta + \beta\right);$$

(c) $\bar{\pi}$ maximizes

$$\pi \cdot \sum_i \left(\hat{x}^i - e^i\right).$$

A fixed point exists and it can be shown to be an abstract equilibrium of the truncated economy. Therefore, in a truncated economy, an abstract equilibrium exists for all arbitrarily chosen strictly positive μ small enough.

Suppose now that no individual holds an initial debt position. Choose any positive transfer, β. Consider the space of all

$$f = \left(\left(\ldots, x^i, \ldots\right), \pi, \mu\right) \in \cdots \times X^i \times \cdots \times \Pi \times M = F,$$

where X^i is consumption space of individual i, Π is the space of normalized present value prices and $M = \{\mu \geq 0\}$. And the correspondence $\hat{f} \twoheadrightarrow \bar{f}$ defined by: (a) $\bar{x}^i$ is an optimal choice subject to

$$\hat{\pi} \cdot \left(x^i - e^i\right) + \left(\frac{r}{1+r}\right) \hat{\pi} \cdot \left(x^i - e^i\right)^- \leq \hat{\mu} \left(\delta^i + \zeta^i \beta\right) - \hat{\pi} \cdot g^i;$$

(b) $\bar{\mu}$ solves

$$\left(\frac{r}{1+r}\right) \hat{\pi} \cdot \sum_i \left(\hat{x}^i - e^i\right)^- + \hat{\pi} \cdot g = \mu \left(\delta + \beta\right);$$

(c) $\bar{\pi}$ maximizes

$$\pi \cdot \sum_i \left(\hat{x}^i - e^i\right).$$

A fixed point exists and it can be shown to be an abstract equilibrium of the truncated economy. Therefore, in a truncated economy, an abstract equilibrium exists for all arbitrarily chosen positive β, allowing for $\mu = 0$.

Limit

For a vector, x, in $\ell_\infty(\mathcal{S}\times\mathcal{N})$ $(\ell_1(\mathcal{S}\times\mathcal{N}))$, let x^t denote its truncation at t. That is, $x^t_\sigma = x_\sigma$, if $0 \le t_\sigma \le t$, and $x^t_\sigma = 0$, otherwise.

We make now truncation explicit. A t-truncated economy is constructed as follows: preferences on the consumption space, X^i, the positive cone of $\ell_\infty(\mathcal{S}\times\mathcal{N})$, are recovered using $x^i \succeq^i_t z^i$ if and only if $x^{i,t} + \left(e^i - e^{i,t}\right) \succeq^i z^{i,t} + \left(e^i - e^{i,t}\right)$; truncated present value prices of commodities are elements of $\Pi_t = \left\{\pi \in \Pi : \pi^t = \pi\right\}$, where Π contains all positive vectors, π, in $\ell_1(\mathcal{S}\times\mathcal{N})$ such that $\|\pi\|_1 = 1$.

Consider a sequence of abstract equilibria of t-truncated economies: allocations are $\left(\ldots, x^i_t, \ldots\right)$, present value prices of commodities are π_t, transfers are β_t and indexes for the overall price level are μ_t. To simplify, write $\alpha^i_t = \mu_t\left(\delta^i + \zeta^i\beta_t\right)$. Notice that, along the sequence, one controls either for μ_t or for β_t, as made clear by the proof of existence of abstract equilibria in truncated economies. Furthermore, in every truncated economy,

$$\alpha^i_t + \left(\frac{1}{1+r}\right)\pi_t \cdot e^i - \pi_t \cdot g^i \ge \epsilon > 0.$$

This, indeed, follows from interiority assumptions and boundedness of nominal rates of interest. We refer to such inequalities as solvency conditions.

Letting

$$\varphi_t = (\ldots, \varphi_t(\sigma), \ldots) = \left(\ldots, \left(\frac{1}{1+r(\sigma)}\right)\pi_t(\sigma), \ldots\right),$$

π_t and φ_t can be viewed as elements of $ba(\mathcal{S}\times\mathcal{N})$, the norm dual of $\ell_\infty(\mathcal{S}\times\mathcal{N})$ consisting of all finitely additive set functions on $\mathcal{S}\times\mathcal{N}$ and endowed with the dual norm $\|\cdot\|_{ba}$ (the norm of bounded variation). Let $\sigma(ba, \ell_\infty)$ denote the weak* topology of $ba(\mathcal{S}\times\mathcal{N})$. Since

$$\|\varphi_t\|_1 = \|\varphi_t\|_{ba} \le \|\pi_t\|_{ba} = \|\pi_t\|_1 = 1$$

and since, by Alaoglu Theorem, the unit sphere in $ba(\mathcal{S}\times\mathcal{N})$ is $\sigma(ba, \ell_\infty)$ compact, without loss of generality, $\{\pi_t\}$ and $\{\varphi_t\}$ converge to π and φ, respectively, in the $\sigma(ba, \ell_\infty)$ topology. Moreover, both π and φ, as well as $\pi - \varphi$, are positive elements of $ba(\mathcal{S}\times\mathcal{N})$ and $0 < \|\varphi\|_{ba} \le \|\pi\|_{ba} = 1$.

By Tychonov Theorem, without loss of generality, every $\left\{x^i_t\right\}$ converges to x^i in the product topology. Since the product and the Mackey topology coincide on norm-bounded subsets of $\ell_\infty(\mathcal{S}\times\mathcal{N})$, it follows that every $\left\{x^i_t\right\}$ converges to x^i in the Mackey topology.

As sequences $\{\beta_t\}$ and $\{\mu_t\}$ can be assumed to be bounded, without loss of generality, they converge to β and μ, respectively. Defining $\alpha^i = \lambda\left(\delta^i + \zeta^i\beta\right)$, it follows that every α^i_t converges to α^i.

The proof, which is presented in a sequence of steps, uses standard arguments.

1

Decomposition. Since $\pi(\varphi)$ is a positive linear functional, it follows from Yosida-Hewitt Theorem that there is a unique decomposition $\pi = \pi_f + \pi_b$ $(\varphi = \varphi_f + \varphi_b)$, where π_f (φ_f) is a positive functional in $\ell_1(\mathcal{S} \times \mathcal{N})$, the Mackey-topology dual of $\ell_\infty(\mathcal{S} \times \mathcal{N})$, and π_b (φ_b) is a positive finitely additive measure (a pure charge) vanishing on all vectors having only a finite number of non-zero components.

2

$z^i \succeq^i x^i$ *implies*

$$\pi \cdot g^i + \pi \cdot \left(z^i - e^i\right)^+ \geq \alpha^i + \varphi \cdot \left(z^i - e^i\right)^- .$$

For a strictly positive real number, λ, $z^i + \lambda e^i \succ_t^i x_t^i$ for all t large enough, which implies that

$$\pi_t \cdot g^i + \pi_t \cdot \left(z^i - (1-\lambda) e^i\right)^+ \geq \alpha_t^i + \varphi_t \cdot \left(z^i - (1-\lambda) e^i\right)^- .$$

Taking the limit, one obtains

$$\pi \cdot g^i + \pi \cdot \left(z^i - (1-\lambda) e^i\right)^+ \geq \alpha^i + \varphi \cdot \left(z^i - (1-\lambda) e^i\right)^- .$$

As lattice operations are continuous in the norm topology and π and φ are norm-continuous linear functionals, letting λ go to zero, the claim is proven.

3.

$z^i \succ^i x^i$ *implies*

$$\pi \cdot g^i + \pi \cdot \left(z^i - e^i\right)^+ > \alpha^i + \varphi \cdot \left(z^i - e^i\right)^- .$$

Continuity of preferences implies that $\lambda z^i \succ^i x^i$ for some $0 < \lambda < 1$. Since

$$\pi \cdot g^i + \lambda \pi \cdot \left(z^i - e^i\right) + (\pi - \varphi) \cdot \left(\lambda z^i - e^i\right)^- \geq$$

$$\alpha^i + (1-\lambda) \pi \cdot e^i,$$

and

$$\lambda \left(z^i - e^i\right)^- + (1-\lambda) e^i \geq$$

$$\left(\lambda z^i - e^i\right)^- ,$$

one obtains

$$\pi \cdot g^i + \pi \cdot \left(z^i - e^i\right)^+ \geq$$

$$\alpha^i + \varphi \cdot \left(z^i - e^i\right)^- + \left(\tfrac{1-\lambda}{\lambda}\right) \left(\varphi \cdot e^i - \pi \cdot g^i + \alpha^i\right) \geq$$

$$\alpha^i + \varphi \cdot \left(z^i - e^i\right)^- + \left(\tfrac{1-\lambda}{\lambda}\right) \epsilon,$$

where the last inequality follows from solvency conditions.

4.

$\pi_b = \varphi_b = 0$ *and*

$$\pi \cdot g^i + \pi \cdot \left(x^i - e^i\right)^+ = \alpha^i + \varphi \cdot \left(x^i - e^i\right)^- .$$

By our assumptions of interiority,

$$\varphi_t \cdot e^i - \pi_t \cdot g^i = \pi_t \cdot \left(\left(\frac{1}{1+r}\right) e^i - g^i\right) \geq \eta > 0.$$

In the limit,

$$\varphi \cdot e^i - \pi \cdot g^i = \pi \cdot \left(\left(\frac{1}{1+r}\right) e^i - g^i\right) \geq \eta > 0$$

and, using truncations, one can show that

$$\varphi_b \cdot e^i - \pi_b \cdot g^i = \pi_b \cdot \left(\left(\frac{1}{1+r}\right) e^i - g^i\right).$$

It follows that $\pi_b = 0$ implies $\varphi_b = 0$.

Suppose that $\pi_b > 0$, so that, by interiority assumptions, $\varphi_b \cdot e^i - \pi_b \cdot g^i = \xi > 0$. Since $x^{i,t} + \lambda e^{i,t} \succ^i x^i$ for all t large enough and all strictly positive real numbers, λ,

$$\pi \cdot g^i + \pi \cdot \left(\left(x^i - e^i\right)^+\right)^t + \lambda \pi \cdot e^{i,t} \geq$$

$$\pi \cdot g^i + \pi \cdot \left(x^{i,t} + \lambda e^{i,t} - e^i\right)^+ \geq$$

$$\alpha^i + \varphi \cdot \left(x^{i,t} + \lambda e^{i,t} - e^i\right)^- \geq$$

$$\alpha^i + \varphi \cdot \left(\left(x^i - e^i\right)^-\right)^t - \lambda \varphi \cdot e^{i,t} + \varphi \cdot \left(e^i - e^{i,t}\right).$$

In the limit, one obtains

$$\pi_f \cdot g^i + \pi_f \cdot \left(x^i - e^i\right)^+ + \lambda \left(\pi_f - \varphi_f\right) \cdot e^i \geq$$

$$\alpha^i + \varphi_f \cdot \left(x^i - e^i\right)^- + \varphi_b \cdot e^i - \pi_b \cdot g^i \geq$$

$$\alpha^i + \varphi_f \cdot \left(x^i - e^i\right)^- + \xi.$$

Thus,

$$\pi_f \cdot g^i + \pi_f \cdot \left(x^i - e^i\right)^+ \geq \alpha^i + \varphi_f \cdot \left(x^i - e^i\right)^- + \xi.$$

To prove equality, notice that, for all $0 \leq s \leq t$,

$$\pi_t \cdot g^i + \pi_t \cdot \left(\left(x_t^i - e^i\right)^+\right)^s \leq \alpha^i + \varphi_t \cdot \left(\left(x_t^i - e^i\right)^-\right)^s + \varphi_t \cdot \left(e^i - e^{i,s}\right).$$

Therefore, in the limit,

$$\pi_f \cdot g^i + \pi_f \cdot \left(x^i - e^i\right)^+ \leq \alpha^i + \varphi_f \cdot \left(x^i - e^i\right)^- + \xi.$$

Summing over individuals,

$$\pi_f \cdot g^i + (\pi_f - \varphi_f) \cdot \sum_i \left(x^i - e^i\right)^- > \sum_i \alpha^i.$$

Observe that, for all $0 \leq s \leq t$,

$$\pi_t \cdot g^s + (\pi_t - \varphi_t) \cdot \left(\sum_i \left(x_t^i - e^i\right)^-\right)^s \leq \sum_i \alpha^i.$$

In the limit,

$$\pi_f \cdot g + (\pi_f - \varphi_f) \cdot \sum_i \left(x^i - e^i\right)^- \leq \sum_i \alpha^i,$$

a contradiction.

5

Limit is an abstract equilibrium. By pointwise limits, one obtains

$$\varphi = \left(\frac{1}{1+r}\right)\pi,$$

thus proving the claim.

Equilibrium

Under conditions stated in section 5, we show that, for given monetary, fiscal and portfolio policies, when $\mu > 0$, abstract equilibria correspond to sequential equilibria under the following qualifications:

(a) for some transfer policy;

(b) for some positive transfer policy, when $\beta \geq 0$;

(c) without transfer policy, when $\beta = 0$.

(a)

Along the sequence of abstract equilibria of truncated economies, one can assume that $\mu_t = \mu > 0$ for all t and, possibly rescaling present value prices of commodities, that $\mu = 1$. At every date-event, let

$$u_\sigma = \sum_{\tau \in \mathcal{S}_\sigma} \left(\frac{r_\tau}{1+r_\tau}\right)\pi_\tau \cdot \sum_i \left(x_\tau^i - e_\tau^i\right)^- + \sum_{\tau \in \mathcal{S}_\sigma} \pi_\tau \cdot g_\tau,$$

which implies

$$u_\sigma = \sum_{\tau \in \sigma_+} u_\tau + \left(\frac{r_\sigma}{1+r_\sigma}\right) \pi_\sigma \cdot \sum_i \left(x^i_\sigma - e^i_\sigma\right)^- + \pi_\sigma \cdot g_\sigma.$$

For arbitrarily set state prices, a, consistent with nominal rates of interest, and a strictly positive real number, ρ, by induction, define a real valued map on $\mathcal{S}$, b, as follows: if b_σ is defined at all date-events, σ, in $\mathcal{S}^t$, at all date-events, τ, in $\mathcal{S}_{t+1}$ let

$$a_\tau b_\tau = \left(\frac{a_\tau \Theta_\tau}{\sum_{\xi \in \sigma_+} a_\xi \Theta_\xi}\right) \left(\left(\frac{1}{1+\rho}\right) a_\sigma b_\sigma - \sum_{\xi \in \sigma_+} u_\xi\right) + u_\tau;$$

at the initial date-event, ϕ, set $b_\phi = \beta$. It follows that

$$\sum_{\tau \in \sigma_+} a_\tau b_\tau = \left(\frac{1}{1+\rho}\right) a_\sigma b_\sigma$$

and, therefore, that

$$\sum_{\tau \in \mathcal{S}_{\sigma,t}} a_\tau b_\tau = \left(\frac{1}{1+\rho}\right)^t a_\sigma b_\sigma,$$

where $\mathcal{S}_{\sigma,t} = \{\tau \succeq \sigma : t_\tau - t_\sigma = t\}$ and $\mathcal{S}^t_\sigma = \{\tau \succeq \sigma : 0 \leq t_\tau - t_\sigma \leq t\}$. Setting transfers so as to satisfy, at every date-event,

$$h_\sigma = \left(\frac{\rho}{1+\rho}\right) b_\sigma,$$

one verifies that

$$a_\sigma b_\sigma = \textstyle\sum_{\tau \in \mathcal{S}^t_\sigma} a_\tau h_\tau + \left(\frac{1}{1+\rho}\right) \sum_{\tau \in \mathcal{S}_{\sigma,t}} a_\tau b_\tau =$$

$$\textstyle\sum_{\tau \in \mathcal{S}^t_\sigma} a_\tau h_\tau + \left(\frac{1}{1+\rho}\right)^t a_\sigma b_\sigma =$$

$$\textstyle\sum_{\tau \in \mathcal{S}_\sigma} a_\tau h_\tau,$$

which ensures consistency of the construction.

At given state prices, a, prices of commodities are $a_\sigma p_\sigma = \pi_\sigma$ and, at every date-event, balances and assets holdings of individuals are, respectively,

$$m^i_\sigma = p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right)^-$$

and

$$w^i_\sigma = a_\sigma^{-1} \sum_{\tau \in \mathcal{S}_\sigma} a_\tau \left(p_\tau \left(x^i_\sigma - e^i_\sigma + g^i_\sigma\right) + \left(\frac{r_\tau}{1+r_\tau}\right) m^i_\tau - h^i_\tau\right),$$

At every date-event, thus, summing across individuals, market clearing on assets markets is given by

$$a_\sigma\left(w_\sigma + b_\sigma\right) = u_\sigma.$$

By induction, assuming that market clearing holds at all date-events, σ, in $\mathcal{S}^t$, then, at all date-events, τ, in $\mathcal{S}_{t+1}$,

$$a_\tau w_\tau = \left(\tfrac{a_\tau \Theta_\tau}{\sum_{\xi\in\sigma_+} a_\xi \Theta_\xi}\right)\left(a_\sigma w_\sigma - a_\sigma p_\sigma \cdot g_\sigma - \left(\tfrac{r_\sigma}{1+r_\sigma}\right) a_\sigma m_\sigma + a_\sigma h_\sigma\right) =$$

$$\left(\tfrac{a_\tau \Theta_\tau}{\sum_{\xi\in\sigma_+} a_\xi \Theta_\xi}\right)\left(u_\sigma - a_\sigma p_\sigma \cdot g_\sigma - \left(\tfrac{r_\sigma}{1+r_\sigma}\right) a_\sigma m_\sigma + \left(\tfrac{1}{1+\rho}\right) a_\sigma b_\sigma\right) =$$

$$\left(\tfrac{a_\tau \Theta_\tau}{\sum_{\xi\in\sigma_+} a_\xi \Theta_\xi}\right)\left(\textstyle\sum_{\xi\in\sigma_+} u_\xi - \left(\tfrac{1}{1+\rho}\right) a_\sigma b_\sigma\right) =$$

$$u_\tau - a_\tau b_\tau,$$

thus proving the claim.

(b)

We first show that, for all small enough $\mu > 0$, there is an abstract equilibrium with associated transfer $\beta_\mu > 0$. Indeed, for every small enough $\mu > 0$, there is an abstract equilibrium with associated transfer β_μ. Suppose that, letting μ vanish, there is a sequence of abstract equilibria with associated transfers $-\delta \leq \beta_\mu \leq 0$. One can show that, possibly using subsequences, the limit is also an abstract equilibrium with $\mu = 0$, which implies

$$\left(\frac{r}{1+r}\right)\pi \cdot \sum_i \left(x^i - e^i\right)^- + \pi \cdot g = 0.$$

Since present value prices of commodities are strictly positive in every abstract equilibrium, one is to assume that fiscal policy is zero, for, otherwise, a contradiction would emerge. If monetary policy is strictly positive, however, the limit allocation, $\left(\ldots, x^i, \ldots\right)$, does not involve trade, that is, it coincides with the initial allocation, $\left(\ldots, e^i, \ldots\right)$. By the condition on trade at equilibrium, there exists an allocation, $\left(\ldots, z^i, \ldots\right)$, which Pareto dominates the initial allocation, $\left(\ldots, e^i, \ldots\right)$, and satisfies

$$\sum_i z^i + \left(\frac{r}{1+r}\right)\sum_i \left(z^i - e^i\right)^- \leq \sum_i e^i.$$

By strict monotonicity of preferences, one can also assume that, for every individual, $z^i \succ^i e^i$. It follows that, for every individual,

$$\pi \cdot \left(z^i - e^i + \left(\tfrac{r}{1+r}\right)\left(z^i - e^i\right)^-\right) =$$

$$\pi \cdot \left(z^i - e^i\right) + \left(\tfrac{r}{1+r}\right)\pi \cdot \left(z^i - e^i\right)^- > 0,$$

which, summing over individuals, implies

$$\pi \cdot \left(\sum_i z^i - \sum_i e^i + \left(\frac{r}{1+r} \right) \sum_i \left(z^i - e^i \right)^- \right) > 0,$$

a contradiction. Therefore, as μ vanishes, associated transfers are strictly positive.

For given strictly positive portfolio policy (if portfolio policy is not strictly positive, some minor modifications are required), when taxes are strictly positive, one modifies the preceding proof so as to take into account positivity of transfers. If state prices are consistently defined at all date-events, σ, in $\mathcal{S}^t$, define state prices at all date-events, τ, in $\mathcal{S}_{t+1}$, so as to satisfy,

$$\frac{u_\tau}{\sum_{\xi \in \sigma_+} u_\xi} = \frac{a_\tau \Theta_\tau}{\sum_{\xi \in \sigma_+} a_\xi \Theta_\xi}.$$

This can be done since, by strict positivity of fiscal policy, u is strictly positive and, by strict positivity of portfolio policy, state prices result uniquely determined under consistency with nominal rates of interest. It follows that

$$a_\tau b_\tau = \left(\frac{a_\tau \Theta_\tau}{\sum_{\xi \in \sigma_+} a_\xi \Theta_\xi} \right) \left(\left(\frac{1}{1+\rho} \right) a_\sigma b_\sigma \right),$$

which shows that transfers are positive at all date-events, since β can be assumed to be positive. When β is strictly positive, there are infinitely many perturbations of these values of state prices which still respect positivity of transfers.

(c)

When there are no initial nominal debt, one can assume that, in an abstract equilibrium, $\beta = 0$. To verify that $\mu > 0$, one uses either the fact that taxes are nonzero or the condition on trade at equilibrium, as done for case (b) above. Moreover, when taxes are strictly positive, for given portfolio policy, state prices are, by induction, uniquely determined by conditions

$$\left(\frac{u_\tau}{\sum_{\xi \in \sigma_+} u_\xi} = \frac{a_\tau \Theta_\tau}{\sum_{\xi \in \sigma_+} a_\xi \Theta_\xi} : \tau \in \sigma_+ \right),$$

in addition to consistency with nominal rates of interest.

Proof of Lemma 7.1

Let $\left(\ldots, x^i, \ldots\right)$ be a weakly supportable feasible allocation and suppose that there is an allocation $\left(\ldots, z^i, \ldots\right)$ which Pareto dominates $\left(\ldots, x^i, \ldots\right)$ and satisfies adapted feasibility. In particular, assume that $z^j \succ^j x^j$, so that $\theta =$

$\left(z^j - x^j\right)^+ > 0$ by the strict monotonicity of preferences. For a positive small real number α, define

$$y^j = z^j - \alpha\theta = x^i + (1-\alpha)\left(z^j - x^j\right)^+ - \left(z^j - x^j\right)^- \geq 0$$

and

$$y^i = z^i + \alpha\,(n-1)^{-1}\left(\frac{1}{1+r}\right)\theta > z^i,$$

where n is the number of individuals. The number α can be chosen so small as to satisfy $y^i \succ^i x^i$, for all individuals, because preferences are Mackey-continuous and strictly monotone. Notice that

$$\sum_i y^i = \sum_i z^i - \alpha\theta + \alpha\left(\frac{1}{1+r}\right)\theta = \sum_i z^i - \alpha\left(\frac{r}{1+r}\right)\theta$$

and

$$\sum_i \left(\frac{r}{1+r}\right)\left(y^i - e^i\right)^- \leq \sum_i \left(\frac{r}{1+r}\right)\left(z^i - e^i\right)^- + \alpha\left(\frac{r}{1+r}\right)\theta.$$

This contradicts the fact that $\left(\ldots, x^i, \ldots\right)$ is a supportable feasible allocation.

Concerning the second statement, suppose that $\left(\ldots, x^i, \ldots\right)$ is supportable feasible allocation and

$$-\theta = \sum_i \left(x^i - e^i\right) < 0.$$

Define, for each individual, $z^i = x^i + n^{-1}\theta$, where n is the number of individuals, and notice that, by the strict monotonicity of preferences, $z^i \succ^i x^i$. Summing over individuals, we have that

$$\sum_i z^i + \sum_i \left(\frac{r}{1+r}\right)\left(z^i - e^i\right)^- \leq \theta + \sum_i x^i + \sum_i \left(\frac{r}{1+r}\right)\left(x^i - e^i\right)^-,$$

which contradicts supportability.

Proof of Proposition 7.1

To obtain a contradiction, assume $\left(\ldots, x^i, \ldots\right)$ is an equilibrium allocation and is not weakly supportable. Therefore, there is a weakly Pareto improving allocation, $\left(\ldots, z^i, \ldots\right)$, which satisfies adapted feasibility. Equilibrium implies

$$\pi \cdot \left(z^i - x^i\right) > \left(\frac{r}{1+r}\right)\pi \cdot \left(\left(x^i - e^i\right)^- - \left(z^i - e^i\right)^-\right)$$

and, summing over all individuals,

$$\pi \cdot \left(\sum_i z^i + \sum_i \left(\frac{r}{1+r}\right)\left(z^i - e^i\right)^- - \sum_i e^i - \sum_i \left(\frac{r}{1+r}\right)\left(x^i - e^i\right)^-\right) > 0.$$

Since π is a positive linear functional, a contradiction is obtained.

Proof of Proposition 7.2

Because of Lemma 7.1, one can assume that $\left(\ldots, x^i, \ldots\right)$ is a weakly supportable allocation. For every individual, define

$$F^i = \left\{ z^i + \left(\frac{r}{1+r}\right)\left(z^i - e^i\right)^- \in X^i : z^i \in X^i \text{ and } z^i \succ^i x^i \right\}.$$

Clearly, F^i, by the strict monotonicity of preferences, has a nonempty interior in the norm topology. We then show that every F^i is convex.

Consider

$$f_0^i = z_0^i + \left(\frac{r}{1+r}\right)\left(z_0^i - e^i\right)^-,$$

with $z_0^i \succ^i x^i$, and

$$f_1^i = z_1^i + \left(\frac{r}{1+r}\right)\left(z_1^i - e^i\right)^-,$$

with $z_1^i \succ^i x^i$. For every $0 < \lambda < 1$, define $f_\lambda^i = (1-\lambda) f_0^i + \lambda f_1^i$ and $z_\lambda^i = f_\lambda^i - r\left(f_\lambda^i - e^i\right)^-$, so that

$$z_\lambda^i + \left(\frac{r}{1+r}\right)\left(z_\lambda^i - e^i\right)^- = z_\lambda^i + r\left(f_\lambda^i - e^i\right)^- = f_\lambda^i.$$

Since

$$z_\lambda^i =$$

$$f_\lambda^i - r\left(f_\lambda^i - e^i\right)^- \geq$$

$$f_\lambda^i - (1-\lambda) r\left(f_0^i - e^i\right)^- - \lambda r\left(f_1^i - e^i\right)^- =$$

$$f_\lambda^i - (1-\lambda)\left(\tfrac{r}{1+r}\right)\left(z_0^i - e^i\right)^- - \lambda\left(\tfrac{r}{1+r}\right)\left(z_1^i - e^i\right)^- =$$

$$(1-\lambda) z_0^i + \lambda z_1^i \geq 0,$$

it follows that z_λ^i belongs to the consumption space X^i. Convexity and monotonicity of preferences then guarantee that $z_\lambda^i \succeq^i (1-\lambda) z_0^i + \lambda z_1^i \succ^i x^i$, thus implying that f_λ^i is an element of F^i.

Consider the convex set

$$F = \sum_i F^i - \sum_i x^i - \left(\frac{r}{1+r}\right)\sum_i \left(x^i - e^i\right)^-$$

and notice that $0 \notin F$, since $\left(\ldots, x^i, \ldots\right)$ is a weakly supportable feasible allocation. One can then apply the Separating Hyperplane Theorem, which gives a norm-continuous non-zero linear functional π on $\ell_\infty(\mathcal{S} \times \mathcal{N})$ such that, for all f in F, $\pi \cdot f \geq 0$. Since F contains the positive cone, π is a positive functional. Therefore, Yosida-Hewitt Decomposition Theorem allows one to write $\pi = \pi_f + \pi_b$, where π_f is a norm-continuous positive linear functional on $\ell_\infty(\mathcal{S} \times \mathcal{N})$ admitting a sequence representation (thus, a Mackey-continuous positive linear functional on $\ell_\infty(\mathcal{S} \times \mathcal{N})$) and π_b is a positive purely finitely additive measure. We show that π_f is non-zero and separates.

Fix any f in F. There is an allocation, $\left(\ldots, z^i, \ldots\right)$, weakly Pareto-improving upon $\left(\ldots, x^i, \ldots\right)$, such that

$$f = \sum_i z^i + \sum_i \left(\frac{r}{1+r}\right)\left(z^i - e^i\right)^- - \sum_i x^i - \sum_i \left(\frac{r}{1+r}\right)\left(x^i - e^i\right)^-$$

Mackey-continuity of preferences implies that $z^{i,t} + \left(e^i - e^{i,t}\right) \succ^i x^i$, for t large enough, and, therefore,

$$\sum_i \left(z^i - e^i\right)^t + \sum_i \left(\frac{r}{1+r}\right)\left(\left(z^i - e^i\right)^-\right)^t - \sum_i \left(\frac{r}{1+r}\right)\left(x^i - e^i\right)^-$$

is also an element of F. It follows that

$$\pi_f \cdot \left(\sum_i \left(z^i - x^i\right)^t + \sum_i \left(\tfrac{r}{1+r}\right)\left(\left(z^i - e^i\right)^-\right)^t - \sum_i \left(\tfrac{r}{1+r}\right)\left(x^i - e^i\right)^-\right) \geq$$

$$\pi_b \left(\sum_i \left(\tfrac{r}{1+r}\right) \pi \cdot \left(x^i - e^i\right)^-\right) \geq 0.$$

Taking the limit and using the Mackey continuity of π_f, one establishes that $\pi_f \cdot f \geq 0$.

Suppose now that $\pi_b = 0$. Since, for each t large enough, $\left(x^i + e^i\right)^t \succ^i x^i$, it follows that

$$\sum_i \left(x^i + e^i\right)^t + \sum_i \left(\tfrac{r}{1+r}\right)\left(e^i - e^{i,t}\right) - \sum_i x^i - \sum_i \left(\tfrac{r}{1+r}\right)\left(x^i - e^i\right)^- =$$

$$\sum_i x^{i,t} + \sum_i \left(\tfrac{1}{1+r}\right) e^{i,t} - \sum_i \left(\tfrac{1}{1+r}\right) e^i - \sum_i \left(\tfrac{r}{1+r}\right)\left(x^i - e^i\right)^-$$

is an element of F. Separation, therefore, gives

$$0 \geq \pi_b \cdot \left(\sum_i \left(\frac{1}{1+r}\right) e^i + \sum_i \left(\frac{r}{1+r}\right)\left(x^i - e^i\right)^-\right) > 0,$$

where the last strict inequality follows from interiority assumptions and boundedness of nominal rates of interest. By contradiction, this proves that $\pi_f > 0$.

Fix an individual j and suppose that $z^j \succeq^j x^j$. Define, for all other individuals, i, $z^i = x^i$. Observe that, for all strictly positive real numbers λ,

$$f_\lambda = z^j + \lambda \sum_i e^i + \left(\frac{r}{1+r}\right) \sum_i \left(z^i - (1-\lambda) e^i\right)^- - x^j - \left(\frac{r}{1+r}\right) \sum_i \left(x^i - e^i\right)^-$$

is an element of F. As f_λ converges to f_0 in the Mackey-topology (lattice operations are Mackey-continuous) and π_f is a Mackey-continuous linear functional, one obtains

$$\pi_f \cdot f_0 = \pi \cdot \left(z^j - x^j\right) + \left(\frac{r}{1+r}\right) \pi \cdot \left(\left(z^j - e^j\right)^- - \left(x^j - e^j\right)^-\right) \geq 0,$$

thus establishing the claim.

Proof of Proposition 7.3

Suppose that $(\ldots, z^i, \ldots)$ Pareto-dominates $(\ldots, x^i, \ldots)$ and satisfies the inequality in the Proposition. Since $\left(z^i - e^i\right)^- \leq \left(z^i - x^i\right)^- + \left(x^i - e^i\right)^-$ and r is positive, it follows that

$$\sum_i z^i + \sum_i \left(\frac{r}{1+r}\right) \left(\left(z^i - e^i\right)^- - \left(x^i - e^i\right)^-\right) \leq \sum_i x^i = \sum_i e^i,$$

which contradicts supportability.