

A SHORT NOTE ON THE SOLUTION PROCEDURE OF BARRO AND SALA-I-MARTIN FOR RESTORING CONSTANCY CONDITIONS

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Abstract

The so-called AK models (and models that reduce to AK models without generating transitional dynamics) give rise to a very special property that is called constancy conditions. These conditions impose fix ratios among quantities of the model from the start. Hence, knowing one of the initial values of stock variables becomes sufficient to derive time paths of other variables, given constancy conditions. One source of upsetting these conditions is physical shocks. When a shock disturbs these conditions, preserving intertemporal maximization requires restoring them, preferably immediately. This can be done only by employing a temporary maximization problem, in general. Barro and Sala-i-Martin (1995, pp.172-9) offer a solution procedure based on the idea that the abundant variable has to be kept constant while the scarce variable is let to grow till the condition is satisfied. This note contributes to the discussion in two ways. First, it shows that the solution procedure suggested by Barro and Sala-i-Martin (1995) contains flaws. Second, it shows the right solution procedure that restores constancy conditions.

Keywords: Constancy conditions, economic growth, natural shocks, physical shocks

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1 Introduction

In some growth problems, two or more quantities are ‘forced’ to keep a *constant ratio* among them *from the start* that we call them constancy conditions. In such cases, if the path of one variable is known, then, necessarily, the time-paths of the rest are also known. What makes these models interesting is the observation that these constant ratios are *not tolerant* to disturbances. In other words, the conditions need to be restored as quickly as possible (preferably immediately), if an unexpected shock (*e.g.*, an earthquake) causes a deviation from these conditions because otherwise intertemporal maximization cannot be sustained.

It would not be wrong to say that constancy conditions in growth models have been rarely paid attention. There are two reasons behind this: first, constancy conditions do arise only in a limited number of growth-modeling approaches. Indeed, to our knowledge, the so-called *AK* models are the only frames that generate constancy conditions from the start. Second, the issue of shocks itself has been rarely studied in deterministic growth modeling approaches.¹ Hence, neither constancy conditions nor the question of how to restore these conditions after a shock has been studied sufficiently in deterministic growth models.

One exception to our argument above is Chapter 5 of Barro and Sala-i-Martin (1995). In that chapter, Barro and Sala-i-Martin (henceforth BSM) discuss, in an *extended AK* model, how to restore a constancy condition between the physical capital and human capital after a physical shock on capital (*e.g.*, a war) or on human capital (*e.g.*, an epidemic). In their study, BSM argue that, after a shock, a temporary optimization policy that restricts the growth of the abundant quantity while letting the scarce variable to grow is sufficient to restore a constancy condition. This paper agrees with their intuition but objects to the solution procedure they offer. In particular, this paper aims to (i) show that BSM’s solution procedure is ad hoc and contains technical and conceptual caveats, (ii) denote the right solution mechanism that eradicates the technical and conceptual mistakes of the BSM procedure.

¹ These few works include Oulton (1993), Selcuk and Yeldan (2001), Kepenek et al. (2001), and Yetkiner (2003).

The organization of the paper is as follows. The next section is all about restoring constancy conditions. We first discuss the basic AK model, aiming at introducing the concept. Next, we discuss the solution procedure of BSM (1995) when these conditions are upset. We show that their solution procedure contains flaws. Third, we present our solution mechanism that restores constancy conditions. The main contribution of this section is that it advances our understanding on adjustment dynamics after a shock. The last section is reserved for concluding remarks.

2 Restoring Optimality Conditions

2.1 The Basics

Constancy conditions arise in AK type models or in models that ultimately reduce to AK form without generating transitional dynamics. The basic AK model is the natural starting point for familiarizing with the condition. Define the overall utility as

$$U(C) = \int_0^{\infty} e^{-\rho t} \frac{C^{1-\theta} - 1}{1-\theta} dt \quad (1)$$

where C is aggregate consumption, ρ is the discount rate, and θ is the (absolute) value of elasticity of marginal utility. We assume that $\rho > 0$ and $\theta > 0$, and that the population is normalized to one and does not grow.

The production function is defined as

$$Y = AK \quad (2)$$

where Y is aggregate output, A is the exogenous technology parameter, and K is the aggregate physical capital stock. The model is closed by the macroeconomic budget constraint

$$\dot{K} = AK - C - \delta K \quad (3)$$

where \dot{K} is the instantaneous rate of change in the capital stock and δ is the rate of depreciation of capital. The solution of this problem is part of many textbooks (*e.g.*, BSM (1995)) and we will not elaborate it here. The system generates steady state growth without transitional dynamics:²

$$g^* = \hat{C} = \hat{K} = \frac{A - \rho - \delta}{\theta}. \quad (4)$$

In (4), g^* is the rate of growth (a hat over a variable indicates the rate of change of the respective variable). A steady state growth without transitional dynamics entails also that variables of the system, namely consumption C and physical capital K , hold a constant ratio between them from the start. In particular, it is straightforward to show that

$$\frac{C(t)}{K(t)} = A - \delta - g^*. \quad (5)$$

Hence, there is a constant ratio between capital and consumption, starting from initial values $K(0)$ and $C(0)$. Consequently, consumption is not a free choice but a function of initial capital stock, given parameter values. This is called “closed-form policy function” (see BSM, 1995, p.143, footnote 3). Furthermore, the condition is “binding” not only once-and-for-all but permanently, implying that the constant ratio between consumption and capital must be satisfied at all times.³ Finally, it is worth to note that a change on the right hand side of equation (5) does not violate the condition but just alters ‘the rule’ in accordance with the change. A violation arises if any of the quantities on the left-hand side (*i.e.*, physical quantities) is upset. BSM offered a solution procedure in chapter 5 of

² Note that this result can be derived after the usual transversality condition on capital is applied.

³ Perhaps an association can be made between constancy conditions and the saddle-path stability (*e.g.*, Cass-Koopmans framework) in the sense that the value of the initial control variable is dependent on the initial value of state variable in the case of saddle-path stability as well. The difference is that saddle-path stability does not require a constant ratio between quantities at all time points.

their 1995 book for restoring constancy conditions after a disturbance. We next look at their solution procedure in detail.

2.2 The BSM Example

BSM (1995, pp.172-9) is a two-sector growth model, which reduces to an AK model. Details of the model are as follows. BSM (1995) assume a Cobb-Douglas production function that exhibits constant returns to physical capital K and human capital H :

$$Y = AK^\alpha H^{1-\alpha} \quad (6)$$

where $0 \leq \alpha \leq 1$. Output can be used for consumption or investment in physical or human capital. The economy's resource constraint is

$$Y = AK^\alpha H^{1-\alpha} = C + I_K + I_H \quad (7)$$

where I_K and I_H are gross investment in physical and human capital, respectively. The changes in the two capital stocks are given by

$$\dot{K} = I_K - \delta K \quad (8)$$

$$\dot{H} = I_H - \delta H \quad (9)$$

BSM assume that, for matter of clarity purposes, the stocks of physical and human capital depreciate at the same rate, δ . The Hamiltonian is

$$J = e^{-\rho t} u(C) + v \{I_K - \delta K\} + \mu \{I_H - \delta H\} + \omega \{Y - C - I_K - I_H\} \quad (10)$$

where $u(C)$ is momentary utility (*cf.* equation (1)), v and μ are shadow prices associated with state variables K and H , and ω is the Lagrange multiplier associated with the budget constraint (*cf.* equation (7)). The familiar first order conditions yield that

$$K/H = \alpha/(1-\alpha) \quad (11)$$

and that *all* quantities grow at the constant rate

$$g = \hat{C} = \hat{K} = \hat{H} = \frac{A\alpha^\alpha(1-\alpha)^{1-\alpha} - \rho - \delta}{\theta}. \quad (12)$$

In (12), g is the rate of growth. Next, BSM (1995) question what happens if the K/H ratio deviates from the value $\alpha/(1-\alpha)$ due to a shock on one of the quantities. They state that the constancy condition dictates adjustments in the two stocks, *preferably instantaneously*, in order to attain the value $\alpha/(1-\alpha)$. BSM (1995) add that instantaneous adjustment (“reversible investment”) is not viable because “it depends on the possibility of an infinite positive rate of investment in one form of capital and an infinite negative rate of investment in the other form” (BSM, 1995, p.175). They argue that a more realistic assumption is to limit the growth of the abundant stock variable while the scarce stock variable is allowed to grow.

For matter of illustration, let us continue with one of the examples in their analysis. Suppose that a war destructed part of the capital stock and hence the constancy condition between capital and human capital has been upset. Since human capital becomes abundant compared to physical capital, BSM’s solution procedure proposes to limit the growth of the abundant stock: human capital in that case. From equation (9), we observe that keeping gross investment in human capital I_H at zero implies $H(t) = H(0)e^{-\delta t}$. BSM’s interpretation is that the social planner realizes that the economy has too much H in relation to K , but since it is infeasible to have negative gross investment in H , they allow H to depreciate at the exogenously given rate δ .

BSM state that, given that $I_H = 0$, the social planner has to solve a temporary optimization problem, which is nested in equation (10)

$$J = e^{-\rho t} u(C) + v \{ AK^\alpha \bar{H}^{1-\alpha} - C - \delta K \} \quad (13)$$

where ν is costate variable and a bar on top of a variable shows that its value is given exogenously. Noticeably, this set up is equivalent to the standard Cass-Koopmans framework, where the rate of exogenous technological change is $-(1-\alpha)\delta$, as H depreciates at the rate of δ . BSM argue that the K/H ratio will rise and reach the value $\alpha/(1-\alpha)$ in finite time and thereafter the system will return to the pre-shock equilibrium.

BSM has two minor fallacies in their analysis. First, BSM (1995, p.176) use the expression that “(...) the constraint of nonnegative gross investment in human capital becomes nonbinding” to refer to the time point that the pre-shock condition is recovered. It is not correct to use this phrase to describe what is happening there. State-space and control constraints are used to restrict the movement of the respective variable above or below certain values *at all times*. For instance, referring to BSM’s example, the nonnegativity constraint on human capital investment is “active” at all times, even though the constraint may not be binding at all (indeed, since the undisturbed model (*cf.* equation (10)) generates endogenous growth, the nonnegativity constraints are not binding at all before the shock). When the shock hits, the constraint becomes binding in a temporary optimization condition; however, the nonnegativity constraint does not imply any termination rule in the temporary problem. In that sense, it is wrong to state that the nonnegativity constraint will become nonbinding.

The second fallacy of equation (13) is that it assumes the shock hits the model economy at time zero, which invalidates, by definition, the very existence of optimality conditions unless constancy conditions are taken as initial values.

BSM (1995) have one big caveat in their analysis. We know from the basic AK model that *all* quantities in such models grow at the same rate from the start, and keep constant ratios among them. In particular, the BSM model (*cf.* equation (10)) yields that there is indeed a second condition in addition to equation (11):

$$\frac{K(t)}{C(t)} = \frac{\alpha}{(\theta-1)g + \rho}. \quad (14)$$

Therefore, if constancy conditions between quantities have to be ever restored, it requires one to consider not only the condition between K and H , but also the one between K

and C . Evidently, the temporary problem suggested by BSM (*cf.* equation (13)) ignores the second constancy condition. It is easy to see from equation (13) that consumption will change over time during the temporary optimization, and that there is no rule that secures the second constancy condition at the terminal time of the temporary problem. We may conclude that a more thorough thinking on the question of restoring constancy conditions is needed. In the next subsection, we elaborate our solution procedure.

2.3 The Simple-Response Procedure and Others

We conjecture that there are infinite ways of restoring constancy conditions. For example, first the constancy condition between physical capital and consumption and next the constancy condition between human capital and physical capital can be restored. Evidently, any combination of the abovementioned program such as restoring halfway the constancy condition between human capital and physical capital following a halfway in restoring the constancy condition between physical capital and consumption after completing the first halfway in restoring the constancy condition between human and physical capitals and so on can also be setup. Recall however that the aim of the social planner in this temporary optimization program is to restore the original programme in the shortest time. In that respect, we *speculate* that the *immediate* restriction of growth of *all undisturbed* variables in the model seems to be the best policy in order to restore constancy conditions, given that discrete adjustment is not possible. We do not have any theoretical proof to this argument but intuition dictates that a program with multiple stages should prolong the duration for restoring the conditions. This policy can be called as ‘simple response policy’ in the sense that the policy-maker follows a very simple scheme in order to restore optimality conditions. Figure 1 below illustrates the simple response policy:

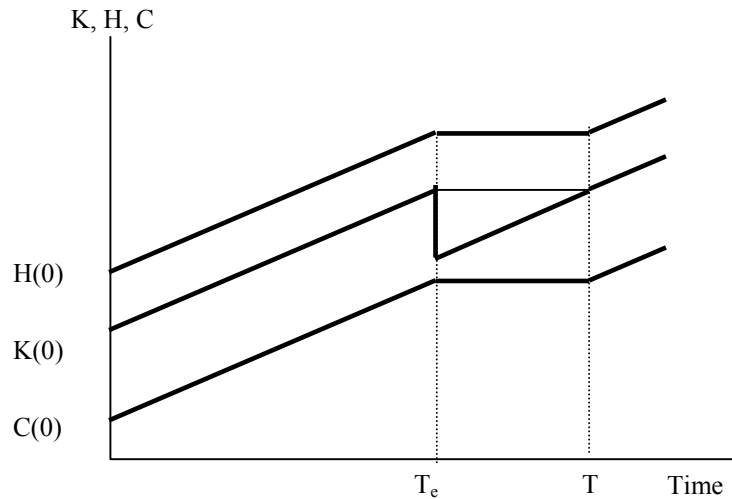


Figure 1. Simple response policy:
(drawn linear for matter of presentation)

In figure 1, at time T_e , a physical shock hits, say, the capital sector, and thus the constancy between K and H , and C and H are disturbed. Since there are two constancy conditions, it is *not* possible to restore them without constraining the growth of both of the undisturbed variables. As figure 1 shows, we must restrict then the growth of H and C immediately after the shock and release them to grow at the point that K reaches the pre-shock level. Before presenting the analytical solution, let us illustrate what we mean by “infinite ways of restoring constancy conditions” in a relevant example.

Note that we assume in figure 1 that the housing stock does not change throughout the temporary problem. This implies that gross investment I_H is equal to δH , or net investment is zero. Had we assumed that gross investment in human capital was zero, as BSM did, then human capital would grow at the rate $-\delta$. In that case, the solution procedure would involve two steps (*cf.* figure 2):

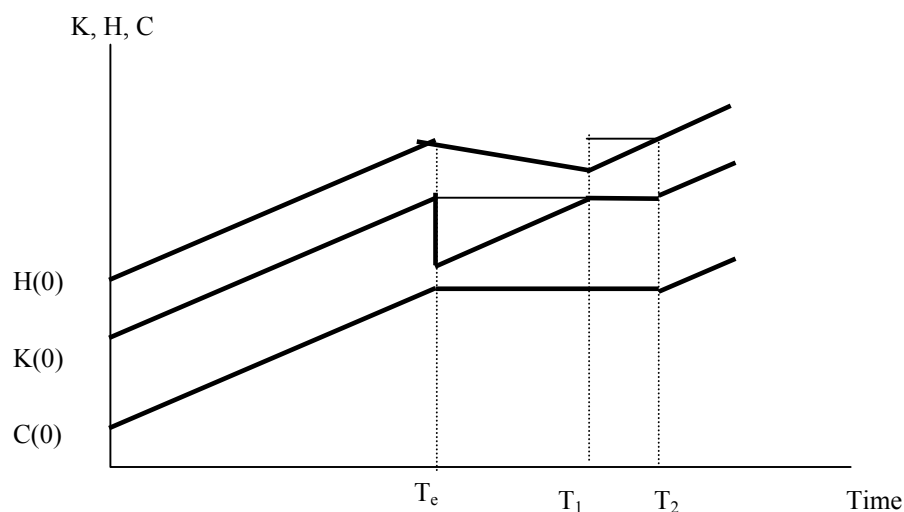


Figure 2. A 2-Step response policy:

In the first step, the social planner fixes consumption and *gross investment* in human capital and lets physical capital grow with the released resources. At time T_1 , the physical stock ‘catches up’ with the consumption, but the system fails to satisfy the constancy condition with respect to human capital. In the second step, physical capital and consumption are kept constant, while human capital is set to grow. At time T_2 , all constancy conditions are satisfied and hence the temporary problem terminates. Which one does imply the shortest time interval? It is not possible to give a precise answer to this question without solving the two programs, though intuition may read out that $T - T_e < T_2 - T_e$.

The Algebraic Representation

Suppose that the social planner agrees to restrict undisturbed variables immediately after the disturbance. The restriction implies setting up a temporary optimization problem, where the only unknowns are capital stock and ‘restoration’ time. The temporary problem starts at T_e^+ and ends at time T , in which the social planner maximizes

$$\begin{aligned}
& \text{Max} \quad \int_{T_e^+}^T e^{-\rho t} \frac{\bar{C}^{1-\theta} - 1}{1-\theta} dt \\
& \text{s.t.} \quad \dot{K} = AK^\alpha \bar{H}^{1-\alpha} - \delta K - \delta \bar{H} - \bar{C}
\end{aligned} \tag{15}$$

$$\text{where } \bar{C} = (C(0))e^{gT_e}, \quad \bar{H} = (H(0))e^{gT_e},$$

$$K(T_e^+) = (1-\kappa)K(T_e^-), \quad K(T) = K(T_e^-)$$

where κ indicates the rate of destruction due to the shock. In the specific example of BSM (1995), the maximization problem degenerates because the temporary maximization problem reduces to a single capital accumulation function when both C and H are kept constant at their just-before-the-shock values.⁴ In particular, the maximization problem above reduces to

$$\dot{K} = AK^\alpha \bar{H}^{1-\alpha} - \bar{C} - \delta K - \delta \bar{H} \tag{16}$$

Unfortunately, it is not possible to solve algebraically the nonlinear differential equation given in equation (16). We run a small experiment for a set of hypothetical parameter values.⁵ Our numerical experiment shows that it approximately takes 12 “years” to recover a 50 percent reduction in the capital stock due to a physical shock that hits the sector in “year” 80.

2.4 Conclusion

In this study, we first showed that AK models (and frames that reduce to AK model without generating transitional dynamics) have a very special property that they generate constant ratios among quantities. This property implies that the time paths of all

⁴ See Yetkiner (2003) for another application of such temporary problems. It may be also useful to make clear that this temporary optimization problem does not fit into the so-called maximin criterion (Rawlsian criterion) because consumption (as a parameter) is given to the problem in our case.

⁵ The hypothetical values are as follows. $T_e = 80$, $\theta = 2$, $A = 0.2$, $\rho = 0.02$, $H(0) = 1$, $\delta = 0.05$, $\alpha = 0.3$, and $\kappa = 0.5$. Note that these values imply $K(0) = 0.428$, $C(0) = 0.05$, $g = 0.019$, $H(T_e) = 4.67$, $K(T_e) = 2.0$, $C(T_e) = 0.26$.

quantities are interdependent in such systems and that they are not tolerant to being put off the equilibrium. Next, we elaborate the solution procedure proposed by Barro and Sala-i-Martin (1995) for restoring constancy conditions. We show that their solution procedure contains flaws. Finally, we indicate the solution procedure in order to restore constancy conditions and speculate among the best. The main premise of the solution procedure remained to be the argument that all quantities that are not exposed to the shock have to be kept constant while the shocked variable is let to grow to its pre-shock value.

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