האוניברסיטה העברית בירושלים
The Hebrew University of Jerusalem

המרכז למחקר בכלכלה חקלאית
The Center for Agricultural Economic Research

המחלקה לכלכלה חקלאית ומנהל
The Department of Agricultural Economics and Management

Discussion Paper No. 8.02

On Knowledge-Based Economic Growth

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November 2002

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* Running Title: Knowledge-Based Growth

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Abstract: Complete time paths of growth processes are derived for economies with an endogenous labor-augmenting knowledge sector. Depending on the production technology, the effect of knowledge on labor productivity, time preferences and capital endowment, a variety of optimal growth patterns emerges, ranging from knowledge-based balanced growth paths, to no knowledge accumulation at all. It is found that long run considerations alone may not be sufficient to determine endogenous growth prospects, as short-term concerns may prevent an economy from ever growing large enough to realize its long-run growth potential.

JEL Classification: C61, O38, O41

Keywords: Endogenous Growth; Knowledge, Technological Progress, Turnpike

1. Introduction

The various mechanisms that give rise to endogenous growth, in light of the large variations in growth rates observed across national economies, have attracted much attention since Solow's [1,2] seminal contributions. Early attempts to endogenize Solow's technological progress residuals include Arrow's [3] learning-by-doing approach and Shell's [4,5,6] treatment of knowledge asset as an additional sector subject to policy decisions. The abundance of models that continued this line of research emerged some two decades later, following the works of Romer [7,8], Lucas [9], Grossman and Helpman [10,11,12], Aghion and Howitt [13] and others. Detailed accounts of the different approaches, as well as exhaustive surveys of the relevant literature, can be found in Aghion and Howitt [14], Solow [15], Mundlak [16] and Lucas [17].

The present effort adopts Shell's [4,5] framework of a two-sector economy with the traditional consumption/capital-commodity sector and a knowledge-asset sector. Knowledge is taken to represent the stock of intangible (know-how, education, human capital) goods and services that enhance labor productivity. It does not fall as manna from heaven nor obtained indirectly through accumulated experience. Rather, knowledge is generated through intentional learning activities that consume resources. Society, then, must decide at each time period how to allocate its available resources among consumption, capital accumulation and knowledge accumulation, thereby determining its growth prospects. Our framework bears some similarity to Lucas' [9], in that both models analyze resource allocation decisions between production and knowledge (or human capital) accumulation.

In the conclusion of his illuminating exposition of growth theory, Solow [15] stresses the importance of short and medium run considerations that are often overlooked. Indeed, we find that assessing long-run endogenous growth prospects may require information on decisions made during the entire time span.

We derive complete time profiles of optimal capital accumulation (growth) and knowledge accumulation processes and examine the conditions that support endogenous growth. We find that growth processes exhibit a *turnpike* property, in that they reach a certain (turnpike) path as rapidly as possible (in a sense precisely defined in the text) and proceed along it thereafter. When the effect of knowledge on labor productivity is linear the condition for growth depends on the production technology, time preferences and a scale parameter. If this condition is met, the economy grows along the turnpike at a constant endogenous growth rate. When knowledge affects labor productivity nonlinearly, whether or not endogenous growth prevails depends also on capital endowment. It is possible that long-run considerations support a balanced (endogenous) growth path, but short-term concerns prevent the economy from ever growing large enough to realize its long-run growth potential.

The analysis is carried out in the (knowledge-capital) state space and makes extensive use of geometric arguments. The methodology has recently been used by Tsur and Zemel [18] to study R&D in backstop substitutes for scarce resources when the latter are essential factors of production. It is applied here to study the details of endogenous growth mechanisms. In essence, we identify four prototypical economies, based on the production technology, time preferences and how knowledge affects labor productivity, and show that the optimal growth policy depends on the economy's *type* and its capital endowment. A wide range of growth patterns emerges, explaining why knowledge-based sustained growth is optimal for some economies but not for others. We also find large variations in the time profiles of knowledge accumulation policies, ranging from early vigorous accumulation (when capital endowment is large enough) to delayed learning that allows capital buildup early on. The optimal processes themselves, thus, can be used to address policy questions such as what part of income should be devoted to learning at different growth stages.

The next section sets up the economic environment and lists necessary conditions for the optimal capital and knowledge accumulation processes. In Sections (3) and (4) optimal growth processes are characterized for linear and nonlinear knowledge effects, respectively. Section 5 concludes and the appendix contains technical derivations and proofs.

2. The economy

A single composite good is produced at the rate $Y = F(K, A(N)L)$, using capital (K), labor (L) and knowledge (N). Knowledge, thus, enhances the efficiency of labor according to an increasing productivity function $A(N)$. The common assumptions that F is linearly homogeneous in its respective arguments, i.e., $F(K,L) = Lf(k)$, $k = K/L$, with f satisfying $f(0) = 0, f(\infty) = \infty, f'(k) > 0, f'(0) = \infty, f'(\infty) = 0$ and $f''(k) < 0$ are maintained. To focus attention on endogenous growth, increase in the labor force and exogenous technological progress are assumed away.

At each time instant t, a fraction $\alpha_t \in [0,1]$ of the income Y is allocated to finance knowledge accumulation activities, so that $dN_t / dt \equiv \dot{N}_t = \alpha_t Y$. Knowledge itself is taken to represent the (intentional) accumulation of labor-productivity-enhancing activities, such as education and training. These activities will be referred to generically as "learning" (learning-by-doing is not considered here). With increasing $A(N)$, the dynamics of N implies

$$\dot{A}_t = G(A_t)\alpha_t L y(k_t, A_t), \tag{2.1}$$

where

$$y(k, A) = Af(k/A) \tag{2.2}$$

is income per capita and $G(A)$ is the marginal knowledge productivity $A'(N)$ expressed as a function of A (when $A(N) = N^{\xi}$, for example, $G(A) = \xi A^{(\xi-1)/\xi}$).

The remaining income, $(1-\alpha)Y$, is used for consumption and investment in physical capital. Thus, $(1-\alpha_t)Y_t = \dot{K}_t + C$, which in terms of the per-capita variables $k = K/L$, $y = Y/L$ and $c = C/L$ becomes

$$\dot{k}_t = (1-\alpha_t)y(k_t, A_t) - c_t. \tag{2.3}$$

Utility is derived from consumption according to $u(c) = (c^{1-\sigma}-1)/(1-\sigma)$ with the intertemporal elasticity of substitution parameter $1/\sigma$ assumed not to exceed unity (this ensures that capital will not be completely consumed at a finite time).

The optimal growth policy consists of the trajectories of income fraction devoted to knowledge generation (α_t) and consumption (c_t) that solve

$$V(k_0, A_0) = Max_{\{c_t, \alpha_t\}} \left\{ \int_0^\infty Lu(c_t)e^{-\rho t} dt \right\} \tag{2.4}$$

subject to (2.1), (2.3), $k_t \geq 0$, $c_t \geq 0$ and $0 \leq \alpha_t \leq 1$, given the endowments k_0 and A_0 and the utility rate of discount ρ. For convenience, knowledge endowment is normalized such that $A_0 = 1$.

With λ and γ representing the current-value costate variables of k and A, respectively, the current-value Hamiltonian is

$$H_t = Lu(c_t) + \lambda_t[(1-\alpha_t)y(k_t,A_t) - c_t] + \gamma_t LG(A_t)\alpha_t y(k_t,A_t). \tag{2.5}$$

Necessary conditions for optimum include

$$u'(c_t^*) = \lambda_t / L, \tag{2.6}$$

$$\alpha_t^* = \begin{cases} 1 & if\ \lambda_t < LG(A_t)\gamma_t \\ 0 & if\ \lambda_t > LG(A_t)\gamma_t\ , \\ \alpha_t^S & if\ \lambda_t = LG(A_t)\gamma_t \end{cases} \tag{2.7}$$

where α_t^S is the singular solution (characterized below), and λ and γ evolve according to

$$\dot{\lambda}_t - \rho\lambda_t = -y_1(k_t, A_t)[(1-\alpha_t)\lambda_t + \alpha_t LG(A_t)\gamma_t], \tag{2.8}$$

$$\dot{\gamma}_t - \rho\gamma_t = -y_2(k_t, A_t)[(1-\alpha_t)\lambda_t + \alpha_t LG(A_t)\gamma_t] - y(k_t, A_t)\alpha_t LG'(A_t)\gamma_t \tag{2.9}$$

(y_1 and y_2 denote the derivatives of y with respect to k and A, respectively). The transversality conditions are,

(a) $lim_{t\to\infty}\{k_t\lambda_t e^{-\rho t}\} = 0$ and (b) $lim_{t\to\infty}\{A_t\gamma_t e^{-\rho t}\} = 0.$ (2.10)

We turn now to characterize the optimal growth (capital accumulation) and knowledge accumulation processes for linear (Section 3) and nonlinear (Section 4) knowledge productivity functions $A(N)$. A few words on terminology are in order. In the growth literature, steady state growth is sometimes used to represent exponential growth. Here we reserve the term steady state to situations in which the optimal processes converge to finite values and the economy approaches stagnation. Situations in which the processes increase indefinitely are referred to as unbounded growth, and the special case of constant endogenous growth rate is called explicitly exponential growth.

3. **Linear *A*(*N*)**

We characterize here the complete time evolution of the optimal knowledge and capital accumulation processes when $A(N) = N$. Our analysis is based on two characteristic lines defined in the (A–k) state space and extends straightforwardly to general A functions (see the following section).

With $A = N$, the function $G(A)$ of (2.1) equals unity for all A and can be dropped. Condition (2.7) identifies three possible knowledge accumulation (learning) policies: no learning ($\alpha = 0$), denoted *o*-policy; maximal learning efforts ($\alpha = 1$), denoted *m*-policy; and singular learning ($0 \le \alpha \le 1$), denoted *s*-policy. The optimal policy consists of selecting among these three possibilities at different phases of the planning horizon. Given the learning regime, only capital remains as an independent state variable and the optimal growth policy reduces to a single-state problem. The selection among the three learning policies, thus, reduces the two-state problem (2.4) into a series of single-state problems.

The selection task, it turns out, depends on the relative positions of two characteristic lines defined in the (A–k) state space. The first line corresponds to the singular *s*-policy of (2.7). Implementing the singular policy during a finite period of time requires that both $\lambda = L\gamma$ and $\dot{\lambda} = L\dot{\gamma}$ hold during that period, which, noting (2.8)-(2.9), implies

$$y_1(k_t, A_t) = Ly_2(k_t, A_t)\,. \tag{3.1}$$

Noting that y_1/L represents the marginal productivity of K (aggregate capital), we see that condition (3.1) requires that along the singular line capital and knowledge are equally productive at the margin, i.e., their marginal rate of substitution, $y_2/(y_1/L)$, equals unity. With $x = k/A$, it is seen that $y_2/y_1 = f(x)/f'(x) - x \equiv z(x)$, where $z(x)$ is increasing and $z(0) = 0$. Thus, condition (3.1) implies $z(k_t/A_t) = 1/L$ and defines a straight line in the A–k plane, $k^S(A) = \chi_s A$, with a slope χ_s given by

$$\chi_s = z^{-1}(1/L). \tag{3.2}$$

We call this line the *singular line* and show below that it serves as a *turnpike* process for growing economies.

The tendency to equate the marginal productivity of capital and knowledge is intuitively obvious when we note that they both compete for shares of the same income source. If capital is more productive at the margin, i.e., $y_1/L > y_2$, the loss in income associated with the decrease in capital due to learning exceeds the income gain associated with the additional knowledge generated by the same learning effort, hence this effort is not warranted. Indeed, we verify in the appendix that the *o*-policy ($\alpha = 0$) is the optimal policy for this case. Conversely, when $y_1/L < y_2$, increasing capital is less beneficial than increasing knowledge, hence the capital process must decrease in this case, either via the *m*-policy ($\alpha = 1$) or via the *o*-policy with $c > y$.

An additional line in the A–k plane is defined by the steady state conditions $\dot{A} = \dot{\lambda} = 0$, or, in view of (2.1) and (2.8), by

$$y_1(k,A) = \rho. \tag{3.3}$$

This equation relates the marginal productivity of capital to its cost—the utility discount rate—and yields a unique solution for k, denoted $k(A)$, which is recognized as the optimal steady state of k when knowledge is fixed at A. We refer to $k(A)$ as the k–line and note that (2.2) and (3.3) reduce to $f'(k_t/A_t) = \rho$, yielding again a straight line $k(A) = \chi_k A$ with the slope

$$\chi_k = f'^{-1}(\rho). \tag{3.4}$$

Since (3.3) is derived from steady state conditions, it implies that a steady state of the two-state problem (2.4) must fall on the k–line. In fact, we establish (proofs and technical derivations are presented at the appendix)

Property 3.1: The optimal knowledge and capital processes corresponding to (2.4) must either (i) converge to a steady state on the k–line, or (ii) grow indefinitely along the singular line.

Which of these two possibilities is realized depends on the relative location of the two characteristic lines. In particular, we find (see appendix)

Property 3.2: (i) If the k–line lies below the singular line for all feasible knowledge states, the economy converges to a steady state. (ii) If the k–line lies above the singular line for all feasible knowledge states, the economy grows indefinitely.

The linear forms derived above for the k– and singular lines render it easy to detect which of these two cases prevails: $\chi_s > \chi_k$ implies convergence to a steady state while $\chi_s < \chi_k$ implies sustained growth.

A growing economy, with $\chi_s < \chi_k$, first approaches the singular line at a *most rapid learning rate*: no learning ($\alpha = 0$) and capital buildup while below the singular line, or maximal learning ($\alpha = 1$) and decreasing capital while above the line. Once the singular line is reached, the learning rate is so tuned as to drive the economy on a sustained growth path along it (Figure 3.1). Thus, when $k_0 < \chi_s$, learning is delayed under the *o*-policy while capital is increased until it reaches χ_s, following which the *s*-policy is implemented and the economy keeps growing along the singular line (Figure 3.2a). When $k_0 > \chi_s$, a vigorous knowledge accumulation policy is initiated under the *m*-policy ($\alpha = 1$) until the singular line is reached, at which time the singular policy is adopted (Figure 3.2b). For a growing economy, then, the singular line serves as a *turnpike* which the economy approaches at a most rapid learning rate and along which it grows (exponentially, as we show below).

Figure 3.1 presents optimal k and A trajectories in the state space for a growing economy. Arrows indicate the direction of evolution over time. The corresponding time trajectories of α, A and k when $k_0 < \chi_s$ and $k_0 > \chi_s$ are depicted in Figures 3.2a-b.

Figures 3.1, 3.2a-b

Depending on the initial capital, the converging economy (with $\chi_s > \chi_k$) may or may not support knowledge accumulation. The optimal initial learning policy is determined in terms of some threshold capital level $k^1 > \chi_s$ as follows: When $k_0 \le k^1$, no learning is ever warranted, implying an *o*-policy that leads to a steady state with $A = A_0 = 1$ and $k = k(1) = \chi_k$. If $k_0 > k^1$, the entire income is initially devoted to knowledge accumulation ($\alpha = 1$) under the *m*-policy, followed by an *o*-policy of no learning and decreasing capital, leading to an eventual steady state on the *k*–line with $A > 1$. The switch from the *m*-policy to the *o*-policy occurs above the singular line, which is crossed as the process evolves toward the steady state below. Thus, when $\chi_s > \chi_k$, regardless of the initial endowment, the economy will eventually stagnate.

It remains to characterize the turnpike growth processes for a growing economy with $\chi_s < \chi_k$:

Property 3.3: The turnpike income process y_t^S grows exponentially according to

$$\frac{\dot{y}_t^S}{y_t^S} \equiv g = \frac{f'(\chi_s) - f'(\chi_k)}{\sigma}. \tag{3.5}$$

The corresponding capital (k_t^S), knowledge (A_t^S) and consumption (c_t^S) processes are proportional to income, hence grow at the same rate g. The fraction of income devoted to knowledge accumulation along the turnpike is the constant

$$\alpha^S = g/[Lf(\chi_s)] \tag{3.6}$$

(it is verified in Appendix B that α^S lies between 0 and 1).

The turnpike growth rate $g = [f'(\chi_s) - f'(\chi_k)]/\sigma$ of (3.5) suggests an interest-rate interpretation for $f'(\chi_s)$. To see this, consider the problem of allocating consumption over time according to $Max_{\{c\}}\left\{\int_0^\infty u(c_t)e^{-\rho t}dt\right\}$ subject to

$\dot{k}_t = y_t - c_t + rK_t$, $k_\infty \geq 0$ and k_0 given, where ρ is the utility discount rate, r is the interest rate and y_t is an exogenous income stream. Standard optimization yields $\dot{c}/c = (r-\rho)/\sigma$. In our case (cf. (3.4) and (3.5)) $\dot{c}/c = (f'(\chi_s)-\rho)/\sigma$. $f'(\chi_s)$, then, plays the role of an interest rate: Along the turnpike, investing in knowledge yields a return $f'(\chi_s)$. The slope of the singular line is thus related to an effective interest rate associated with knowledge buildup due to technological progress. For the economy to grow, the interest rate must exceed impatience—the utility discount rate $\rho = f'(\chi_k)$.

The Cobb-Douglass production technology $f(k) = ak^\beta$, for example, yields $\chi_s = \beta/(L(1-\beta))$ and $\chi_k = (a\beta/\rho)^{1/(1-\beta)}$. The condition $f'(\chi_k) < f'(\chi_s)$ for growth becomes $\rho < a(1-\beta)[\beta/(1-\beta)]^\beta L^{1-\beta}$. The scale ($L$) effect is evident: there exists a critical population L_c corresponding to $\chi_s = \chi_k$ such that small economies (with $L < L_c$) converge to a steady state whereas larger economies grow indefinitely at a rate that increases with L. The economy must be large enough (or impatience small enough) for the public good nature of knowledge to have an impact that justifies sacrifice today for future benefits.[1]

The endogenous exponential growth of (3.5) is directly linked to the assumption that $A(N)$ is linear. Linear $A(N)$ implies constant returns to A in the knowledge generation equation (2.1) along the turnpike (where k is proportional to A). Solow [15] identifies equivalent assumptions in a variety of endogenous growth models and argues that deviations from these assumptions give rise to economies that either diverge to infinity at finite time or fail to exhibit long-run endogenous growth.

[1] Positive scale effects were common in earlier endogenous growth models, but have recently lost some appeal due to lack of empirical support [19-24].

For example, Lucas' [9] model gives rise to endogenous exponential growth only under the linear human capital generation process assumed in his equation (13) [9, p. 19]. Solow's critique applies to the present model too, in that $A(N)$ that gives rise to increasing returns to N in knowledge generation can yield a diverging economy, while decreasing returns force the economy to stagnate at a (static) steady state, as we shall see below. The two models differ, however, in one important aspect: whereas in Lucas [9] the linear relation between $\dot{A}$ and A is assumed, here this relation is a property of the singular line and it holds since it is optimal for the economy to grow along this line.

Neither the diverging nor the stagnating scenarios appear promising. Does that mean that only the linear specification for $A(N)$ is of interest? Not necessarily; there is another possibility, namely, a general specification of $A(N)$ approaching constant returns for large N. It might appear that this case merely reproduces the linear productivity of the previous section in the long run and the economy must eventually grow exponentially. But this conclusion can be misleading. The reason is that the long-run behavior of an economy depends on whether knowledge and capital will ever grow large enough to move the economy into the growth regime. Thus, it is possible that based on long run considerations the economy should eventually evolve along a balanced growth path, yet short run considerations prevent it from ever getting there; the economy stagnates along the way with no (or too little) knowledge to be able to pull itself from poverty.

In the conclusion of his exposition, Solow [15] stresses the importance of short run considerations that are often overlooked by growth theories. The results of the following section put forward another argument supporting this view: evaluating long run growth may require knowledge of the entire growth process, since long-run behavior may not be path-independent.

4. Nonlinear *A*(*N*)

The analysis of the linear case in Section 3 characterizes the complete time evolution of growth paths. It is extended here to provide a full characterization of growth processes under more general specifications, assuming that $A(N)$ is increasing and $A(N_0) = 1$. Prospects for endogenous growth are then assessed. Further generalizations might replace the labor-augmenting (Harrod-neutral) form assumed here by other forms of technological progress, such as Solow- or Hicks-neutral. The shape and location of the characteristic k– and singular lines should be modified accordingly, but the analysis remains the same.

The derivation of Section 3 is repeated, taking account of the terms involving $G(A)$ in the dynamic equations for A, λ and γ. For the k–line we find $k(A) = \chi_k A$ where the slope χ_k is again given by (3.4). Thus, the k–line retains its linear form.

The modification of the singular line is more subtle. Condition (2.7) for the singular solution is $\lambda = LG(A)\gamma$, hence (2.8)-(2.9) yield $y_1 = LG(A)y_2$ as the generalization of (3.1) and define the singular line $k^S(A)$ as the solution of

$$z(k^S(A)/A) = 1/(LG(A)), \tag{4.1}$$

where it is recalled that $z(x) = f(x)/f'(x) - x$. One sees that the ratio $\chi_s(A) = k^S(A)/A$ now depends on A, with $\chi_s'(A) = -G'(A)z(\chi_s)/(G(A)z'(\chi_s))$. Thus, if $G(A)\to\infty$ for large A (e.g., when $A(N)=N^{\xi}$ and $\xi > 1$) the singular line must lie below the k–line above a certain A value, while when $G(A)\to 0$ the singular line must lie above the k–line for large A. Finally, when $A(N)$ is asymptotically linear, so that $G(A)\to G_\infty = lim\ A(N)/N$ and the latter limit obtains a finite positive value, the relative location of the k– and singular lines at large A depends on whether the asymptotic slope $\chi_s(\infty)$ corresponding to $z^{-1}(1/(LG_\infty))$ exceeds or falls short of the slope χ_k of the k–line.

These considerations give rise to four prototypical economies:

Type 1: The singular line always lies below the k–line;

Type 2: The singular line always lies above the k–line;

Type 3: The singular line crosses the k–line once from below;

Type 4: The singular line crosses the k–line once from above.[2]

The optimal processes for Type 1 and Type 2 economies have been characterized in Property 3.2 and the discussion following it (the linearity of the lines is irrelevant for this property). We consider now the other two types, in which the characteristic lines intersect at some point, denoted $(\hat{A},\hat{k}) = (\hat{A},\chi_k\hat{A})$. For a formal derivation, we refer to the appendix.

Optimal processes for Type 3 economies

In general (see exception below) the optimal (A,k) processes of Type 3 economies approach the singular line at a most rapid learning rate and move along it to a steady state at the intersection point $(\hat{A},\hat{k})$. Thus, when $k_0 < k^S(1) = \chi_s$, the o-policy ($\alpha = 0$) is initially implemented, allowing capital to build up until it reaches $k^S(1)$, at which time the s-policy is implemented and the economy evolves along the singular line, approaching a steady state at $(\hat{A},\hat{k})$ (Figure 4.1). If $k_0 = k^S(1)$, the s-policy is immediately implemented and the (A,k) process evolves along the singular line towards the same steady state. For larger capital endowments, k_0 is compared with a critical capital stock $k^3 > k^S(1)$ defined by the property that the m-policy ($\alpha = 1$)

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[2] Multiple crossing of the characteristic lines cannot, in general, be ruled out. This introduces some ambiguity regarding the identification of the optimal steady state, but otherwise yields no further insight and is therefore ignored. Similarly, we consider here only non-decreasing singular lines.

brings the (A,k) process to the intersection point $(\hat{A},\hat{k})$. When $k^S(1) < k_0 \leq k^3$, the m-policy is initially implemented to increase knowledge and decrease capital until the singular line is reached, at which time the s-policy takes over to stir the (A,k) processes along the singular line to a steady state at the intersection point (Figure 4.1). Thus, when $k_0 \leq k^3$, the singular line is a turnpike to which the (A,k) processes proceed at a most rapid learning approach ($\alpha = 0$ while below the singular line and $\alpha = 1$ while above it) and along which they evolve toward the steady state $(\hat{A},\hat{k})$.

If $k_0 > k^3$, the m-policy is first implemented, followed by a switch to the o-policy at some knowledge level $A > \hat{A}$ and above the singular line. The o-policy involves decreasing capital, moving the system towards a steady state on the k–line segment below the singular line (see Figure 4.1). While Type 3 economies encourage some knowledge accumulation (A is increased to $\hat{A}$ or above), economies of this type do not support sustained endogenous growth.

Figure 4.1

Notice that $A(N)/N \rightarrow 0$ can give rise to Type 2 or Type 3 economies only, implying that no endogenous growth is possible in the long run. Consider, for example, the Cobb-Douglas technology $f(k) = ak^{\beta}$ and $A(N)=N^{\xi}$ with $\xi < 1$. The singular line is given by $k^S(A) = \beta/[L\xi(1\text{-}\beta)]A^{1/\xi}$ and must lie above the linear k–line for large A. It will start at $A = 1$ above the k–line if $\beta/[L\xi(1-\beta)] > \chi_k = (a\beta/\rho)^{1/(1-\beta)}$, yielding a Type 2 economy. Otherwise, the characteristic lines cross and the economy must be of Type 3. Whether or not learning activities will be undertaken and the eventual knowledge level depend on the economy type and on its capital endowment.

Optimal processes for Type 4 economies

For Type 4 economies the intersection point $(\hat{A}, \hat{k})$ cannot be a steady state and the optimal policy depends on the endowment k_0 vis-à-vis two threshold capital levels $k^{4a} \le k^{4b}$ (with $k^S(1) \le k^{4b}$) as follows (see Figure 4.2): (i) When $k_0 \le k^{4a}$, no learning is ever warranted and the *o*-policy is implemented to drive capital to a steady state at χ_k. (ii) When $k_0 > k^{4b}$, the *m*-policy is initially adopted until the (A,k) process reaches the singular line, at which time the singular policy is implemented to drive the process along the singular line in an unbounded growth path.

For $k^{4a} < k_0 < k^{4b}$ the optimal policy depends also on the relative positions of k^{4a} and $k^S(1)$ in the following way: if $k^S(1) \le k^{4a}$, (as in Figure 4.2), then initially the *m*-policy is carried out, followed by a switch to the *o*-policy at some (A,k) point above the singular line with $A < \hat{A}$. The *o*-policy drives the system towards a steady state on the *k*–line below the singular line. If $k^S(1) > k^{4a}$, then the (A,k) process proceeds towards the singular line at a most rapid learning approach. Upon reaching the singular line, the *s*-policy is adopted, yielding a path of unbounded growth thereafter.

Figure 4.2

Unlike Type 3 economies, where the turnpike process converges to a finite steady state at the intersection point, here the turnpike involves growth along the singular line. On the other hand, Type 4 economies allow situations in which no learning is warranted (when $k_0 \le k^{4a}$) whereas Type 3 economies always call for some learning efforts.

5. Concluding comments

This paper addresses a long-standing economic problem, namely, the tradeoff between consumption/saving of a material commodity on the one hand, and accumulating knowledge to enhance future productivity, on the other. The problem was first considered by Shell [4,5] and has recently regained interest following Lucas' work [9]. Here we adopt Shell's framework of deciding, at each point of time, on the desirable rate of consumption and on the fraction of income devoted to support knowledge accumulation activities.

The analysis is carried out in the (knowledge-capital) state space in terms of two characteristic lines dividing this space: the k–line, representing the steady-state capital for any *fixed* knowledge level, and the singular line, along which marginal gains and losses of capital and knowledge due to learning just balance. Depending on capital endowment, the four prototypical economies defined by these lines display an assortment of growth patterns:

(*i*) Type-1 and Type-4 economies have the potential to sustain long-run growth. If they realize this potential, they display turnpike characteristics: the optimal processes reach the singular line—the turnpike—at a most-rapid-learning-rate (maximal or delayed learning while above or below the turnpike, respectively), and grow along it thereafter.

(*ii*) Endowed with any positive capital, Type-1 economies always grow. A Type-4 economy will not realize its growth potential if its capital endowment is too small. In such a case the economy stagnates at a finite state.

(*iii*) Type-2 and Type-3 economies may grow for a while but eventually stagnate at a steady state, with a knowledge level depending on capital endowment.

Assessing the prospects for endogenous growth, then, requires information on both the type and capital endowment, so that the entire growth process can be derived. The complete time profiles of the optimal capital and knowledge processes can be used to address policy questions also in the short run, such as what fraction of national income should be devoted to learning activities at any given stage of economic development. We find that regardless of what happens in the long run, an economy need not accumulate knowledge from the outset. With small capital endowments, it is typically preferable to delay knowledge accumulation until enough capital had been generated. Large capital endowments, on the other hand, call for vigorous initial learning activities at the maximal affordable rate, to be later followed by a moderate learning rate or no learning at all.

The type classification reveals that some economies are capable of sustaining endogenous growth, pending sufficient capital endowment, whereas others need a structural change (i.e., a change of type) to gain this capability. External infusion of capital—a common means of foreign aid for stagnating economies—can move those of the favorable types unto a path of self-sustained growth, whereas for the latter types it can have only short-term effects.

The approach taken here is normative, in that we look at socially optimal outcomes, rather than at actual outcomes under various political and industrial organizations, market conditions and incentive structures. The same methodology can be used to study the learning decisions of a (dynastic) family, distinguishing the effect on productivity of the family's own knowledge from that of aggregate knowledge, which is external to the family (as in Lucas [9]). Other extensions consider different forms of technological progress, e.g. Hicks-neutral and Solow-neutral, as well as allowing for exogenous technical change and population growth. Since our results are scale-sensitive, introducing exogenous growth is expected to give rise to a wealth of interesting phenomena.

Acknowledgement: We are grateful to Yair Mundlak for helpful discussions. This work has been supported by the Paul Ivanier Center of Robotics and Production Management, Ben Gurion University of the Negev, and by the Center for Agricultural Economic Research.

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Appendix A: Optimal policies for the four economy types

We derive the optimality of the policies for the four economy types introduced in Section 4. The analysis is carried out in the (A-k) state space and makes use of the characteristic k– and singular lines. The optimal (A,k) process departs from (1,k_0) and its evolution depends on its position vis-à-vis the characteristic lines. We shall frequently use terms like "above the singular line," meaning "when $k > k^S(A)$."

The k–line $k(A)$, introduced in (3.3) as the solution of $y_1(k,A) = f'(k/A) = \rho$, obtains the linear form $k(A) = \chi_k A$ for every specification of the function $A(N)$. Since the line is defined by the steady-state conditions $\dot{A} = \dot{\lambda} = 0$, it follows that

Claim 1: An optimal steady state (A^*,k^*) falls on the k–line, i.e., $k^* = k(A^*)$. ■

Indeed, Claim 1 is consistent with the identification of $k(A)$ as the steady state of k when knowledge is fixed.

Since f' is decreasing, $\rho < y_1$ holds below the k–line. Thus, using (2.7) and (2.8), $\dot{\lambda} < 0$ below the k–line. This, together with (2.6) and $u''(c) < 0$, implies that $\dot{c} > 0$ holds below the k-line as well. The reverse relations hold above the k–line under the o- and s-policies, (with $\alpha = 0$ and $\alpha = \alpha^S$, respectively), yielding

Claim 2: The optimal consumption process increases in time below the k–line under all learning policies and decreases in time above it under the o- and s-policies. ■

Turning to properties of the singular line, we recall that this line is defined by the requirement that condition $\lambda_t = LG(A_t)\gamma_t$ of (2.7) holds during a time interval. Using (2.8) and (2.9) for the time derivatives of the shadow prices and (2.1) for the corresponding derivative of A, we determine the singular line as the trajectory $\Lambda(k^S(A),A) = 0$, where $\Lambda(k,A) = y_1(k,A) - LG(A)y_2(k,A)$. Using (2.2), we obtain

$z(k^S(A)/A) = 1/(LG(A))$, where $z(x) = f(x)/f'(x) - x$. When $A(N)$ is linear, $G(A)$ reduces to a constant and the latter result implies that the singular line is linear, $k^S(A) = \chi_s A$. For more general specifications of $A(N)$, we consider in this work only situations in which the singular line increases.

Using the properties $y_{11}(k,A) < 0$ and $y_{21}(k,A) > 0$ we find that

Claim 3: $\Lambda(k,A) > 0$ below the singular line and $\Lambda(k,A) < 0$ above it. ■

According to (2.7), the optimal learning rate is determined by $\zeta = LG\gamma - \lambda$: $\alpha = 1$ when $\zeta > 0$ (the *m*-policy); $\alpha = 0$ when $\zeta < 0$ (the *o*-policy), and $\alpha = \alpha^S$ when $\zeta = \dot{\zeta} = 0$ (the *s*-policy). Using (2.8) and (2.9), one finds

$$\dot{\zeta} = [(1-\alpha)\lambda + LG(A)\alpha\gamma]\Lambda(k,A) + \rho\zeta. \tag{A.1}$$

Since the shadow prices are positive, we conclude:

Claim 4: (a) When the *m*-policy holds below the singular line, $\zeta e^{-\rho t} \to \infty$. (b) When the *o*-policy holds above the singular line, $\zeta e^{-\rho t} \to -\infty$. ■

Observe that allowing the faster-than-exponential divergence of Claim 4 to proceed permanently is inconsistent with the transversality conditions (2.10). Since a steady state above the singular line involves an *o*-policy, Claim 4b implies

Claim 5: A steady state cannot fall above the singular line. ■

Claim 4 entails restrictions also on the dynamic processes. For example, if the capital-decreasing *m*-policy is adopted at or below the singular line, the sub-optimal behavior of Claim 4a will be followed permanently. Hence,

Claim 6: An *m*-policy can hold only above the singular line. ■

In fact, an m-policy can hold only during a finite period, after which it must be replaced by either an o-policy (above the singular line) or an s-policy (on the singular line).

As long as an o-policy holds, the capital process is monotonic in time because the problem is essentially one-dimensional (knowledge remains constant under the o-policy). If an o-policy holds above the singular line, it must involve decreasing capital until the singular line is reached, for otherwise the sub-optimal behavior of Claim 4b will be followed permanently. Now, ζ must be negative when the singular line is reached from above by an o-policy. Since no other policy can hold below the singular line (Claim 6), this k-decreasing, constant-knowledge policy must converge to a steady state on the k–line segment below the singular line.

Initiated below the singular line, the (A,k) process under an o-policy cannot cross it. Neither can it switch to another policy below the singular line (an s-policy can hold only on the singular line and Claim 6 precludes the m-policy below the singular line). The only two possibilities left are to converge to a steady state below the singular line or to reach the singular line (with $\zeta = 0$) and switch to the s-policy. We summarize these considerations in

Claim 7: (a) When initiated above the singular line, an o-policy continues permanently and the (A,k) process (with A remaining constant) converges to a steady state on the k–line segment below the singular line. (b) When initiated below the singular line, an o-policy either converges to a steady state below this line or reaches the singular line where it switches to the s-policy. ■

Turning to the s-policy, we recall that it can proceed only along the singular line. Moreover, using (A.1) we find that once a singular policy has been initiated (with $\dot{\zeta} = \zeta = 0$), the (A,k) process cannot leave the singular line without violating Claim 6 or 7 (in other words, the *s*-policy is trapping). In view of Claim 1, the following characterization holds:

Claim 8: An *s*-policy either converges to a steady state on the intersection point of the characteristic lines or grows indefinitely along the singular line. ■

To decide between the two options offered in Claim 8, consider an *s*-policy that grows permanently along a singular line segment above the *k*–line. According to Claim 2, this policy involves a decreasing consumption process. However, the policy of staying at the initial state (diverting to consumption the funds allocated by the singular policy to increase the capital and knowledge stocks), is feasible and yields a higher utility. Therefore, the singular policy cannot be optimal. Of course, a singular policy that drives the (A,k) process along a segment above the *k*–line during a finite period, and upon reaching the intersection point moves on to a singular segment below the *k*–line cannot be ruled out. These considerations imply

Claim 9: An *s*-policy cannot be confined to an increasing segment of the singular line above the *k*–line. ■

We apply these results to characterize the optimal processes corresponding to the four economy types introduced in Section 4. It turns out that the steady states themselves, as well as whether the economy converges to a steady state, depend on the initial capital level.

Type 1: Here the *k*–line is always above the singular line. Claims 1 and 5 forbid the existence of any steady state, hence the economy must grow permanently along the singular line (Figure 3.1). When $k_0 < k^S(1)$ the *o*-policy is invoked, increasing capital until $k^S(1)$ is reached (Claim 7b), following which the process evolves along the singular line (Figure 3.2a). In contrast, when $k_0 > k^S(1)$ the *o*-policy is not allowed (Claim 7a) and the *m*-policy is followed until the singular line is reached and the *s*-policy takes over (Figure 3.2b).

Type 2: In Type 2 economies the *k*–line always lies below the singular line. In this case, no point along the *k*–line can be ruled out as a steady state. According to Claim 9, unbounded growth along the singular line cannot be optimal. Therefore when $k_0 \leq k^S(1)$, Claim 7b implies an *o*-policy leading to the steady state $(1,\chi_k)$. Moreover, since ζ is negative at $(1,k^S(1))$ (otherwise an *o*-policy is not optimal), $\zeta < 0$ also for some capital stocks above $k^S(1)$, implying the same policy from these states as well (Claim 7a). For even larger capital stocks, however, the associated value of ζ must turn positive. To see this, we solve (A.1) backwards in time (using the reversed-time $\tau = -t$ normalized such that $\tau = 0$ indicates the time when the singular line is crossed and $\zeta_0 < 0$ and $k^S(1)$ are the corresponding values of ζ and k at that time) with $\alpha = 0$ and $A_\tau \equiv 1$ and find $\zeta_\tau = [\zeta_0 - \int_0^\tau \Lambda(k_s,1)\lambda_s e^{\rho s} ds]e^{-\rho\tau}$. Above the singular line $\Lambda < 0$ and for sufficiently large τ, this result entails $\zeta_\tau > 0$ which is inconsistent with the *o*-policy. Thus, there exists a threshold level $k^1 > k^S(1)$ (corresponding to $\zeta_\tau = 0$) such that the *o*-policy leading to $(1,\chi_k)$ is adopted only when the initial capital does not exceed it.

If, however, $k_0 > k^1$, it is desirable to initially activate learning at full capacity under the *m*-policy. The singular policy is not favored by Type 2 economies (Claim 9), hence the *m*-policy cannot extend to the singular line. The variable ζ, therefore, must decrease and vanish at some (A,k) state above the singular line. This implies a switch to the *o*-policy which leaves A constant and decreases capital towards a steady state on the *k*–line, as Claim 7a implies.

Type 3: A Type-3 economy is characterized by the property that the k–line crosses the singular line from above (Figure 4.1). It follows from Claims 1 and 5 that an optimal steady state must lie on the k–line segment with $A \geq \hat{A}$. Suppose $0 < k_0 < k^S(1)$. Claim 6 forbids the m-policy while the s-policy can be adopted only on the singular line, hence it must be optimal to initially delay knowledge accumulation and apply the o-policy. Since $k(1) > k^S(1)$ Claim 7b implies that it is optimal to delay learning (keeping $\alpha = 0$) and increase capital until k_t reaches $k^S(1)$, and proceed thereafter along the singular line towards the intersection point $(\hat{A}, \hat{k})$.

According to Claim 9, it cannot be optimal to continue the singular policy past the intersection point (where the singular line lies above the k–line). The only steady state allowed on the singular line by Claim 1 is the intersection point. Thus, we deduce from Claim 8 that the optimal (A,k) process must converge to the steady state $(\hat{A}, \hat{k})$.

With larger endowment $k_0 > k^S(1)$, delaying learning is no longer advantageous (Claim 7a) and the optimal policy is to initially set $\alpha = 1$, increasing knowledge and decreasing capital until the (A,k) process reaches the singular line at some time. From that time on, α is reduced to the singular value, and the process continues along the singular line to the steady state $(\hat{A}, \hat{k})$.

Evidently, the higher the initial endowment k_0, the higher is the point at which the singular line is reached. In fact, there exists some threshold initial stock $k^3 > k^S(1)$ such that the (A,k) process initiated from $(1,k^3)$ under the m-policy meets the singular line exactly at $(\hat{A}, \hat{k})$ (Figure 4.1). To see this, we solve the dynamic equations backwards in time, setting $\alpha = 1$ and using the reversed time τ and the initial values $k_{\tau=0} = \hat{k}$, $A_{\tau=0} = \hat{A}$ and $\lambda_{\tau=0} = Lu'(\hat{c})$ where $\hat{c}$ is the consumption rate at the steady state. The threshold stock k^3 is determined from the solution as the state k_τ corresponding to the reversed time τ when $A_\tau = 1$. Using Claim 3 and the time-reversed version of (A.1) with $\zeta_{\tau=0} = 0$, it is verified that $\zeta_\tau = -\int_0^\tau \Lambda(k_s, A_s)\lambda_s e^{\rho(s-\tau)} ds > 0$ along the solution and the m-policy is indeed optimal all the way back to $(1,k^3)$. When $k_0 > k^3$, the m-policy brings the process to a point $(\hat{A}, k)$ above the singular line. In such cases, the m-policy continues to higher knowledge stocks, but it cannot meet the singular line above the k–line (Claims 6 and 9). At some point above the singular line the variable ζ vanishes and learning activities abruptly cease, switching to an o-policy that leads the process to a steady state on the k–line segment below the singular line. Thus, $(\hat{A}, \hat{k})$ is the optimal steady state whenever $k_0 \le k^3$, while larger endowments imply higher asymptotic knowledge and capital stocks.

Type 4: Here the k–line crosses the singular line from below (Figure 4.2). Claims 1 and 5 restrict optimal steady states to lie on the k–line segment with $1 \le A \le \hat{A}$. In contrast to Type 3 economies, Claim 9 forbids the optimal process to converge to the intersection point $(\hat{A}, \hat{k})$ along the singular line. However, unbounded growth along the singular line cannot be ruled out. The dynamic behavior, then, depends on two critical capital stocks defined by the following properties: k^{4a} is the maximum endowment for which it is optimal to avoid learning altogether and approach the steady state $(1,\chi_k)$. (If the endowment $k_0 = \chi_k$ implies approaching the singular line, set $k^{4a} = 0$.) Obviously, for all $0 < k_0 < k^{4a}$ it is optimal to follow the o-policy to $(1,\chi_k)$. k^{4b} is the minimum endowment in excess of $k^S(1)$ for which eventual growth along the singular line is optimal (Figure 4.2). (If the endowment $k_0 = k^S(1)$ implies unbounded singular growth, set $k^{4b} = k^S(1)$.) To find k^{4b} we note, using Claims 1 and 5, that there must exist a minimal level $1 \le A \le \hat{A}$ such that initiated from the state $(A,k^S(A))$ on the singular line, the optimal process must follow the s-policy of unbounded growth. The critical level k^{4b} is obtained by solving the dynamic equations, initiated at $(A,k^S(A))$, backwards in time with $\alpha = 1$ until $A = 1$ is reached. Evidently, $k^{4b} \ge k^{4a}$, and for all $k_0 > k^{4b}$ it is optimal to initially implement the m-policy that will drive the process to the singular line and then switch to unbounded growth under the s-policy. To characterize the behavior for intermediate endowments with $k^{4a} < k_0 < k^{4b}$ we distinguish between two cases: (i) $k^{4a} \ge k^S(1)$ and (ii) $k^{4a} < k^S(1)$. In the former case (depicted in Figure 4.2) an m-policy is initially adopted, to be replaced upon the vanishing of ζ at some point above the singular line by an o-policy that crosses the singular line and drives the process to a steady state on the k–line below. The latter case implies $k^{4b} = k^S(1)$ because any point on the singular line gives rise to a growing s-policy. Delayed learning under the o-policy leads the process to $(1,k^S(1))$. Once the singular line is reached, the s-policy of unbounded growth takes over.

Appendix B: Proof of Property 3.3

We derive the optimal growth rate under linear productivity, with $G(A) = 1$. The condition $\chi_s < \chi_k$ gives rise to sustained growth along the singular line with $k_t = \chi_s A_t$, hence (2.1) and (2.3) imply $c = (1-\alpha B)y$, where $B = 1+\chi_s L = Lf(\chi_s)/f'(\chi_s)$ is a constant depending only on L (cf. (3.2)). The singular policy, then, is determined by the one-dimensional optimization problem

$$V^S(A_0) = Max_{\{\alpha\}} \left\{ \int_0^\infty Lu[(1-\alpha_t B) y(\chi_s A_t, A_t)] e^{-\rho t} dt \right\} \tag{B.1}$$

subject to (2.1), $A_t \geq 0$ and $0 \leq \alpha_t \leq 1/B$, where A_0 is the knowledge productivity state at which the s-policy begins. (For convenience, we reset the time at which the singular process starts to $t = 0$.) With the current-value Hamiltonian given by

$$H = Lu[(1-\alpha B) y(\chi_s A, A)] + \gamma \alpha L y(\chi_s A, A), \tag{B.2}$$

the necessary conditions include

$$u'(c) = \gamma / B, \tag{B.3}$$

and, observing that (3.1) implies $dy(\chi_s A, A)/dA = y_1 B/L = f'(\chi_s)B/L$,

$$\dot{\gamma} = \rho\gamma - \partial H / \partial A = \gamma(\rho - y_1) = -\gamma\Phi, \tag{B.4}$$

where, according to (3.4)

$$\Phi \equiv f'(\chi_s) - f'(\chi_k) > 0. \tag{B.5}$$

Thus, $\gamma = \gamma_0 \exp(-\Phi t)$ and (B.3) and the specification $u(c) = (c^{1-\sigma} - 1)/(1-\sigma)$ imply

$$c = (B/\gamma_0)^{1/\sigma} \exp(\Phi t/\sigma) = c_0 \exp(gt). \tag{B.6}$$

Thus, consumption grows exponentially at the rate $0 < g < \Phi$.

On the singular line, $y = Af(\chi_s)$, hence (2.1) reduces to $\dot{A} = \alpha LAf(\chi_s) = \alpha BAf'(\chi_s)$ and c is expressed as $c = (1-\alpha B)Af(\chi_s)$. Taking the time derivative and using (B.6) we find

$$B\dot{\alpha} / g = (1 - B\alpha)(\delta B\alpha - 1), \tag{B.7}$$

where $\delta = f'(\chi_s)/g > 1$. Equation (B.7) is readily integrated, yielding $(\delta B\alpha - 1)/(1 - B\alpha) = \psi exp[g(\delta - 1)t]$, or

$$B\alpha = \frac{1 + \psi exp[g(\delta - 1)t]}{\delta + \psi exp[g(\delta - 1)t]}. \tag{B.8}$$

The choice of the integration constant ψ requires some care: with $\delta > 1$, any non vanishing value that avoids divergence at finite time implies that $B\alpha$ converges to unity in the long run with $1 - B\alpha \approx exp[-g(\delta - 1)t]$ (The notation $a_t \approx b_t$ signifies that the ratio a_t/b_t approaches a constant as $t \to \infty$). It follows that $A = c/[f(\chi_s)(1 - B\alpha)] \approx exp[g\delta t] = exp[f'(\chi_s)t]$ hence $A\gamma exp(-\rho t) \approx exp[(f'(\chi_s) - \Phi - \rho)t] = 1$ (cf. (B.5)), violating the transversality condition (2.10b). Thus one must choose $\psi = 0$, reducing α^S to the constant $1/(\delta B)$ (which gives 3.6) and ensuring that $1 - B\alpha > 0$. Moreover, a constant learning fraction implies the y and A (and therefore k as well) are all proportional to the consumption c and grow exponentially at the same rate g.

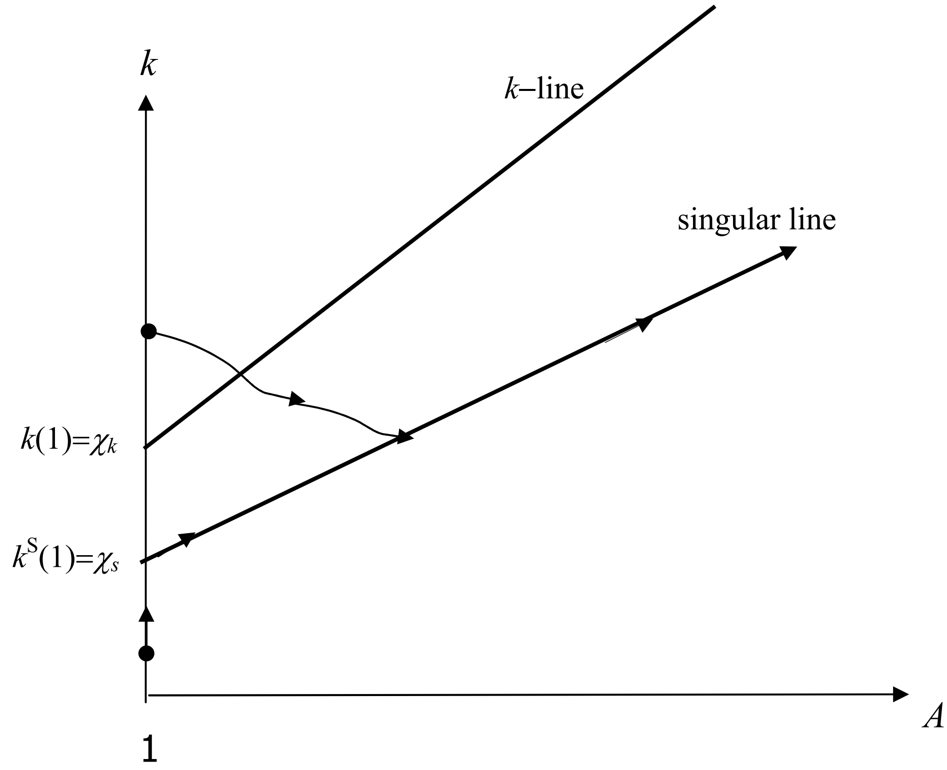

Figure 3.1: Optimal capital and knowledge accumulation when the singular line lies below the *k*–line. Two processes, with capital endowments smaller and larger than $k^S(1)$, respectively, are displayed. The arrows indicate the direction of evolution over time.

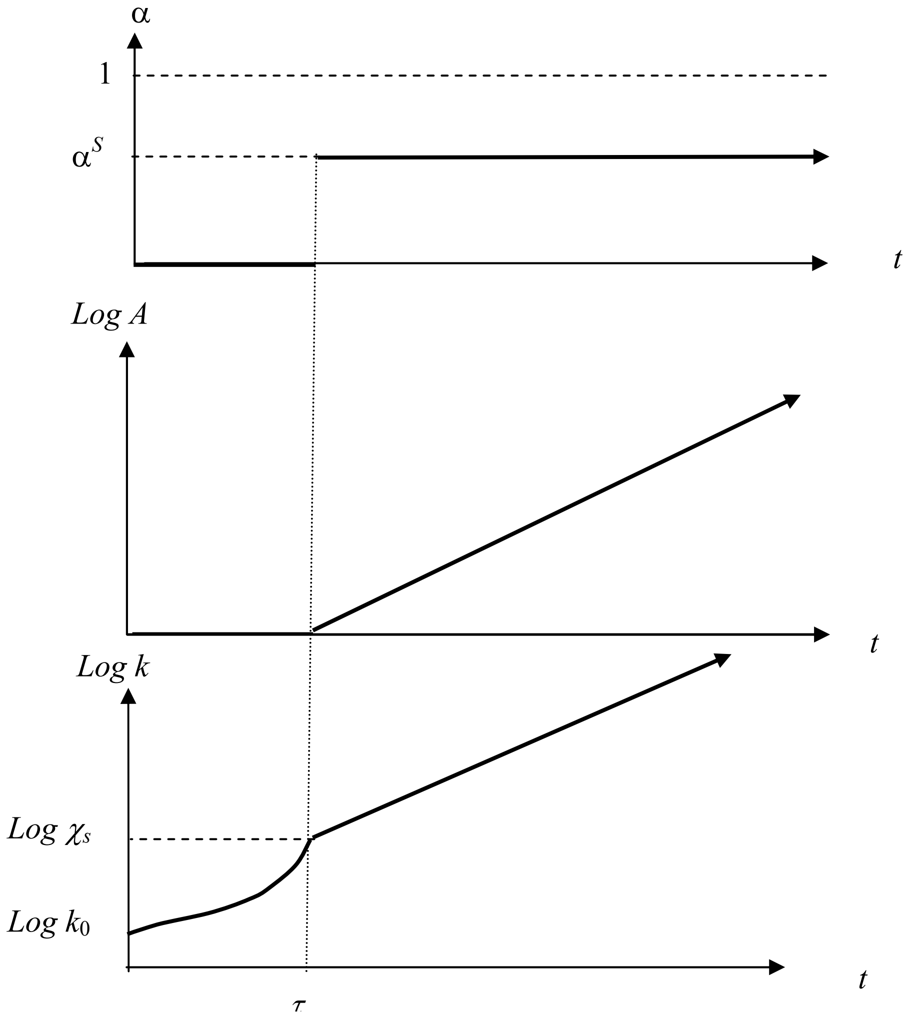

Figure 3.2a: Time profiles of the optimal learning income share (top), knowledge (center) and capital (bottom) processes when capital endowment is smaller than $k^S(1)$. The stock processes are displayed on a logarithmic scale. The processes arrive at the singular line at time τ.

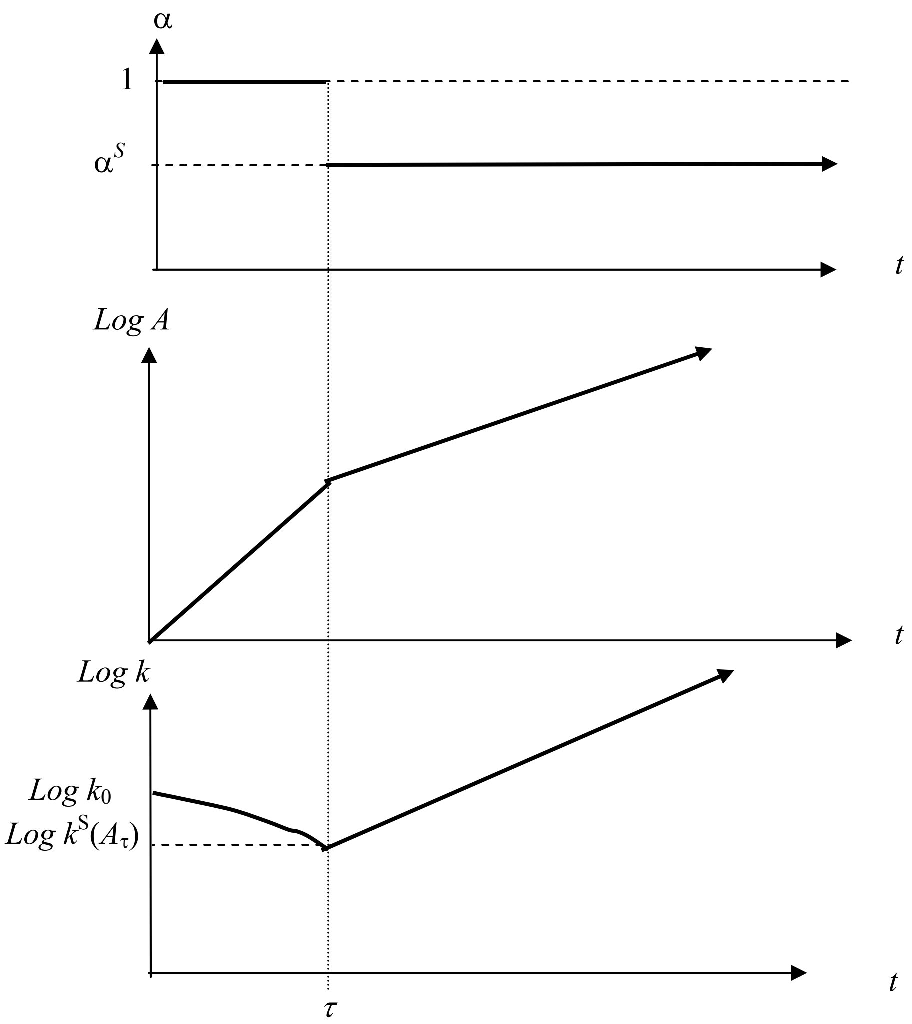

Figure 3.2b: Time profiles of the optimal learning income share (top), knowledge (center), and capital (bottom) processes when capital endowment is larger than $k^S(1)$. The stock processes are displayed on a logarithmic scale. The processes arrive at the singular line at time τ.

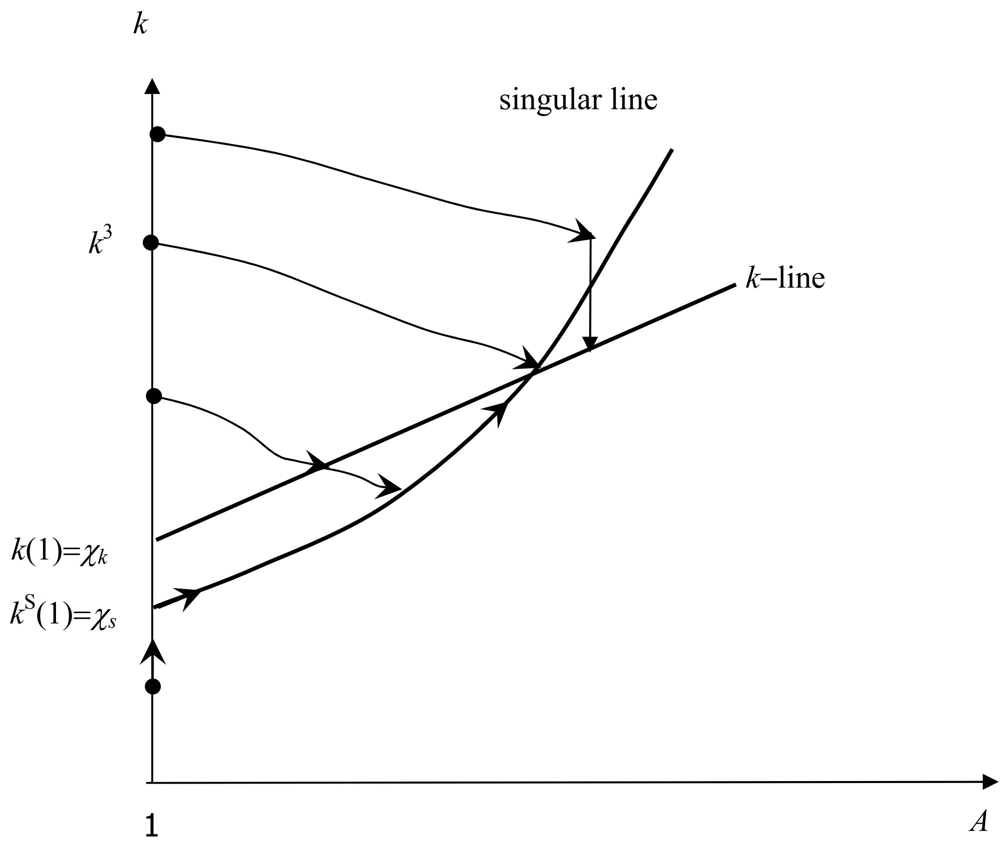

Figure 4.1: Optimal capital and knowledge accumulation processes for Type-3 economies. The arrows indicate the direction of evolution over time. Economies with capital endowments below k^3 approach the singular line and proceed along it to a steady state at the intersection point. A larger endowment implies a steady state higher up on the k–line.

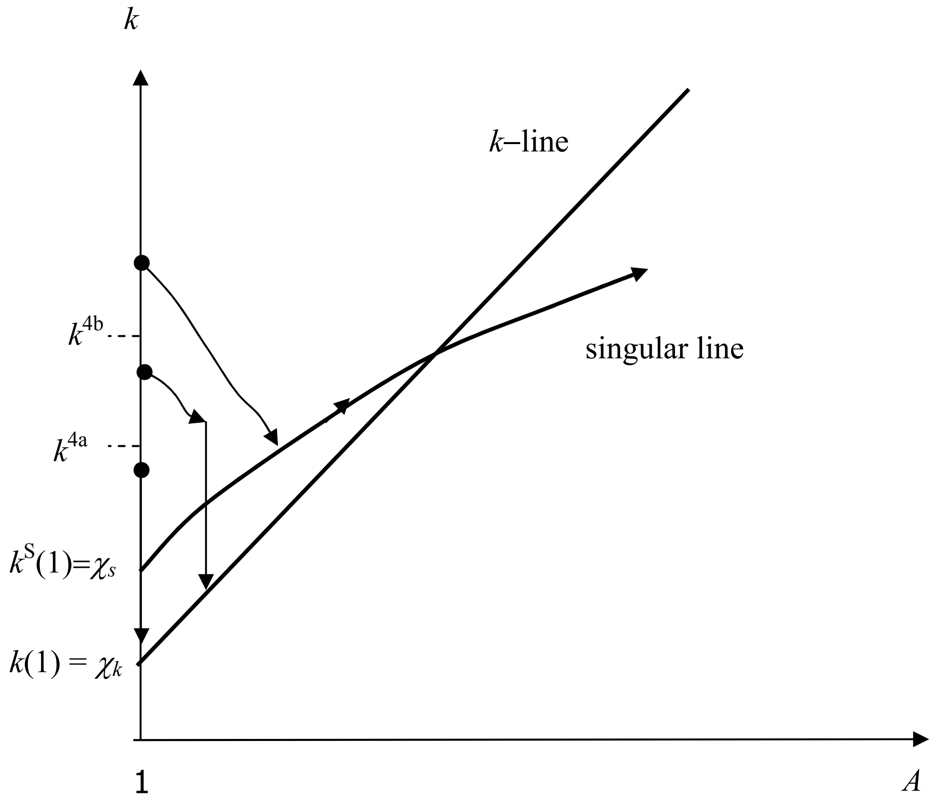

Figure 4.2: Optimal capital and knowledge accumulation processes for Type-4 economies. The arrows indicate the direction of evolution over time. Economies with capital endowments below k^{4a} warrant no learning. Economies with capital endowments below k^{4b} converge to a steady state on the k–line. Economies with $k_0 > k^{4b}$ approach the singular (turnpike) line and grow along it thereafter.