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1. Introduction

It is hoped that this paper will prove a useful start in developing expected utility analysis of the choices faced by various participants in futures markets and eventually some reconsideration of theories of futures markets from this point of view.

The model analyzed represents circumstances as faced by a grain farmer when his harvest is known and he is making storage and hedging decisions. The scope of the analysis is limited in several respects. A one-period model is employed; only a limited number of options are recognized; and interrelations between the grain enterprise and other economic activities of the decision maker are neglected.

Within this limited framework an effort has been made to chooose relevant options and to provide a reasonably comprehensive qualitative analysis. This should aid in the development of needed extensions of the present model. To furnish help with practical decisions, two important extensions would seem to be the incorporation of marketing and hedging decisions made during the growing season, and consideration of a variety of circumstances regarding the availability of credit to the decision maker (see [8]). In the present analysis receipts at different dates are compared by simply applying a known interest charge.

Results reported here are obtained using only very general assumptions (e.g., risk aversion) about the decision makers preferences (utility function) and beliefs (personal probabilities). This seems a desirable way to start in order to know to what extent later conclusions based on more specific assumptions depend on the more specific assumptions.

A more complete description of the circumstances envisaged in the present model is given in Section 2 along with an informal statement of the results. The results are established in Section 3 and discussed in Section 4. The Appendix contains proofs of some propositions needed in Section 3.

2. Circumstances and Main Results

A grain farmer has just harvested and has n bushels on hand. He must decide how much to sell now, how much to store, how much to constract for future delivery, and what position, if any, to take on the futures market.

Suppose that time τ some months ahead is the time of year when this grain usually attains its seasonal peak price and that his opportunity to sell forward would involve delivery at τ.[1] Let a represent his current cash price and c (known) the price at which he can contract for τ-delivery. Let m $(0 \leq m \leq n)$ be the amount he decides to store and g $(0 \leq g \leq m)$ be the quantity he decides to sell forward.

He can also take a hedging (short) position on the futures market. Let b be the current price (per bushel) for futures contracts maturing at τ. Assume that any physical grain stored and not covered by a forward contract will be sold at τ and any short futures position will be closed at τ.[2] His return will then be -

$$\pi = r[a(n-m) - fs] + cg - dm + A(m-g) + (b + f - q - B)s$$

where:

r: cumulation factor converting current dollars to dollars at time τ ; equals $1 + (\frac{j}{12}$ times interest rate) where j is number of months until time τ .

f: margin requirement per bushel for futures transactions.

d: marginal cost of storage.

A: a random variable, unknown price to be realized for local cash grain at τ .

B: a random variable, unknown price of maturing futures contracts at time τ .

q: commission on futures contracts.

s: size (bushel) of short position in futures market $(0 \leq s)$.

m, g, s are the decision variables. Rewriting -

$$\pi = ran + (A - ra-d)m + (b - (r-1)f - q - B)s + (c-A)g$$

$$= k_0 + (A-k_1)m + (k_2-B)s + (k_3 - A)g \ .$$

The k_i are known when m, s, g must be decided. In the formal analysis which follows in Section 3, it is assumed that the farmer acts as though he has a subjective probability distribution of unknown A and B and acts to maximize expected utility of return or gain with respect to that subjective distribution. It is also assumed that he is a risk averter (would demand favorable odds to participate in a pure game of chance, has concave utility function) and that his utility function and personal probabilities satisfy certain mathematical regularity conditions. These assumptions are sufficient to determine a number of conclusions that will be summarized after a little additional terminology is noted.

Call $(A-k_1)$, (k_2-B), (k_3-A) the respective returns to storing, "futuring," and "forwarding." Futuring will be a brief synonym for "taking a short position on the futures market;" forwarding will mean "contracting for delivery at time τ of grain already stored."

Let $A - B = H$, the farmer's basis at time τ (see [1] for a discussion of basis). $(A-k_1) + (k_2-B) = k_2 + A - B - k_1 = k_2 + H - k_1$ will be called the return to futured storage. It represents the effect on final return of simultaneously placing a bushel in storage and increasing one's short futures position by a bushel. $(A-k_1) + (k_3-A)$ $= k_3 - k_1$ represents the effect of simultaneously adding a bushel to storage and selling an additional bushel forward, and will sometimes be called the return to forwarded storage.

If X is any quantity unknown when decisions are made, let EX be the expected or mean value of X computed from the decision maker's subjective probability distribution. Thus EA is his expected cash price at τ, $EA - k_1$ is his expected return to storing, $k_2 + EH - k_1$ is his expected return to futured storage, etc. We shall say the farmer is over, fully, or under hedged according to whether $s + g > m$, $= m$, or $< m$.

The principal results of the next section are -

Storage: Some grain should be stored if and only if at least one of the three returns: return to forwarded storage $(k_3 - k_1)$, expected return to futured storage $(k_2 + EH - k_1)$, expected return to storing $(EA - k_1)$, is positive. If return to forwarded storage is positive, the entire supply should be stored.

Total Hedging: The farmer should overhedge if and only if expected return to futuring $(k_2 - EB)$ is positive. He should fully hedge if $k_2 - EB = 0$, or if $(k_2 - EB) < 0$ and expected return to forwarding $(k_3 - EA)$ is nonnegative. He should underhedge if some grain is stored and both $(k_2 - EB)$ and $(k_3 - EA)$ are negative.

Forwarding: There should be no forward sales unless the return to forwarded storage $(k_3 - k_1)$ is positive. If $k_3 - k_1 > 0$ and if expected return to forwarding $(k_3 - EA)$ is greater than or equal to expected return to futuring $(k_2 - EB)$, i.e., if $k_3 - EH - k_2 \geq 0$, then the entire supply should be forwarded. If $(k_3 - EH - k_2) < 0$ then an amount less than the entire supply should be forwarded (possibly none).

Futuring: If expected return to futuring $(k_2 - EB)$ is positive, a short position in excess of the stored grain uncovered by forward sales should be taken. If $k_2 - EB < 0$, any short position taken should be less than the physical quantity stored and no short position should be taken unless two conditions hold - (a) expected return to forwarding is less than expected return to futuring and (b) expected return to futured storage is positive, i.e., $(EA - k_1) + (k_2 - EB) = (k_2 + EA - k_1) > 0$.

If $(k_2 - EB) = 0$, any stored grain should be fully hedged; the hedging should be entirely by futuring if the return to forwarded storage $(k_3 - k_1)$ is negative; entirely by forwarding if return to forwarding $(k_3 - EA)$ is nonnegative; and by some of each if return to forwarded storage is positive while return to forwarding is negative.

3. Derivation of Results

Mathematically, the decision maker's problem is to find values $\hat{m}$, $\hat{s}$, $\hat{g}$ which maximize the function

$$(1) \qquad \eta(m,s,g) = E\psi[(A-k_1)m + (k_2-B)s + (k_3-A)g]$$

subject to $0 \le m \le n$, $0 \le s$, $0 \le g \le m$.

ψ is the decision maker's utility function for gain,[3] η is his expected utility function. The other symbols were defined in the previous section. The task for this section is to relate the maximizers $\hat{m}$, $\hat{s}$, $\hat{g}$ to some circumstances and expectations of the farmer. Except when the contrary is stated, the following conditions are assumed -

(a) $\psi' > 0$, $\psi'' < 0$, $\lim_{x \to \infty} \psi'(x) = 0$

(b) A, B have finite means and variances; $H = A - B$ is statistically independent of B; any linear combination of A and B is nontrivial

(c) $E|\psi[(A-k_1)m + (k_2-B)s + (k_3-A)g]| < \infty$ and $E|Y\psi'[(A-k_1)m + (k_2-B)s + (k_3-A)g]| < \infty$ for all m, s, g in R^3.

(d) $P(k_2-B \ge 0) < 1$.

(e) $k_3 \ne k_1$

(a) is a standard assumption in expected utility theory. $\psi' > 0$ means that larger gains are preferred. $\psi'' < 0$ implies risk aversion. $\lim_{x \to \infty} \psi'(x) = 0$ is a weaker condition than bounded utility which has sometimes been assumed. (b), (c) are mathematical regularities which seem plausible.[4] A trivial random variable is one that is constant with probability one. If a linear combination of A and B were trivial, one could be written as a linear function of the other and eliminated from the problem. (d) says that futuring is not a sure thing, i.e., it does not offer positive probability of gain with zero probability of loss. Inspection of grain market data (examples are offered in Section 4, page 19) suggests that, typically, $P(k_2 - B \geq 0)$ should be less than one-half. (e) is initially assumed for convenience. $k_3 = k_1$ is highly unlikely and will be seen to cause no difficulty if it should occur. However, the development is simplified by deferring this case to the end of the section.

It is shown in the Appendix that

(i) (a) and (c) imply that η has continuous partial derivatives which may be obtained by differentiation under the expectation.

(ii) The set of assumptions implies that η is strictly concave and has a unique maximum over the admissible set.

(i) follows from a proposition proved in [3, page 3] and (ii) is essentially due to Leland [7]. In both cases, there are minor differences in context which probably justify restatement of proofs.[5]

In this section we shall repeatedly want to determine the sign of a product of random variables of the form -

(2) $$EY\ \varphi(X) = (EY)(E\ \varphi(X)) + \text{Cov}\ (Y, \varphi(X)) = \text{Ex} + \text{Cov}$$

where $X, Y, \varphi(X)$ have finite means and variances and φ is a positive, strictly decreasing function. φ positive means

$$\text{Ex} \overset{s}{=} EY$$

where $\overset{s}{=}$ means "agrees in sign with" in the strict sense that $x \overset{s}{=} y$ if and only if ($x \gtreqless 0$ if and only if $y \gtreqless 0$) . The following proposition [4, page 6] will be useful -

(iii) Let X, Y be random variables with finite means and variances. Suppose $Y = f(W, V)$, $X = g(W, Z)$ where f and g are strictly monotonic in their respective first arguments; W is nontrivial; and V, Z, W are independent.

Let φ be strictly decreasing and such that $\varphi(X)$ has finite mean and variance. Then $\text{Cov}\ (Y, \varphi(X))$ is negative if f, g are of the same monotonicity (both increasing or both decreasing) and $\text{Cov}\ (Y, \varphi(X))$ is positive if f, g are of opposite monotonicity.

Now to justify the conclusions on optimal choice that were stated in Section 2, it is helpful to start with a simple transformation of variables.

Let[6] $w = m - g$ and let

(3) $\gamma(m,s,w) = \eta(m,s,m-w) = E\psi[(k_3-k_1)m + (k_2-B)s + (A-k_3)w]$

where $0 \le m \le n$, $0 \le s$, $0 \le w \le m$.

Since the transformation is 1 - 1 onto, $\hat{m}$, $\hat{s}$, $\hat{m}-\hat{w}$ maximizes η if and only if $\hat{m}$, $\hat{s}$, $\hat{w}$ maximizes γ . Suppose $\hat{s}$, $\hat{w}$ were known, consider -

(4) $$\gamma'_m(m,\hat{s},\hat{w}) = E(k_3-k_1)\ \psi'[(k_3-k_1)m + (k_2-B)\hat{s} + (A-k_3)\hat{w}]$$

$$= (k_3-k_1)\ E\psi'[(k_3-k_1)m + (k_2-B)\hat{s} + (A-k_3)\hat{w}]\ .$$

Since $\psi' > 0$, $E\psi' > 0$ and $\gamma'_m \overset{s}{=} (k_3-k_1)$ regardless of $\hat{s}$, $\hat{w}$. Thus $k_3 - k_1 > 0$ implies that γ can be maximized by assigning m its highest admissible value (n) and $k_3 - k_1 < 0$ indicates that γ can be maximized by assigning m its least admissible value $(\hat{w})$. Hence -

(5) $$(k_3 - k_1) > 0 \Rightarrow \hat{m} = n,\ (k_3 - k_1) < 0 \Rightarrow \hat{m} = \hat{w}\ .$$

These two cases are examined separately. Consideration of the highly unlikely case that $k_3 = k_1$ is deferred to the end of the section.

<u>Case I</u>: $k_3 < k_1$

By the second part of (5), there are no forward sales if $k_3 < k_1$, $(\hat{m} = \hat{w} \Rightarrow \hat{g} = \hat{m} - \hat{w} = 0)$ so there are just two decisions to be made, namely m and s . Let, recalling $A = B + H$,

(6) $$\mu(m,s) = \gamma(m,s,m) = E\psi[(B + H - k_1)m + (k_2-B)s]$$

$0 \le s$, $0 \le m \le n$ be the expected utility function obtained by recognizing the equality of m and w . To investigate possible optimal values of the short futures position, note

$$\mu_s' = E(k_2-B) \ \psi'[(B+H-k_1)m + (k_2-B)s] \tag{7}$$

Equation (7) is of the same form as Equation (2), page 8, if we let

$$Y = (k_2-B), \ X = (B+H-k_1)m + (k_2-B)s \qquad \text{and}$$

$$\mu_s' = Ex + Cov \qquad \text{where}$$

$$Ex = [E(k_2-B)] \ [E\psi'[(B+H-k_1)m + (k_2-B)s]]$$

$$Cov = Cov \ [(k_2-B), \ \psi'[(B+H-k_1)m + (k_2-B)s]] \ .$$

Since $\psi' > 0$, the second factor of Ex is positive and

$$Ex \overset{s}{=} k_2 - EB \ . \tag{8}$$

Recall that ψ' is strictly decreasing $(\psi'' < 0)$ and H is independent of B . Thus Proposition (iii), page 8, applies and

$$Cov \overset{s}{=} m - s \ . \tag{9}$$

Suppose $\hat{m}$ were known. Then $\mu(\hat{m},s)$ and $\mu_s'(\hat{m},s)$ are functions of a single variable s . From the strict concavity of η (Proposition (ii), page 7) it follows that $\mu(\hat{m},s)$ is strictly concave. This implies that $\mu(\hat{m},s)$ has a unique unrestricted maximum, say $\hat{s}_u$; that $\hat{s}_u - s \overset{s}{=} \hat{\mu}_s'$; and that μ can be increased by moving s toward $\hat{s}_u$ from either side. These circumstances are illustrated in Figure 1.

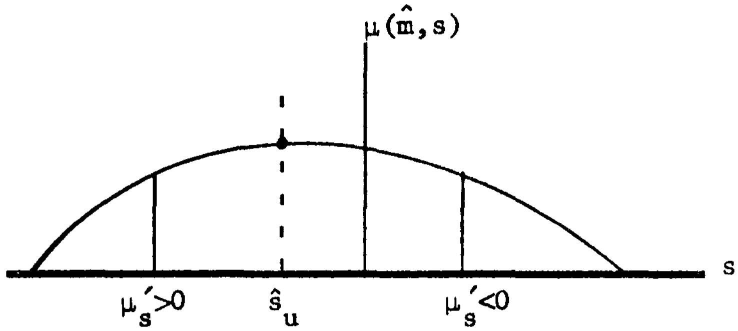

Figure 1

Thus (8) and (9) imply $\mu_s'(\hat{m},\hat{m}) \overset{s}{=} k_2 - EB$ and

(10) $$\hat{s}_u - \hat{m} \overset{s}{=} k_2 - EB \ .$$

Recognizing the restriction $0 \le s$ yields

(11) $$\hat{s} - \hat{m} \overset{s}{=} k_2 - EB \text{ except that } k_2 - EB < 0 , \quad \hat{m} = 0 \Rightarrow \hat{s} = 0 \ .$$

Thus, if something is stored and forward sales are not attractive, the decision maker will over, under, or fully hedge in futures according to whether expected return to a short position is positive, negative or zero. This partly covers the conclusions stated under Total Hedging and under Futuring in Section 1.

Now consider possible choices of m assuming $\hat{s}$ known $(0 \le \hat{s})$.

$$\mu_m' = E(B+H-k_1) \ \psi'[(B+H-k_1)m + (k_2-B)s] \ .$$

Again, this is of the form of Equation (2) with X as before and $Y = B + H - k_1$. Therefore,

$$Ex = EB + EH - k_1 = EA - k_1$$

and if $m = 0$, $\text{Cov} \overset{s}{=} \hat{s}$.

Accordingly,

(12)
$$EA - k_1 > 0 \Rightarrow \mu_m'(0,\hat{s}) > 0 \Rightarrow \hat{m} > 0$$

$$EA - k_1 = 0,\ \hat{s} > 0 \Rightarrow \mu_m'(0,\hat{s}) > 0,\ \mu_m'(\hat{s},\hat{s}) < 0$$

$$\Rightarrow 0 < \hat{m} < \hat{s}$$

$$EA - k_1 \leq 0,\ \hat{s} = 0 \Rightarrow \mu_m'(0,0) \leq 0 \Rightarrow \hat{m} = 0 \quad .$$

Together (12) and (11) make a number of qualitative assertions about optimal storage and futures hedging for various circumstances regarding expected return to futuring and to storage. Before summarizing these, it seems useful to supplement them with a further result obtained by another change of variable.

Let $z = m - s$. Then z (which might be negative) represents unhedged storage. Rewrite expected utility -

$$\zeta(m,z) = \mu(m,m-z) = E\ \psi[(k_2 + H - k_1)m + (B-k_2)z]$$
$$0 \leq m \leq n,\ z \leq m$$

If $\hat{z}$ is temporarily regarded as fixed, we may obtain

$$\zeta_m' = E(k_2 + H - k_1)\ \psi'[(k_2+H-k_1)m + (B-k_2)z]$$

$$Ex \gtreqless k_2 + EH - k_1$$

$$Cov \gtreqless - m$$

$$\hat{m}_u \gtreqless k_2 + EH - k_1$$

(13) $$k_2 + EH - k_1 > 0 \Rightarrow \hat{m} > 0$$

$$k_2 + EH - k_1 \le 0 \Rightarrow \hat{m} = \max \{\hat{z}, 0\} \quad .$$

The reader can readily verify that examining ζ_z' merely reproduces the results already obtained from μ_s' .

Expected utility may also be written -

$$\theta(s,z) = E\psi[(A-k_1)z + (k_2+H-k_1)s]$$

$$0 \le s \quad - s \le z \le n - s$$

Again, regarding $\hat{z}$ as fixed,

$$\theta_s' = E(k_2+H-k_1)\ \psi'[(B+H-k_1)z + (k_2+H-k_1)s]$$

$$Ex \overset{s}{=} k_2 + EH - k_1$$

$$Cov \overset{s}{=} - (\hat{z} + s) \quad . \qquad \text{Therefore}$$

$$\hat{s}_u - \hat{z} \overset{s}{=} k_2 + EH - k_1 \qquad \text{or}$$

(14) $$k_2 + EH - k_1 > 0 \Rightarrow \hat{s} > - \hat{z}$$

$$k_2 + EH - k_1 \le 0 \Rightarrow \hat{s} = \max \{0, - \hat{z}\} \quad .$$

The relations given in (11), (12), (13), (14) are summarized in Table 1.

Table 1

NEGATIVE RETURN TO FORWARDED STORAGE ($k_1 > k_3$, $\hat{g}=0$)

Circumstances of Expected Returns			Optimal Decisions	
1	2	3 (2-1)		
Futuring	Futured Storage	Storing	Amount Stored	Short Future
k_2 - EB	k_2 + EH - k_1	EA - k_1	$\hat{m}$	$\hat{s}$
+	+		+	$> \hat{m}$
+	θ	-	0	+
0	+	+	+	$= \hat{m}$
θ	θ	θ	0	0
-	+	+	+	$< \hat{m}$
-	θ	+	+	0

+, -, 0 indicate respectively positive, negative or zero values for the quantity specified at the top of the column.

θ indicates a nonpositive value.

The blank in the second column indicates that return to futured storage need not be specified to obtain the implications for $\hat{m}$, $\hat{s}$ that appear in the same row.

<u>Case II</u>: $k_3 > k_1$

The procedure follows the pattern of Case I and will be presented more briefly. Case II assumes $k_3 > k_1$ which implies $\hat{m} = n$. Rewriting (1)

$$\eta(n,s,g) = E\psi[(A-k_1)n + (k_2-B)s + (k_3-A)g]$$
$$0 \le s, \quad 0 \le g \le n$$

$$\eta_s' = E(k_2-B)\ \psi'[(A-k_1)n + (k_2-B)s + (k_3-A)g]$$
$$Ex \overset{s}{=} k_2 - EB$$
$$Cov \overset{s}{=} n - s - g$$

(15) $\hat{s} + \hat{g} - n \overset{s}{=} k_2 - EB$ except that $\hat{s} = 0$ if $k_2 - EB < 0$, $\hat{g} = n$

$$\eta_g' = E(k_3-B-H)\ \psi'[(B+H-k_1)n + (k_2-B)s + (k_3-B-H)g]$$
$$Ex \overset{s}{=} k_3 - EB - EH$$
$$n \ge \hat{s} + g \Rightarrow Cov > 0 \quad \text{except} \quad n = g,\ \hat{s} = 0 \Rightarrow Cov = 0$$

(16) $k_3 - EA \ge 0 \Rightarrow \hat{g} > n - \hat{s}$ except that $k_3 - EA \ge 0,\ \hat{s} = 0 \Rightarrow \hat{g} = n$

$k_3 - EA \le 0 \Rightarrow \hat{g} < n$ except that $k_3 - EA = 0,\ \hat{s} = 0 \Rightarrow \hat{g} = n$

Again, the conclusions can be supplemented by a change of variable. Let $v = g + s$ and let

$$\zeta(v,g) = \eta(n,v-g,g) = E\psi[(A-k_1)n + (k_2-B)v + (k_3-H-k_2)g]$$

$$0 \le g \le n \qquad g \le v$$

$$\zeta_g' = E(k_3-H-k_2)\ \psi'[(B+H-k_1)n + (k_2-B)v + (k_3-H-k_2)g]$$

$$\text{Ex} \overset{s}{=} k_3 - EH - k_2$$

$$\text{Cov} \overset{s}{=} n - g$$

$$\hat{g}_u - n \overset{s}{=} k_3 - EH - k_2$$

(17) $$k_3 - EH - k_2 \ge 0 \Rightarrow \hat{g} = \min\{\hat{v},\ n\}$$

$$k_3 - EH - k_2 < 0 \Rightarrow \hat{g} < n$$

Note that $k_3 - H - k_2$ is the excess of return to forwarding over return to futuring. It is the effect on final return of simultaneously reducing the short future by a bushel while increasing forward sales by a bushel.

Alternatively, expected utility is

$$\lambda(s,v) = \eta(n,s,v-s) = E\psi[(A-k_1)n + (k_2+H-k_3)s + (k_3-A)v]$$

$$0 \le s \qquad v - n \le s \le v$$

$$\lambda_s' = E(k_2+H-k_3)\ \psi'[(B+H-k_1)n + (k_2+H-k_3)s + (k_3-A)v]$$

$$\text{Ex} \overset{s}{=} k_2 + EH - k_3$$

$$\text{Cov} \overset{s}{=} \hat{v} - n - s$$

$$\hat{s}_u - (\hat{v}-n) \overset{s}{=} k_2 + EH - k_3$$

(18) $$k_2 + EH - k_3 > 0 \Rightarrow \hat{s} > v - n$$

$$k_2 + EH - k_3 \le 0 \Rightarrow \hat{s} = \max\{0,\ \hat{v}-n\}$$

Implications of (15), (16), (17), (18) are summarized in Table 2.

Now consider the case that $k_3 = k_1$. Write expected utility as -

$$(3') \qquad \gamma(m,s,w) = E\psi[(k_3-k_1)m + (k_2-B)s + (A-k_3)w]$$

$$= E\psi[(k_2-B)s + (A-k_3)w] = \theta(s,w)$$

$$0 \le s \qquad 0 \le w \le n$$

Varying m while holding s and w constant does not affect the argument of ψ and therefore does not affect expected utility. Clearly, optimal choice is not generally unique if $k_3 = k_1$. Recall that $w = m - g$ so if $\hat{w} = n$, then $\hat{m} = n$, $\hat{g} = 0$. If $\hat{w} = 0$, then $\hat{m} = \hat{g} = 0$. However, if $0 < \hat{w} < n$, there is a range of variation for m, g (specifically $\hat{w} \le m \le n$ with $g = m - \hat{w}$) that corresponds to maximum expected utility.

Note that setting $m = n$ does not restrict the range of variation of s, w . Since, in this case, expected utility may be stated as a function of s, w ; this means that no expected utility is lost if the decision maker sets $m = n$ and then proceeds as in Case II. Alternatively he could set $g = 0$ and proceed as in Case I. Thus Tables 1 and 2 are also relevant to $k_3 = k_1$, but in this latter case the decision maker can proceed either as though $k_3 > k_1$ or as though $k_3 < k_1$ without loss of expected utility.

Table 2

POSITIVE RETURN TO FORWARDED STORAGE ($k_3 > k_1$, $\hat{m} = n$)

Circumstances of Expected Returns

1 Futuring $k_2 - EB$	2 Futuring over Forwarding $k_2 + EH - k_3$	3 (1-2) Forwarding $k_3 - EA$	Total Hedge $\hat{s} + \hat{g}$	Forward Sales $\hat{g}$	Short Future $\hat{s}$
+	+		> n	< n	> n - $\hat{g}$
+	θ	+	> n	n	+
0	+	-	n	< n	n - $\hat{g}$
θ	θ	⊕	n	n	0
-	+	-	< n	< n	< n - $\hat{g}$
-	θ	-	< n	< n	0

+, -, 0 indicate respectively positive, negative or zero values for the variable specified at the top of the column. θ indicates nonpositive, ⊕ nonnegative.

The blank in row 1, column 3 indicates that, in this instance, the expected return to forwarding need not be specified to obtain the results in the final three entries of that row.

4. Some Discussion of Assumptions and Results

In this model the farmer has four ways to market his grain. Two are riskless: sell now and sell forward. Two are risky: store unhedged and store with a futures hedge. He can use any combination of alternatives and he can also speculate on futures if he wishes.[7] For each bushel he sells for current delivery, his return in dollars at τ is ra where a is his current cash price and r allows for interest to time τ.

His possible gains (positive or negative) from selecting one of the other alternatives are

Unhedged storage:	$A - k_1$
Storing and hedging in futures:	$k_2 + H - k_1$
Storing and selling forward:	$k_3 - k_1$

Unhedged storage exposes him to the random variable A, unknown cash price at τ. Hedging in futures makes his outcome depend on the random variable H. Selling forward avoids randomness altogether.

Recall that $H = A - B$ was called the basis and assumed independent of B. Since H is the only random component introduced in storing and hedging in futures, someone who chooses this alternative is said to be "gambling on the basis." It is also frequently said that "the basis is more predictable than the price" as an advantage of futures hedging over unhedged storage. This is borne out by general experience and by the data in Table 3. However, we clearly need conceptual and empirical studies of just what might be meant by "more predictable" in an expected utility approach and what reasonable criteria might be

Table 3

DECEMBER PRICES AND BASIS AT SEVERAL LOCATIONS

	Hard Red Winter Wheat					Soft Red Winter Wheat				
	Kansas City	Omaha		St. Louis		Chicago	Toledo		St. Louis	
	(Dec. Future)	Cash	Basis	Cash	Basis	(Dec. Fut.)	Cash	Basis	Cash	Basis
1950	230					234			238	4
1951	252					264			265	1
1952	239					232			234	2
1953	213					203			211	8
1954	237					227			233	6
1955	210					208			214	6
1956	229					239	238	- 1	242	3
1957	215					219	219	0	225	6
1958	192					194	188	- 6	200	6
1959	200					197	198	1	205	8
1960	199					205	199	- 6	210	5
1961	203					205	204	- 1	209	4
1962	218			229	11	208	210	2	215	7
1963	214			228	14	216	217	1	224	6
1964	162	161	- 1	169	7	150	147	- 3	155	5
1965	156	158	2	169	13	165	169	4	170	5
1966	182	183	1	188	6	177	180	3	188	11
1967	151	154	3	159	8	145	144	- 1	150	5
1968	136	137	1	143	7	128	131	3	138	10
1969	141	145	4	150	9	144	145	1	150	6
1970	154	156	2	164	10	168	172	4	168	0
1971	152	155	3	157	5	173	157	-16	157	-16
1972	257	251	- 6	254	- 3	260	264	4	259	- 1
1973	514	511	- 3	524	10	534	550	16	546	8
1974	469	464	- 5	457	-12	469	459	-10	457	-12
1975	346	346	0	336	-10	335	328	- 7	336	1
-u MEAN	195	156	2	176	9	196	182	- 1	200	4
71 STANDARD DEVIATION	35	13	2	30	3	35	33	5	35	5
-u MEAN	226	235	0	238	5	227	226	- 1	231	4
75 STANDARD DEVIATION	90	132	3	120	8	92	107	7	92	6

Sources: Futures prices are from the Kansas City Board of Trade and Chicago Board of Trade Statistical Annuals. Cash prices are from the USDA: Grain Market News.

developed for prediction, or more accurately, for subjective probability formulation.

Table 3 shows average prices of December wheat futures in the delivery month at Kansas City and Chicago for the crop years 1950-1975 along with average December cash prices at various locations in the winter wheat area and the resulting basis for crop years for which quotations are available. Means and standard deviations are calculated for years prior to 1971 as well as for all available years. Calculations through 1971 are included because of the extreme fluctuations in grain markets since 1972. Looking at both sets of calculations, standard deviations of 2 to 8 for basis are a different order of magnitude from standard deviations of cash prices of 13 to 132. Of course, an individual farmer's distribution of basis in a given year will not be the same as the frequency distribution for a nearby market, but the much lower observed standard deviation of basis as compared to price does confirm our prior belief that he is justified in having a more concentrated subjective distribution for basis than for price.

The only properties of the basis used in obtaining the qualitative results of the preceding section were its expected value and its independence of B . The determination of B , the futures price in a given year, depends on national and international supply and demand; whereas the basis H is determined by such things as local transportation costs, the quality of the farmer's grain that year, and the effectiveness of arbitrage between the local and central markets. While these sets of considerations may not be entirely unrelated, possible relations do not seem a priori strong and various small interconnections need not all work in the same direction. It must be borne in mind that the independence being assumed is between the farmer's subjective distribution of B and his subjective distribution of H, both conditional on information available at harvest. The assumption of independence should be tested (not easy) as our knowledge of the expectation formation process develops.

Perusing Tables 1 and 2, it is interesting that use or nonuse of futures can usually be indicated without raising questions of variability. As more complete models are developed to get more precise conclusions, low variability of the basis may be expected to be important in determining the magnitude of the short position in circumstances in which some futures position is indicated.

In the real world, forward contracting does not completely eliminate uncertainty as presumed in our idealized model. Some forward contracts make price dependent on a market quotation at some future date ([1], pages 15-18) and reports of occasional defaults by either buyer or seller do circulate, [1], pages 4, 5 and 19-22. Casual inquiry suggests that most forward contracting is at a specific price. For this reason and also since contracting at an as yet unknown market price blurs the distinction between forward and futures trading it seemed reasonable to take the forward price as fixed in an initial study. As extended models are developed, a variety of assumptions should be explored concerning possible terms of forward and futures contracts and associated uncertainties.

Within the framework of the present model, Tables 1 and 2 of Section 3 show how certain circumstances determine restrictions on optimal choices. It is natural to inquire which circumstances are likely or relevant. This could be answered precisely only by knowing the subjective probability distributions of a number of actual decision makers.

However, we can observe what has happened over a number of crop years and it seems reasonable to suppose that subjective distributions will typically reflect this history to some extent. Unfortunately, we do not have data on prices for forward transactions so our present record is confined to unhedged storage and to returns to short futures positions.

Table 4 shows the returns to futuring $(k_2 - B)$ actually realized on the Kansas City and Chicago wheat markets (July quotation for December wheat less December quotation less commission less interest on margin) along with returns to unhedged storage $(A - k_1)$, December cash price less July price less five months' interest on July price less five months' storage) at various locations, and the return to futured storage $(k_2 - B + A - k_1 = k_2 + H - k_1)$ at these locations for crop years in which data are available.

The frequencies of various observed circumstances in the 72 observed instances of Table 4 are shown in Table 5. Columns 5 - 7 of Table 5 show the restrictions on optimal decisions implied by assuming: (1) that expected returns follow the pattern indicated in the first three columns for observed returns and (2) that $k_3 < k_1$. These restrictions are found by consulting Table 1, page 14.

Table 4

HISTORICAL RETURNS TO UNHEDGED AND FUTURED STORAGE

Locations

Year	Kansas City (k_2-B)	Omaha (A-k_1)	St. Louis (Hard Winter) (A-k_1)	Omaha (k_2+H-k_1)	St. Louis (Hard Winter) (k_2+H-k_1)	Chicago (k_2-B)	Toledo (A-k_1)	St. Louis (Soft Red Winter) (A-k_1)	Toledo (k_2+H-k_1)	St. Louis (Soft Red Winter) (k_2+H-k_1)
1950	- 3.9					- 4.3		8.8		4.5
1951	- 17.6					- 25.4		27.8		2.4
1952	- 6.6					5.6		8.0		13.6
1953	- 2.1					- 0.4		13.3		12.9
1954	- 15.7					- 16.4		22.2		5.8
1955	7.9					- 4.4		6.6		2.2
1956	- 15.8					- 22.4	27.6	28.6	5.2	6.2
1957	- 2.3					0.5	7.1	2.9	7.6	3.4
1958	- 2.8					- 1.4	5.4	9.3	4.0	7.9
1959	- 7.6					- 2.5	8.5	8.3	6.0	5.8
1960	- 5.4					- 14.4	18.1	16.8	3.7	2.4
1961	- 2.2					- 3.4	10.4	8.2	7.0	4.8
1962	2.7		- 1.8		0.9	12.6	- 7.6	- 8.7	5.0	3.9
1963	- 18.3		22.5		4.3	- 28.4	34.0	32.8	5.6	4.4
1964	- 10.6	1.1	6.0	- 9.5	- 4.6	- 2.5	- 0.7	3.2	- 3.2	0.7
1965	- 10.3	2.9	8.8	- 7.4	- 1.5	- 15.6	18.0	16.0	2.4	0.4
1966	9.8	- 16.0	- 16.1	- 6.2	- 6.3	16.1	- 12.8	- 8.9	3.3	7.2
1967	10.8	- 13.7	- 11.8	- 2.9	- 1.0	14.2	- 8.3	- 5.4	5.9	8.8
1968	1.9	- 4.8	- 1.9	- 2.9	0	6.2	0.6	2.4	6.8	8.6
1969	- 13.1	8.1	11.0	- 5.0	- 2.1	- 11.9	11.2	12.1	- 0.7	0.2
1970	- 15.1	11.0	11.7	- 4.1	- 3.4	- 19.9	19.7	16.8	- 0.2	- 3.1
1971	- 4.3	- 1.9	- 2.0	- 6.2	- 6.3	- 17.8	3.1	5.2	-14.7	-12.6
1972	-100.7	88.5	94.5	-12.2	- 6.2	-101.7	113.8	105.7	2.1	4.0
1973	-242.1	207.0	209.6	-35.1	-32.5	-245.7	223.1	238.9	-22.6	- 6.8
1974	- 16.1	10.2	- 9.5	- 5.9	-25.6	- 17.4	2.0	- 6.3	-15.4	-23.7
1975	19.1	- 31.4	- 8.4	-12.3	10.7	27.6	- 14.4	- 8.4	13.2	19.2

Sources: Future prices: Kansas City and Chicago Board of Trade annual summaries
Cash prices: Grain Market News
Commissions, Margin Requirements, Storage Costs: Conversations with brokers and millers
Interest rates: 1% plus Prime Commercial Paper Rate, Survey of Current Business

Table 5

FREQUENCY OF VARIOUS CIRCUMSTANCES

(k_2-B)	(k_2+H-k_1)	$(A-k_1)$	Number of Instances	Optimal Decisions $k_3 < k_1$			Optimal Decisions $k_3 > k_1$		
				$\hat{m}$	$\hat{g}$	$\hat{s}$	$\hat{m}$	$\hat{g}$	$\hat{s}$
+	+	+	5	+	0	$>\hat{m}$	n	$<n$	$>n-\hat{g}$
+	+	-	10	+	0	$>\hat{m}$	n	$<n$	$>n-\hat{g}$
+	-	-	6	0	0	+	n	n	+
-	+	+	24	+	0	$<\hat{m}$	n	$<n$	$<n-\hat{g}$
-	-	+	21	+	0	0	n	$<n$	0
-	-	-	5	0	0	0	n	n	0
+	0	-	1	0	0	+	n	n	+
			72						

The last three columns of Table 5 show restrictions on optimal choice that follow if we assume: (1) the circumstances given in the first three columns with k_3 substituted for k_1, (2) $k_3 > k_1$. These are from Table 2, page 18. The data must be supplemented with returns from other locations, especially locations interior to the growing regions before conclusions are drawn. However, the suggestions from this preliminary look are of some interest.

The fact that return to short futures is usually negative checks with traditional futures theory that speculators who bear the risk of price fluctuations (by holding long positions) when much of the crop is unallocated require a normal premium. As a matter of incidental interest the historical premiums are shown in Table 6. The columns headed Kansas City and Chicago show net returns to long positions in these markets from July to December. Each entry is the negative of the corresponding entry in Table 4, less two commissions less twice the interest charge on required margin.

It seems reasonable that a farmer's mean expected return to futuring $(k_2 - EB)$ should typically be negative and when this is so, expected return to storage $(EA - k_1)$ should typically be positive (since A and B are known to be highly correlated) consistent with the high frequencies of circumstances in rows 4 and 5 of Table 5. Thus, his possible use of the futures market depends on his expectations regarding the basis, the only unknown in column 2. The basis must be compared with $k_2 - k_1$ which may be stated as price received on a futures contract (net of transactions cost) less opportunity cost of storing. As noted earlier, it will be possible to analyze this choice more completely when more specific assumptions are made and properties of personal distributions in addition to means are used.

Table 6

HISTORICAL RETURNS TO LONG FUTURES POSITIONS (JULY-DECEMBER)

Year	Kansas City	Chicago
1950	3.1	3.7
1951	16.8	24.6
1952	5.8	-6.4
1953	1.3	-.4
1954	14.9	15.6
1955	-8.7	3.6
1956	15.0	21.6
1957	1.3	-.5
1958	2.0	.6
1959	6.6	1.5
1960	4.6	13.6
1961	1.4	2.6
1962	-3.5	-13.4
1963	17.5	27.6
1964	9.6	1.5
1965	9.1	14.4
1966	-11.6	-18.0
1967	-12.4	-15.8
1968	-3.5	-7.8
1969	11.3	10.1
1970	13.3	18.1
1971	2.7	16.2
1972	99.3	100.3
1973	248.7	242.3
1974	11.3	12.6
1975	-16.3	-30.4

FOOTNOTES

1. For Minnesota crops, wheat and oats would typically be harvested in August with a peak price in January, the respective months would be November and June for corn, and October and June for soybeans. See Houck [5]. In the winter wheat belt, harvest falls in June or July and the typical peak price is in December or January.

2. If he stores uncontracted grain it is natural to contemplate selling at the time of the usual seasonal peak unless the farmer feels he has special knowledge that affects his expectation of the year's seasonal price pattern. Futures hedging contracts are usually closed simultaneously with the offsetting transaction in the physical commodity. A more complete model would permit the farmer to reconsider his uncontracted grain and futures position occasionally during the season.

3. Using the utility function for gain implicitly assumes that the random variables affecting returns from gain are statistically independent of random components of return from other ventures. See Hildreth [2], pages 101-104.

4. Independence between B and H is perhaps a little hard to judge and is discussed more fully on page 21.

5. Leland's assertion [7, footnote 3, page 38] that $E|W| < \infty$ implies $E|\psi(W)| < \infty$ for ψ as above is not correct. Let $\psi(x) = - e^{-x}$ be a utility function exhibiting constant absolute risk aversion and let $P(W = - n) = \frac{1}{2^n}$ for $n = 1, 2, \ldots$. Then $E|W| = |$ while $E|\psi(W)| = \sum_{1}^{\infty} (\frac{e}{2})^n = \infty$. In economic contexts I think we typically want to assume $E|\psi| < \infty$ anyway, making his proof applicable.

6. w represents grain stored but not sold forward. Increasing m while holding s and w constant corresponds to simultaneously increasing storage and forward sales.

7. Only short positions are considered in the present paper. A farmer could, of course, take a long position if his expectations and utility justified such action. In view of his natural long position in physical grain it seems unlikely that an additional long position in futures would often be optimal. However, this possibility should be added to later models.

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APPENDIX

Let $\eta: R^N \to R$ be an expected utility function written

$$\eta(\alpha) = E\psi\left(\sum_1^N \alpha_n Y_n\right) = E\psi(\alpha Y)$$

where $Y_1 \cdot\cdot Y_N$ are random variables such that any linear combination of them is nontrivial and $\alpha_1 \cdot\cdot \alpha_N$ are decision variables.

Suppose

(I) $\mathcal{Q} \subset R^N$ is closed, convex with $0 \in \mathcal{Q}$

(II) $\psi' > 0$, $\psi'' < 0$, $\lim_{x \to \infty} \psi'(x) = 0$

(III) $E|\psi(\alpha Y)| < \infty$, $E|Y_n \psi'(\alpha Y)| < \infty$
for $n = 1 \cdot\cdot N$ and all $\alpha \in R^N$

(IV) $\exists L \ni [\alpha^o \in \mathcal{Q},\ P(\alpha^o Y \geq 0) = 1 \Rightarrow \|\alpha^o\| < L]$

Then

(V) η is strictly concave

(VI) η has continuous partial derivatives which are obtained by differentiation under the expectation

(VII) η assumes a unique maximum on $\mathcal{Q}$

<u>Proof</u>

(V) Let $(\Omega, \mathcal{F}, P)$ be the probability space of Y.

Let $0 < \lambda < 1$, $\lambda^* = 1 - \lambda$, $\beta \neq \alpha$. Then

$$\eta(\lambda\alpha + \lambda^*\beta) = E\psi(\lambda\alpha Y + \lambda^*\beta Y) = \int \psi(\lambda\alpha Y(\omega) + \lambda^*\beta Y(\omega))\, dP$$

$$> \int (\lambda\psi(\alpha Y(\omega)) + \lambda^*\psi(\beta Y(\omega))dP = \lambda E\psi(\alpha Y) + \lambda^* E\psi(\beta Y)$$

$$= \lambda\eta(\alpha) + \lambda^*\eta(\beta)$$

where the strict concavity of ψ (Assumption II) implies that

$\psi(\lambda\alpha Y(\omega) + \lambda^*\beta Y(\omega)) > \lambda\psi(\alpha Y(\omega)) + \lambda^*\psi(\beta Y(\omega))$ on $\{\omega | \alpha Y(\omega) \neq \beta Y(\omega)\}$

and nontriviality of linear combinations of components of Y guarantees that the latter set has positive probability.

(VI) $$\frac{\partial\eta}{\partial\alpha_1} = \lim_{h\to 0} \frac{\eta(\alpha Y + hY_1) - \eta(\alpha Y)}{h} = \lim_{h\to 0} E\, \frac{\psi(\alpha Y + hY_1) - \psi(\alpha Y)}{h}$$

By the Mean Value Theorem

$\psi(\alpha Y + hY_1) = \psi(\alpha Y) + hY_1\ \psi'(\alpha Y + hKY_1)$

where K is a random variable $\ni 0 \leq K \leq 1$. Thus

$$\frac{\partial\eta}{\partial\alpha_1} = \lim_{h\to 0} E\, Y_1\ \psi'(\alpha Y + hKY_1) \ .$$

Let $Y_1^+ = \max\{Y, 0\}$, $Y_1^- = \max\{-Y, 0\}$,

$W^+ = Y_1^+\ \psi'(\alpha Y - |h|Y_1)$, $W^- = Y_1^-\ \psi'(\alpha Y - |h|Y_1)$.

Since ψ' is strictly decreasing

$|Y_1\ \psi'(\alpha Y + hKY_1)| \leq W^+ + W^- = \overline{W}$ and $\overline{W}$ is integrable since integrability of W^+, W^- is assured by (III). Hence, by Lebesgue's Convergence Theorem

$$\frac{\partial\eta}{\partial\alpha_1} = \lim_{h\to 0} EY_1\psi'(\alpha Y + hKY_1) = E \lim_{h\to 0} Y_1\psi'(\alpha Y + KY_1 h) = EY_1\ \psi'(\alpha Y) \ .$$

By a result of Fenchel (see Katzner [6], page 198), called to my attention by M. Richter, a partial derivative of a concave function is continuous wherever it exists.

(VII) Let $C = \eta^{-1}([\eta(0), \infty))$; C is the inverse image of a closed set under a continuous function and therefore closed. From the strict concavity of η, C is strictly convex, and, by definition, C contains any maximizers of η.

Thus $C \cap \mathcal{A}$ is closed and convex. It suffices to show that $C \cap \mathcal{A}$ is bounded since compactness immediately follows and Weierstrass' Theorem assures a maximum. If there were two maximizers, the line segment joining them would lie in $C \cap \mathcal{A}$ and contain higher values of η (η strictly concave).

To show $C \cap \mathcal{A}$ bounded let $B_L = \{\alpha \mid \|\alpha\| \le L\}$ with boundary S_L and L the limit postulated in (IV). Define $\mathcal{J} = C \cap \mathcal{A} \cap S_L$. $\mathcal{J}$ is compact.

From $0 \in C \cap \mathcal{A}$, $C \cap \mathcal{A}$ convex it follows that if $\alpha \in C \cap \mathcal{A} \cap B_L^c$, then $\frac{L}{\|\alpha\|}\alpha \in \mathcal{J}$. If $\mathcal{J}$ is empty, $C \cap \mathcal{A} \subset B_L$ and we are through, so consider $\mathcal{J} \neq \phi$. Also, $C \cap \mathcal{A} \cap B_L^c \subset \text{Cone } \mathcal{J} = \{\alpha \mid \alpha = \lambda g,\ \lambda \ge 0,\ g \in \mathcal{J}\}$.

Choose any $g \in \mathcal{J}$ and for $\lambda \ge 0$, define $\mu(\lambda) = \eta(\lambda g) = E\psi(\lambda g Y) = E\psi(\lambda X)$. From (VI) μ is continuously differentiable and, from (V), strictly concave. Note

$$\mu' = EX\psi'(\lambda X) = EX^+\psi'(\lambda X^+) - EX^-\psi'(-\lambda X^-) = a(\lambda) - b(\lambda).$$

Since $\psi' > 0$, $a(\lambda) > 0$ and $b(\lambda) > 0$. From $\psi'' < 0$ and $\lim_{x\to\infty} \psi'(x) = 0$, one sees that $a(\lambda) \downarrow 0$ as $\lambda \to \infty$

while $b(\lambda)$ increases with λ . Thus $\mu'(\lambda)$ becomes and remains negative as λ increases so $\mu(\lambda)$ sometime returns to the value $\mu(0) = \eta(0)$ for a $\lambda \geq 1$, say $\lambda(g)$. Since $g \in \mathcal{S}$ was arbitrary we observe that $\forall\, g \in \mathcal{S}\ \exists\, \lambda(g) \geq 1 \ni \eta(\lambda(g)g) = \eta(0)$ and $\eta(\lambda g) < \eta(0)$ for $\lambda > \lambda(g)$.

Furthermore, since $\lambda(g)$ must solve $\eta(\lambda g) = \eta(0)$ and η is continuously differentiable with $\eta'_\lambda(\lambda(g)g) \neq 0$, $\lambda(g)$ is continuous (and indeed differentiable) by the Implicit Function Theorem. Therefore $\lambda(g)$ assumes a maximum, say λ^*, on compact $\mathcal{S}$. By this construction, $\alpha \in \text{Cone}\ \mathcal{S} \cap B^c_{\lambda^*} \Rightarrow \eta(\alpha) < \eta(0)$ or $C \cap \text{Cone}\ \mathcal{S} \cap B^c_{\lambda^*}$ is empty. It was observed above that $C \cap \mathcal{A} \cap B^c_L \subset \text{Cone}\ \mathcal{S}$ so we conclude $C \cap \mathcal{A} \cap B^c_L \subset B_{\lambda^*}$ bounded. The rest of $C \cap \mathcal{A}$ is contained in B_L and hence bounded so $C \cap \mathcal{A}$ is bounded.

In the storage-hedging problem,

$$\alpha = (m,\ s,\ g),\ Y = [(A - k_1),\ (k_2 - B),\ (k_3 - A)]\ ,$$

$$\mathcal{A} = \{m,\ s,\ g \mid 0 \leq m \leq n,\ 0 \leq s,\ 0 \leq g \leq m\}\ .$$

Clearly (I) above is satisfied and Conditions (a) and (c) of Section 3, page 6 duplicate (II) and (III) of this Appendix. It remains to show that Condition (IV) is satisfied for the problem as specified in Section 3. Note that (IV) says that the admissible amount of any sure thing (a combination of random variables or ventures that can win but can't lose) is bounded. We must show that $\exists\, L$ such that

$$P[(A - k_1)m + (k_2 - B)s + (k_3 - A)g \geq 0] = 1 \Rightarrow (m^2 + s^2 + g^2)^{\frac{1}{2}} < L .$$

Since admissibility requires $m \leq n$, $g \leq n$ it suffices to show that s is bounded for any sure thing. Write

$$(A - k_1)m + (k_2 - B)s + (k_3 - A)g = Xw + Ys + kg$$

where

$$w = m - g,\ 0 \leq w \leq n,\ X = A - k_1,\ Y = k_2 - B,\ k = k_3 - k_1 .$$

Condition (d), page 6 requires that $P(Y < 0) > 0$. It follows via Lebesgue's Convergence Theorem that $\exists\ \varepsilon > 0 \ni P(Y < -\varepsilon) > 0$. Condition (b), page 6 requires that $E|X| < \infty$ which requires $\lim_{j\to\infty} P(X < j) = 1$. Thus if we choose a positive ε so that $P(Y < -\varepsilon) > 0$ and define $M_j = (X < j) \cap (Y < -\varepsilon)$, then $P(M_j) > 0$ must hold for sufficiently large j . Now if $Xw + Ys + kg \geq 0$ a.s. then, in particular, the inequality must hold on almost all of M_j . But $Xw + Ys + kg \geq 0$ for $\omega \in M_j \Rightarrow jw - \varepsilon s + kg \geq 0 \Rightarrow \varepsilon s \leq jw + kg \Rightarrow s \leq \frac{(j + |k|)n}{\varepsilon}$. Hence

$$L = n\sqrt{2 + \frac{1}{\varepsilon^2}(j + |k|)^2}$$

is an upper bound for amounts of sure things. This completes the arguments that Conditions (a) - (e) of page 6 with $\mathcal{A}$ as above imply Conditions (I) - (IV) of the Appendix.

It may be worth noting that if $k_3 = k_1$, η remains concave but not strictly concave. If $k_3 > k_1$ then $(A - k_1) + (k_3 - A)$ is a sure thing, but it is bounded by n. Whether or not there are other sure things cannot be said from the assumptions of the model, but the argument given assures that admissible amounts of any which might exist are bounded by L.