Staff Paper Series

Staff Paper P75-30 December 1975

GRAIN RESERVES AND PRICE STABILIZATION

by

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An Introductory Statement

This staff paper describes research on grain reserve stocks as a means of achieving price stability. It is assumed that price stabilization is desirable and the question of desirability is not investigated in this study. The paper is divided into two parts:

Part I: Concept and Measurement of Price Instability and a Model of Price-Stocks Relations

Part II: A Model of Optimal Buffer Stocks for Price Stabilization--Theory and Computation

Part I

1. The Concept and Measurement of Price Instability

The paper begins with a discussion of the concept of price instability and its measurement. The valuation of price instability is a subjective matter. However, for a quantitative analysis it is necessary to define some quantitative instability index which has certain characteristics. The one that is suggested in this study is the mean of square deviations of a series of unknown future prices from a series of target prices.

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*The author is a Lecturer in the Department of Agricultural Economics, Hebrew University of Jerusalem, Rehovot, Israel. This paper summarizes part of a study on grain reserve stocks and price stabilization undertaken jointly by Professor Willard W. Cochrane and the author when the latter was a Post-Doctoral Fellow in the Department of Agricultural and Applied Economics, University of Minnesota, 1975.

2. A Simple Model of Price and Stocks Relations and Computations of Probabilities

The problem of grain reserves as a means for price stabilization is stated. Methods of evaluating an adequate quantity of grain reserves and proposals for a stocks policy are briefly surveyed. It is argued that an optimal stocks policy for price stabilization should take into account current prices as well as the current level of existing stocks as indicators in a price stabilization rule for changes in stocks. The latter indicator is important because it affects the potential of reducing price instability in future periods.

A simple stochastic model of demand, supply, and price determination is formulated. An intervention by acquiring or selling stocks is introduced and the relations between stocks and prices are analyzed. In addition it is shown how to calculate the probability distributions of prices and stocks in the model under any specific stocks policy. These probability distributions are the basic data one needs in order to compare the outcomes of different stocks policies.

3. Minimizing Price Instability by Buffer Stocks--a Simplified Example

Part I is concluded with a very simplified example which demonstrates the main considerations in the search for an optimal stocks policy, i.e., the one which minimizes the instability index of future prices. In particular, it is shown that there is a substitution between current and future price stabilization. It follows that, in general, the change in stocks should be greater than the change in stocks which stabilizes current prices, in order to accumulate reserves for future contingencies.

Part II

1. A Model of Optimal Buffer Stocks for Price Stabilization

Based on the market model presented in Part I, an optimization model is formulated in an attempt to reduce the price instability index. Some propositions which characterize the optimal stocks policy are stated and proved. In fact, the procedure developed here enables one to obtain a whole set of efficient stocks policies. By "efficient" is meant minimizing instability for a given mean stock, or equivalently, minimizing mean stocks for a given level of instability index. From the set of efficient policies a policymaker can choose the one that is compatible with his subjective preferences.

2. Computation Procedure

Based on the analysis of section 1, a computer program was written to compute a price minimization stocks policy. The program also computes the probability distributions of prices and stocks under any stocks policy. In particular, a program which computes the implications of a specific stocks rule was written. This specific rule is the "bounded price rule" proposed by Professor W. W. Cochrane. A range of prices is defined, and, as directed by the rule, stocks are acquired whenever the market price is below the lower boundary of the range, and stocks are sold whenever the market price is above the upper boundary (if there are enough stocks to release). Using the probability distributions, the program calculates the following indicators by which one can compare different stocks policy proposals: the mean price, coefficient of variation of price, the coefficient of variation around the target price, and the mean and coefficient of variation of stocks.

3. Application to Grain Reserves Problem

The procedure described above was applied to two cases: (1) world grains and (2) U.S. wheat. The purpose was to demonstrate how the procedure can be used to evaluate the order of magnitude of grain reserves needed for price stabilization and to compare the expected implications of different stocks policies. However, it should be noted that the empirical work in the present study is a preliminary one. No econometric work has been done to estimate the parameters of the assumed models. Those were evaluated by judgment and information from other studies. This calls for further research.

Part I: Concept and Measurement of Price Instability and a Model of Price-Stocks Relations

The subject of this paper is the instability of grain prices and a policy of reserve stocks to reduce the instability. There has been increasing interest in the problem of grain price instability in recent years following a worldwide shortage of grains in 1972-73, the increase in purchases in the world market by the Soviet Union in 1973, and the depletion of U.S. stocks where, before 1972, reserves had stabilized the market.

The desirability of price stabilization and the questions of who benefits or loses from this stabilization are controversial and have been discussed intensively in the literature (for examples see Waugh (1944), (1961), Oi (1961), Samuelson (1972), Turnovsky (1974)). Related to this are the questions of what is to be stabilized--prices, quantities, consumer expenditures, or farmer incomes? (See Subotnik and Houck (1975) [6].) In this study it is assumed that price stabilization is socially desirable.

There are many ways by which prices can be stabilized: e.g., buffer stocks, production control, taxes and subsidies, export and import control, etc., and each has its advantages and disadvantages. In the present study only buffer stocks are considered.

The order of the paper is as follows:

Section 1 discusses the concept of price instability and its measurement from ex post and ex ante points of view. A criterion for instability in the context of a stochastic world is suggested. A procedure for analyzing the effect of a price stabilization stock policy within the framework of a simple supply and demand model is the content of section 2.

Based on the instability criterion suggested in section 1, a primitive two-period optimization model is presented in section 3. Some important propositions which characterize the general problem of buffer stocks for price stabilization are analyzed within the framework of this simple model.

Part II presents a more general optimization model, a stochastic dynamic programming computation procedure, and some empirical results.

1. The Concept and Measurement of Price Instability

Before any discussion of price instability can be made, one has to clarify exactly what is meant by this concept. In particular, in a quantitative analysis aimed at evaluating the performance of a system or testing the implications of some stabilization policy, it is important to define a quantitative criterion for instability. However, instability may be defined in different ways since it is a subjective matter.

In what follows it will be convenient to distinguish between an ex post and an ex ante point of view. Consider the ex post attitude first, in which a historical time path of price is to be qualified as stable or unstable. The simplest case is when price remains constant all of the time, a case most, if not all, people would probably define as stable. However, some would probably also define as stable a price time path that coincides with some regular monotonic curve of time (like a linear time trend or a constant proportional rate of change), and as unstable one that fluctuates up and down from the curve. This concept is adopted in the present study, with the following definition and notation:

Let $\{p_t^*\}$ be some regular monotonic function of time t $(1 \leq t \leq T)$, to be called "reference or target price," and let $\{P_t\}$ be an actual historical time path of price. $\{P_t\}$ is defined as stable, relative to $\{p_t^*\}$ if $P_t = p_t^*$ for all t, and as unstable otherwise.

To measure the degree of instability many indices can be used and it is a matter of personal taste which one to choose. However, it seems reasonable that an instability index $I(\{P_t\})$ should fulfill at least two characteristics:

(1) I = 0 if and only if $P_t = p_t^*$ for all t, i.e., $\{P_t\}$ is stable relative to p_t^* if it coincides with p_t^* all of the time.

(2) I is an increasing function of the deviations (of the same direction) from the reference price, i.e.: Let $\{P_t^{(1)}\}$ and $\{P_t^{(2)}\}$ be two time paths of P such that:

$$P_t^{(1)} = P_t^{(2)} \text{ for all } t \neq t^o$$

and such that:

$$\text{sign } (P_{t^o}^{(1)} - p_{t^o}^*) = \text{sign } (P_{t^o}^{(2)} - p_{t^o}^*)$$

and

$$|P_{t^o}^{(1)} - p_{t^o}^*| > |P_{t^o}^{(2)} - p_{t^o}^*|$$

then: $$I(\{P_t^{(1)}\}) > I(\{P_t^{(2)}\}) \ .$$

In this paper $I(\{P_t\})$ is defined to be the sum of square deviations of P_t from the target p_t^*, that is:

$$I(\{P_t\}) = \frac{1}{T} \sum_{t=1}^{T} (P_t - p_t^*)^2 \tag{1.1}$$

This index, which is similar to the variance, has the characteristic of increasing the marginal penalty for deviations from the reference path. However, it is not argued that this index is in any sense the best one. Many indices which have the above two conditions can be applied[1/] and it is hard to say which is better.

Let us turn now to the <u>ex ante</u> case in which the instability of a future time path of price is defined. Its main difference from the <u>ex post</u> case is, of course, the uncertainty about future prices (the unrealistic case of certainty can be treated exactly as the <u>ex post</u> case). In the <u>ex ante</u> case only the probability distribution of future prices might be known or assessed. A new element of instability is added, namely, the

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[1/] A few examples of possible indices are:

$$\frac{1}{T} \sum_{t=1}^{T} |P_t - p_t^*| \qquad \frac{1}{T} \sum_{t=1}^{T} \frac{|P_t - p_t^*|}{p_t}$$

$$\frac{1}{T} \sum_{t=1}^{T} (P_t - p_t^*)^2 \qquad \frac{1}{T} \sum_{t=1}^{T} \frac{(P_t - p_t^*)^2}{p_t}$$

$$\left[\frac{1}{T} \sum_{t=1}^{T} (P_t - p_T^*)^2\right]^{1/2} \qquad \left[\frac{1}{T} \sum_{t=1}^{T} \left(\frac{P_t - p_t^*}{p_t}\right)^2\right]^{1/2}$$

$$\frac{1}{T-1} \sum_{t=2}^{T} \frac{|P_t - P_{t-1}|}{P_{t-1}} \qquad \text{etc.}$$

dispersion of the probability distribution at a point in time, in addition to the deviations of the whole time path from the reference path, which was the source of instability in the ex post case. Again, there are many possible indices to measure the dispersion of a probability distribution; probably the most in use is the variance, i.e., the mean of square deviation from the mean. To include the notion of reference or target price, an index of dispersion is used here that is similar to the variance and is the mean of square deviation from the target price (at some time t), i.e.:

$$E[(P_t - p_t^*)^2]$$

where E is the expectation, or mean, operator. To combine the two elements of instability, the dispersion of the probability distribution at each point in time and the deviations of the whole time path from the reference path, the following instability index is defined and used in the rest of the paper:

$$(1.2) \qquad I(\{P_t\}) = E\left[\frac{1}{T}\sum_{t=1}^{T}(P_t - p_t^*)^2\right] = \frac{1}{T}\sum_{t=1}^{T}[E(P_t - p_t^*)^2] .$$

It should be clear that this index should not be considered as ideal and many others may be applied. A possible modification is to apply different weights to different periods of time. For example, one may consider instability at near future to be more important than instability at far future and introduce a discount factor δ to modify the index into

$$(1.3) \qquad I(\{P_t\}) = E\left[\frac{1}{T}\sum_{t=1}^{T}(P_t - p_t^*)^2 \cdot \frac{1}{(1 + \delta)^t}\right] .$$

In addition, one might want to consider relative deviations rather than absolute ones and to modify the instability index into:

$$I(\{P_t\}) = E\left[\frac{1}{T}\sum_{t=1}^{T}\left(\frac{P_t - P_t^*}{P^*}\right)^2 \cdot \frac{1}{(1+\delta)^t}\right] \tag{1.4}$$

This is the index that will be used later in a model of optimal buffer stocks.

2. A Simple Model of Price and Stocks Relations and Probabilities Computations

2.1 Background

Since 1972 the world has experienced a chain of events that caused a steep rise in grain prices, a depletion of stocks, and serious problems of hunger in some developing countries in South and Southeast Asia and in Africa. The United States, as a major exporter of wheat and feed grains and a major supplier of food aid to needy countries, found these worldwide problems reflected in its grain market through its open trade relations with the rest of the world. Thus, there is increasing interest in the question of adequate grain reserves both within the United States and in international discussions.

Some proposals have been suggested concerning the evaluation of adequate reserves and the policy to be implemented in order to meet the two main aspects of grain stocks, food security and price stability.

One method to evaluate the quantity of needed stocks, initiated by Waugh (1967) [11] and followed by Bailey, Kutish and Rojko (1974) [1], can be called "shortfall analysis." Generally, a time-trend curve of production is estimated and the deviations of production from that curve are measured. Assuming that past patterns and trend will continue in the future, the problem is to find the level of stocks that is needed to meet an accumulated deficit (i.e., negative deviation from the trend line), or some percentage of it, with some probability. The deficiency of this method is that it does not give a definite rule on how to build and dispose of stocks, except the general obvious idea that stocks should be accumulated in good years and disposed of in bad ones.

More specific proposals on a stocks policy can be grouped under the title "bounded price" policy. W. W. Cochrane (1974) [2] suggested defining an upper and lower price boundary and buying stocks whenever the market price falls below the lower boundary and selling stocks whenever the market price is above the upper boundary insofar as there are positive stocks. The deficiency of this proposal is that it is based completely on price signals and does not take into account the existing level of inventory; the same reaction to price is proposed when stocks are in abundance or in shortage. Later, in section 3 it is shown that even if the only goal is price stabilization, quantity considerations should also be taken into account.

Other proposals combine the basic bounded price rule with some target stocks level and modify the rule when stocks are above or below the target.

In order to evaluate the feasibility and desirability of any specific proposal, it is important to estimate its implications in attaining the objectives for which it was designed and the means needed to implement it. In designing and implementing a stocks rule, consideration should be given to its effect on the probability distribution of future prices as well as the quantities of stocks and their probability distribution. For example, the following questions might be asked: What is the probability that the price will be in some interval? What is the probability of being out of stocks? What is the mean price and the mean of stocks, etc.? In addition one might decide on the general formula of a stocks rule, but want to test different values of its parameters (e.g., different boundaries of the bounded price rule). Unfortunately, the grain economy is very complicated. A complete stochastic and dynamic analysis must include cross effects among the main crops on the demand side as well as on the supply side and relations between different sources of supply and demand within a country (interregional) and among countries (international). The problem is complicated by stochastic disturbances in the various relations, especially in a dynamic analysis such as required here. In view of this complexity, it seems inevitable that many simplifying assumptions must be made if one wants to do a quantitative analysis.

Until now some studies on grain stocks have been done by simulation procedures (Tweeten, Kalbfleish and Lu (1971)[8], Sharples and Walker (1974)[5]). In these studies a simple model is assumed that includes basic demand and supply equations with stochastic terms that are assumed to have known probability distributions. A sequence of drawings from the probability distributions is made by a computer generator corresponding to a sequence of time periods. The whole system is then simulated by the computer, resulting in the determination of a sequence of equilibrium values of the main variables (prices, quantities, stocks, etc.). This procedure is repeated many times and the average results printed, including the average yield, production prices and stocks, as well as their variances. Any specific stocks rule can be introduced into such a model and comparisons made between different rules and the "free market" solution.

The method used in the present study is similar to the simulation studies in its main assumptions about the economic model. However, instead of repeatedly drawing sequences of disturbances from their probability distributions, the probability distributions of the main variables (prices and stocks) are directly computed using the internal relationships in the model. Practically, a method of approximation is used by discretionizing the probability distributions (i.e., allowing the variables to obtain only isolated values). The basic model is presented in the following subsection.

2.2 The basic model

The model is a partial one that includes demand and supply equations of only one commodity (which may be one grain or an aggregate of some crops). Price and quantity are determined by the model. All other factors are assumed to be exogenous and included either in systematic "shifters" which shift the demand and supply functions with time or in stochastic disturbances that shift these functions randomly, according to some known probability law. Formally, the model is as follows:

Demand function at time t:

(2.1) $$Y_t = [\phi(P_t) + e_{dt}](1 + g_d)^t$$

where: Y_t = quantity demanded at time t

P_t = price at time t

$\phi(P_t)$ = mean of demand function at t = 0, $\phi'(P) > 0$.

e_{dt} = random disturbance term

g_d = demand's rate of growth

It is assumed that e_{dt} is distributed according to a known probability law, represented by $f_d(e_{dt})$, which is a probability mass function (in the case that e_{dt} is a discrete random variable) or a probability density function (if e_{dt} is continuous). It is assumed that $f_d(e_{dt})$ is identical for all t, that e_{dt} has zero mean, and that any two disturbances of two periods, e_{dt} and $e_{dt'}$, are mutually independent.

All the exogenous factors that cause a systematic shifting of the demand function are assumed to be included in g_d (e.g., population and income effects), while all other exogenous determinants of the demand are included in e_d. (For example, if the demand includes a demand for export, the randomness might be due to weather effect on yield abroad, or e_d might be caused by unsystematic changes in prices of complements and substitutes, etc.)

Supply function at time t:

(2.2) $$X_t = [\psi(P_{t-1}) + e_{st}](1 + g_s)^t$$

where: X_t = quantity supplied from production at time t

P_{t-1} = one-period-lagged price at time t

$\psi(P_{t-1})$ = mean of supply function at t = 0

$\psi'(P_{t-1}) \geqq 0$

e_{st} = random disturbance term

g_s = supply's rate of growth

e_{st} is distributed according to a probability law represented by $f_s(e_{st})$, which is a probability mass function or a probability density function (if e_{st} is discrete or continuous, respectively). $f_s(e_{st})$ is identical for all t and for any two periods t t', e_{st}, and $e_{st'}$ are mutually independent. In addition, e_{st} and $e_{dt'}$ are independent for all t t'.

As in the demand case, it is assumed that all of the exogenous factors of supply that are changed systematically with time are included in g_s (e.g., systematic technical change), and all other exogenous determinants of supply are included in the random disturbance e_s (e.g., weather effect on yields).

In summary, it is a simple "cobweb" economy. For convenience, let us use the following notation:

$$\phi_t(P_t) \equiv \phi(P_t)\cdot(1 + g_d)^t$$

$$\psi_t(P_{t-1}) \equiv \psi(P_{t-1})\cdot(1 + g_s)^t$$

$$\varepsilon_{dt} \equiv e_{dt} \cdot (1 + g_d)^t$$

$$\varepsilon_{st} \equiv e_{st} \cdot (1 + g_s)^t$$

and rewrite the demand and supply functions:

$$Y_t = \phi_t(P_t) + \varepsilon_{dt} \tag{2.1'}$$

$$X_t = \psi_t(P_{t-1}) + \varepsilon_{st} \tag{2.2'}$$

Let us denote the stocks at the end of period t by C_t (the beginning stocks at t are C_{t-1}).

Given the lagged price P_{t-1}, and the values of the random disturbances ε_{dt} and ε_{st} at time t, the price is determined by the equilibrium condition:

$$Y_t + C_t - C_{t-1} = X_t \tag{2.3}$$

or

$$\phi_t(P_t) + \varepsilon_{dt} + \Delta c_t = \psi_t(P_{t-1}) + \varepsilon_{st} \tag{2.3'}$$

where: $\Delta c_t \equiv C_t - C_{t-1}$ is the change of stocks at t.

Define the "free market price" to be the equilibrium price when $\Delta c_t = 0$ and denote it by $\tilde{P}_t$.

$\tilde{P}_t$ is defined by:

$$\phi_t(\tilde{P}_t) + \varepsilon_{dt} = \psi_t(P_{t-1}) + \varepsilon_{st}$$

or:

$$\tilde{P}_t = \phi_t^{-1}[\psi_t(P_{t-1}) + \varepsilon_t] \tag{2.4}$$

where: ϕ_t^{-1} is the inverse demand function, $\varepsilon_t \equiv \varepsilon_{st} - \varepsilon_{dt}$ is the combined disturbance of supply and demand. ε_t is distributed according to a probability law represented by $f_t(\varepsilon_t)$ which is derived from $f_d(e_d)$, $f_s(e_s)$, and the definitions of ε_{dt} and ε_{st}.[2/]

Eq. (2.4) describes the "free price" as a function of the lagged price P_{t-1} and the disturbance ε_t. Let us denote this function by $A_t(P_{t-1}, \varepsilon_t)$.

$$\tilde{P}_t = \phi_t^{-1}[\psi_t(P_{t-1}) + \varepsilon_t] \equiv A_t(P_{t-1}, \varepsilon_t) \tag{2.4'}$$

Figure 1 might help to understand equation (2.4). Prices are measured along the vertical axis and quantities along the horizontal one. The mean of production supply at time t is a function of P_{t-1}: Given P_{t-1}, it is constant and is described by the heavy line in figure 1, designated by $\psi_t(P_{t-1})$. Actual production at time t is obtained by adding the supply disturbance ε_{st}--see the light vertical line designated by $\psi_t(P_{t-1}) + \varepsilon_{st}$. Similarly, the mean demand curve at time t is the heavy curve, designated by ϕ_t in figure 1. The actual demand curve is obtained from the latter by adding

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[2/] For example: If $e_d \sim N(0, \sigma_d^2)$ and $e_s \sim N(0, \sigma_s^2)$ then $\varepsilon_t \sim N(0, \sigma_t^2)$, where $\sigma_t^2 \equiv (1 + g_d)^{2t}\sigma_d^2 + (1 + g_s)^{2t}\sigma_s^2$.

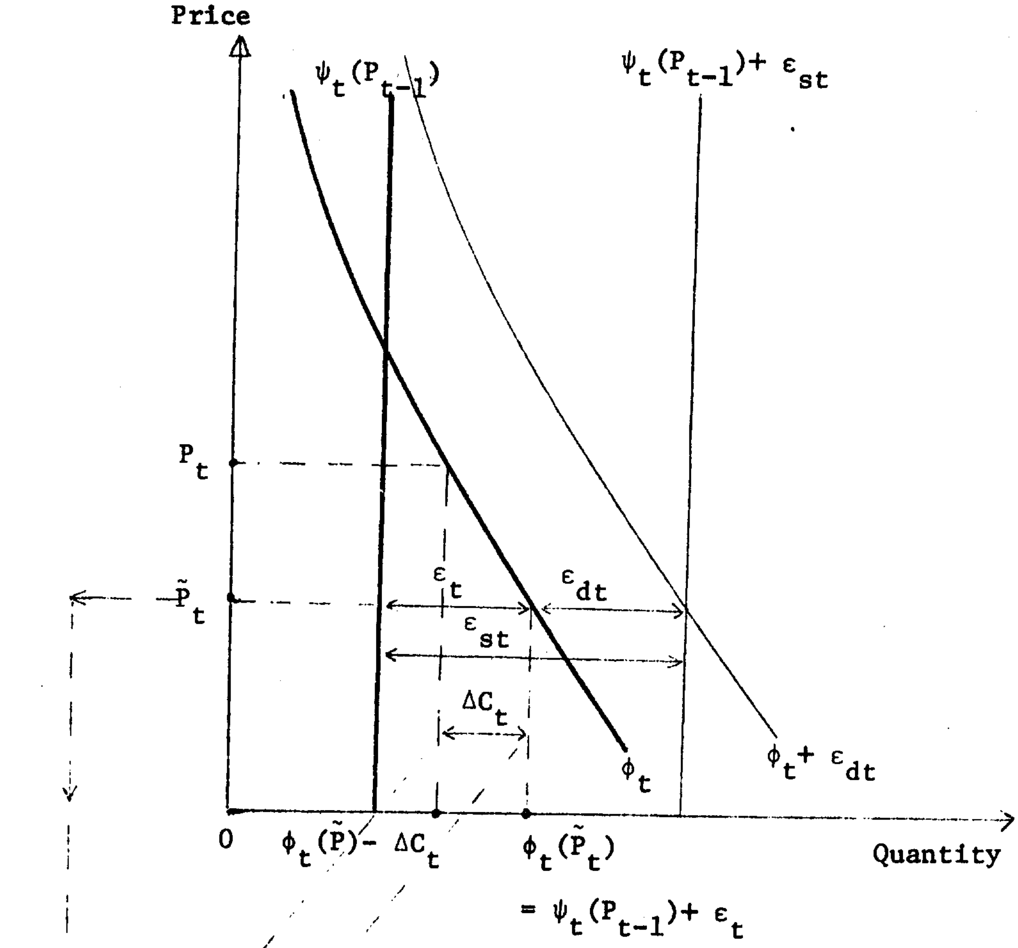


Figure 1

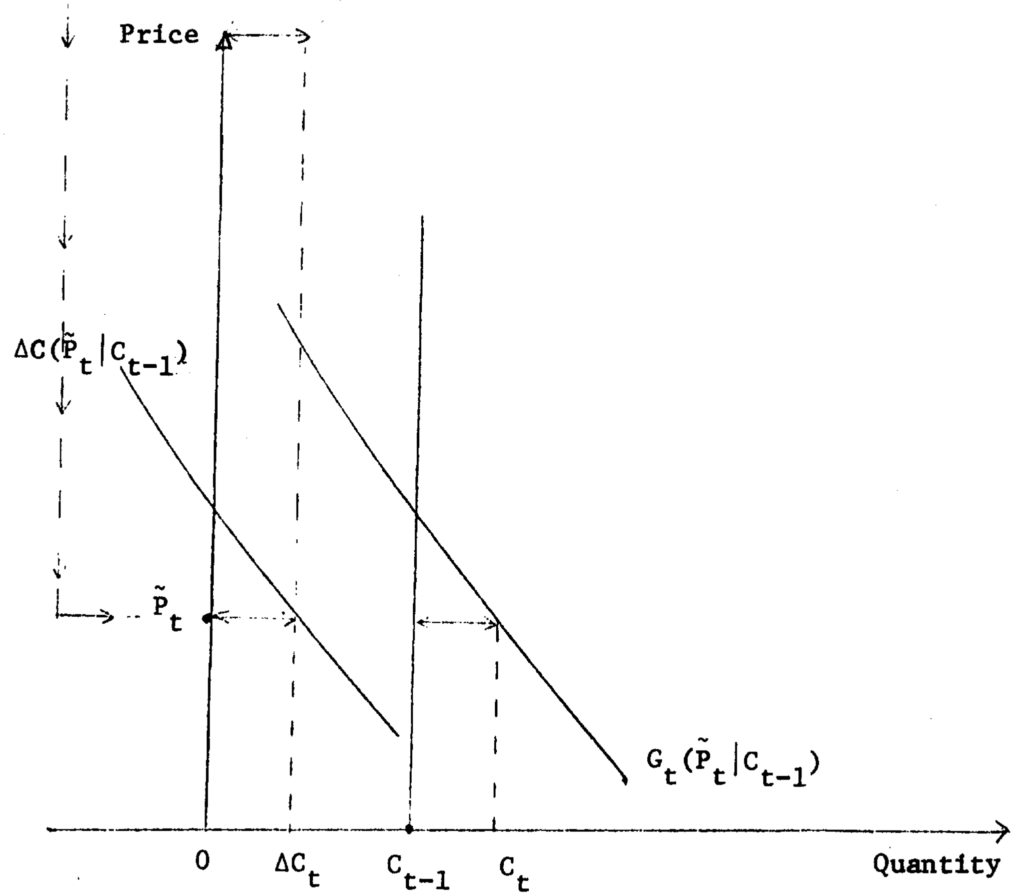


Figure 2

the demand disturbance ε_{dt}--see the light curve designated by $\psi_t + \varepsilon_{dt}$. The "free market price," $\tilde{P}_t$, is determined by the intersection of the actual supply and demand curves. It is clear from the figure that the difference between the mean demand curve and the mean supply curve (i.e., the heavy curves) at price $\tilde{P}_t$ is equal to $\varepsilon_t = \varepsilon_{st} - \varepsilon_{dt}$. Thus, $\tilde{P}_t$ can also be determined by adding ε_t to the mean production $\psi_t(P_{t-1})$ and reading the corresponding price on the mean demand curve ϕ_t (equivalently to ϕ_t^{-1} in equation (2.4)).

Given $\tilde{P}_t$, beginning stocks C_{t-1}, and carryover stocks C_t, the price P_t can be determined by the equilibrium condition (2.3).

Suppose now that a stock rule is to be investigated. Generally this rule describes the quantity of carryout, C_t, which is the policy variable, as a function, say, G_t of the state variables, which are the market price $\tilde{P}_t$ and the beginning stocks C_{t-1}. Formally,

$$C_t = G_t(\tilde{P}_t, C_{t-1}). \tag{2.5}$$

Following this rule let us derive the probability law of the price P_t and the stocks C_t. To do that let us first describe the stocks C_t as a function of P_{t-1}, C_{t-1}, and ε_t:

C_t is a function of C_{t-1} and of $\tilde{P}_t$ (see (2.5)). $\tilde{P}_t$ in turn is a function of P_{t-1} and ε_t(see (2.4')).

Therefore, from (2.4') and (2.5):

$$C_t = G_t[A(P_{t-1}, \varepsilon_t), C_{t-1}] \equiv B_t(P_{t-1}, C_{t-1}, \varepsilon_t) . \tag{2.6}$$

Next we want to express P_t also as a function of these three variables. From (2.3'):

(2.7) $$P_t = \phi_t^{-1}[\psi_t(P_{t-1}) + \varepsilon_t - (C_t - C_{t-1})],$$

and from (2.4) we get:

(2.8) $$\psi_t(P_{t-1}) + \varepsilon_t = \phi_t(\tilde{P}_t) .$$

(This is the mean of the quantity demanded at the "free market price," $\tilde{P}_t$--see figure 1.) Therefore, by (2.5):

(2.9) $$P_t = \phi_t^{-1}\{\phi_t(\tilde{P}_t) - [G_t(\tilde{P}_t, C_{t-1}) - C_{t-1}]\},$$

and by using (2.4') and (2.6):

(2.10) $$P_t = \phi_t^{-1}\{\phi_t[A(P_{t-1}, \varepsilon_t)] - B(P_{t-1}, C_{t-1}, \varepsilon_t) + C_{t-1}\}$$
$$\equiv D(P_{t-1}, C_{t-1}, \varepsilon_t) .$$

Graphically, the stocks rule G_t, given the beginning stocks C_{t-1}, is described by the curve designated by $G_t(\tilde{P}_t|C_{t-1})$ in figure 2, in which prices are measured along the vertical axis and quantities along the horizontal one. The change of stocks as a function of $\tilde{P}_t$, given C_{t-1}, is obtained by the difference between the latter curve and C_{t-1}--see the curve $\Delta C_t(\tilde{P}_t|C_{t-1})$ in figure 2. To determine the price P_t, given P_{t-1}, ε_t, and C_{t-1}, first find $\tilde{P}_t$ in figure 1 as described above. Second, find ΔC_t in figure 2 by using $\tilde{P}_t$ (from figure 1) and C_{t-1}. Return to figure 1, subtract ΔC_t from $\phi_t(\tilde{P}_t)$, and determine P_t by the corresponding point on the mean demand curve ϕ_t.

2.3 Probabilities computations

Assume for a moment that the joint distribution of (P_{t-1}, C_{t-1}) is known and let us derive the joint distribution of (P_t, C_t). Since in the computation procedure the probability distribution $f_t(\varepsilon_t)$ is assumed to be discrete, we describe here only the discrete case. The continuous case is conceptually similar, but the formal notation is much more complicated and will not be presented here. Thus, given the joint probability function of (P_{t-1}, C_{t-1}), i.e.:

$$\text{Prob}\ \{P_{t-1} = p^i,\ C_{t-1} = c^j\} \qquad i = 1, 2, \ldots \qquad j = 1, 2, \ldots$$

and the probability of ε_t, i.e.:

$$\text{Prob}\ \{\varepsilon_t = e^k \qquad k = 1, 2, \ldots ,$$

what is the probability $\text{Prob}\ \{P_t = p,\ C_t = c\}$?

The independence of ε_t and ε_{t-1} implies that ε_t and (P_{t-1}, C_{t-1}) are also independent. Therefore:

$$\text{(2.11)} \quad \text{Prob}\ \{P_t = p,\ C_t = c\}$$

$$= \sum_i \sum_j \sum_k \text{Prob}\ \{P_{t-1} = p^i,\ C_{t-1} = c^j\}\{\text{Prob}\ \varepsilon_t = e^l\}$$

over all i, j, k
such that:

$$B(p^i, c^j, \varepsilon^k) = c$$

$$\text{and } D(p^i, c^j, \varepsilon^k) = p$$

Now, the beginning values of the first period are given since they have already occurred. Suppose that $P_o = p^o$ and $C_o = c^o$ are known; then the probability function of P_o and C_o is also known to be:

$$\text{(2.12)} \qquad \text{Prob}\ \{P_o = p,\ C_o = c\} = \begin{cases} 1 \text{ if } p = p^o \text{ and } c = c^o \\ 0 \text{ otherwise} \end{cases}$$

Applying equation (2.11), the joint probability function of (P_1, C_1) can be computed and from it the probability function of (P_2, C_2), etc.

The sequence of joint probability distributions summarizes the most important information for the analysis of a stocks rule. The (marginal) probability distribution of prices and stocks can be easily calculated by:

$$\text{(2.13)} \qquad \text{Prob } \{P_t = p\} = \sum_{\text{all } j} \text{Prob } \{P_t = p,\ C = c^j\}$$

and

$$\text{(2.14)} \qquad \text{Prob } \{C_t = c\} = \sum_{\text{all } i} \text{Prob } \{P_t = p^i,\ C = c\}$$

and so also the cumulative probability functions:

$$\text{(2.15)} \qquad F_{Pt}(p) \equiv \text{Prob } \{P_t \leq p\} = \sum_{i} \text{Prob } \{P_t = p^i\}$$

over i
such that
$p^i \leq p$.

$$\text{(2.16)} \qquad F_{Ct}(c) \equiv \text{Prob } \{C_t \leq c\} = \sum_{j} \text{Prob } \{C_t = c^j\}$$

such that
$c^j \leq c$.

In particular, by (2.16) one knows the probability of being out of stocks. Having the probability distribution of prices and stocks, one can also compute some summarizing indices such as the mean, the variance, the coefficient of variation, and an instability index around some target price. When a comparison is made between the simulation models mentioned above, one finds that both procedures give the same information; the difference is in the method of deriving it. In the present procedure the basic probability distributions of the disturbances are used explicitly in the computation, even though in an approximative way, while in simulation, repeated samples are drawn from these distributions. Each of the methods can be used to investigate the outcome of some specified buffer stocks policy but does not by itself constitute a rule as a result of the computations. In the next section an optimization model is developed within the framework of the basic model presented here in an attempt to derive a stocks rule which minimizes a price instability index. The model in section 3, which follows, is a simplified example in which the main features of the problem are analyzed and demonstrated. This model is then extended to a more detailed and more realistic model in Part II, together with some empirical experiments.

3. Minimizing Price Instability by Buffer Stocks--A Primitive Example

In the previous section it was argued that a stocks rule should take into account quantity aspects, even if the only goal is price stabilization; that is, in addition to the market price, the quantity of the existing stocks should influence the decision on carryover stocks. Let us demonstrate this by a primitive example that also shows some of the principles on which an optimization model is developed in Part II.

Following the general model of section 2, assume that there are only two periods, t = 1, 2. The demand equations are identical for the two periods and are nonstochastic and linear, i.e.:

$$Y_t = \alpha_o - \alpha P_t \qquad t = 1,\ 2. \tag{3.1}$$

Production of the present period (t = 1) is known and equal to x_1:

$$X_1 = x_1 \tag{3.2}$$

On the other hand, production in year 2 is uncertain. However, its probability function is known and is of the following simple form:

$$\text{Prob }\{X_2 = x\} = \begin{cases} (1-\lambda) & \text{if } x = x_2^{(1)} \\ \lambda & \text{if } x = x_2^{(2)} \\ 0 & \text{otherwise} \end{cases}$$

$$0 < \lambda < 1.$$

The following equations follow from the equilibrium conditions (see (2.3)):

$$P^t = \frac{\alpha_o - X_t}{\alpha} + \frac{\Delta C_t}{\alpha} \qquad t = 1, 2. \tag{3.4}$$

The "free market price," $\tilde{P}_t$, is given by:

$$\tilde{P}_t = \frac{\alpha_o - X_t}{\alpha} \qquad t = 1, 2. \tag{3.5}$$

So:

$$P_t = \tilde{P}_t + \frac{\Delta C_t}{\alpha} \qquad t = 1, 2. \tag{3.6}$$

Using the instability index (1.2), assume that the problem is to find stocks rules that minimize the instability index, given the beginning stocks C_o. Formally the problem is:

Given C_o, find functions

$\hat{C}_1 = G_1(\tilde{P}_1, C_o)$ and $\hat{C}_2 = G_2(\tilde{P}_2, C_1)$ that minimize

$$E[(P_1 - p_1^*)^2 + (P_2 - p_2^*)^2], \tag{3.7}$$

where p_1^* and p_2^* are the target prices and $\hat{C}_1$, $\hat{C}_2$ are the optimal stocks at the end of periods 1 and 2 respectively.

Define $w_t = (P_t - p_t^*)^2$ and rewrite (3.7):

$$I(\{P_t\}) = E(w_1 + w_2). \tag{3.7'}$$

Notice that in order to minimize (3.7), it is necessary that for any given C_1, C_2 will be chosen, such as to

$$\text{minimize } Ew_2 = E(P_2 - p_2^*)^2.$$

By assumption (3.3), using (3.5), the "free market price" may obtain only two possible values, namely:

$$\tilde{P}_2 = \begin{cases} \dfrac{\alpha_o - x_2^{(1)}}{\alpha} \equiv p_2^{(1)} & \text{if } X_2 = x_2^{(1)} \\[2ex] \dfrac{\alpha_o - X_2^{(2)}}{\alpha} \equiv p_2^{(2)} & \text{if } X_2 = x_2^{(2)}. \end{cases} \tag{3.8}$$

Assume that $p_2^{(1)} < p^* < p_2^{(2)}$, see figure 3.

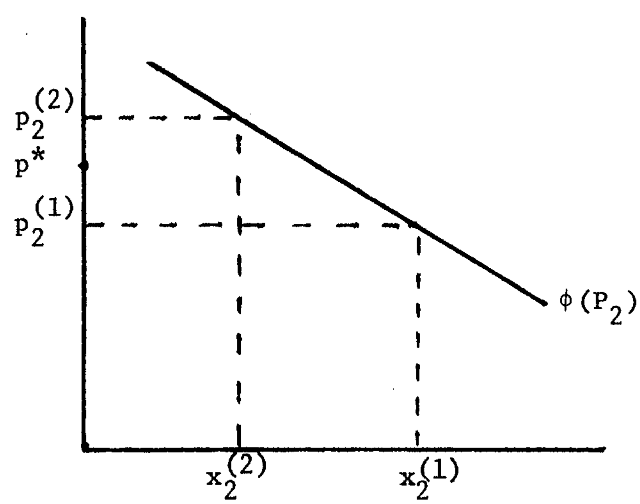


Figure 3.

From the probability function of X_2 (see (3.3)), the probability function of $\tilde{P}_2$ is given by:

$$\text{(3.9)} \qquad \text{Prob}\{\tilde{P}_2 = p\} = \begin{cases} (1-\lambda) & \text{if } p = p_2^{(1)} \\ \lambda & \text{if } p = p_2^{(2)} \\ 0 & \text{otherwise.} \end{cases}$$

It is obvious that in order to minimize $E(P_2 - p_2^*)^2$ one must choose C_2, such that for any given $\tilde{P}_2$, $w_2 = (P_2 - p_2^*)^2$ be minimized, where P_2 is given by (3.6). The rule for $t = 2$ is clearly as follows:

If $\tilde{P}_2 < p_2^*$, increase stocks until $P_2 = p_2^*$.

If $\tilde{P}_2 > p_2^*$, reduce stocks until $P_2 = p_2^*$, if there is enough inventory; if not, reduce stocks to zero. Formally, the optimal rule for $t = 2$ is

$$\text{(3.10)} \quad \hat{C}_2 = G_2(\tilde{P}_2, C_1) = \begin{cases} C_1 + (p_2^* - \tilde{P}_2)\cdot\alpha & \text{if } \tilde{P}_2 \leq p_2^* \\ C_1 + (p_2^* - \tilde{P}_2)\cdot\alpha & \text{if } \tilde{P}_2 \geq p_2^* \text{ and } C_1 \geq (\tilde{P}_2 - p_2^*)\cdot\alpha \\ 0 & \text{if } \tilde{P}_2 \geqq p_2^* \text{ and } C_1 \leqq (\tilde{P}_2 - p_2^*)\cdot\alpha \end{cases}$$

For each possible $\tilde{P}_2$ and given C_1, the optimal carryover for $t = 2$ has been determined. Let us denote by $V_2(\tilde{P}_2, C_1)$, the value of $w_2 = (P_2 - p_2^*)^2$, corresponding to $\tilde{P}_2$ and C_1, when applying $G_2(\tilde{P}_2, C_1)$ from (3.10). Using (3.6):

$$\text{(3.11)} \qquad V_2(\tilde{P}_2, C_1) = [\tilde{P}_2 + \frac{G_2(\tilde{P}_2, C_1) - C_1}{\alpha} - p_2^*]^2.$$

More specifically we have:

$$\text{(3.11')} \quad V_2(\tilde{P}_2, C_1) = \begin{cases} 0 & \text{if } \tilde{P}_2 \leq p_2^* \text{ for any } C_1 \\ 0 & \text{if } \tilde{P}_2 \geqq p_2^* \text{ and } C_1 \geqq \alpha(\tilde{P}_2 - p_2^*) \\ (\tilde{P}_2 - p_2^* - \frac{C_1}{\alpha})^2 & \text{if } \tilde{P}_2 \geqq p_2^* \text{ and } C_1 \leqq \alpha(\tilde{P}_2 - p_2^*) \end{cases}$$

Using the probability function of $\tilde{P}_2$ (3.9), it can be easily shown that EV_2, the expectation of V_2, is a function of C_1, given by:

$$(3.12) \qquad EV_2(C_1) = V_2(p_2^{(1)}, C_1)\cdot(1 - \lambda) + V_2(p_2^{(2)}, C_1)\cdot\lambda$$

and using (3.11'):

$$(3.12') \qquad EV_2(C_1) = \begin{cases} 0 & \text{if } C_1 \geqq \alpha\cdot(p_2^{(2)} - p_2^*) \\ \lambda(p_2^{(2)} - p_2^* - \frac{C_1}{\alpha})^2 & \text{if } C_1 \leqq \alpha\cdot(p_2^{(2)} - p_2^*) \end{cases}$$

Graphically $EV_2(C_1)$ is depicted in figure 4:

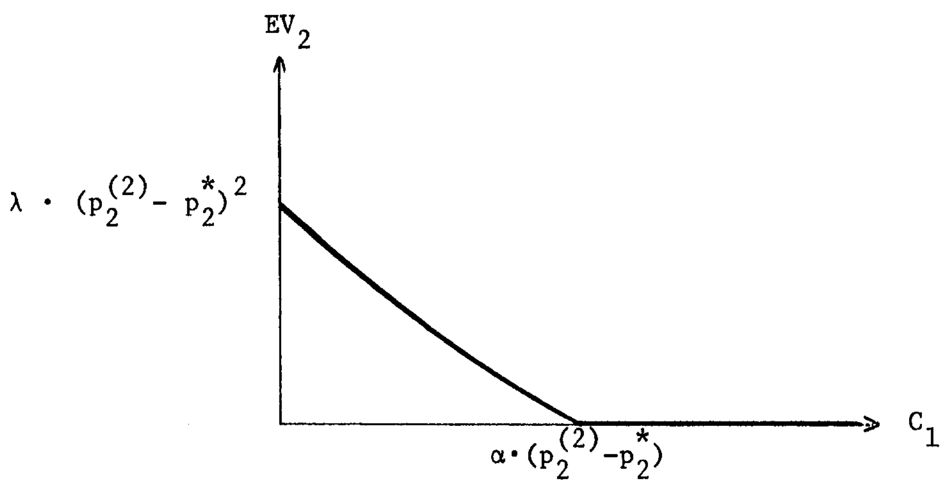


Figure 4.

Note that the maximal quantity of stocks which is needed to stabilize the price of period 2 is $\alpha(p_2^{(2)} - p_2^*)$. Equation (3.12') and figure 4 show that as long as the beginning stocks of t = 2, i.e., C_1, is smaller than $\alpha(p_2^{(2)} - p_2^*)$, EV_2 is positive and decreases with C_1, and EV_2 is zero for C_1, which is greater than or equal to $\alpha(p_2^{(2)} - p_2^*)$. This indicates that the more stocks that are held from period 1, the less instability is expected in period 2. Formally this is expressed by the negativity of the derivative for $C_1 < \alpha(p_2^{(2)} - p_2^*)$:

$$(3.13)\quad 0 \geqq \frac{\partial EV_2(C_1)}{\partial C_1} = \begin{cases} 0 & \text{if } C_1 \geq \alpha(p_2^{(2)} - p_2^*) \\ \frac{2\lambda}{\alpha^2}[C_1 - \alpha(p_2^{(2)} - p_2^*)] & \text{if } C_1 \leq \alpha(p_2^{(2)} - p_2^*) . \end{cases}$$

Let us return now to the original problem and search the optimal carryover rule for $t = 1$, i.e., $\hat{C}_1 = G_1(\tilde{P}_1, C_o)$. It will be shown that the same rule which was applied to $t = 2$ is no more optimal for $t = 1$, and that sometimes it is worthwhile to carry over more than is needed to equate P_1 to p_1^*. The difference between the two periods is that under the assumptions, one does not care what might happen after $t = 2$, thus C_2 is valueless. On the other hand, C_1 is valuable because of its ability to reduce future instability, which is the objective of the problem.

Knowing $G_2(\tilde{P}_2, C_1)$, the problem has been reduced now to the following:

Find a nonnegative function $\hat{C}_1 = G_1(\tilde{P}_1, C_o)$, that minimizes

$$(3.14)\qquad I(\{P_t\}) = E(P_1 - p_1^*)^2 + E(P_2 - p_2^*)^2.$$

At time $t = 1$, $\tilde{P}_1$ is assumed to be known and is not stochastic any more. Therefore, the expectation sign can be taken off from the first term of (3.14), which can be written by:

$$(3.14')\qquad I(\{P_t\}) = w_1 + EV_2(C_1).$$

Using (3.6), w_1 can be written in terms of ΔC_1 and $\tilde{P}_1$:

$$(3.15)\qquad w_1 = (P_1 - p_1^*)^2 = \left(\tilde{P}_1 - p_1^* + \frac{C_1 - C_o}{\alpha}\right)^2 .$$

A simple marginal analysis results in the following rule: Increase C_1 as long as the marginal change in $I(\{P_t\})$ is negative, and reduce C_1 whenever the marginal change in $I(\{P_t\})$ is positive; that is:

$$\text{(3.16)} \qquad \text{If } \frac{\partial I}{\partial C_1} < 0 \quad \text{increase } C_1 \text{ and}$$

$$\text{If } \frac{\partial I}{\partial C_1} > 0 \quad \text{reduce } C_1 \text{ if } C_1 \neq 0.$$

Now, $\frac{\partial I}{\partial C_1} = \frac{\partial w_1}{\partial C_1} + \frac{\partial EV_2}{\partial C_1}$, hence (3.16) can be equivalently written:

$$\text{(3.16')} \qquad \text{If } -\frac{\partial EV_2}{\partial C_1} > \frac{\partial w_1}{\partial C_1} \quad \text{increase } C_1 \text{ and}$$

$$\text{if } -\frac{\partial EV_2}{\partial C_1} < \frac{\partial w_1}{\partial C_1} \quad \text{reduce } C_1 \text{ if } C_1 \neq 0.$$

The details of $\frac{\partial EV_2}{\partial C_1}$ were given in eq. (3.13). $\frac{\partial w_1}{\partial C_1}$ is obtained by differentiation of (3.15):

$$\text{(3.17)} \qquad \frac{\partial w_1}{\partial C_1} = \frac{2}{\alpha^2}\,[C_1 - C_o - \alpha(p_1^* - \tilde{P}_1)] \begin{cases} > 0 & \text{if } C_1 - C_o > \alpha(p_1^* - \tilde{P}_1) \\ = 0 & \text{if } C_1 - C_o = \alpha(p_1^* - \tilde{P}_1) \\ < 0 & \text{if } C_1 - C_o < \alpha(p_1^* - \tilde{P}_1) \end{cases}$$

$\frac{\partial w_1}{\partial C_1}$ is the marginal change of instability in period 1 due to a marginal change in C_1, or the "marginal destabilization of C_1 in period 1", to be denoted by MW. If it is positive (negative) this means that further increase of C_1 will lead to greater (smaller) instability in period 1. Similarly, $\frac{\partial EV_2}{\partial C_1}$ is the marginal change of instability in period 2 due to a marginal change in C_1, or the "marginal destabilization of C_1 in period 2." The negative of the marginal destabilization in period 2, i.e., $-\frac{\partial EV_2}{\partial C_1}$, will be called the "marginal stabilization of C_1 in period 2" and will be denoted by $|MEV_2|$. If $|MEV_2|$ is positive it means that an increase in C_1 will lead to a greater stability in period 2. In this terminology (3.16') states as follows: Increase C_1 as long as the marginal stabilization of C_1 in period 2 is greater than the marginal destabilization of C_1 in period 1, and reduce C_1 in the opposite case if C_1 is not zero.

It follows that if $\hat{C}_1$ is positive, it is necessary[3/] that the marginal destabilization of C_1 in period 1 will be equal to the marginal stabilization of C_1 in period 2, i.e.,

(3.19) $$|MEV_2| = MW_1, \text{ if } \hat{C}_1 > 0.$$

Three cases can be distinguished in the following table.

$\hat{C}_1$ \ $\lvert MEV_2 \rvert$	= 0	> 0
> 0	case (1)	case (2)
= 0	impossible	case (3)

Let us analyze each case.

[3/]This condition is necessary and sufficient for global minimums at $\hat{C}_1$, since I is strictly convex in $\hat{C}_1$, which follows from the positivity of $\frac{\partial^2 I}{\partial C_1^2}$.

<u>Case (1): $\hat{C}_1 > 0$, $|MEV_2| = 0$.</u> In this case it is optimal to carry over a large enough quantity of stocks that the marginal stabilization of C_1 in period 2 is zero, i.e. (see 3.13):

$$\hat{C}_1 \geqq \alpha(p_2^{(2)} - p_2^*) \ .$$

It follows that the only consideration in the determination of $\hat{C}_1$ is its effect on period 1. Hence, to be optimal the change of stocks should be such that the marginal destabilization of C_1 in period 1, MW_1, be zero, i.e. (see 3.17):

$$\Delta\hat{C}_1 = \hat{C}_1 - C_o = \alpha \cdot (p_1^* - \tilde{P}_1) \text{ in case (1).} \tag{3.20}$$

$\alpha(p_1^* - \tilde{P}_1)$ is exactly the change of stocks ΔC_1 which is needed to equate the price P_1 to the target price p_1^* if the free market price is $\tilde{P}_1$. Note that in this case the change of stocks depends only on $\tilde{P}_1$.

Graphically, case (1) is described in figure 5, case (1), in which C_1 is measured along the horizontal axis. Given C_o and $\tilde{P}_1$, the marginal curves MW_1, $|MEV_2|$, and $MI = MW_1 - |MEV_2|$ are depicted in the upper section and the total curves W_1, EV_2, and $I = W_1 + EV_2$ are depicted in the lower section. The optimal $\hat{C}_1$ is the minimum point of I (lower section). At this point MI = 0 (in the upper section). In case (1) the MI and I curves coincide with MW_1 and W_1 curves respectively in the neighborhood of $\hat{C}_1$. Hence the same C_1 is optimal also to the problem without period 2.

It is possible to define a range in the $(\tilde{P}_1, C_o)$ plane which corresponds to case (1): From (3.13), $|MEV_2| = 0$ implies $C_1 \geqq \alpha(p_2^{(2)} - p_2^*)$, and from (3.20) $C_1 = C_o + \alpha(p_1^* - \tilde{P}_1)$. It follows that

(3.21) $$C_o \geqq \alpha(\tilde{P}_1 - p_1^*) + \alpha(p_2^{(2)} - p_2^*) \text{ in case (1).}$$

See figure 6.

Case (2): $\hat{C}_1 > 0,\ |MEV_2| > 0$. In this case, at the margin, the carryover C_1 has an effect on period 2. Hence it is worthwhile to hold more stocks than would be held had only period 1 been taken into account. The optimal carryover is derived by equating the marginal destabilization of C_1 in period 1, MW_1 (from (3.17)) to the marginal stabilization of C_1 in period 2, MEV_2 (from (3.13)), that is:

(3.22) $$MW_1 = \frac{2}{\alpha^2}[\hat{C}_1 - C_o - \alpha(p_1^* - \tilde{P}_1)] = \frac{-2\lambda}{\alpha^2}[\hat{C}_1 - \alpha(p_2^{(2)} - p^*)] = |MEV_2|$$

Hence:

(3.23) $$\hat{C}_1 = \frac{1}{1+\lambda}[C_o + \alpha(p_1^* - \tilde{P}_1) + \lambda\alpha(p_2^{(2)} - p_2^*)] \text{ in case (2)}$$

or:

(3.23') $$\hat{C}_1 = [C_o + \alpha(p_1^* - \tilde{P}_1)] + \frac{\lambda}{1+\lambda}[\alpha(p_2^{(2)} - p_2^*) - C_o - \alpha(p_1^* - \tilde{P}_1)]$$

in case (2).

Define:

(3.24) $$A = \frac{\lambda}{1+\lambda}[\alpha(p_2^{(2)} - p_2^*) - C_o - \alpha(p_1^* - \tilde{P}_1)]$$

Equation (3.22) and the condition $|MEV_2| > 0$ imply that

$$\hat{C}_1 > C_o + \alpha(p_1^* - \tilde{P}_1)$$

i.e., the optimal carryover $\hat{C}_1$ is greater than the amount that would be optimal if there were no period 2. The addition (see 3.23') is A, defined by (3.24). The change of stocks is given by:

(3.25) $$\Delta\hat{C}_1 = \hat{C}_1 - \hat{C}_o = \alpha(p_1^* - \tilde{P}_1) + A$$

(3.25) shows that the optimal change of stocks is equal to the change needed to equate P_1 to p_1^* (that is $\alpha(p_1^* - P_1)$), plus the positive term A. A is the additional stocks due to the effect on period 2. Notice that the first part $[\alpha(p_1^* - \tilde{P}_1)]$ depends only on $\tilde{P}_1$. However, A depends on $\tilde{P}_1$ as well as on C_o. In particular, the effect of C_o on A is negative, which means that smaller quantities of C_o lead to greater accumulation of $\Delta\hat{C}_1$. The interpretation is as follows: The effect of C_1 on period 2 is realized only when $\tilde{P}_2$ is greater than the target price p_2^*, that is, in our example only if it happened that $\tilde{P}_2 = p_2^{(2)}$. In this case a quantity of $\alpha(p_2^{(2)} - p_2^*)$ is needed to reduce the price to p_2^*. However, a quantity of $C_o + \alpha(p_1^* - \tilde{P}_1)$ is secured due to considerations for period 1. The addition is $[\alpha(p_2^{(2)} - p_2^*) - C_o - \alpha(p_1^* - \tilde{P}_1)]$. This addition is weighted in (3.24) by $\frac{\lambda}{1+\lambda}$.

It is worthwhile to note that generally in the context of price stabilization by buffer stocks, the more unstable future price is, the more valuable are the stocks; hence it pays to carry over more stocks. In our primitive example, future price instability is measured by the variance around the target price, which is

$$\lambda(p_2^{(2)} - p_2^*)^2 + (1 - \lambda)(p_2^{(1)} - p_2^*)^2 = (p_2^{(2)} - p_2^*)^2 .$$

(It was assumed that $|p_2^{(2)} - p_2^*| = |p_2^{(1)} - p_2^*|$). This explains why $\Delta\hat{C}_1$ is greater when $p_2^{(2)}$ is higher. In addition, it was stressed that the effect of stocks on future instability is realized when prices are above the target price. The more biased the probability distribution of market prices is toward higher prices in period 2, the more valuable will be the stocks from period 1 and greater carryover will be optimal. In the present primitive example, the bias of the probability distribution of market prices toward higher prices is expressed by the probability of being above p_2^*, i.e., by λ. It can be seen (3.24) that the greater λ is, the greater is A, hence, $\Delta\hat{C}_1$.

Case (2) is demonstrated graphically in figure 5 case (2). In this case, the curves of marginal stabilization in period 2, $|MEV_2|$ and marginal destabilization in period 1, MW_1 intersect above zero (upper section of the figure). The minimum point is at $\hat{C}_1$. The stocks which would be held, had only period 1 existed (ignoring the effect on period 2), are at C_1^o in which W_1 is minimized and $MW_1 = 0$. (This is the level of stocks which leads to $P_1 = p_1^*$).

Let us define the range in $(\tilde{P}_1, C_o)$ plane in which case (2) exists: From (3.13), $|MEV_2| > 0$ implies $C_1 \leq \alpha(p_2^{(2)} - p_2^*)$ and from (3.23), $C_1 = \frac{1}{1+\lambda}[C_o + (p_1^* - \tilde{P}_1) + \lambda\alpha(p_2^{(2)} - p_2^*)]$. It follows that

$$(3.25a) \qquad C_o \leqq \alpha(\tilde{P}_1 - p_1^*) + \alpha(p_2^{(2)} - p_2^*) \text{ in case (2).}$$

In addition, $\hat{C}_1 \geq 0$, hence

$$(3.25b) \qquad C_o \geqq \alpha(\tilde{P}_1 - p_1^*) - \lambda\alpha(p_2^{(2)} - p_2^*) \text{ in case (2).}$$

See figure 6.

<u>Case (3): $\hat{C}_1 = 0$, $|MEV| > 0$</u>. In this case, for a given beginning stocks C_o the free market price $\tilde{P}_1$ is so high that even if all the stocks C_o are withdrawn to the market, still the marginal destabilization of C_1 in period 1, MW_1, is greater than the marginal stabilization of C_1 in period 2, $|MEV_2|$. It is optimal to reduce carryover to zero. This is demonstrated in figure 5, case (3). The range in $(\tilde{P}_1, C_o)$ plane which

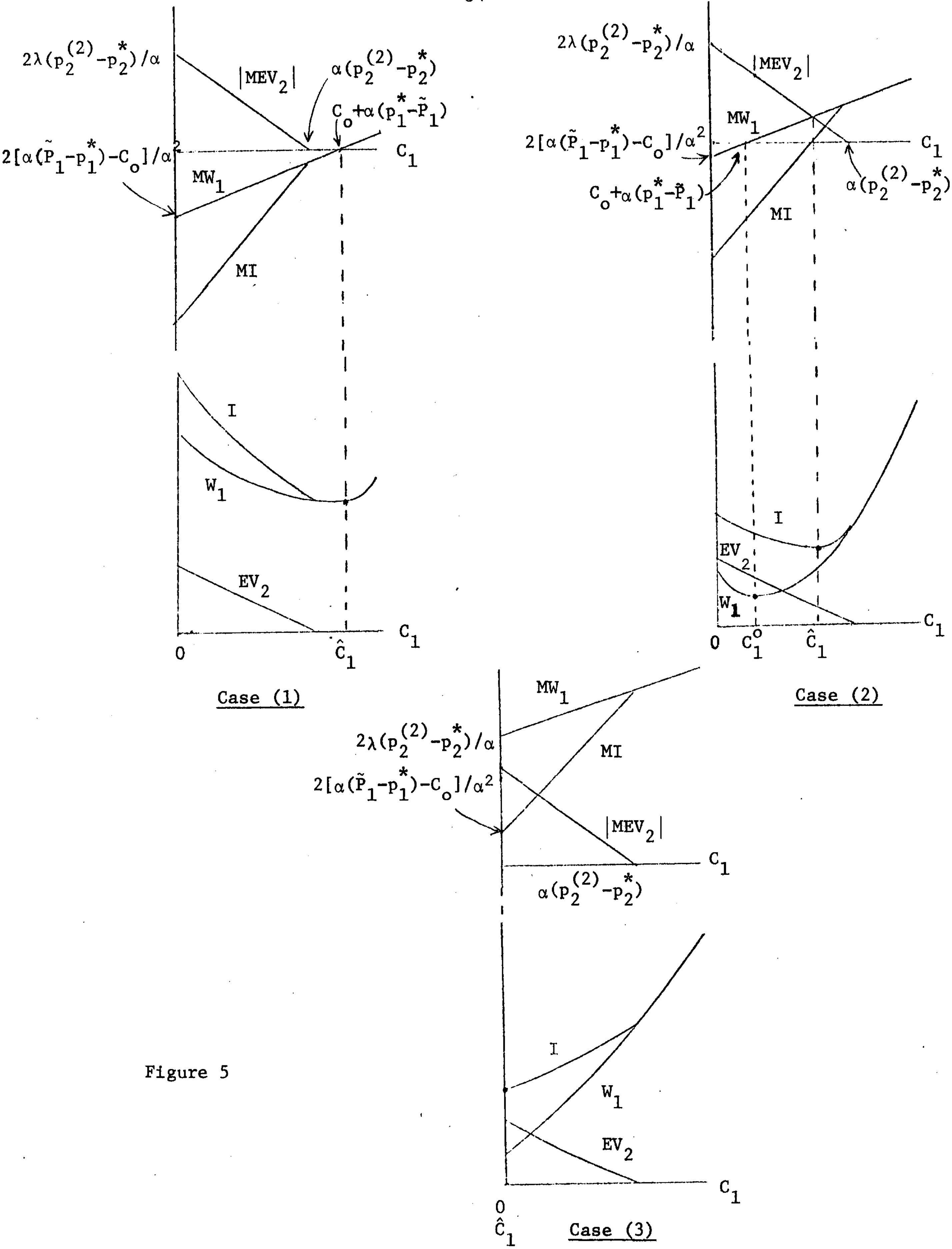

$2\lambda(p_2^{(2)}-p_2^*)/\alpha$
$|MEV_2|$
$\alpha(p_2^{(2)}-p_2^*)$
$C_o+\alpha(p_1^*-\tilde{P}_1)$
$2[\alpha(\tilde{P}_1-p_1^*)-C_o]/\alpha^2$
C_1
MW_1
MI
I
W_1
EV_2
0
$\hat{C}_1$
Case (1)
$2\lambda(p_2^{(2)}-p_2^*)/\alpha$
$|MEV_2|$
MW_1
$2[\alpha(\tilde{P}_1-p_1^*)-C_o]/\alpha^2$
C_1
$C_o+\alpha(p_1^*-\tilde{P}_1)$
$\alpha(p_2^{(2)}-p_2^*)$
MI
I
EV_2
W_1
0
C_1^o
$\hat{C}_1$
Case (2)
MW_1
$2\lambda(p_2^{(2)}-p_2^*)/\alpha$
MI
$2[\alpha(\tilde{P}_1-p_1^*)-C_o]/\alpha^2$
$|MEV_2|$
C_1
$\alpha(p_2^{(2)}-p_2^*)$
I
W_1
EV_2
0
$\hat{C}_1$
Case (3)

Figure 5

corresponds to case (3) can be derived by the complement of (3.25b), i.e.,

$$(3.26) \qquad C_o \leq \alpha(\tilde{P}_1 - p_1^*) - \lambda\alpha(p_2^{(2)} - p_2^*) \text{ in case (3).}$$

See figure 6.

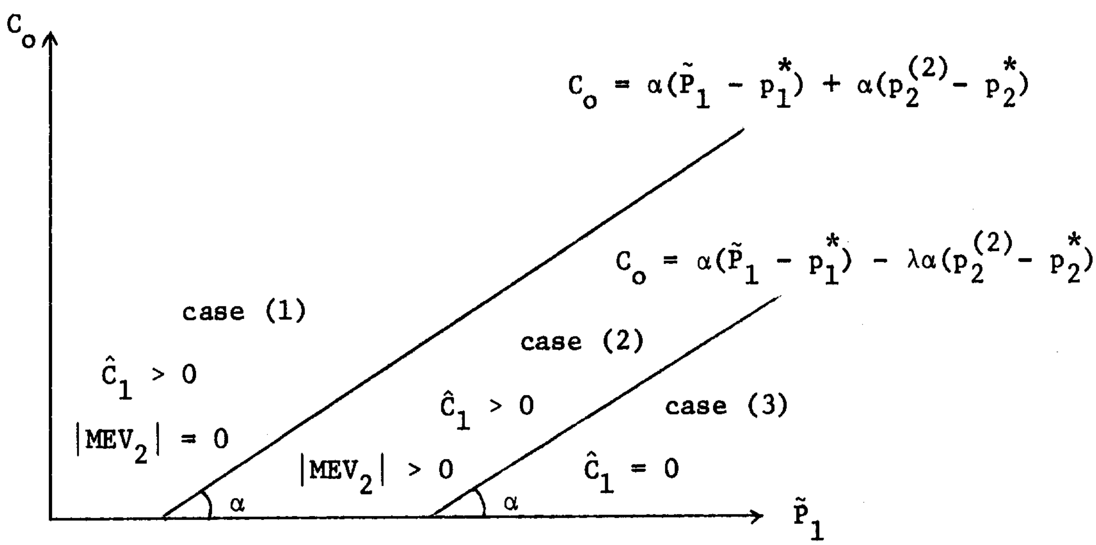


Figure 6

To summarize the discussion on the example let us write the optimal carryover rule of the three cases in the following compact formula.

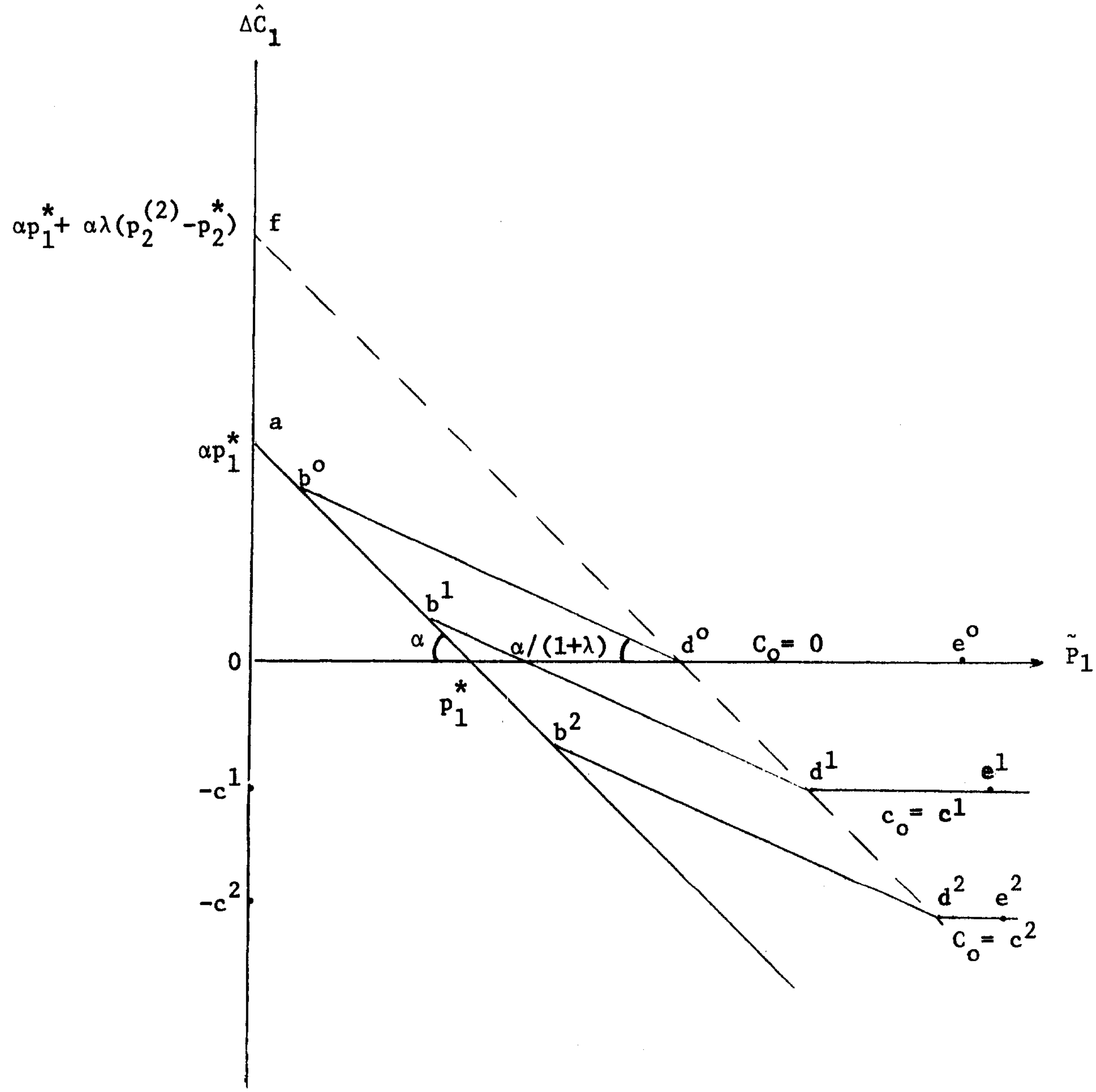
$$\hat{C}_1 = G_1(\tilde{P}_1, C_o)$$

$$(3.27) = \begin{cases} C_o + \alpha(p_1^* - \tilde{P}_1) & \text{if: } C_o \geq \alpha(\tilde{P}_1 - p_1^*) + \alpha(p_2^{(2)} - p_2^*) \\ C_o + \alpha(p_1^* - \tilde{P}_1) + \frac{\lambda}{1+\lambda}\{\alpha(p_2^{(2)} - p_2^*) - [C_o + \alpha(p_1^* - \tilde{P}_1)]\} & \\ \qquad \text{if: } \alpha(\tilde{P}_1 - p_1^*) - \lambda\alpha(p_2^{(2)} - p_2^*) < C_o < \alpha(P_1 - p_1^*) + \alpha(P_2^{(2)} - p_2^*) & \\ 0 \qquad \text{if: } \alpha(\tilde{P}_1 - p_1^*) - \lambda\alpha(p_2^{(2)} - p_2^*) \geq C_o & \end{cases}$$

and in terms of change of stocks:

$$
(3.28)\quad \Delta\hat{C}_1 = \begin{cases} \alpha(p_1^* - \tilde{P}_1) & \text{if: } C_o \geq \alpha(\tilde{P}_1 - p_1^*) + \alpha(p_2^{(2)} - p_2^*) \\ \alpha(p_1^* - \tilde{P}_1) + \frac{\lambda}{1+\lambda}\{\alpha(p_2^{(2)} - p_2^*) - [C_o + \alpha(p_1^* - P_1)]\} & \\ \qquad \text{if: } (\tilde{P}_1 - p_1^*) - \lambda\alpha(p_2^{(2)} - p_2^*) < C_o < \alpha(\tilde{P}_1 - p_1^*) + \alpha(p_2^{(2)} - p_2^*) & \\ 0 \quad \text{if: } \alpha(\tilde{P}_1 - p_1^*) - \lambda\alpha(p_2^{(2)} - p_2^*) \geq C_o & \end{cases}
$$

Figure 7 demonstrates (3.28) graphically: The optimal change of stocks $\Delta\hat{C}_1$ is measured along the vertical axis and $\tilde{P}_1$ is measured along the horizontal one. For a given beginning stocks C_o, $\Delta\hat{C}_1$ is a function of $\tilde{P}_1$ described by (3.28). For example, if $C_o = 0$ the curve a b^o d^o e^o describes the optimal accumulation $\Delta\hat{C}_1$ for different values of $\tilde{P}_1$. ab^o corresponds to case (1), b^od^o corresponds to case (2), and d^oe^o to case (3), in which $\Delta C_1 = - C_o$. Two other optimal curves corresponding to two beginning stocks, $C_o = c_o^1$ and $C_o = c_o^2$ respectively, are drawn in the figure, i.e.: $ab^1d^1e^1$ and $ab^2d^2e^2$ respectively. Again, the ab part corresponds to case (1), the bd to case (2) and de to case (3). The line $ab^ob^1b^2$, etc. . . . represents the carryover rule which does not take into account the effects on period 2. The portion of case (1) coincides with it. However, given the beginning stocks C_o, if $\tilde{P}_1$ is greater than some P the optimal quantity of carryover is greater than implied by the former curve. This is case (2). As $\tilde{P}_1$ obtains higher and higher values, the change of stocks $\Delta\hat{C}_1$ decreases until all the beginning stocks are reduced to zero. From this point on, carryover $\hat{C}_1$ is zero and the change of stocks remains equal to $- C_o$. This is case (3).



$\Delta\hat{C}_1$
$\alpha p_1^* + \alpha\lambda(p_2^{(2)} - p_2^*)$
f
a
αp_1^*
b^o
b^1
α
$\alpha/(1+\lambda)$
d^o
$C_o = 0$
e^o
$\tilde{P}_1$
0
p_1^*
b^2
$-c^1$
d^1
e^1
$c_o = c^1$
$-c^2$
d^2
e^2
$C_o = c^2$

Figure 7

Notice that generally the optimal change of stocks depends not only on the price $\tilde{P}_1$ but also on the beginning stocks C_o.

One can also see from figure 7 the implications of greater future instability, that is, of higher $p_2^{(2)}$, and of greater bias of the probability distribution of future prices toward higher prices, that is of greater λ. Both of them imply wider range of case (2) and greater carry-over $\hat{C}_1$ for each C_o and $\tilde{P}_1$.

Part I

REFERENCES

[1] Bailey, W. R.; Kutish, F. A.; and Rojko, A. S. Grain Stocks Issues and Alternatives--A Progress Report. Economic Research Service, U. S. Department of Agriculture, Washington, D. C. 1974.

[2] Cochrane, W. W. Feast or Famine: The Uncertain World of Food and Agriculture and Its Policy Implications for the United States. Report No. 136, National Planning Association, Washington, D. C. 1974.

[3] Oi, W. Y. "The Desirability of Price Instability under Perfect Competition." Econometrica, 29:58-64. 1961.

[4] Samuelson, P. A. "The Consumer Does Benefit from Feasible Price Stability." Quarterly Journal of Economics, Vol. 86, No. 3, August 1972, pp. 476-493.

[5] Sharples, J. A. and Walker, R. Reserve Stocks of Grain. Research Status Report No. 1, Description of the Price Bounds Model. Economic Research Service, U. S. Department of Agriculture and Department of Agricultural Economics, Purdue University (unpublished). 1974.

[6] Subotnik, A. and Houck, J. P. "An Economic Analysis of Stabilizing Schemes," Staff Paper No. P75-7. Department of Agricultural and Applied Economics, University of Minnesota. 1975.

[7] Turnovsky, S. J. "Price Expectations and the Welfare Gains from Price Stabilization." American Journal of Agricultural Economics, 56(2), November 1974, pp. 706-716.

[8] Tweeten, L.; Kalbfleisch, D.; and Lu, Y. C. An Economic Analysis of Carryover Policies for the United States Wheat Industry. Technical Bulletin T-132, Agricultural Experiment Station, Oklahoma State University. 1971.

[9] Waugh, F. V. "Does the Consumer Benefit from Price Instability?" Quarterly Journal of Economics, 86: 351-365. 1944.

[10] Waugh, F. V. "Consumer Aspects of Price Instability." Econometrica, 29(1):58-64. 1961.

[11] Waugh F. V. "Reserve Stocks of Farm Products." Agricultural Policy: A Review of Programs and Needs, Vol. V. National Advisory Commission on Food and Fiber, Washington, D. C., pp. 3-52. 1967.

Part II: A Model of Optimal Buffer Stocks for Price Stabilization--Theory and Computation

In part I of this paper, the concept of price instability was discussed and an index of price instability in the context of a stochastic world was introduced. This index was defined to be the mean of the sum of square deviations of a series of prices from a series of "target" or "reference" prices (see (1.4) in part I). In addition, a model of price and stock relations within the framework of a simple demand and supply system was presented (section 2.2 in part I). A primitive example was constructed to demonstrate how an optimal buffer stocks model can be applied to the problem of minimizing the price instability index. In part II, a more general case is analyzed and demonstrated. Section 1 presents an optimal buffer stocks model for price stabilization and analyzes some of its characteristics. A summary of a computational procedure is the content of section 2. Detailed computations and a computer program are given in the appendix in a technical form. In section 3, two empirical experiments with the model demonstrate how it can be applied. Finally, section 4 concludes the paper with some comments and recommendations for further research.

1. A Model of Optimal Buffer Stocks for Price Stabilization

In part I, an instability index of a series of future prices $\{P_t\}_1^T$ was defined by (1.4), which is rewritten here:

$$(1.1)\quad I\left(\{P_t\}_1^T\right) \equiv E\left[\frac{1}{T}\sum_{t=1}^{T}\left(\frac{P_t - p_t^*}{p_t^*}\right)^2 \frac{1}{(1+\delta)^t}\right]$$

where t=1, 2..., T is a sequence of periods from 1 to T, $\{p_t^*\}_1^T$ is a series of "target" or "reference" prices, and δ is a discount factor.

In this section we analyze the possibility of reducing future price instability (measured by the instability index (1.1)) by an intervention of a stocks activity in the market of the commodity under discussion. The analysis will be undertaken within the framework of the single commodity model which was discussed in section 2.2 of part I. The basic assumptions will be summarized in 1.1 for convenience.

1.1 <u>The model</u>.

Demand:

$$(1.2)\quad Y_t = \phi_t(P_t) + \varepsilon_{dt} \qquad t = 1, 2, \ldots, T$$

where Y_t = quantity demanded at time t

P_t = price at time t

$\phi_t(P_t)$ = mean of demand function at time t

and ε_{dt} is a stochastic disturbance at time t.

It is assumed that

$$(1.3)\quad \phi_t(P_t) = \phi(P_t)(1 + g_d)^t$$

where $\phi(P_t)$ is the mean of demand function at t=0, and g_d is a (constant) rate of demand growth.

Also (1.4) $\varepsilon_{dt} = e_{dt}(1 + g_d)^t$

where e_{dt} is distributed according to a known probability law represented by $f_d(e_{dt})$, which is a probability density or mass function (in the case that e is continuous or discrete respectively).

Supply:

(1.5) $X_t = \psi_t(P_{t-1}) + \varepsilon_{st}$ $\qquad t=1, 2, \ldots, T$

where X_t = quantity supplied at time t

$\psi_t(P_{t-1})$ = mean of supply function at time t

ε_{st} = stochastic disturbance at time t.

It is assumed that

(1.6) $\psi_t(P_{t-1}) = \psi(P_{t-1})(1 + g_s)^t$

where $\psi(P_{t-1})$ is the mean of supply function at t=0 and g_s is the rate of supply growth.

Also: (1.7) $\varepsilon_{st} = e_{st}(1 + g)^t$

where e_{st} is distributed according to a probability density or mass function $f_s(e_{st})$.

It is assumed that ε_{st}, $\varepsilon_{st'}$, ε_{dt}, $\varepsilon_{dt'}$ are mutually independent for any t and t'.

It is assumed that price is determined by the equilibrium condition:

(1.8) $Y_t + C_t - C_{t-1} = X_t$

where C_t is the stocks held at time t, and by (1.2) and (1.5):

(1.9) $P_t = \phi_t^{-1}[\psi_t(P_{t-1}) + \varepsilon_t - \Delta C_t]$

where ϕ_t^{-1} is the inverse demand function at time t, $\varepsilon_t \equiv \varepsilon_{st} - \varepsilon_{dt}$, and $\Delta C_t \equiv C_t - C_{t-1}$. The "free market" price (or briefly the "market price") is denoted by $\tilde{P}_t$ and defined by

$$\tilde{P}_t = \phi_t^{-1}[\psi_t(P_{t-1}) + \varepsilon_t] \tag{1.10}$$

From (1.9) and (1.10) it follows that:

$$P_t = \phi_t^{-1}[\phi_t(\tilde{P}_t) - \Delta C_t]. \tag{1.11}$$

(1.11) is the basic relation that exists between the final price P_t on the one hand and the market price $(\tilde{P}_t)$, the beginning stocks (C_{t-1}), and the final stocks (C_t) on the other hand.

1.2 The minimization problem. Suppose that the parameters of the model presented in 1.1 are known (this includes the parameters of the supply and demand functions as well as the parameters of the probability distribution functions of ε_t). The objective is to reduce future price instability by an intervention of a stocks activity. The general idea is, of course, to acquire stocks when prices are low and sell stocks when prices are high. However, the possibility of selling is limited to the amount of stocks held at that time, which is the result of previous decisions. Hence, at each time the question is how much to sell or buy to affect the current price in a desired direction as well as to maintain adequate stocks for future contingencies. One also has to take into account that there are costs connected with holding stocks, so in general there is a substitution between cost and price stability. However, the proper rate of substitution between the two is a subjective matter. Nevertheless, to be efficient, cost should be minimized for any given degree of instability (measured by the instability index (4.1)) and vice versa; for any level of cost, instability should be minimized.

Assume that the cost of holding a unit of stocks per unit of time is constant and equal to θ. Assume also that the subjective rate of substitution of instability (index) for storage cost is λ. Then a criterion for a stocks activity might be to minimize

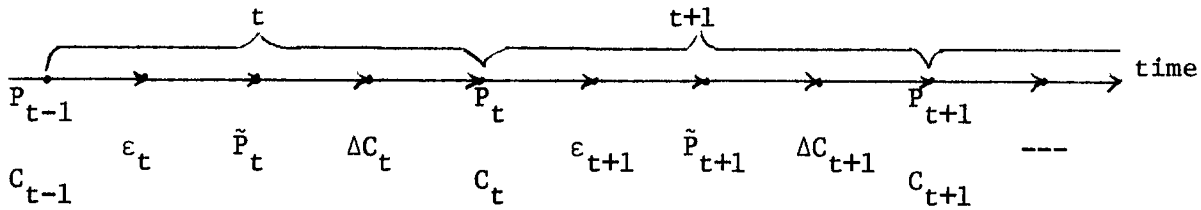
$$(1.12) \qquad E\{\frac{1}{T}\sum_{t=1}^{T}\left[\left(\frac{P_t - p_t^*}{p_t^*}\right)^2 \cdot \lambda + \theta \cdot C_t\right]\frac{1}{(1+\delta)^t}\}$$

$$= I(\{P_t\}_1^T) \cdot \lambda + \theta E\left[\sum_t C_t \frac{1}{(1+\delta)^t}\right]$$

Different results will be obtained with different λ's and by changing λ one can trace a whole set of stock policies.

In each period of time t, a decision has to be made on how much stock to accumulate or dispose of. In a dynamic stochastic system, the sequence of various events is important. As time passes, variables of previous times that were stochastic are realized. Any decision rule at some point in time can depend on realized variables that correspond only to previous time; the other ones remain stochastic. So far we have divided the whole time horizon into a sequence of periods. Let us now assume the following order within any particular period t:

At the beginning of period t, all the events that occurred previously are given. In particular, the beginning stocks are given and equal to the ending stocks of t-1. Also the final price of t-1 is given. Assume now that the decision on change of stocks ΔC_t is made after the realization of the stochastic disturbance of that period. Given P_{t-1}, this means (according to (1.10)) that at the decision time the market price $\tilde{P}_t$ is also given. Then the change of stocks implies the ending stocks as well as the ending price (according to equation (1.11)) and is illustrated by this sequence.



The problem of optimal stocks policy[1/] can now be stated in a formal way as follows:

Given C_o and P_o, find a series of stocks rules

$$0 \leq C_t(\cdot) \qquad t = 1, \ldots, T$$

which minimize

$$(1.12)^{2/} \quad E\left\{\frac{1}{T}\sum_{t=1}^{T}\left[\left(\frac{P_t - p_t^*}{p_t^*}\right)^2 \lambda + \theta C_t\right]\frac{1}{(1+\delta)^t} \;\middle|\; C_o, P_o\right\}$$

subject to:

$$(1.11) \quad P_t = \phi_t^{-1}[\phi_t(\tilde{P}_t) - (C_t - C_{t-1})] \qquad t = 1, 2, \ldots, T$$

[1/] The term "optimal" should be considered only in relation to the present problem of minimizing the objective criterion (1.12) and not in the general meaning of social desirability. As has been stated in the outset of part I, there is no agreement on the desirability of price stability.

[2/] $E\{\cdot \mid C_o, P_o\}$ means the (conditional) expectation, given C_o and P_o. The $\frac{1}{T}$ will be omitted from now on since it does not affect the solution.

$$(1.10) \qquad \tilde{P}_t = \phi_t^{-1}[\psi_t(P_{t-1}) + \varepsilon_t] \qquad t = 1, 2, \ldots, T$$

ε_t is distributed according to $f_t(\varepsilon_t)$

ε_t, $\varepsilon_{t'}$ are independent for any $t \neq t'$.

Before proceeding into the investigation of the problem stated above, let us introduce additional notation for convenience by defining

$$(1.13) \qquad w_t \equiv \left(\frac{P_t - P_t^*}{P_t^*}\right)^2 \lambda + \theta C_t$$

w_t is a function of P_t and C_t. However, P_t is a function of $\tilde{P}_t$, C_t and C_{t-1} (eq. (1.11)) and $\tilde{P}_t$ in turn is a function of P_{t-1} and ε_t; hence in summary, w_t is a function of C_{t-1}, P_{t-1}, C_t and ε_t. The following notation will be used interchangeably to indicate explicitly the dependence of w_t on these variables.

$$(1.13') \quad w_t = w_t(P_t, C_t) = w_t[P_t(C_{t-1}, C_t, \tilde{P}_t), C_t] = w_t(C_{t-1}, \tilde{P}_t, C_t)$$

$$= w_t(C_{t-1}, P_{t-1}, C_t, \varepsilon_t).$$

Using this notation the objective function (1.12) can be written as:

$$(1.12') \qquad \text{Minimize} \quad E\{\sum_{t=1}^{T} w_t \frac{1}{(1+\delta)^t} \mid C_o, P_o\}$$

Generally, there is an interdependence among the decision rules of all the periods and they have to be solved simultaneously. However, the special structure of the problem reduces the complexity of the solution and enables one to solve it in steps, period by period.

The additivity of the objective function (1.12') in $w_t \frac{1}{(1+\delta)^t}$ and the recursive structure that follows in equations (1.11) and (1.10) imply a proposition that is similar to Belman's Optimality Principle of Dynamic Programming. Generally, any stocks rule for some period t^*, i.e., $\hat{C}_{t*}$, may be a function of realized variables of only previous time, and of the probability distribution of future variables.

Suppose that the stock rules for all t are given. Then, by the additivity of (1.12') in $w_t \frac{1}{(1+\delta)^t}$ and by (1.11) and (1.10) it follows that changing the stock rule for t^* does not affect the value of that part of the objective function that corresponds to time before t^*, i.e., it does not affect the value of

$$(1.14) \qquad E\{\sum_{t=1}^{t*-1} w_t \frac{1}{(1+\delta)^t} \mid C_o, P_o\},$$

because no variable in (1.14) depends on C_{t*}. However, changing the stocks rule for t^* does affect the value of the residual part of (1.12'), i.e., of

$$(1.15) \qquad E\{w_{t*} \frac{1}{(1+\delta)^{t*}} + \sum_{t=t*+1}^{T} w_t \frac{1}{(1+\delta)^t} \mid C_o, P_o\}$$

because w_{t*} and w_{t*+1} are functions of C_{t*} (see 1.13')), as well as other future variables (through (1.11) and (1.10)).

Notice that all the terms of (1.15) do not include explicitly any variables of periods $t = 1, 2, \ldots, t*-1$, except w_{t*}, which includes C_{t*-1} and P_{t*-1} (see (1.13')).

Together with the assumption of the independence of the stochastic disturbances from different periods, it follows that the only variables from periods that are previous to t^*, that affect (1.15), are C_{t*-1} and P_{t*-1}; hence (1.15) can be written as

$$(1.15') \quad E\{\sum_{t=t*}^{T} w_t \frac{1}{(1+\delta)^t} \mid C_o P_o\}$$

$$= E_{C_{t*-1}, P_{t*-1}} \{E_{\varepsilon_{t*}, \ldots, \varepsilon_T} [\sum_{t=t*}^{T} w_t \frac{1}{(1+\delta)^t} \mid C_{t*-1}, P_{t*-1}] \mid C_o, P_o\}$$

where the subscripts below the expectation sign indicate the random variables over which the expectation operation is averaged.

(1.15') states that (1.15) can be calculated in two steps: first take the expectation over the random variables of t=t*, t*+1, ..., T, conditioned on C_{t*-1}, P_{t*-1}; then take the expectation over C_{t*-1}, P_{t*-1} (conditioned on C_o, P_o). The whole objective function (1.12') can be written as follows: Minimize

$$(1.12'') \quad E_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_T} \{\sum_{t=1}^{T} w_t \frac{1}{(1+\delta)^t} \mid C_o, P_o\}$$

$$= E_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{t*-1}} \{\sum_{t=1}^{t*-1} w_t \frac{1}{(1+\delta)^t} \mid C_o, P_o\}$$

$$+ E_{C_{t*-t}, P_{t*-1}} \{E_{\varepsilon_{t*}, \varepsilon_{t*+1}, \ldots, \varepsilon_T} [\sum_{t=t*}^{T} w_t \frac{1}{(1+\delta)^t} \mid C_{t*-1}, P_{t*-1}]$$

$$\mid C_o, P_o\}$$

Suppose that the stocks rules of t=1, 2, ..., t*-1 are given; then it is clear that the value of the first part of (1.12"), i.e. of

$$(1.14) \quad E_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{t*-t}} \{\sum_{t=1}^{t*-1} w_t \frac{1}{(1+\delta)^t} \mid C_o, P_o\}$$

is not affected by the determination of the stocks rules for t=t*, t*+1, ..., T. It follows that the stocks rules for t=t*, t*+1, ..., T which minimize the second part of (1.12"), i.e., which minimize

$$(1.15') \quad E_{c_{t^*-1},\, P_{t^*-1}}\{E_{\varepsilon_{t^*},\, \varepsilon_{t^*+1},\, \ldots,\, \varepsilon_T}[\sum_{t=t^*}^{T} w_t \frac{1}{(1+\delta)^t} | C_{t^*-1},\, P_{t^*-1}] C_o,\, P_o\}$$

also minimize (1.12"), for any given stocks rules for t=1, 2, ..., t*-1. The following proposition is therefore true:

Proposition. To be optimal to the problem of minimizing (1.12") it is necessary and sufficient that the stocks rules for t=t*, t*+1, ..., t will be optimal to the following problem:

Given C_{t^*-1}, P_{t^*-1} find stocks rules $C_{t^*}(\cdot)$, $C_{t^*+1}(\cdot)$, ..., $C_T(\cdot)$ which minimize

$$(1.16) \quad E_{\varepsilon_{t^*},\, \varepsilon_{t^*+1},\, \ldots,\, \varepsilon_T}\{\sum_{t=t^*}^{T} w_t \frac{1}{(1+\delta)} | C_{t^*-1},\, P_{t^*-1}\}$$

subject to (1.13'), (1.11) and (1.10).

This proposition enables us to solve the problem in steps, period after period, starting with the last period T and proceeding backwards to the first one, similarly to regular dynamic programming. First the stocks rule which minimizes (1.16) for only one period, i.e., t=T, is solved. Having this, one knows the effect of changing C_{T-1} on the T-th term of the objective function. This can be used in the determination of the stocks rule of T-1 in the subproblem of minimizing (1.16) for t=T-1, T. This procedure can be extended by induction to t=T-2, T-3, ..., 2, 1 and the whole problem is thus solved.

Let us now analyze the solution in more detail. This will provide the background on which the computation procedure, described in section 2, is based.

Starting from t=T, the subproblem to be solved is:

<u>Problem T</u>: Given C_{T-1} and P_{T-1}, find a stocks rule $0 \leq C_T(\cdot)$ which minimizes

$$(1.16)_T \quad E_{\varepsilon_T}\{w_T \frac{1}{(1+\delta)^T} \mid C_{T-1}, P_{T-1}\} = \frac{1}{(1+\delta)^T} E_{\varepsilon_T}\{w_T \mid C_{T-1}, P_{T-1}\}$$

where:

$$(1.13)_T \quad w_T = \left(\frac{P_T - p_T^*}{p_T^*}\right)^2 \lambda + \theta C_T$$

$$(1.11)_T \quad P_T = \phi_T^{-1}\,[\phi_T(\tilde{P}_T) - (C_T - C_{T-1})]$$

$$(1.10)_T \quad \tilde{P}_T = \phi_T^{-1}\,[\psi_T(P_{T-1}) + \varepsilon_T]$$

According to the assumptions on timing within a period (see figure 1.1), ε_T is already given at the time of stocks change ΔC_T, hence $\tilde{P}_T$ (which is a function of P_{T-1} and ε_T (see $(1.10)_T$) is also given at that time. Therefore, to minimize $(1.13)_T$ it is necessary for any given C_{T-1} and $\tilde{P}_T$ to find that stocks level, $\hat{C}_T$, which minimizes w_T. Let us denote the optimal stocks rule by

$$\hat{C}_T = G_T(C_{T-1}, \tilde{P}_T)$$

Also, denote the value of w_T under the optimal rule by V_T. V_T is defined by substituting $G_T(C_{T-1}, \tilde{P}_T)$ for C_T in $(1.13)_T$ and $(1.11)_T$. Clearly, V_T is a function of C_{T-1} and $\tilde{P}_T$ and we write

$$V_T(C_{T-1}, \tilde{P}_T)$$

To obtain the minimum value of $(1.16)_T$, the expectation of V_T has to be taken over ε_T. $\tilde{P}_T$ is a function of P_{t-1} and ε_T (see $(1.10)_T$) so that

V_T is a function of C_{T-1}, P_{T-1} and ε_T. However, after integration, the expected value of $V_T(C_{T-1}, \tilde{P}_T(P_{T-1}, \varepsilon_T))$ is a function of C_{T-1} and P_{T-1}. Let us denote the minimum of $(1.16)_T$, given C_{T-1}, P_{T-1}, by $EV_T(C_{T-1}, P_{T-1})$:

$$(1.17)_T \quad EV_T(C_{T-1}, P_{T-1}) = E_{\varepsilon_T}\{V_T[C_{T-1}, \tilde{P}_T(P_{T-1}, \varepsilon_T)]\}$$

$$= \int V_T[C_{T-1}, \tilde{P}_T(P_{T-1}, \varepsilon_T)]df_T(\varepsilon_T).$$

This is the end of step T.

Proceed now to step T-1:

The problem to be solved is:

Problem T-1: Given C_{T-2} and P_{T-2}, find stocks rules $0 \leq C_{T-1}(\cdot)$, $C_T(\cdot)$ which minimize

$$(1.16)_{T-1} \quad E_{\varepsilon_{T-1}, \varepsilon_T}[w_{T-1}\frac{1}{(1+\delta)^{T-1}} + w_T\frac{1}{(1+\delta)^T} \mid C_{T-2}, P_{T-2}]$$

$$= \frac{1}{(1+\delta)^{T-1}}\{E_{\varepsilon_{T-1}}\{w_{T-1} + \frac{1}{(1+\delta)}E_{\varepsilon_T}[w_T|C_{T-1}, P_{T-1}]|C_{T-2}, P_{T-2}\}\}$$

where:

$$(1.13)_{T-1} \quad w_{T-1} = \left(\frac{P_{T-1} - P^*_{T-1}}{P^*_{T-1}}\right)^2\lambda + \theta C_{T-1}$$

$$(1.11)_{T-1} \quad P_{T-1} = \phi^{-1}_{T-1}[\phi_{T-1}(\tilde{P}_{T-1}) - (C_{T-1} - C_{T-2})]$$

$$(1.10)_{T-1} \quad \tilde{P}_{T-1} = \phi^{-1}_{T-1}[\psi_{T-1}(P_{T-2}) + \varepsilon_{T-1}]$$

The optimal stocks rule for t=T is already known from the previous step.

Substituting it in $(1.16)_T$ and using the notation of $EV_T(C_{T-1}, P_{T-1})$, the problem is reduced to the following:

Find stocks rule $0 \leq C_{T-1}(\cdot)$ which minimizes

$$(1.16')_{T-1} \quad E_{\varepsilon_{T-1}}\{w_{T-1} + \frac{1}{1+\delta} EV_T(C_{T-1}, P_{T-1}) | C_{T-2}, P_{T-2}\}$$

As in problem T, the assumptions on timing within a period imply that the optimal stocks rule for T-1 should be a function of C_{T-2} and $\tilde{P}_{T-1}$, to be denoted by

$$\hat{C}_{T-1} = G_{T-1}(C_{T-2}, \tilde{P}_{T-1}).$$

It is the stocks level $\hat{C}_{T-1}$ which minimizes

$$(1.18)_{T-1} \quad w_{T-1}[C_{T-1}, P_{T-1}(C_{T-2}, \tilde{P}_{T-1}, C_{T-1})] + \frac{1}{(1+\delta)} EV_T[C_{T-1}, P_{T-1}(C_{T-2}, \tilde{P}_{T-1}, C_{T-1})]$$

for given C_{T-2}, $\tilde{P}_{T-1}$, where the dependence of P_{T-1} on C_{T-2}, $\tilde{P}_{T-1}$ and C_{T-1} follows from $(1.11)_T$.

For each given $(C_{T-2}, \tilde{P}_{T-1})$ the minimal value of $(1.18)_{T-1}$ is obtained by substituting $G_{T-1}(C_{T-2}, \tilde{P}_{T-1})$ for C_{T-1} in $(1.18)_{T-1}$. This value is also a function of C_{T-2} and $\tilde{P}_{T-1}$ and will be denoted by

$$V_{T-1}(C_{T-2}, \tilde{P}_{T-1}) .$$

Recall that $\tilde{P}_{T-1}$ is a function of P_{T-2} and ε_{T-1} (see $(1.10)_{T-1}$). The minimum of $(1.16')_{T-1}$ is obtained by taking the expectation of $V_{T-1}[C_{T-2}, \tilde{P}_{T-1}(P_{T-2}, \varepsilon_{T-1})]$ over ε_{T-1}, given C_{T-2}, P_{T-2}, to be denoted by EV_{T-1}, which is clearly a function of C_{T-2} and P_{T-2}:

$$(1.17)_{T-1} \quad EV_{T-1}(C_{T-2},P_{T-2}) = E_{\varepsilon_{T-1}}\{V_{T-1}[C_{T-2}, \tilde{P}_{T-1}(P_{T-2}, \varepsilon_{T-1})]\}$$

$$= \int V_{T-1}[C_{T-2}, \tilde{P}_{T-1}(P_{T-2}, \varepsilon_{T-1})]df_{T-1}(\varepsilon_{T-1}).$$

This is the end of stage T-1 and it is possible now to proceed into stage T-2 and to solve the problem:

Problem T-2: Given C_{T-3} and P_{T-3}, find stocks rules $0 \leq C_{T-2}(\cdot)$, $C_{T-1}(\cdot)$, $C_T(\cdot)$, which minimize

$$(1.16)_{T-2} \quad E_{\varepsilon_{T-2}, \varepsilon_{T-1}, \varepsilon_T} \; [\sum_{t=T-2}^{T} w_T \frac{1}{(1+\delta)} + |C_{T-3}, P_{T-3}]$$

which by using the results of step T-2 is reduced to the problem of finding $0 \leq C_{T-3}(\cdot)$, which minimizes

$$(1.16')_{T-2} \quad E_{\varepsilon_{T-2}}\{w_{T-2} + EV_{T-1}(C_{T-2}, P_{T-2}) \mid C_{T-3}, P_{T-3}\}$$

which is in the same form as $(1.16')_{T-1}$ in step T-1. Thus, by induction the procedure can continue to steps T-3, T-4, ..., 1. Step 1 concludes the computation.

In this study a computational procedure which is based on the last discussion was used. In this procedure all the continuous variables were approximated by sets of discrete values. This will be discussed in the next section (section 2) and in more detail in the appendix. Before discussing this technical aspect let us analyze the optimal stocks rules in some detail.

1.3 An analysis of the optimal stocks rules. In this subsection, the optimal stocks rules will be characterized and some conclusions, which are extensions of those which have been concluded in a primitive example in part I (sec. 3), will be derived. First, the simplest case, which assumes that no costs are attributed to holding stocks and that supply does not depend on lagged price, is analyzed. The case with costs will be discussed later.

Case (a): No cost no lagged price.

Assume that holding stocks costs nothing and that the supply does not depend on the previous year's price. For convenience assume also that $\delta = 0$ and $\lambda = 1$ (this will not affect the qualitative results).

Under the present assumptions, the t-th term of the objective function does not include cost and is defined simply by[3/]

$$(1.20)\quad w_t^* = \left(\frac{P_t - p_t^*}{p_t^*}\right)^2$$

The supply functions are now

$$(1.21)\quad x_t = x_t^o + \varepsilon_{st}$$

where x_t^o is a constant mean of production at time t which does not depend on P_{t-1}.

The price equations (see (1.10), (1.11) above) become

$$(1.22)\quad P_t = \phi_t^{-1}[\phi_t(\tilde{P}_t) - (C_t - C_{t-1})] = \phi_t^{-1}[\phi_t(\tilde{P}_t) - \Delta C_t]$$

$$(1.23)\quad \tilde{P}_t = \phi_t^{-1}(x_t^o + \varepsilon_t)$$

[3/] We reserve the unstarred w_t for the general case, which includes costs. Hence, in general:

$$w_t = w_t^* + \theta C$$

However, in the no-cost case w_t^* and w_t are identical.

The following derivatives will be useful throughout the analysis and therefore are derived at the outset:

From (1.20)

$$(1.24)\quad \frac{\partial w_t^*}{\partial \Delta C_t} = \frac{2}{p_t^{*2}} (P_t - p_t^*) \frac{\partial P_t}{\partial \Delta C_t} ,$$

and from (1.22)

$$(1.25)\quad \frac{\partial P_t}{\partial \Delta C_t} = - \frac{1}{\phi_t'(P_t)} > 0$$

where $\phi_t'(P_t)$ is the derivative of the demand function with respect to price.

The sign of $\frac{\partial w_t^*}{\partial \Delta C_t}$ depends on P_t and p_t^*.

$$(1.26)\quad \text{sign } \frac{\partial w_t^*}{\partial \Delta C_t} = \text{sign } (P_t - p_t^*) = \text{sign } \{\phi_t^{-1}[\phi_t(\tilde{P}_t) - \Delta C_t] - p_t^*\},$$

which states simply that when $P_t > p_t^*$, increasing the stocks' change increases the gap between P_t and p_t^* , and the opposite is true when $P_t < p^*$.

The second derivatives are:

From (1.24)

$$\frac{\partial^2 w_t^*}{\partial \Delta C_t^2} = \frac{2}{p_t^{*2}} \{\left(\frac{\partial P_t}{\partial \Delta C_t}\right)^2 + (P_t - p_t^*) \frac{\partial^2 P_t}{\partial \Delta C_t^2}\}$$

and from (1.22) and (1.25)

$$(1.27)\quad \frac{\partial^2 P_t}{\partial \Delta C_t^2} = \frac{1}{[\phi_t'(P_t)]^2} \phi_t''(P_t) \cdot \frac{\partial P_t}{\partial \Delta C_t} = \left(\frac{\partial P_t}{\partial \Delta C_t}\right)^3 \phi_t''(P_t) ,$$

where: $\phi''_t(P_t)$ is the second derivative of the demand function with respect to P.

It follows that:

$$(1.28)\quad \frac{\partial^2 w_t^*}{\partial \Delta C_t^2} = \frac{2}{p_t^{*2}} \left(\frac{\partial P_t}{\partial \Delta C_t}\right)^2 \{1 + (P_t - p_t^*) \frac{\phi_t''(P_t)}{[-\phi_t'(P_t)]}\} \ .$$

The sign of (1.28) is not a priori determined because ϕ_t'' can in general be either negative or positive or zero. However, many common demand functions, often in use, are convex and have $\phi_t'' \geq 0$; (e.g., linear, constant elasticity, semi-log, quadratic, etc.), therefore we assume:

$$(1.29)\quad \Phi_t''(P_t) \geqq 0 \ .$$

It follows that

$$(1.30)\quad \frac{\partial^2 w_t^*}{\partial \Delta C_t^2} > 0 \quad \text{if } P_t > p_t^* \ .$$

This property means that when $P_t > p_t^*$, the marginal effect of ΔC_t on w_t^* is positive and increasing, i.e., the more ΔC_t, the greater is its marginal effect in increasing w_t^*. On the other hand, the decreasing part of w_t^* with respect to ΔC_t might still have $\frac{\partial^2 w_t^*}{\partial \Delta C_t}$ positive or negative. However, it turns out that the positiveness of the second derivative is important only in regard to the increasing part (this point will be clear later on). Equipped with this formal discussion, let us start with the analysis of the stocks rule of the last period T, following the procedure of the last subsection (sec. 1.2). The problem to be solved is problem T of subsection 1.2.

It is more convenient to transform it into terms of stocks change (ΔC_T) instead of stocks (C_T). Also, since the lagged price (P_{T-1}) does not play any role in the present case it will be omitted from the formulation. Restated, the problem of period T is:

<u>Problem T</u>: Given the beginning stocks C_{T-1} find a stocks rule $\Delta \hat{C}_T(\cdot)$ which minimizes

$$(1.31)_T \quad E_{\varepsilon_T}(w_T^* | C_{T-1})$$

subject to the constraint:

$$(1.32)_T \quad \Delta C_T \geq - C_{T-1}$$

where the last constraint is implied by the nonnegativity of C_T:

$$0 \leq C_T = C_{T-1} + \Delta C_T \ .$$

w_T^* depends on ΔC_T through equation (1.20), (1.22) for t=T. Recall from subsection 1.2 that in order to minimize the expected value $(1.31)_T$ it is necessary to minimize w_T^* for any given beginning stocks and market price i.e., given C_{T-1} and $\tilde{P}_T$

$$(1.33)^{4/} \quad \underset{\Delta C_T}{\text{minimize}} \ (w_T^* | C_{T-1}, \tilde{P}_T)$$

subject to the constraint (1.32) and equations (1.20)(1.22) for t=T.

First ignore the constraint and find the unconstrained minimum of w_T^* with respect to ΔC_T. From (1.26) it is clear that given $\tilde{P}_T$, w_T has

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4/ The notation of C_{T-1}, $\tilde{P}_T$ to the right hand side of the vertical line in (1.33) indicates that C_{T-1} and $\tilde{P}_{T-1}$ are given.

a unique minimum where $P_T = p_T^*$. Let us denote the unconstrained minimizing stocks change by $\Delta C_T^o(C_{T-1}, \tilde{P}_T)$ to indicate its dependence on C_{T-1} and $\tilde{P}_T$. The condition $P_T = p_T^*$ implies, through equation (1.22) that

$$(1.34)_T \quad \Delta C_T^o(C_{T-1}, \tilde{P}_T) = \phi_T(\tilde{P}_T) - \phi_T(p_T^*)$$

which states simply that to minimize w_T^*, ΔC_T should be equal to the amount which is needed to close the gap between the demand at $\tilde{P}_T$ and the demand at the target price p_T^*. Figure 1 may help to demonstrate the determination of ΔC_T^o.

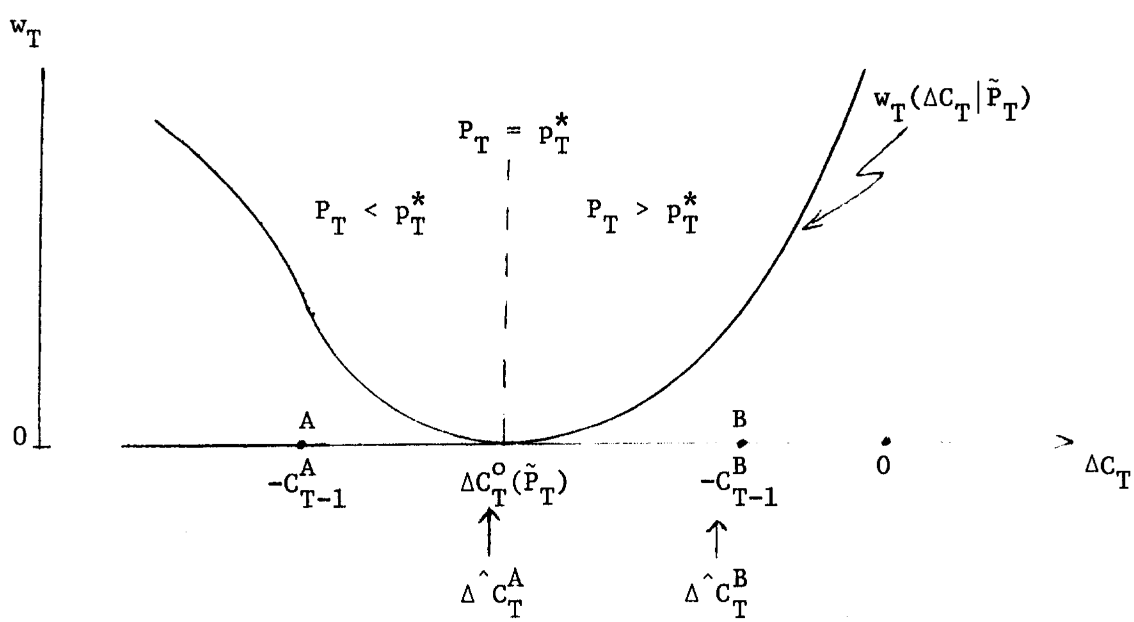

Figure 1

Notice that although formally ΔC_T^o is a function of both the beginning stocks (C_{T-1}) and the market price ($\tilde{P}_T$), in fact only the last one affects ΔC_T^o. This is typical only to the last period since it is assumed that it does not matter what happens after T, hence the ending stocks C_T is valueless by itself. The only consideration in period T is to put the price P_T as close as possible to p_T^*. Later on it will be shown that in general, the amount of beginning stocks at some period t, i.e., C_{t-1}, affects ΔC_t^o, because the ending stocks C_t are valuable in their potential to reduce future instability. This consideration, which other buffer-stocks proposals (see Cochrane [1] and part I, sec. 2.1) fail to take into account, does not exist in period T.

So far, the constraint (1.2) has been ignored. However, $\Delta C_T^o(C_{T-1}, \tilde{P}_T)$ might violate it. In this case, the minimal feasible value of w_T^* is obtained with

$$\Delta C_T = - C_{T-1}$$

i.e., dispose of all the existing stocks. Let us denote by ^ the optimal value of a variable. In particular $\Delta\hat{C}_T$ denotes the optimal stocks rule for t=T:

$$(1.35)_T \quad \Delta\hat{C}_T(C_{T-1}, \tilde{P}_T) = \begin{cases} - C_{T-1} & \text{if } C_{T-1} \leq \phi_T(p_T^*) - \phi_T(\tilde{P}_T) = \Delta C_T^o \\ \phi_T(\tilde{P}_T) - \phi_T(p_T^*) & \text{if } C_{T-1} \geq \phi_T(p_T^*) - \phi_T(\tilde{P}_T) = \Delta C_T^o \end{cases}$$

It is obvious that the first line of $(1.35)_T$ can be relevant only in case that the market price $\tilde{P}_T$ is greater than the target price p_T^*, since only in this case the change of stocks is negative (i.e., selling) and might violate the constraint. Refer back to figure 1: two cases of beginning stocks are presented by A and B. When $C_{T-1} = C_{T-1}^A$ the amount of beginning stocks is OA. In this case there are enough stocks to dispose of and to put the price equal to the target price. The optimal change of stocks is

$\Delta\hat{C}_T^A$ which is identical to the unconstrained one (ΔC_T^o). When $C_{T-1} = C_{T-1}^B$ the amount of beginning stocks is represented by OB. In this case ΔC_T^o is not feasible and the optimal stocks change is $\Delta\hat{C}_T^B$ which is identical to $- C_{T-1}^B$.

Let us summarize the characterization of $\Delta\hat{C}_T$ in the following Proposition.

Proposition 1_T (Optimal stocks change for t=T). The optimal stocks change for the last period ($\Delta\hat{C}_T$) is generally a function of the beginning stocks (C_{T-1}) and the market price ($\tilde{P}_T$). However the optimal stocks rule for period T is to set the final price (P_T) as close as possible to the target price (p_T^*). The beginning stocks (C_{T-1}) affects $\Delta\hat{C}_T$ only in the sense that it might be smaller than is needed for disposals of stocks in order to equate $P_T = p_T^*$. Otherwise $\Delta\hat{C}_T$ is not a function of C_{T-1}.

The ending price which results from the application of the optimal stocks rule for period T, given C_{T-1} and $\tilde{P}_T$, is denoted by $\hat{P}_T(C_{T-1}, \tilde{P}_T)$. It is derived by substituting the optimal stocks change in the price equation (1.22)

$$(1.36)_T \quad \hat{P}_T(C_{T-1}, \tilde{P}_T) = \begin{cases} \phi_T^{-1}[\phi_T(\tilde{P}_T) + C_{T-1}] & \text{if } C_{T-1} \leqq \phi_T(p_T^*) - \phi_T(\tilde{P}_T) = -\Delta C_T^o \\ \\ p_T^* & \text{if } C_{T-1} \geqq \phi_T(p_T^*) - \phi_T(\tilde{P}_T) = -\Delta C_T^o \end{cases}$$

The feasible minimum value of w_T^*, i.e., the value of w_T^* under the application of the optimal stocks rule for period T is denoted by $V_T^*(C_{T-1}, \tilde{P}_T)$. Using $(1.36)_T$ and the definition of w_T^* in (1.20),

$$(1.37)_T \quad V_T^*(C_{T-1}, \tilde{P}_T) = \begin{cases} \left(\dfrac{\hat{P}_T - p_T^*}{p_T^*}\right)^2 & \text{if } C_{T-1} \leq \phi_T(p_T^*) - \phi_T(\tilde{P}_T) = -\Delta C_T^o \\ 0 & \text{if } C_{T-1} \geq \phi_T(p_T^*) - \phi_T(\tilde{P}_T) = -\Delta C_T^o \end{cases}$$

where: $\hat{P}_T$ is defined by $(1.36)_T$

Let us characterize the dependence of V_T^* on the beginning stocks by the partial derivative with respect to C_{T-1}:

$$\frac{\partial V_T^*(C_{T-1}, \tilde{P}_T)}{\partial C_{T-1}} = \frac{\partial w_T^*}{\partial \Delta C_T} \cdot \frac{\partial \Delta \hat{C}_T}{\partial C_{T-1}}$$

By (1.35) it follows that

$$(1.38)_T \quad \frac{\partial \Delta \hat{C}_T(C_{T-1}, \tilde{P}_T)}{\partial C_{T-1}} = \begin{cases} -1 & \text{if } C_{T-1} \leqq \phi_T(p_T^*) - \phi_T(\tilde{P}_T) \\ 0 & \text{if } C_{T-1} \geqq \phi_T(p_T^*) - \phi_T(\tilde{P}_T) \end{cases}$$

and using (1.24) and (1.25),

$$(1.39)_T \quad \frac{\partial V_T^*(C_{T-1}, \tilde{P}_T)}{\partial C_{T-1}} = \begin{cases} \dfrac{2}{p_T^{*2}} (\hat{P}_T - p_T^*) \dfrac{1}{\phi_T'} < 0 & \text{if } C_{T-1} \leq \phi_T(p_T^*) - \phi_T(\tilde{P}_T) \\ 0 & \text{if } C_{T-1} \geq \phi_T(p_T^*) - \phi_T(\tilde{P}_T) \end{cases}$$

$(1.39)_T$ states that for a given $\tilde{P}_T$, V_T^* decreases with C_{T-1} as long as C_{T-1} is less than a certain quantity which by itself increases with $\tilde{P}_T$(i.e., $\phi_T(p_T^*) - \phi_T(\tilde{P}_T)$). Let us denote this quantity by $\bar{C}_{T-1}(\tilde{P}_T)$.[5/]

[5/] $\bar{C}_{T-1}(\tilde{P}_T)$ is in fact identical to the minus unconstrained minimum ($-\Delta C_T^o = \phi_T(p_T^*) - \phi(\tilde{P}_T)$). In other periods, $t \neq T$, it will be seen that

(continued)

When C_{T-1} is greater than $\bar{C}_T(\tilde{P}_T)$, V_T^* does not change and from (1.36) and (1.37) it is known that in this situation the final price is equal to the target price ($P_T = p_t^*$) and V_T^* is equal to zero.

Turn now to the second derivative of V_T^* with respect to C_{T-1}:

$$\frac{\partial^2 V_T^*(C_{T-1}, \tilde{P}_T)}{\partial C_{T-1}^2} = \frac{\partial^2 w_T^*}{\partial \Delta C_T^2}\left(\frac{\partial \Delta \hat{\ } C_T}{\partial C_{T-1}}\right)^2 + \frac{\partial w_T^*}{\partial \Delta C_T} \cdot \frac{\partial^2 \Delta \hat{\ } C}{\partial C_{T-1}^2}$$

$(1.38)_T$ implies that $\dfrac{\partial^2 \Delta C_T}{\partial C_{T-1}^2} = 0$, hence, using (1.28) and $(1.38)_T$

$$(1.40)_T \quad \frac{\partial^2 V_T^*(C_{T-1}, \tilde{P}_T)}{\partial C_{T-1}^2} = \begin{cases} \dfrac{2}{p_T^{*2}} \dfrac{1}{[\phi_T'(\hat{P}_T)]^2} [1 + (\hat{P}_T - p_T^*) \dfrac{\phi_T''(\hat{P}_T)}{-\phi_T'(\hat{P}_T)}] > 0 & \\ \qquad\qquad \text{if } C_{T-1} \leqq \phi_T(p_T^*) - \phi_T(\tilde{P}_T) & \\ 0 \qquad\qquad \text{if } C_{T-1} \geqq \phi_T(p_T^*) - \phi_T(\tilde{P}_T) & \end{cases}$$

The positiveness of the first line of (1.40) is implied by the assumption that $\phi_T'' > 0$ (i.e., the demand function is convex) and by the fact that, when $C_T \leq \bar{C}_{T-1}(P_T) = \phi_T(p_T^*) - \phi_T(\tilde{P}_T)$, the price $\hat{P}_T$ is greater than the target price p_T^* (see $(1.36)_T$). The following Proposition summarizes the discussion on the effects of the beginning stocks for period T.

<u>Proposition 2_T</u> (The effect of beginning stocks, given the market price, for t=T). Given the market price ($\tilde{P}_T$), there is a level of

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(footnote 5/ continued)

(a similar value $\bar{C}_{t-1}(\tilde{P}_t)$ is not identical to the unconstrained minimum $-\Delta C_t^o$. We keep the special notation also for T for later comparisons.

beginning stocks $\overline{C}_{T-1}(\tilde{P}_T)$ which is an increasing function of $\tilde{P}_T$ such that:

(1) Whenever the beginning stocks (C_{T-1}) are smaller than $\overline{C}_{T-1}(\tilde{P}_T)$, then the optimal stocks change $(\Delta\hat{C}_T)$ is a strictly decreasing function of the beginning stocks (C_{T-1}), and $V_T^*(C_{T-1}, \tilde{P}_T)$, the optimal value, is positive, and is a strictly decreasing and convex function of C_{T-1}.

(2) Whenever $C_{T-1} \geq \overline{C}_{T-1}(\tilde{P}_T)$, then $\Delta\hat{C}_T$ equals constantly to the unconstrained minimum point, and $V_T^*(C_{T-1},\tilde{P}_T)$ is constantly 0 (i.e., complete stabilization: $\hat{P}_T = p_T^*$).

Figure 2 illustrates this. As $\tilde{P}_T$ approaches p_T^* from above, the turning point $\overline{C}_{T-1}(\tilde{P}_T)$ tends to zero. Whenever $\tilde{P}_T \leq p_T^*$, V_T^* is constantly zero.

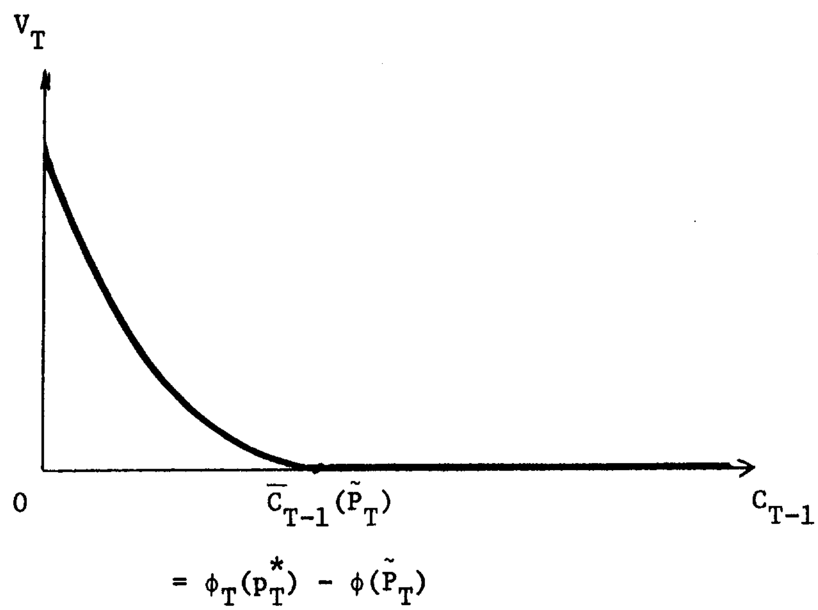


Figure 2

To conclude the analysis of the last period consider the expected value of $V_T^*(C_{T-1}, \tilde{P}_T)$, recalling that $\tilde{P}_T$ is a function of the compound disturbance ε_T. (See 1.23.)

$$(1.41)_T \quad EV_T^*(C_{T-1}) = \int V_T^*[C_{T-1}, \phi_T^{-1}(x_T^o + \varepsilon_T)] \cdot df_T(\varepsilon_T).$$

The linearity of the expectation operation and of the derivative implies that the derivative and second derivative of the expected value are equal to the expected value of the derivatives. Hence the qualitative results which have been stated for V_T^*, given $\tilde{P}_T$, are valid for EV_T^* also. Theroetically, it might be that for any level of C_{T-1} there is a market price $\tilde{P}_T$ which is high enough so that $C_{T-1} < \overline{C}_{T-1}(\tilde{P}_T) = P_T(p_T^*) - \phi_T(\tilde{P}_T)$. In this case EV_T will always have a negative derivative. However, practically it can be assumed that there is a maximal price $\overline{P}$ such that any market price higher than $\overline{P}$ has probability zero, i.e.:

$$\text{Prob. } \{\tilde{P} > \overline{P}\} = 0 \quad \text{and} \quad \text{Prob. } \{\tilde{P} = \overline{P}\} > 0$$

Define $\overline{\overline{C}}_{T-1} \equiv \overline{C}_{T-1}(\overline{P})$. Then for $C_{T-1} < \overline{\overline{C}}_{T-1}$, EV_T^* decreases in C_{T-1} and for $C_{T-1} \geq \overline{\overline{C}}_{T-1}$, EV_T^* is zero. The following Proposition follows directly from $(1.37)_T$, $(1.39)_T$ and $(1.40)_T$ (like Proposition 2) and characterizes $EV_T^*(C_{T-1})$ as a function of C_{T-1}.

<u>Proposition 3_T</u> (The effect of the beginning stocks on the expected value for t=T): There exists a $\overline{\overline{C}}_{T-1}$ such that: whenever $C_{T-1} < \overline{\overline{C}}_{T-1}$, $EV_T^*(C_{T-1})$ is positive, strictly decreases in C_{T-1}, and is convex in C_{T-1}. Whenever $C_{T-1} \geq \overline{C}_{T-1}$, $EV_T^*(C_{T-1})$ constantly equals to zero. Translated into terms of derivatives: $dEV_T^*/dC_{T-1} < 0$ and $d^2EV_T^*/dC_{T-1}^2 > 0$ whenever $C_{T-1} < \overline{\overline{C}}_{T-1}$. $dEV_T^*/dC_{T-1} = 0$ whenever $C_{T-1} \geq \overline{\overline{C}}_{T-1}$. Figure 3, which is

similar to figure 2, illustrates this Proposition: The upper part shows the course of $EV_T^*(C_{T-1})$ whereas the lower graph, denoted by MEV_T^* depicts the marginal effect of C_{T-1} on EV_T^*. MEV_T^* is the derivative of $EV_T^*(C_{T-1})$. Note that in the no-cost case EV_T^* is the minimal value of the index of price instability for period T, given C_{T-1}, so MEV_T^* can be called the marginal destabilization of ΔC_{T-1} in period T. $|MEV_T^*|$ will be called

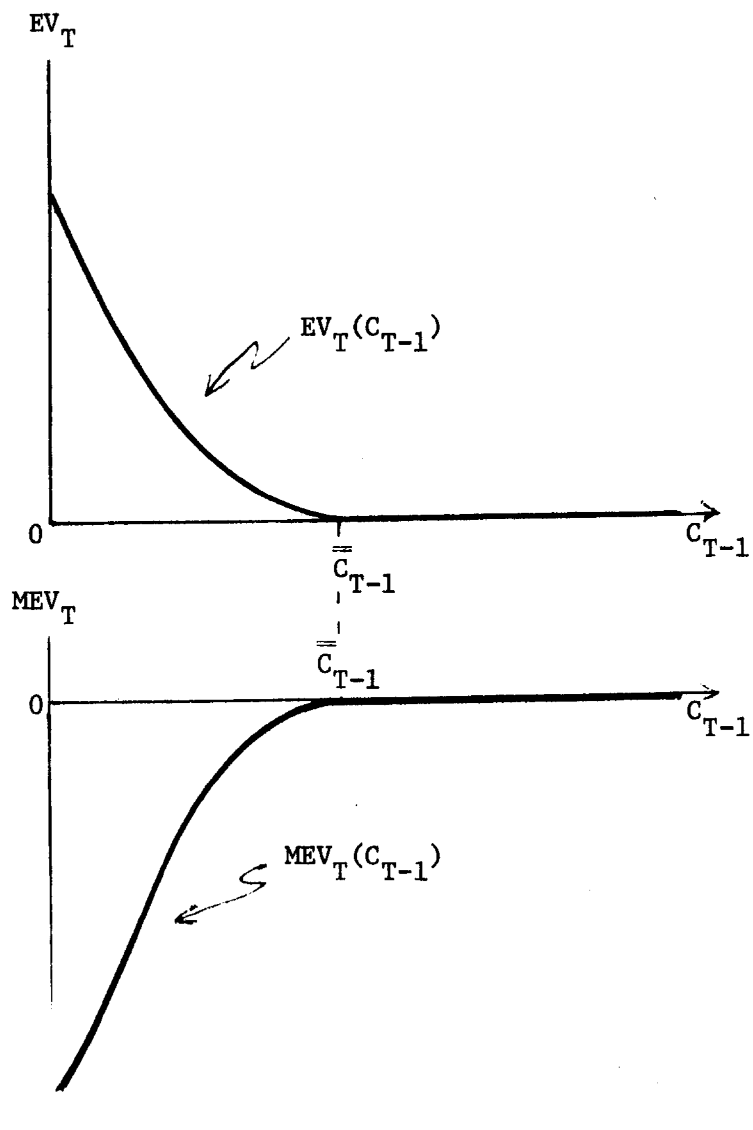


Figure 3

respectively, the marginal stabilization of C_{T-1} in period T. Proposition 3 and figure 3 show the stabilization effect of C_{T-1} on period T up to a point, $\bar{\bar{C}}_{T-1}$, in which complete stability is reached. This effect is the factor which makes the difference between the determination of the optimal stocks rule for the last period T and the optimal rule of period T-1 as well as other periods. Let us analyze now the stocks rule for period T-1.

The stocks rule for period T-1 has to solve problem T-1 of subsection 1.2. Restated in terms of stocks change and under the special assumption of the present case (no-cost, no-lagged price), the problem is as follows:

<u>Problem T-1</u>: Given the beginning stocks for period T-1, i.e. given C_{T-2}, find stocks rule $\Delta\hat{}C_{T-1}(\cdot)$ which minimizes

$$(1.31)_{T-1} \qquad E_{\varepsilon_{T-1}}(w^*_{T-1} + EV^*_T | C_{T-2})$$

subject to the constraint

$$(1.32)_{T-1} \qquad \Delta C_{T-1} \geq - C_{T-2} \ .$$

w^*_{T-1} depends on ΔC_{T-1} through equations (1.20), (1.22) for t=T-1; EV^*_T is a function of $C_{T-1}((1.41)_T)$ and therefore, given C_{T-2} and the definition of $\Delta C_{T-1} \equiv C_{T-1} - C_{T-2}$, it is also a function of ΔC_{T-1}.

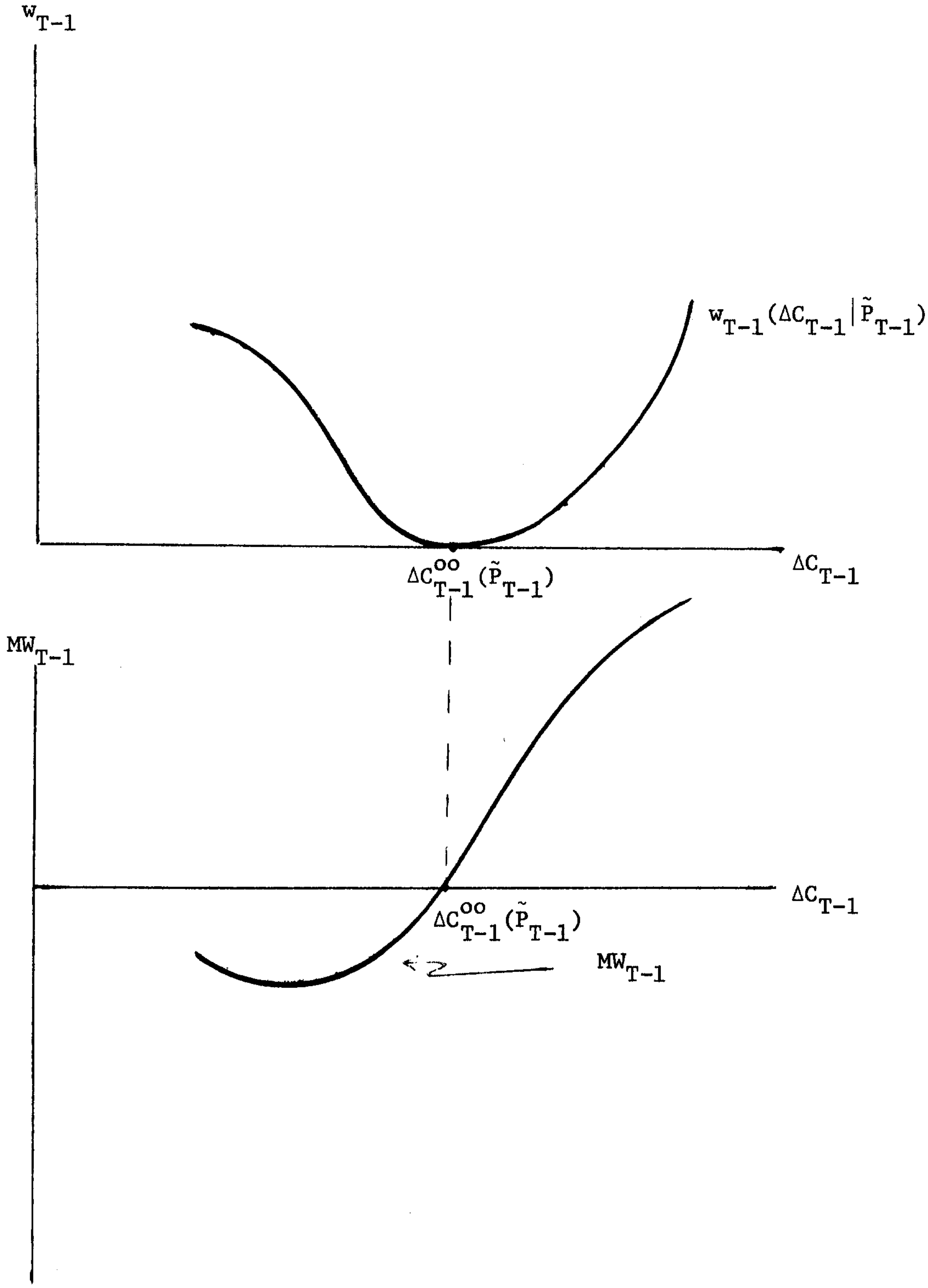
Notice that in comparison to $(1.31)_T$, $(1.31)_{T-1}$ includes two terms, i.e., w^*_{T-1} and EV^*_T, instead of w_T only in $(1.31)_T$. w^*_{T-1} is a measure of price instability of the current period T-1, while EV^*_T is a measure of instability of future time (in this case t=T). This fact makes the difference between the two problems. Recall from subsection 1.2 that in order to minimize the expected value $(1.32)_{T-1}$ it is necessary to minimize $w^*_{T-1} + EV^*_T$ for any given beginning stocks and market price, i.e., given C_{T-2} and $\tilde{P}_{T-1}$

$$(1.33)_{T-1} \quad \underset{\Delta C_{T-1}}{\text{minimize}} \quad (w^*_{T-1} + EV^*_T | C_{T-2}, \tilde{P}_{T-1})$$

subject to: the constraint $(1.32)_{T-1}$, eq. (1.20)(1.22) for t=T-1 and eq. $(1.37)_T$, $(1.41)_T$, which define $EV^*_T(C_{T-1})$. As before, first ignore the constraint and find the unconstrained minimum of $w^*_{T-1} + EV^*_T$ with respect to ΔC_{T-1}. In the discussion for period T, w^*_T as a function of ΔC_T was characterized. This can be applied also to the relation between w^*_t and ΔC_t of any period t. In particular $w^*_{T-1}(\Delta C_{T-1}|C_{T-2}, \tilde{P}_{T-1})$ has a unique minimum at some ΔC_{T-1} which equates the final price P_{T-1} to the target price p^*_{T-1}. Denote the stocks change which minimizes w^*_{T-1} by $\Delta C^{oo}_{T-1}(\tilde{P}_{T-1})$.[6/] In addition, for any ΔC_{T-1} which is less than $\Delta C^{oo}_{T-1}(\tilde{P}_{T-1})$, w^*_T decreases monatonically with ΔC_{T-1} and for any ΔC_{T-1} which is greater than $\Delta C^{oo}_{T-1}(\tilde{P}_{T-1})$, w^*_{T-1} increases with ΔC_{T-1}. Furthermore, the <u>marginal change</u> of w^*_{T-1} increases with ΔC_{T-1} that is greater than $\Delta C^{oo}_{T-1}(\tilde{P}_{T-1})$ (this follows from (1.30).

Figure 4 demonstrates this characterization. In the upper section of the figure, the course of w^*_{T-1} as a function of ΔC_{T-1} is depicted, given $\tilde{P}_{T-1}$. The minimum point is at ΔC^{oo}_{T-1}. In the lower section of figure 4 the marginal curve of w^*_{T-1} as a function of ΔC_{T-1} is drawn.

[6/] It was shown that ΔC^o_T was a function of only $\tilde{P}_T$ and not of C_{T-1}. This is applied here to C^{oo}_{T-1} which is a function of only $\tilde{P}_{T-1}$.



w_{T-1}
$w_{T-1}(\Delta C_{T-1}|\tilde{P}_{T-1})$
$\Delta C^{oo}_{T-1}(\tilde{P}_{T-1})$
ΔC_{T-1}
MW_{T-1}
ΔC_{T-1}
$\Delta C^{oo}_{T-1}(\tilde{P}_{T-1})$
MW_{T-1}

Figure 4

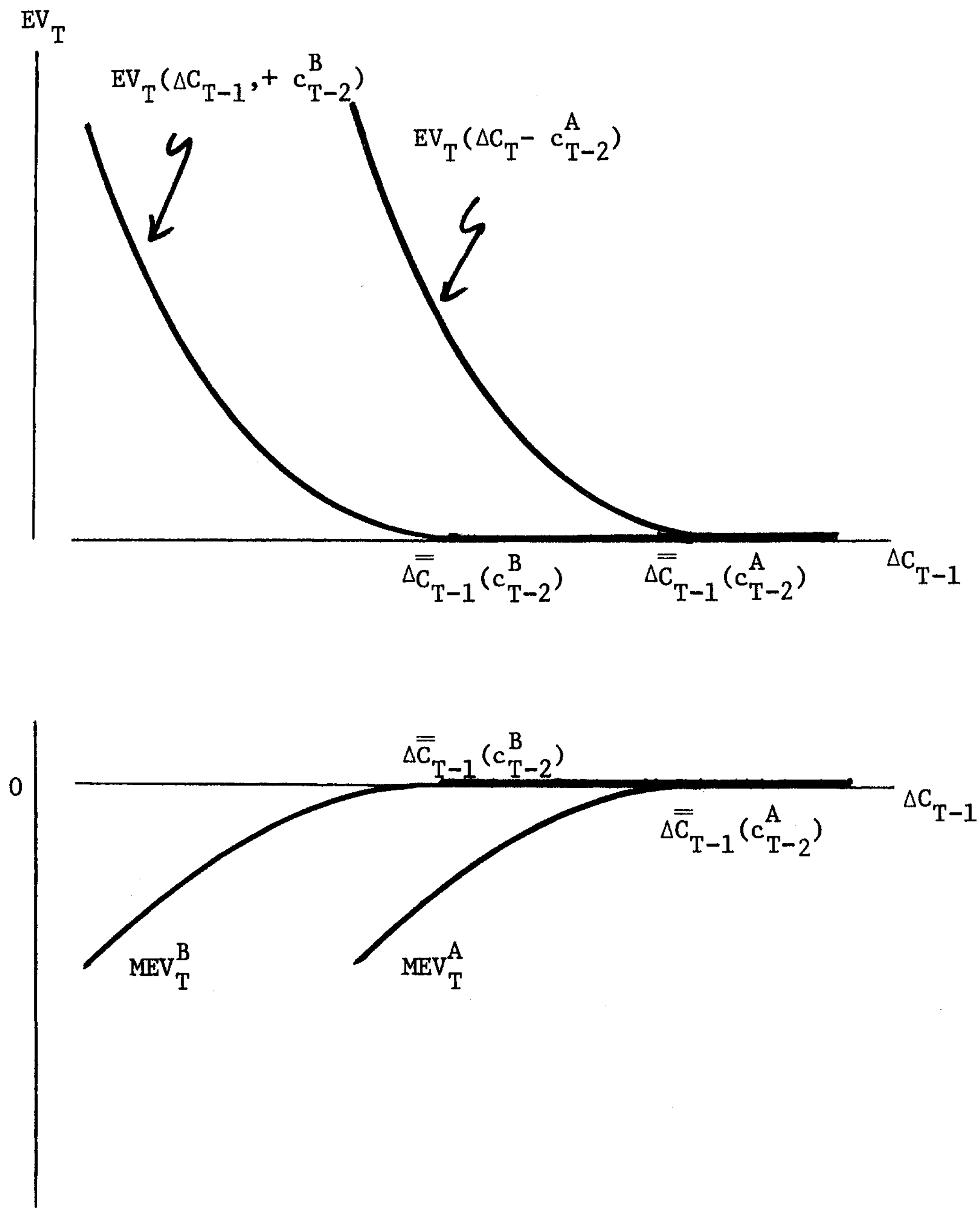
This curve is denoted by MW^*_{T-1} (marginal of w^*_{T-1}). Recall that w^*_{T-1} in the case of no cost is a measure of current price instability (for period T-1), so MW^*_{T-1} can be called the <u>current marginal destabilization of ΔC_{T-1}</u>.

Let us now add the other term, i.e., the measure of future instability EV^*_T. Given C_{T-2}, C_{T-1} is a direct function of ΔC_{T-1} and it is easy to translate the dependence of EV^*_T on C_{T-1} into a dependence on ΔC_{T-1}. Proposition 3_T stated that there is $\overline{\overline{C}}_{T-1}$ such that for $C_{T-1} < \overline{\overline{C}}_{T-1}$, EV^*_T is a strictly decreasing convex function of C_{T-1} , and for $C_{T-1} \geq \overline{\overline{C}}_{T-1}$, $EV^*_T = 0$ constantly. Equivalently, <u>given C_{T-2}</u> there exists $\overline{\overline{\Delta C}}_{T-1}(C_{T-2})$ such that whenever $\Delta C_{T-1} < \overline{\overline{\Delta C}}_{T-1}(C_{T-2})$, EV^*_T is a strictly decreasing convex function of ΔC_{T-1}, and whenever $\Delta C_{T-1} \geq \overline{\overline{\Delta C}}_{T-1}(C_{T-2})$, $EV^*_T = 0$. It is obvious that:

$$\overline{\overline{\Delta C}}_{T-1}(C_{T-2}) \equiv C_{T-2} - \overline{\overline{C}}_{T-1} .$$

Graphically the presentation of $EV^*_{T-1}(\Delta C_{T-1} + C_{T-2})$, given C_{T-2}, is almost identical to figure 3; one only has to translate C_{T-1} into ΔC_{T-1}. As C_{T-2} increases, the same value of EV^*_T is obtained with smaller stocks change, so that an increase in C_{T-2} shifts the EV_T curve parallel to the left. In figure 5, two $EV^*_{T-1}(\Delta C_{T-1} + C_{T-2})$ curves are depicted in the upper section, corresponding to two levels of C_{T-2}, namely, C^A_{T-2} and C^B_{T-2}, such that $C^A_{T-2} < C^B_{T-2}$. Their marginal curves are drawn in the lower section of figure 5.

Let us now study the behavior of the sum $w^*_{T-1} + EV^*_T$ as a function of the stocks change ΔC_{T-1}, given the beginning stocks C_{T-2} and the market price $\tilde{P}_{T-1}$. It is clear that the minimum of the sum cannot be with ΔC_{T-1} smaller than $\Delta C^{oo}_{T-1}(\tilde{P}_{T-1})$, because for such a level of ΔC_{T-1}, w^*_{T-1} strictly



$c^A_{T-2} < c^B_{T-2}$

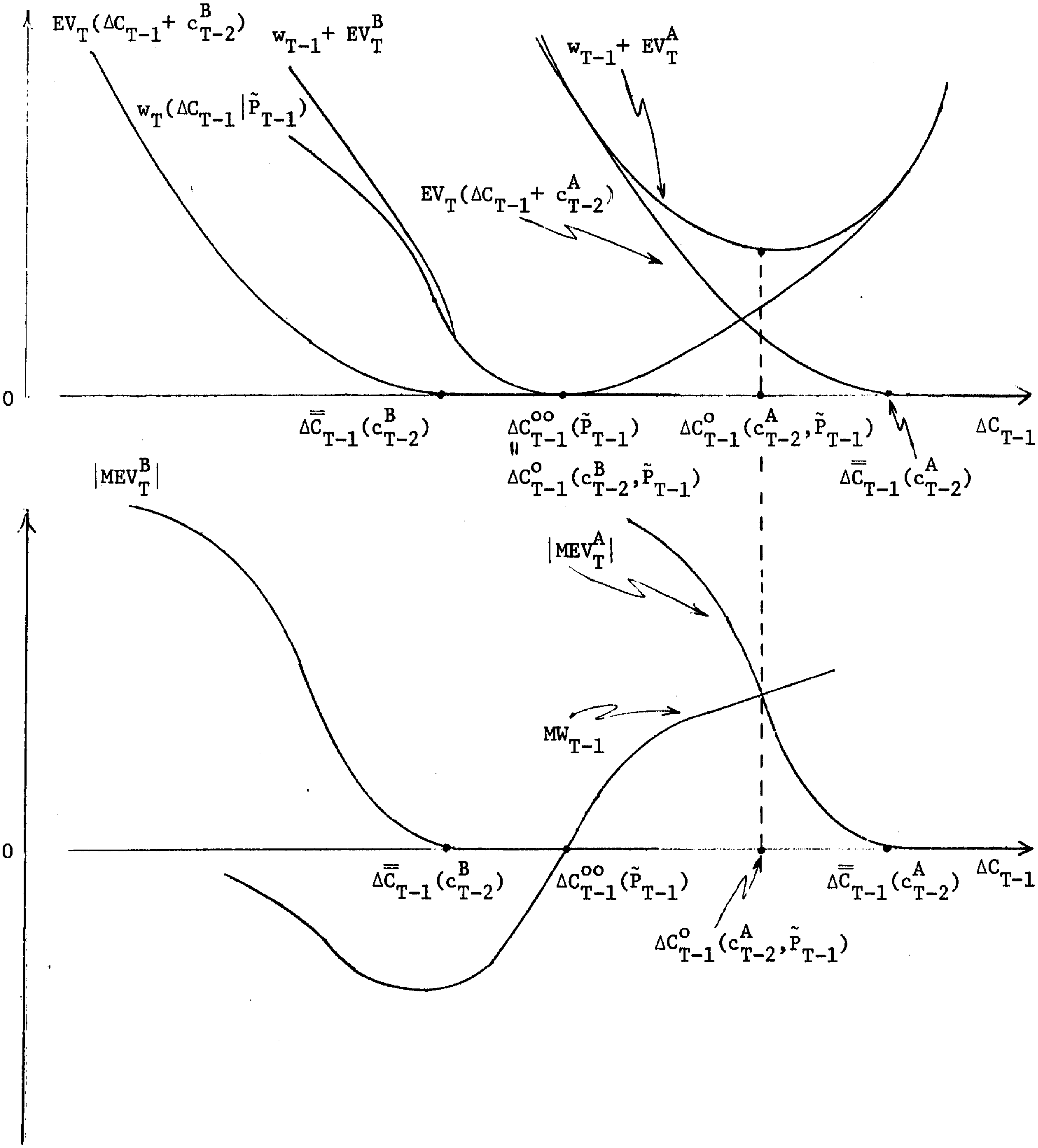
Figure 5

decreases with ΔC_{T-1} and EV_T^* is decreasing or equals 0. If there is a relatively small amount of beginning stocks (C_{T-2}), EV_T^* might still be decreasing at $\Delta C_{T-1}^{oo}(\tilde{P}_{T-1})$, i.e., w_{T-1}^* is at its minimum but EV_T can be reduced more by increasing the stocks change ΔC_{T-1}. This means that at $\Delta C_{T-1}^{oo}(\tilde{P}_{T-1})$ current price (P_{T-1}) is completely stabilized and equals the target price (p_{T-1}^*). However, future price instability can be reduced more. It is worthwhile to increase ΔC_{T-1} as long as the marginal increase of current instability is smaller than the marginal reduction of future instability. Using the terminology defined above: It is worthwhile to increase ΔC_{T-1} as long as the marginal current destabilization (MW_{T-1}^*) is smaller than the marginal future stabilization ($|MEV_T^*|$). The minimum of the sum $w_{T-1}^* + EV_T^*$ is obtained where $MW_{T-1}^* = |MEV_T^*| \equiv - MEV_T^*$ (i.e., the derivative of w_T^* with respect to ΔC_{T-1} is equal to minus the derivative of EV_T^*. Let us denote the unconstrained minimum point of $w_{T-1}^* + EV_T^*$ by $\Delta C_{T-1}^{o}(C_{T-2}, \tilde{P}_{T-1})$ to indicate that it depends on C_{T-1} as well as on $\tilde{P}_{T-1}$ (compare to ΔC_T^o which is a function of only $\tilde{P}_T$). The convexity of w_{T-1}^* in ΔC_{T-1} for $\Delta C_{T-1} > \Delta C_{T-1}^{oo}$ and the convexity of EV_T^* in C_{T-1} ensure that there is a unique minimum ΔC_{T-1}^o.

Figure 6 demonstrates the determination of $\Delta C_{T-1}^o(C_{T-2}, \tilde{P}_{T-1})$ in the case which has just been discussed, i.e., when

$$\Delta C_{T-1}^o(C_{T-2}, \tilde{P}_{T-1}) > \Delta C_{T-1}^{oo}(\tilde{P}_{T-1}):$$

The curve of $w_{T-1}^*(\Delta C_{T-1}|\tilde{P}_{T-1})$ and its marginal curve MW_{T-1}^* are drawn in the upper and lower sections of figure 6, respectively, exactly as in figure 4. The curve $EV_T^*(\Delta C_{T-1} + c_{T-2}^A)$ is an EV_T^* curve corresponding to a relatively small amount of beginning stocks $C_{T-2} = c_{T-2}^A$, as in figure 5.



EV_T(ΔC_T-1 + c^B_T-2)
w_T-1 + EV^B_T
w_T-1 + EV^A_T
w_T(ΔC_T-1 | P̃_T-1)
EV_T(ΔC_T-1 + c^A_T-2)
0
ΔC̿_T-1(c^B_T-2)
ΔC^oo_T-1(P̃_T-1)
ΔC^o_T-1(c^B_T-2, P̃_T-1)
ΔC^o_T-1(c^A_T-2, P̃_T-1)
ΔC̿_T-1(c^A_T-2)
ΔC_T-1
|MEV^B_T|
|MEV^A_T|
MW_T-1
0
ΔC̿_T-1(c^B_T-2)
ΔC^oo_T-1(P̃_T-1)
ΔC̿_T-1(c^A_T-2)
ΔC_T-1
ΔC^o_T-1(c^A_T-2, P̃_T-1)

Figure 6

Instead of drawing the marginal curve of EV_T^*, the negative of it is drawn in the lower section of figure 6, denoted by $|MEV_T^{*A}|$. The curve of the sum $w_{T-1}^* + EV_T^{*A}$ is also drawn in the upper section. Its minimum is at $\Delta C_{T-1}^o(c_{T-2}^A, \tilde{P}_{T-1})$ where the marginal current destabilization curve (MW_{T-1}^*) intersects the marginal future stabilization curve $(|MEV_T^*|)$. Suppose now that there is a relatively large beginning stocks (C_{T-2}) such that at $\Delta C_{T-1}^{oo}(\tilde{P}_{T-1})$, EV_T^* is 0. In this case it is obvious that the minimum of the sum $w_{T-1}^* + EV_T^*$ is obtained at $\Delta C_{T-1}^{oo}(\tilde{P}_{T-1})$, i.e.:

$$\Delta C_{T-1}^o(C_{T-2}, \tilde{P}_{T-1}) = \Delta C_{T-1}^{oo}(\tilde{P}_{T-1}) .$$

There are enough stocks to completely stabilize both current and future prices. The last case is demonstrated also in figure 6. $EV_T(\Delta C_{T-1} + c_{T-2}^B)$ corresponds to beginning stocks $C_{T-2} = c_{T-2}^B$ greater than c_{T-1}^A. $w_{T-1}^* + EV_T^{*B}$ is the curve of the sum in case B and $|MEV_T^{*B}|$ is the marginal future stabilization corresponding to c_{T-2}^B.

Notice that in general the change of stocks ΔC_{T-1}^o which minimizes $w_{T-1}^* + EV_{T-1}^*$, given C_{T-2} and $\tilde{P}_{T-1}$, does not necessarily minimize current price instability but may keep the end price (P_{T-1}) greater than the target price (p_{T-1}^*). This is so in order to accumulate adequate stocks for future contingencies, expressed here by EV_T^*. The need to do that is weakened the more beginning stocks (C_{T-2}) exist. Therefore, the more C_{T-2} is, the closer is the minimizing stocks change ΔC_{T-1}^o to ΔC_{T-1}^{oo} and the closer the final price (P_{T-1}) to the target price (p_{T-1}^*) and the smaller is the minimum value of $w_{T-1}^* + EV_T^*$. Given any market price $\tilde{P}_{T-1}$, there is a beginning stocks level, call it $\overline{C}_{T-2}(\tilde{P}_{T-1})$, such that if $C_{T-2} \geq \overline{C}_{T-2}(\tilde{P}_{T-1})$, then the minimum of $w_{T-1}^* + EV_T^*$ is 0 (i.e., complete stabilization of current and future prices). $\overline{C}_{T-2}(\tilde{P}_{T-1})$ is the amount of C_{T-2} for which the EV_T^* curve intersects the w_{T-1}^* curve at $w_{T-1}^* = EV_T^* = 0$ (also $MW_{T-1}^* = |MEV_T^*| = 0$). Hence, $\overline{C}_{T-2}(\tilde{P}_{T-1})$ is defined by

$$\overline{\overline{C}}_{T-1} - \overline{C}_{T-2}(\tilde{P}_{T-1}) \equiv \Delta C_{T-1}^{oo}(\tilde{P}_{T-1})$$

or

$$\overline{C}_{T-2}(\tilde{P}_{T-1}) \equiv \overline{\overline{C}}_{T-1} - \Delta C_{T-1}^{oo}(\tilde{P}_{T-1}) \; .$$

So far, we ignored the constraint $(1.32)_{T-1}$. Let us consider it now. If it happens that the unconstrained minimum, $\Delta C_{T-1}^{o}(C_{T-2}, \tilde{P}_{T-1})$ is smaller than $- C_{T-2}$, ΔC_{T-1}^{o} is not feasible and the best which can be done is to reduce stocks to zero, i.e., the optimal stocks change in this case is $- C_{T-1}$. To summarize, let us denote the optimal stocks rule for T-1 by $\Delta\hat{C}_{T-1}(C_{T-2}, \tilde{P}_{T-1})$.

$(1.35)_{T-1}$ $\quad \Delta\hat{C}_{T-1}(C_{T-2}, \tilde{P}_{T-1})$

$$= \begin{cases} - C_{T-1} & \text{if } C_{T-2} \leq - \Delta C_{T-1}^{o}(C_{T-2}, \tilde{P}_{T-1}) \\ \Delta C_{T-1}^{o}(C_{T-2}, \tilde{P}_{T-1}) > C_{T-1}^{oo}(\tilde{P}_{T-1}) & \text{if } - \Delta C_{T-1}^{o}(C_{T-2}, \tilde{P}_{T-1}) \\ & \qquad \leq C_{T-2} \leq \overline{C}_{T-2}(\tilde{P}_{T-1}) \\ C_{T-1}^{oo}(\tilde{P}_{T-1}) & \text{if } \overline{C}_{T-2}(\tilde{P}_{T-1}) \leq C_{T-2} \end{cases}$$

It is obvious that the first line of $(1.35)_{T-1}$ can be valid only if the market price $\tilde{P}_{T-1}$ is greater than the target price p^*_{T-1} because only when $\tilde{P}_{T-1} > p^*_{T-1}$, $\Delta C^o_{T-1}(C_{T-2}, \tilde{P}_{T-1})$ can be negative. Recall again that although the optimal stocks rule of the last period T was also a function of the beginning stocks (C_{T-1}), it depended on it only through the constraint $(1.32)_T$, i.e., when there are not enough stocks to dispose of in order to get the unconstrained minimum. Otherwise, $\Delta\hat{C}_T$ was a function only of the market price $\tilde{P}_T$. However, for T-1 the optimal stocks rule $\Delta\hat{C}_{T-1}$ is a function of the beginning stocks C_{T-2} even if the constraint is not violated by the unconstrained minimum. This is true even if the market price is less than the target price within a certain boundary. This is due to the need to accumulate stocks for the future (i.e., for T). Only when there are enough stocks to completely stabilize both current and future prices, then the optimal stocks depend on the market price and the only consideration then is to equate the price to the target price. Recall that the minimal level of beginning stocks which enables the current price to be equated to the target price, as well as all possible future prices to be equated to the target price, was denoted by $\overline{C}_{T-2}(\tilde{P}_{T-1})$.

The following Proposition summarizes the characterization of $\hat{C}_{T-1}(\cdot)$:

<u>Proposition 1_{T-1}</u> (Optimal stocks change for t=T-1): Generally the optimal stocks change for period t=T-1 is a function of the beginning stocks (C_{T-2}) and the market price $\tilde{P}_{T-1}$. (1) For a given $\tilde{P}_{T-1} > p^*_t$, there exists a level of beginning stocks, $\underline{C}_{T-2}$, such that whenever $C_{T-2} \leq \underline{C}_{T-2}$ all the existing stocks will be disposed of in order to reduce the price P_{T-1} as close as possible to the target price p^*_{T-1} , and nothing will be carried out to period T.[7/]

[7/] In fact, $\underline{C}_{T-2} \equiv \text{Max}\,\{0, -\Delta C^o_{T-1}(C_{T-2}, \tilde{P}_{T-1})\}$

(2) For any given $\tilde{P}_{T-1}$ there is also a quantity of beginning stocks $\overline{C}_{T-2}(\tilde{P}_{T-1})$ which enables us to stabilize completely prices of both T-1 and T. Whenever $C_{T-2} \geqq \overline{C}_{T-2}(\tilde{P}_{T-1})$ the optimal stocks change is constantly equal to that amount which is needed to equate the current price to the target price. In this case, and only in this case, the optimal stocks change depends on price only.

(3) Whenever the beginning stocks are between $\underline{C}_{T-2}$ and $\overline{C}_{T-2}$ the optimal stocks change is a decreasing function of C_{T-2}, given $\tilde{P}_{T-1}$.

Denoting the end price when applying the optimal stocks rule for t=T-1 by $\hat{P}_{T-1}(C_{T-2}, \tilde{P}_{T-1})$, it can be concluded that

$$(1.36)_T \quad \hat{P}_{T-1}(C_{T-2}, \tilde{P}_{T-1}) \begin{cases} > \hat{p}^*_{T-1} & \text{if } C_{T-2} \leq \overline{C}_{T-1}(\tilde{P}_{T-1}) \\ \\ = p^*_{T-1} & \text{if } C_{T-2} \geq \overline{C}_{T-1}(\tilde{P}_{T-1}) \end{cases}$$

Turn now to the value of $w^*_{T-1} + EV^*_T$ under the optimal stocks rule, which is denoted by $V^*_{T-1}(C_{T-2}, \tilde{P}_{T-1})$

$$V^*_{T-1} = \hat{w}^*_{T-1} + E\,\hat{V}^*_T$$

From the above discussion it follows that

$$(1.37)_{T-1} \quad V^*_{T-1}(C_{T-2}, \tilde{P}_{T-1})$$

$$= \begin{cases} w^*_{T-1}(-\,C_{T-2}|\tilde{P}_{T-1}) + EV^*_T(0) & \text{if } C_{T-2} \leq -\,\Delta C^o_{T-1}(C_{T-2}, \tilde{P}_{T-1}) \\ w^*_{T-1}(\Delta C^o_{T-1}|\tilde{P}_{T-1}) + EV^*_T(C_{T-2} + \Delta C^o_{T-1}) & \text{if } -\,\Delta C^o_{T-1}(C_{T-2}, \tilde{P}_{T-1}) \\ & \leq C_{T-2} \leq \overline{C}_{T-2}(\tilde{P}_{T-1}) \\ 0 & \text{if } \overline{C}_{T-2}(\tilde{P}_{T-1}) \leq C_{T-2} \end{cases}$$

To characterize the relation between V^*_{T-1} and C_{T-2}, given $\tilde{P}_{T-1}$, let us derive the partial derivative with respect to C_{T-2}. For $C_{T-2} \leq -\,\Delta C^o_T(C_{T-2}, \tilde{P}_{T-1})$ it is obvious that the derivative of $\Delta\hat{\ } C_{T-1}$ with respect to C_{T-2} is negative, since in this case it is optimal to dispose of all the stocks. In the case of $\overline{C}_{T-2} > C_{T-2} \geq \Delta C^o_{T-1}(C_{T-2}, \tilde{P}_{T-1})$ the derivative is also negative, i.e., the more beginning stocks, the less acquiring of stocks (or more selling from stocks). Formally this can be shown as follows:

The condition for the optimum is:

$$\frac{\partial w^*_{T-1}(\Delta\hat{\ } C_{T-1}|\tilde{P}_{T-1})}{\partial \Delta C_{T-1}} = -\,\frac{\partial EV^*_T(C_{T-2} + \Delta\hat{\ } C_{T-1})}{\partial C_{T-1}} \,.$$

Hence:

$$\frac{\partial^2 w^*_{T-1}}{\partial \Delta C^2_{T-1}}\, d\Delta\hat{\ } C_{T-1} = -\,\frac{\partial^2 EV^*_T}{\partial C^2_{T-1}}\,(d\Delta\hat{\ } C_{T-1} + dC_{T-2})$$

$$\frac{\partial^2 w^*_{T-1}}{\partial \Delta C^2_{T-1}} > 0 \quad \text{for } \Delta C_{T-1} > C^{oo}_{T-1}(\tilde{P}_{T-1}). \qquad \text{(See (1.30))}$$

$$\frac{\partial^2 EV^*_T}{\partial C^2_{T-1}} \geqq 0 \quad \text{by Proposition } 3_T$$

therefore

$$\frac{d\hat{\Delta} C_{T-1}}{dC_{T-2}} = \frac{-\dfrac{\partial^2 EV^*_T}{\partial C^2_{T-1}}}{\dfrac{\partial^2 w^*_{T-1}}{\partial \Delta C^2_{T-1}} + \dfrac{\partial^2 EV^*_T}{\partial C^2_{T-1}}} < 0$$

For $C_{T-2} \geq \bar{C}_{T-2}(\tilde{P}_{T-1})$ $\quad \hat{\Delta} C_{T-1} = \Delta C^{oo}_{T-1}(\tilde{P}_{T-1})$ and the derivative equals 0.

In summary:

$$(1.38)_{T-1} \quad 0 \geq \frac{\partial \hat{\Delta} C_{T-1}(C_{T-2}, \tilde{P}_{T-1})}{\partial C_{T-2}}$$

$$= \begin{cases} -1 & \text{if } C_{T-2} \leq -\Delta C^{o}_{T-1}(C_{T-2}, \tilde{P}_{T-1}) \\[2ex] \dfrac{-\partial^2 EV^*_T/\partial C^2_{T-1}}{\dfrac{\partial^2 w^*_{T-1}}{\partial \Delta C^2_{T-1}} + \dfrac{\partial^2 EV^*_T}{\partial C^2_{T-1}}} & \text{if } -\Delta C^{o}_{T-1}(C_{T-2}, \tilde{P}_{T-1}) \leq C_{T-2} \leq \bar{C}_{T-2}(\tilde{P}_{T-1}) \\[2ex] 0 & \text{if } \bar{C}_{T-2}(\tilde{P}_{T-1}) \leq C_{T-2} . \end{cases}$$

The marginal change in $V^*_{T-1}(C_{T-2}, \tilde{P}_{T-1})$ is given by

$$(1.39')_{T-1} \quad \frac{\partial V^*_{T-1}(C_{T-2}, \tilde{P}_{T-1})}{\partial C_{T-2}}$$

$$= \frac{\partial w^*_{T-1}(\Delta C_{T-1} | \tilde{P}_{T-1})}{\partial \Delta C_{T-1}} \frac{\partial \Delta \hat{} C_{T-1}}{\partial C_{T-2}} + \frac{\partial EV^*_T(C_{T-1})}{\partial C_{T-1}} \left(1 + \frac{\partial \Delta \hat{} C_{T-1}}{\partial C_{T-2}}\right)$$

where: $\partial w^*_{T-1}/\partial \Delta C_{T-1}$ is given by (1.24) and is positive, except when $\overline{C}_{T-1}(\tilde{P}_{T-1}) \leq C_{T-2}$, in which case it is 0 ,

$\dfrac{\partial EV^*_T(C_{T-1})}{\partial C_{T-1}}$ is negative except when $\overline{C}_{T-1}(\tilde{P}_{T-2}) \leq C_{T-2}$, in which case it is 0,

and $0 \geqq \dfrac{\partial \Delta \hat{C}_{T-1}}{\partial C_{T-2}} \geqq -1$ is given by $(1.38)_{T-1}$.

Using the details of $(1.38)_{T-1}$ we summarize:

$$(1.39)_{T-1} \quad \frac{\partial V^*_{T-1}(C_{T-2}, \tilde{P}_{T-1})}{\partial C_{T-2}}$$

$$= \begin{cases} \dfrac{-\partial w^*_{T-1}}{\partial \Delta C_{T-1}} & \text{if } C_{T-2} \leq -\Delta C^o_{T-1}(C_{T-2}, \tilde{P}_{T-1}) \\[2em] \dfrac{-\partial w^*_{T-1}}{\partial \Delta C_{T-1}} \dfrac{\partial^2 EV^*_T/\partial C^2_{T-1}}{\dfrac{\partial^2 w^*_{T-1}}{\partial \Delta C^2_{T-1}} + \dfrac{\partial^2 EV^*_T}{\partial C^2_{T-1}}} + \dfrac{\partial EV^*_T}{\partial C_{T-1}} \dfrac{\partial^2 w^*_{T-1}/\partial \Delta C^2_{T-1}}{\dfrac{\partial^2 w^*_{T-1}}{\partial \Delta C^2_{T-1}} + \dfrac{\partial^2 EV^*_T}{\partial C^2_{T-1}}} & \text{if } -\Delta C^o_{T-1}(C_{T-2}, \tilde{P}_{T-1}) \\ & \leq C_{T-2} \leq \overline{C}_{T-2}(\tilde{P}_{T-1}) \\[2em] 0 & \text{if } \overline{C}_{T-2}(\tilde{P}_{T-1}) \leq C_{T-2} \end{cases}$$

The meaning of $(1.39)_{T-1}$ is as follows. The first line of $(1.39)_{T-1}$ refers to the case in which the constraint $(1.32)_{T-1}$ is effective, no stocks are carried out to period T and all C_{T-2} is sold at period T-1 to reduce the price to as close as possible to the target price p^*_{T-1}. The second line of $(1.39)_{T-1}$ refers to the case in which both the current and future instability measures are affected by C_{T-2}: On the one hand, the more C_{T-2}, the less will be the stocks change $\Delta\hat{C}_{T-1}$(see $(1.38)_{T-1}$) and the closer will be the current price $(\hat{P}_{T-1})$ to the target price (p^*_{T-1}). (Recall that $\partial w^*_{T-1}/\partial\Delta C_{T-1}$ is positive when $P_{T-1} > p^*_T$.) On the other hand, although $\Delta\hat{C}_{T-1}$ is reduced when C_{T-2} increases, it decreases less than the increase in C_{T-2} (recall from $(1.38)_{T-1}$ that $\partial\Delta\hat{C}_{T-1}/\partial C_{T-2} < -1$). Therefore in total, $C_{T-1} = C_{T-2} + \Delta C_{T-1}$, increases with C_{T-2} and reduces the measure EV^*_T of future instability. Finally, the third line of $(1.33)_{T-1}$ corresponds to the case where both current and future prices are already stabilized, hence additional beginning stocks make no difference; the sum of $w^*_{t-1} + EV^*_T$ remains constantly zero. To complete the characterization of the optimal stocks rule the convexity of $V^*_{T-1}(C_{T-2}, \tilde{p}_{T-1})$ in C_{T-2} will be proved:

It was shown that at the optimal state, given $\tilde{p}_{T-1}$ and C_{T-2}, the final price (P_{T-1}) is always greater or equal to p^*_{T-1}; hence w^*_{T-1} is convex in ΔC_{T-1} (since $\partial^2 w^*_{T-1}/\partial\Delta C^2_{T-1} \geq 0$ when $P_{T-1} > p^*_{T-1}$). In addition EV_T is convex in C_{T-1}, hence it is convex in ΔC_{T-1} and C_{T-2} $(C_{T-1} = C_{T-2} + \Delta C_{T-1})$. Altogether, $w^*_{T-1} + EV^*_T$ is convex in ΔC_{T-1} and C_{T-2}. The following lemma can be proved:

Lemma: Let $Z(x, a)$ be convex in x and a.

Let x(a) be a solution of the following minimization problem:

Given a,

$$\operatorname*{Min}_{\substack{x \\ -a \leq x}} \mathcal{Z}(x,\ a)$$

and let $\hat{\mathcal{Z}}(a)$ be the minimum value of $\mathcal{Z}$ in this problem, i.e.:

$\hat{\mathcal{Z}}(a) = \mathcal{Z}(\hat{x}(a),\ a) \leq \mathcal{Z}(x,\ a)$ for all x such that $-a \leq x$.

Then $\mathcal{Z}(a)$ is convex in a.[8/]

Apply the lemma to the problem of minimizing $(1.33)_{T-1}$ subject to $(1.32)_{T-1}$ for a given $\tilde{P}_{T-1}$ by substituting ΔC_{T-1} for x, C_{T-2} for a, $w_T(\Delta C_{T-1}|\tilde{P}_{T-1}) + EV_T(C_{T-2} + \Delta C_{T-1})$ for $\mathcal{Z}(x,\ a)$ and $V_{T-1}(C_{T-2},\ \tilde{P}_{T-1})$ for $\mathcal{Z}(a)$. The consequence is that $V_{T-1}(C_{T-2},\ \tilde{P}_{T-1})$ is convex in C_{T-2}, for any given $\tilde{P}_{T-1}$.

Let us summarize the effect of C_{T-2} on $V_{T-1}(C_{T-2},\ \tilde{P}_{T-1})$ by the following Proposition, which is very similar to Proposition 2_T.

Proposition 2_{T-1} (The effect of beginning stocks, given the market price for t=T-1). Given the market price ($\tilde{P}_{T-1}$) there is a level of beginning stocks, $\overline{C}_{T-2}(\tilde{P}_{T-1})$ which is an increasing function of $\tilde{P}_{T-1}$, such that:

(1) Whenever the beginning stocks (C_{T-2}) are smaller than $\overline{C}_{T-2}(\tilde{P}_{T-1})$,

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[8/] Proof of the lemma: We have to prove that:

$$\hat{\mathcal{Z}}[\lambda a' + (1-\lambda)a''] \leq \lambda\hat{\mathcal{Z}}(a') + (1-\lambda)\hat{\mathcal{Z}}(a''),\ 0 \leq \lambda \leq 1.$$

(a', a'' are the two values of a.)

Define $x \equiv \lambda\hat{x}(a') + (1-\lambda)\hat{x}(a'')$.

$\hat{x}(a') \geq -a',\ \hat{x}(a'') > -a'' \implies x \geq \lambda a' + (1-\lambda)a''$.

(continued)

then the optimal stocks change $(\Delta\hat{C}_T)$ is a strictly decreasing function of C_{T-2}, and $V^*_{T-1}(C_{T-2}, \tilde{P}_{T-1})$, the optimal value, is positive and is a strictly decreasing and convex function of C_{T-2}.

(2) Whenever $C_{T-2} \geq \bar{C}_{T-2}(\tilde{P}_{T-1})$, $\Delta\hat{C}_{T-1}$ equals constantly to the unconstrained minimum of w_T(i.e., to $\Delta C^{oo}_{T-1}(\tilde{P}_{T-1})$) and $V^*_{T-1}(C_{T-2}, \tilde{P}_T)$ is constantly 0 (i.e., complete stabilization of both current and future prices).

Returning to the original problem T-1 in which it has to minimize the expected value $E(w^*_{T-1} + EV^*_T)$, let us calculate the expectation of $V^*_{T-1}(C_{T-2}, \tilde{P}_{T-1})$(recall that $\tilde{P}_{T-1}$ is a function of ε_{T-1}(1.23)).

$$(1.41)_{T-1} \quad EV_{T-1}(C_{T-2}) = \int V^*_{T-1}[C_{T-2}, \phi^{-1}_{T-1}(X^o_{T-1} + \varepsilon_{T-1})]df_{T-1}(\varepsilon_{T-1}).$$

As in period T, all the qualitative results which characterize the relation between V^*_{T-1} and C_{T-2}, for any $\tilde{P}_{T-1}$, are translated to the relation between EV^*_{T-1} and C_{T-2}. In addition the assumption that there exists $\bar{P}$ such that $\quad$ Prob $\{P > \bar{P}\} = 0$, Prob $\{P = \bar{P}\} > 0$
implies that $EV^*_{T-1}(C_{T-2})$ reaches its minimum at 0 (i.e., complete stabilization for a large enough C_{T-2}). In short,

<u>Proposition 3_{T-1}</u>. Proposition 3_T which was stated for t=T is true for T-1.

(footnote <u>8</u>/ continued)

$$\hat{z}[\lambda a' + (1-\lambda)a''] \overset{(1)}{\leq} z[x, \lambda a' + (1-\lambda)a''] \equiv z[\lambda\hat{x}(a') + (1-\lambda)\hat{x}(a''), \lambda a' + (1-\lambda)a''] \overset{(2)}{\leq} \lambda z[\hat{x}(a'), a'] + (1-\lambda)z[\hat{x}(a'', a'')] \equiv \lambda\hat{z}(a') + (1-\lambda)\hat{z}(a'')$$

(1) is implied by the fact that $z(\cdot)$ is a minimum, (2) is implied by the convexity of z. The other equalities are followed by definition Q.E.D.

It is now possible to characterize the solution for t=T-2, T-3, ..., 1. All the subproblems are qualitatively similar because of Proposition 3_{T-1}. Hence

<u>Proposition 4</u>. Propositions 1_{T-1}, 2_{T-1} and 3_{T-1} are true for all t=1, 2, ..., T-1.

Summarizing the analysis of the no-cost case, it was shown that the optimal change of stocks is in general a function of both the market price and the quantity of beginning stocks. Only when there are enough stocks for complete stabilization for the whole planning period does the optimal stocks change depend only on price. Besides the last case, the optimal change of stocks is a decreasing function of the beginning stocks; nevertheless, the carryover increases with the quantity of beginning stocks. In addition, in the no-cost case it is optimal in general to accumulate more than is needed to stabilize current price. Hence in general, the mean price is higher than the target price.

Next we analyze the case with storage cost. Some of the results of case (a) are still valid but some of them are not. In particular it will be shown that it is not generally true that it is optimal to accumulate more than is needed to stabilize current price and that if costs are relatively high, it might be optimal to accumulate less than that.

<u>Case (b): Storage cost, no lagged price</u>. Let us assume now that unit storage cost is θ per period of time. Suppose also that the valuation of instability (as measured by the instability index) in terms of cost is λ. The objective to be minimized by the stocks policy is

$$E\{[\sum\left(\frac{P_t - P_t^*}{P_t^*}\right)^2 \lambda + \theta C_t]\frac{1}{(1+\delta)^t}\}$$

For convenience let us continue to assume that the discount rate is zero (no effect on the qualitative results). Defining the t-th term of the objective function by:

$$w_t = w_t^* \cdot \lambda + C_t \cdot \theta.$$

w_t is decomposed into instability (w_t^*) and cost (C_T). The objective function can be rewritten as:

$$E\{\sum_{t=1}^{T} w_t\} = E\{\sum_{t=1}^{T} w_t^* \cdot \lambda + C_t \cdot \theta\}. \tag{1.42}$$

Equations (1.20) through (1.30) are valid also in the present case and we continue to assume no lagged price.

Starting with the last period's stocks rule:

<u>Problem T</u>: Given C_{T-1}, find stock rule $\Delta\hat{C}_T(C_{T-1}, \tilde{P}_T)$ that minimizes

$$E\{w_T\} = E\{w_T^*\lambda + C_T \cdot \theta\} \tag*{$(1.43)_T$}$$

subject to the constraint

$$\Delta C_T \geq - C_{T-1} \, . \tag*{$(1.44)_T$}$$

The stocks rule is the solution of the following problem:

Given C_{T-1} and $\tilde{P}_T$

$$\begin{aligned} \underset{\Delta C_T}{\text{minimize}}\ w_T & \\ &= w_T^*\lambda + \theta C_T \\ &= w_T^*\lambda + \theta\Delta C + \theta C_{T-1} \end{aligned} \tag*{$(1.45)_T$}$$

subject to the constraint $(1.44)_T$.

As in the no-cost case, first we analyze the unconstrained minimum of w_T. The necessary condition is to equate the derivative with respect to ΔC_T to zero, i.e.:[9/]

$$(1.46)_T \qquad \frac{\partial w_T}{\partial \Delta C_T} = \frac{\partial w_T^*(\Delta C_T | \tilde{P}_T)}{\partial \Delta C_T} \lambda + \theta = \frac{2\lambda}{p_T^*} (P_T - p_T^*) \frac{-1}{\phi_T'(P_T)} + \theta = 0 \ .$$

Recall that P_T is a function of $\tilde{P}_T$ and ΔC_T (see (1.22)). $(1.46)_T$ can be solved for P_T. Then, given $\tilde{P}_T$, the minimizing unconstrained stocks change, $\Delta C_T^o(\tilde{P}_T)$ can be solved.

The price which solves $(1.46)_T$ does not depend on the beginning stocks C_{T-1}, hence the unconstrained minimizing change of stocks, $\Delta C_T^o(\tilde{P}_T)$, is a (decreasing) function of only the market price $(\tilde{P}_T)$ and not of the beginning stocks (C_{T-1}). It should be noted that, contrary to the no-cost case, the price which solves $(1.46)_T$ is not equal to the target price but is lower than the target price. Denote this price by P_T^o.

$$P_T^o < p_T^*$$

The second derivative of w_T with respect to ΔC_T is identical to that of w_T^* (see (1.28)). In the present case the convexity of the demand function is not sufficient to ensure the positiveness of $\partial^2 w_T / \partial \Delta C_T^2$ because when $P_T < p_T^*$, the second derivative may be positive or negative or interchangeable. This has implications for the computational procedure, since it might not be sufficient to find a local minimum to ensure globality. However, linear

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[9/] See (1.24)(1.25) for $\partial w_t^* / \partial \Delta C_t$.

demand function and others (e.g., constant elasticity) do imply uniqueness. To avoid complexity, let us assume that the unconstrained minimum is unique and that in the relevant range (i.e., with high probability), w_t is convex in ΔC_t.

Figure 7 demonstrates the determination of the unconstrained minimum. The curve of $w_T^*(\Delta C_T|\tilde{P}_T)$, given $\tilde{P}_T$, is drawn in the upper section, and its marginal curve is denoted by MW_T^* in the lower section. Given C_{T-1}, it can be seen in $(1.45)_T$ that the cost part (C_T) can be separated into variable cost ($\theta\Delta C_T$) and fixed cost (θC_{T-1}); only the first one affects the solution and is drawn (after dividing by λ) in the upper section. The negative of its marginal is depicted in the lower section (the horizontal line through $\frac{\theta}{\lambda}$). In addition, in the upper section the curves of instability plus variable cost ($w_T^* + \frac{\theta}{\lambda}\Delta C_T$) and plus total cost ($w_{TT}^* + \frac{\theta}{\lambda} C_T$) are depicted (the vertical distance between them is $\frac{\theta}{\lambda} C_{T-1}$). Both of them have, of course, the same minimum at $\Delta C_T^o(\tilde{P}_T)$ where the marginal curve Mw_T^* intersects the marginal cost curve in the lower section. Notice that increasing the beginning stocks (C_{T-1}) only shifts the upper curve upwards but does not change its minimum. Notice also that the minimum of $\lambda w_T + \theta C_T$ is to the left of the minimum instability (i.e., minimum of w_T^*), so that $P_T^o < p_T^*$.

The value of the instability measure w_T^* at the unconstrained minimizing stocks change, $\Delta C_T^o(\tilde{P}_T)$, is denoted by V_T^{*o}. Contrary to the no-cost case, V_T^{*o} is positive (i.e., price is not completely stabilized).

Increasing $\tilde{P}_T$ causes a parallel shift to the left of all the curves which include w_T^*. $\Delta C_T^o(\tilde{P}_T)$ also decreases but the value of instability measure at ΔC_T^o (i.e., V_T^{*o}) is not changed.

As in the no-cost analysis, $\Delta C_T^o(\tilde{P}_T)$ may not be feasible, in which case

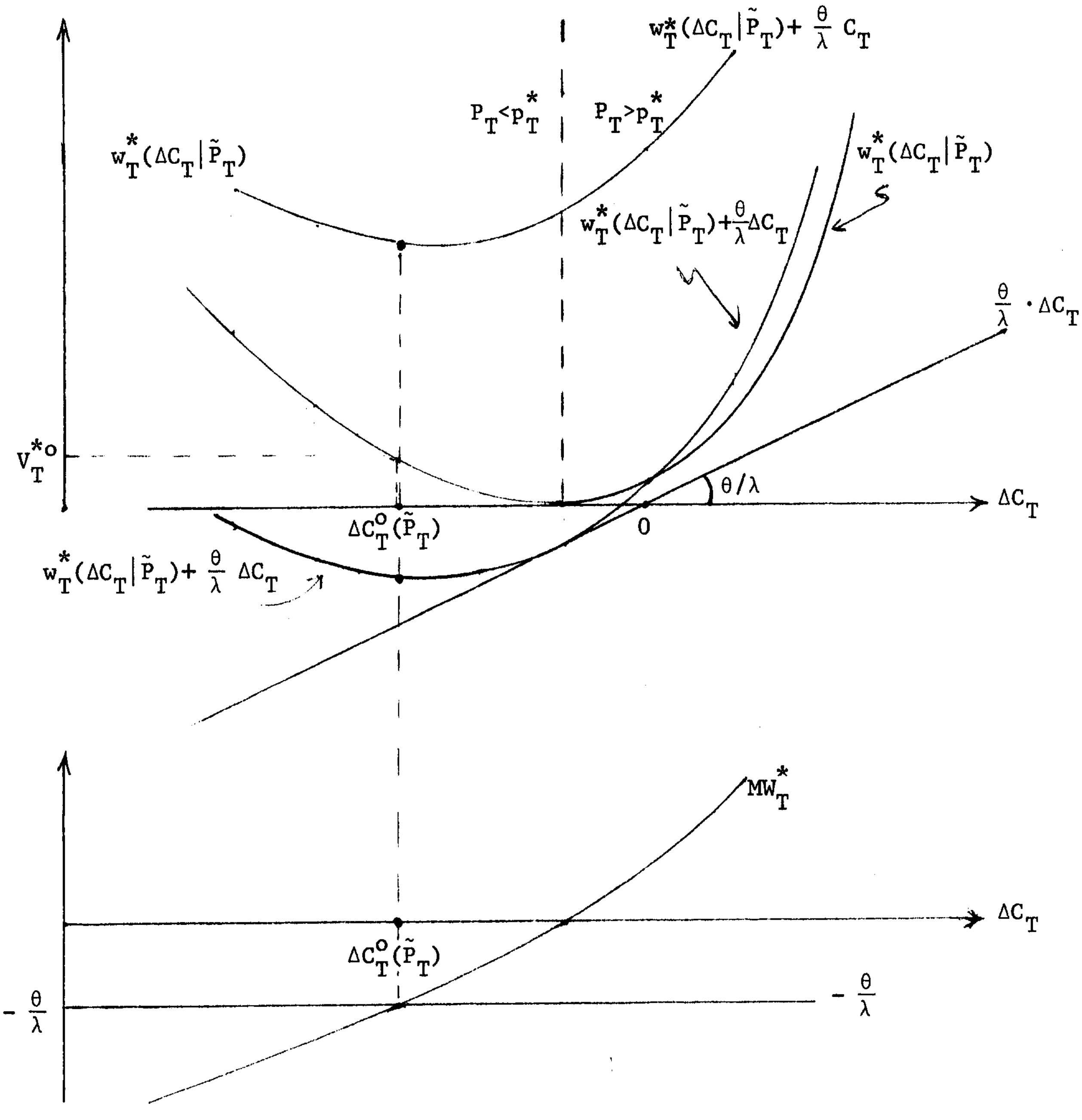

$w_T^*(\Delta C_T|\tilde{P}_T) + \frac{\theta}{\lambda} C_T$
$P_T < p_T^*$
$P_T > p_T^*$
$w_T^*(\Delta C_T|\tilde{P}_T)$
$w_T^*(\Delta C_T|\tilde{P}_T)$
$w_T^*(\Delta C_T|\tilde{P}_T) + \frac{\theta}{\lambda}\Delta C_T$
$\frac{\theta}{\lambda} \cdot \Delta C_T$
V_T^{*o}
θ/λ
ΔC_T
$\Delta C_T^o(\tilde{P}_T)$
0
$w_T^*(\Delta C_T|\tilde{P}_T) + \frac{\theta}{\lambda}\ \Delta C_T$
MW_T^*
ΔC_T
$\Delta C_T^o(\tilde{P}_T)$
$-\frac{\theta}{\lambda}$
$-\frac{\theta}{\lambda}$

Figure 7

the optimal rule is to set the price as close as possible to P_T^o, i.e., to dispose of all stocks.

Define $\overline{C}_{T-1}(\tilde{P}_T)$ by

$$\overline{C}_{T-1}(\tilde{P}_T) \equiv - \Delta C_T^o(\tilde{P}_T)$$

the optimal stocks change for T is:

$$(1.47)_T \quad \Delta\hat{C}_T(C_{T-1}, \tilde{P}_T) = \begin{cases} - C_{T-1} & \text{if } C_{T-1} \leq \overline{C}_{T-1}(\tilde{P}_T) \equiv - \Delta C_T^o(\tilde{P}_T) \\ \\ \Delta C_{T-1}^o(\tilde{P}_T) & \text{if } C_{T-1} \geq \overline{C}_{T-1}(\tilde{P}_T) \equiv - \Delta C_T^o(\tilde{P}_T) \end{cases}$$

The value of w_T under the optimal rule is denoted by $V_T(C_{T-1}, \tilde{P}_T)$ and its corresponding instability component by $V_T^*(C_{T-1}, \tilde{P}_T)$. They can be obtained by substituting $\Delta\hat{C}_T$ in w_T and w_T^* respectively. Given $\tilde{P}_T$ higher than P_T^*, a typical course of $V_T^*(C_{T-1}, \tilde{P}_T)$ is as follows: Starting from zero beginning stocks ($C_{T-1} = 0$), V_T^* decreases until it reaches its minimum, $V_T^* = 0$. Then it increases with C_{T-1} as long as $C_{T-1} \leq \overline{C}_{T-1}(\tilde{P}_T)$. For $C_{T-1} \geq \overline{C}_{T-1}(\tilde{P}_T)$, $V_T^*(C_{T-1}, \tilde{P}_T)$ is constantly equal to V_T^{*o} (because the stocks change is constant in this range (see $(1.47)_T$). A typical curve of $V_T^*(C_{T-1}, \tilde{P}_T)$ (given $\tilde{P}_T > p_T^*$) is depicted in the upper section of figure 8 with its marginal curve in the lower section (denoted by MV_T^*).

As to the course of $V_T(C_{T-1}, \tilde{P}_T) = V_T^*(C_{T-1}, \tilde{P}_T) + \theta C_T$, up to $\overline{C}_{T-1}(\tilde{P}_T)$ it coincides with $V_T^*(C_{T-1}, \tilde{P}_T)$ because no stocks C_T are carried over. However, for $C_{T-1} > \overline{C}_{T-1}(\tilde{P}_T)$, the change of stocks constantly equals $\Delta C_T^o(\tilde{P}_T)$, so that

$$V_T(C_{T-1}, \tilde{P}_T) = \lambda \cdot V_T^{*o} + \theta[C_{T-1} + \Delta C_T^o(\tilde{P}_T)]$$

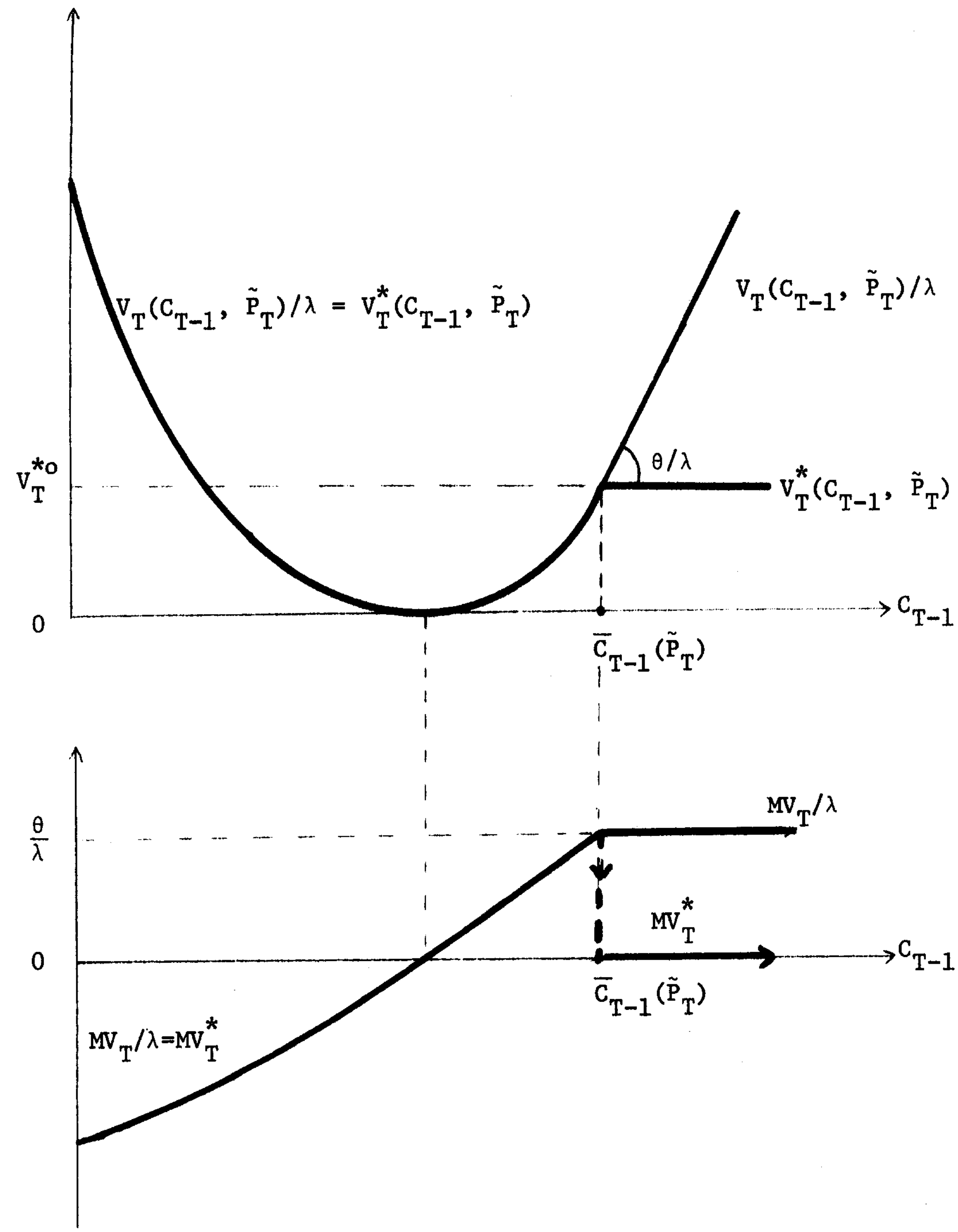

$V_T(C_{T-1}, \tilde{P}_T)/\lambda = V_T^*(C_{T-1}, \tilde{P}_T)$
$V_T(C_{T-1}, \tilde{P}_T)/\lambda$
V_T^{*o}
θ/λ
$V_T^*(C_{T-1}, \tilde{P}_T)$
0
C_{T-1}
$\bar{C}_{T-1}(\tilde{P}_T)$
$\frac{\theta}{\lambda}$
MV_T/λ
MV_T^*
0
C_{T-1}
$\bar{C}_{T-1}(\tilde{P}_T)$
$MV_T/\lambda = MV_T^*$

Figure 8

i.e., it is increasing linearly with C_{T-1}.

Dividing through by λ, the curve of V_T/λ and its marginal curve MV_T/λ are drawn in figure 8.

The effect of a higher market price is to shift the $w_T^*(\Delta C_T|\tilde{P}_T)$ curve to the left (i.e., to set the same price one must decrease the change of stocks). $\Delta C_T^o(\tilde{P}_T)$ also moves to the left (and by definition $\overline{C}_{T-1}(\tilde{P}_T)$ increases). However, V_T^{*o} is not affected because the price P_T^o, which solves $(1.46)_T$, is not changed. It follows that the effect of increasing $\tilde{P}_T$ is to shift the curves of figure 8 to the left. In figure 9, three curves of V_T/λ and V_T^* are drawn for three different market prices $\tilde{P}^A$, $\tilde{P}^B$, and $\tilde{P}^C$ such that:

$$\tilde{P}^A > \tilde{P}^B = p^* > \tilde{P}^C .$$

Their corresponding marginal curves are drawn in the lower section of figure 9.

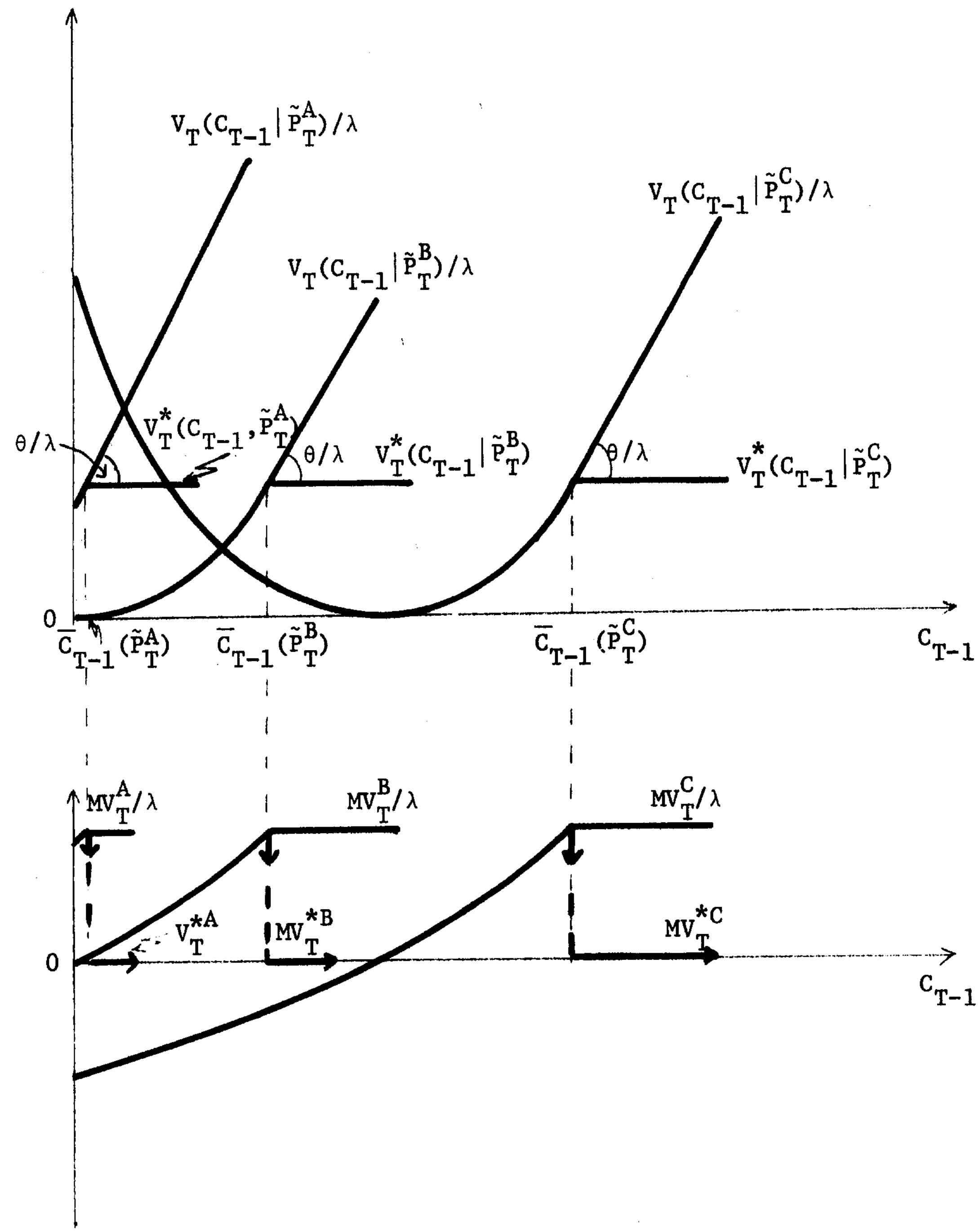
The following proposition is analogous to Propositions 1_T and 2_T in the no-cost case and summarizes the above analysis.

<u>Proposition 5_T</u> (Optimal stocks change for t=T).

(1) The optimal stocks change for the last period ($\Delta\hat{C}_T$) is generally to set the price as close as possible to P_T^o, which is lower than the target price p_T^*. The beginning stocks (C_{T-1}) affects $\Delta\hat{C}_T$ only in the sense that it might be smaller than the quantity needed for disposal in order to equate $P_T = p_T^o$. Otherwise $\Delta\hat{C}_T$ depends only on $\tilde{P}_T$.

(2) Given the market price ($\tilde{P}_T$), there is a level of beginning stocks, $\overline{C}_{T-1}(\tilde{P}_T)$, which by itself is an increasing function of $\tilde{P}_T$, such that:

(a) Whenever $C_{T-1} \leq \overline{C}_{T-1}(\tilde{P}_T)$, the optimal stocks change is equal to $- C_{T-1}$; $V_T(C_{T-1}, \tilde{P}_T)$ coincides with $\lambda V_T^*(C_{T-1}, \tilde{P}_T)$ and may



$V_T(C_{T-1}|\tilde{P}_T^A)/\lambda$
$V_T(C_{T-1}|\tilde{P}_T^B)/\lambda$
$V_T(C_{T-1}|\tilde{P}_T^C)/\lambda$
θ/λ
$V_T^*(C_{T-1},\tilde{P}_T^A)$
θ/λ
$V_T^*(C_{T-1}|\tilde{P}_T^B)$
θ/λ
$V_T^*(C_{T-1}|\tilde{P}_T^C)$
0
$\bar{C}_{T-1}(\tilde{P}_T^A)$
$\bar{C}_{T-1}(\tilde{P}_T^B)$
$\bar{C}_{T-1}(\tilde{P}_T^C)$
C_{T-1}
MV_T^A/λ
MV_T^B/λ
MV_T^C/λ
V_T^{*A}
MV_T^{*B}
MV_T^{*C}
0
C_{T-1}

Figure 9

first decrease with C_{T-1}, but when closer to $\overline{C}_T(\tilde{P}_T)$ it increases with C_{T-1}.

(b) Whenever $C_{T-1} > \overline{C}_{T-1}(\tilde{P}_T)$, the change of stocks $(\Delta \hat{C}_T)$ is constantly equal to the unconstrained minimum point $(\Delta C_T^o(\tilde{P}_T) = -\overline{C}_{T-1}(\tilde{P}_T))$; $V_T^*(C_{T-1}, \tilde{P}_T)$ is constantly equal to $V_T^{*o} > 0$ (compare to 0 in the no-cost case), and $V_T(C_{T-1}, \tilde{P}_T)$ increases linearly in C_{T-1}.

Before proceeding to problem T-1 the expectation of $V_T(C_{T-1}, \tilde{P}_T)$ must be computed. It is denoted by $EV_T(C_{T-1})$. According to our notation EV_T can also be separated into instability and cost terms, i.e.:

$$(1.48)_T \qquad EV_T(C_{T-1}) = EV_T^*(C_{T-1}) \cdot \lambda + \theta \cdot EC_T$$

$$= EV_T^*(C_{T-1}) \cdot \lambda + \theta C_{T-1} + \theta E\Delta C_{T-1}(C_{T-1}) \ .$$

Generally EV_T and EV_T^* are weighted sums of the individual curves corresponding to different market prices, where the weights are the probabilities (or densities in the continuous case) of $\tilde{P}_T$. From the analysis of $V_T^*(C_{T-1}, \tilde{P}_T)$ it follows that the greater C_{T-1} is, the larger is the set of market prices $(\tilde{P}_T)$ for which C_{T-1} is greater than $\overline{C}_{T-1}(\tilde{P}_T)$; hence the probability of $V_T^*(C_{T-1}, \tilde{P}_T)$ being equal to V_T^{*o} increases with C_{T-1}. Hence, $EV_T^*(C_{T-1})$ approaches V_T^{*o} when C_{T-1} increases (compare to 0 in the no-cost case). Its shape depends on the probability distribution of $\tilde{P}_T$. It might be a decreasing function of C_{T-1} for all C_{T-1}, but also may first decrease up to a minimum point and then increase and approach V_T^{*o}. However, it is never zero.

In figure 10 some V_T/λ and V_T^* curves corresponding to different market prices $(\tilde{P}^A > \tilde{P}^B > \ldots > \tilde{P}^F)$ are drawn as well as the EV_T^* and EV_T^*/λ curves.

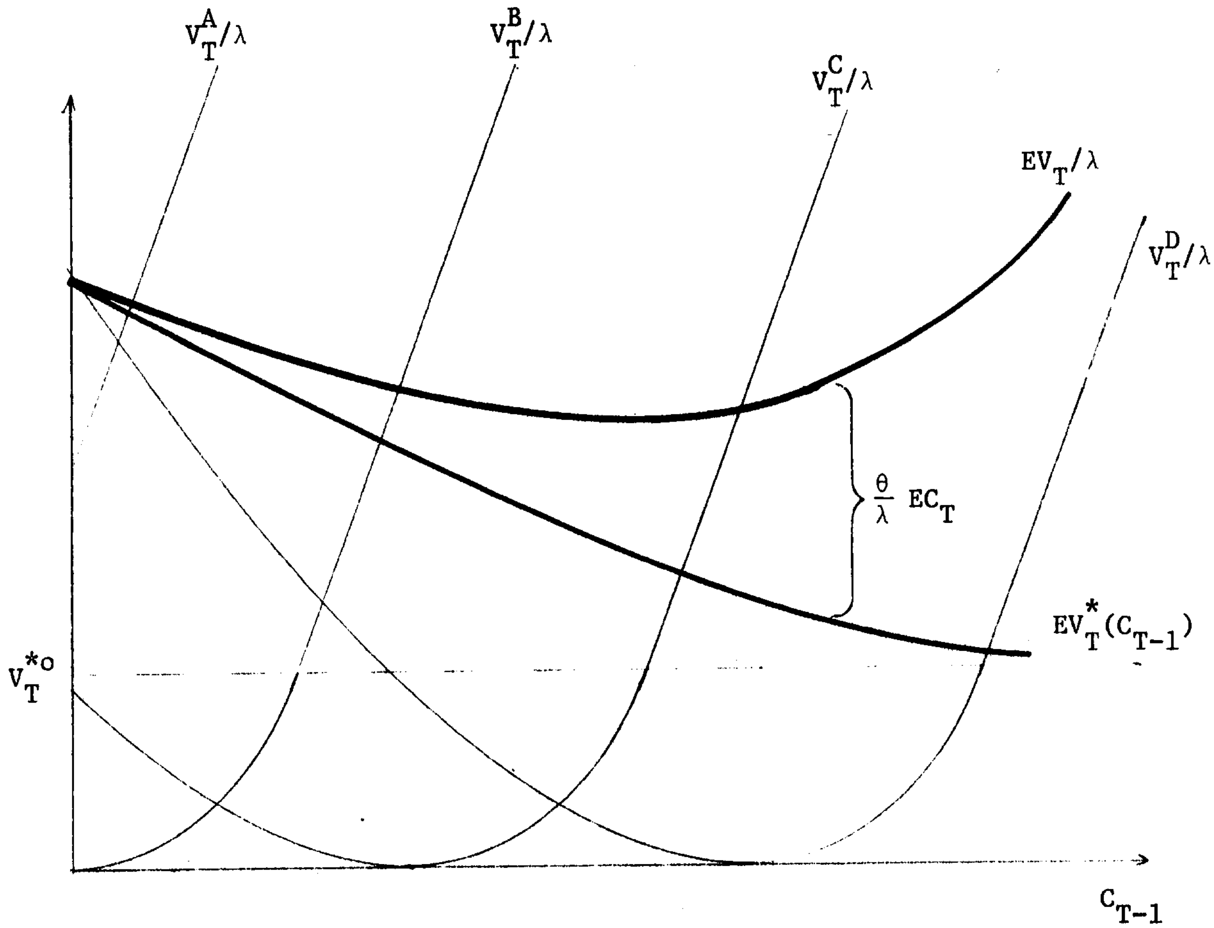

V_T^A/λ
V_T^B/λ
V_T^C/λ
EV_T/λ
V_T^D/λ
$\frac{\theta}{\lambda}\ EC_T$
$EV_T^*(C_{T-1})$
V_T^{*o}
C_{T-1}

Figure 10

The $EV^*_T(C_{T-1})$ curve is a weighted average of the V^*_T parts of these curves. It has a negative slope but as was stated, it might also have an increasing portion.

In a similar way $EV_T(C_{T-1})$ is a weighted average of the $V_T(C_{T-1}, \tilde{P}_T)$ functions. As C_{T-1} increases, $V_T(C_{T-1})$ approaches a slope of θ because of the increasing weight of the cost term $\theta EC_T(C_{T-1})$. A typical EV_T/λ curve is drawn in figure 10. It always has an increasing portion as C_{T-1} increases and it might also have a decreasing portion when C_{T-1} is relatively small.

Let us summarize the last analysis in the following proposition.

<u>Proposition 6_T</u> (Effect of beginning stocks on EV and EV^* for t=T). The shape of the EV_T and EV^*_T curves depends in general on the probability distribution of $\tilde{P}_T$. The instability measure $EV^*_T(C_{T-1})$ is always positive. As C_{T-1} increases, $EV^*_T(C_{T-1})$ approaches V^{*o}_T. $EV^*_T(C_{T-1})$ always has a decreasing portion when beginning stocks are relatively small. However, it might reach a minimum and then increase with C_{T-1}.

The whole term

$$EV_T(C_{T-1}) = \lambda EV^*_T(C_{T-1}) + \theta EC_T(C_{T-1})$$

approaches a slope of $\theta > 0$ when C_{T-1} increases. However, with small beginning stocks, $EV_T(C_{T-1})$ might decrease with C_{T-1}.

To conclude the discussion of the last period, let us investigate the effect of changing λ, i.e., the value of instability in terms of costs.

Given $\tilde{P}_T$, for $C_{T-1} < \bar{C}_{T-1}(\tilde{P}_T)$ we know from the previous analysis that the optimal change of stocks is $\Delta\hat{C}_T = -C_{T-1}$, hence a marginal change in λ does not affect ΔC_T. It also does not affect the instability measure under the optimal rule, i.e., $V_T^*(C_{T-1}, \tilde{P}_T)$. Formally, we can write:

$$(1.49)_T \qquad \frac{d\Delta\hat{C}_T}{d\lambda} = \frac{dV_T^*}{d\lambda} = 0 \quad \text{for } C_{T-1} < \bar{C}_{T-1}(\tilde{P}_T).$$

For $C_{T-1} \geq \bar{C}_{T-1}(\tilde{P}_T)$ the effect of λ on $\Delta\hat{C}_T$ can be analyzed by differentiating the condition for minimum $(1.46)_T$.

$$(1.50)_T \qquad \frac{\partial w_T^*(\Delta C_T \quad | \tilde{P}_T)}{\partial \Delta C_T} d\lambda + \frac{\partial^2 w_T^*(C_T \quad | \tilde{P}_T)}{\partial \Delta C_T^2} \lambda d\Delta\hat{C}_T = 0$$

or

$$(1.50')_T \quad \frac{d\Delta\hat{C}_T}{d\lambda} = - \frac{\partial w_T^*(\Delta C_T \quad | \tilde{P}_T)/\partial \Delta C_T}{\lambda \cdot \partial^2 w_T^*(C_T \quad | \tilde{P}_T)/\partial \Delta C_T^2} > 0 \quad \text{for } C_{T-1} > \bar{C}_{T-1}(\tilde{P}_T)$$

The numerator is negative since the minimum of w_T is when $P_T < p_T^*$, i.e., in the decreasing portion of w_T^*. The denominator is positive because of the 2nd order condition for the minimum. Hence

$$\left.\frac{d\Delta\hat{C}_T}{d\lambda}\right|_{\tilde{P}_T} > 0 \quad \text{for} \quad C_{T-1} > \bar{C}_{T-1}(\tilde{P}_T)$$

which means that it is optimal to accumulate more stocks when λ is greater.

The effect of λ on the instability measure under the optimal rule can be characterized by

$$(1.51)_T \qquad \frac{dV_T^*}{d\lambda} = \frac{\partial w_T^*}{\partial \Delta C_T} \cdot \frac{d\Delta\hat{C}_T}{d\lambda} < 0 \quad \text{for } C_{T-1} > \bar{C}_{T-1}(\tilde{P}_T).$$

i.e., the instability measure decreases as λ increases.

Given $\tilde{P}_T$ and C_{T-1}, one can trace efficient combinations of instability (V_T^*) and cost (θC_T)(i.e., given V_T^* minimize θC_T or given θC_T minimize V_T^*) by changing λ.

The marginal rate of substitution of instability V_T^* for cost (i.e., the slope of the efficiency frontier, given C_{T-1} and $\tilde{P}_T$) is $-\lambda$.

This is followed by $(1.51)_T$:

$$(1.52)_T \qquad \left.\frac{d\theta\hat{C}_T}{dV_T}\right|_{C_{T-1},\ \tilde{P}_T} = \frac{\theta}{\partial w_T^*/\partial\Delta C_T} = -\lambda\ .$$

The last equality is implied by $(1.46)_T$.

So far we held $\tilde{P}_T$ as given. However, $(1.50)_T$ and (1.51) are true for any $\tilde{P}_T$ provided $C_{T-1} > \overline{C}_{T-1}(\tilde{P}_T)$.[10] On the other hand, when $C_{T-1} < \overline{C}_{T-1}(\tilde{P}_T)$, $(1.43)_T$ holds and dV_T^*, $d\hat{C}_T$ both equal 0. Therefore, $(1.52)_T$ is true also for the expectation, i.e.,

$$(1.53)_T \qquad \left.\frac{d\theta E\hat{C}_T}{dEV_T^*}\right|_{C_{T-1}} = -\lambda\ .$$

The meaning is straightforward: EV_T^* measures instability of period T's price, $\theta E\hat{C}_T$ measures the mean cost to reduce instability. To be efficient the mean cost for a given level of instability must be minimized. Equivalently, instability must be minimized for a given level of mean cost. Being on the efficiency frontier there is a substitution between instability and cost: $-\lambda$ is the marginal rate of this substitution. Later on we shall see that this is true for the whole planning period (t=1, ..., T).

---

[10] At $C_{T-1} = \overline{C}_{T-1}(\tilde{P}_T)$ the derivatives do not exist. However, $(1.49)_T$, $(1.50)_T$ and $(1.51)_T$ hold for the proper right hand and left hand derivatives.

Let us now analyze the optimal stocks rule for t=T-1, which is the general case that also fits the other periods t=1, ..., T-1.

<u>Problem T-1</u>: Given C_{T-2}, find the stocks rule $\hat{C}_{T-1}(\cdot)$ that minimizes

$$(1.43)_{T-1} \qquad E\{w_{T-1} + EV_T\} = E\{\lambda w^*_{T-1} + \theta C_{T-1} + \lambda EV^*_T + \theta EC_T\}$$

subject to the constraint

$$(1.44)_{T-1} \qquad \Delta C_{T-1} \geq - C_{T-2}$$

where EV_T is obtained by the solution of problem T.

The stock rule $\hat{C}_{T-1}(C_{T-2}, \tilde{P}_{T-1})$ is the solution of the following problem:

Given C_{T-2} and $\tilde{P}_{T-1}$,

$$(1.45)_{T-1} \qquad \underset{\Delta C_{T-1}}{\text{minimize}} \; w_{T-1} + EV_T$$

$$= \lambda \cdot w^*_{T-1}(\Delta C_{T-1}|\tilde{P}_{T-1}) + \theta C_{T-1} + \lambda EV^*_T(C_{T-1}) + \theta EC_T(C_{T-1})$$

$$= \lambda w^*_{T-1}(\Delta C_{T-1}|P_{T-1}) + \theta C_{T-2} + \theta \Delta C_{T-1} + \lambda EV^*_T(C_{T-2} + \Delta C_{T-1})$$

$$+ \theta EC_T(C_{T-2} + \Delta C_{T-1})$$

subject to the constraint $(1.44)_{T-1}$.

In the last row of $(1.45)_{T-1}$, we indicated explicitly that given $\tilde{P}_{T-1}$, w^*_{T-1} is a function of the stocks change ΔC_{T-1}. Also we separated C_{T-1} to $C_{T-2} + \Delta C_{T-1}$ such that given C_{T-2}, only ΔC_{T-1} plays a role.

As before, first analyze the unconstrained minimization, for which it is necessary that the derivative with respect to ΔC_{T-1} vanishes, i.e.:

$$(1.46)_{T-1} \quad [\lambda \frac{\partial w^*_{T-1}(\Delta C_{T-1}|\tilde{P}_{T-1})}{\partial \Delta C_{T-1}} + \theta] + [\lambda \frac{\partial EV^*_T(C_{T-2} + \Delta C_{T-1})}{\partial C_{T-1}} + \theta \frac{\partial EC_T(C_{T-2} + \Delta C_{T-1})}{\partial C_{T-1}}] = 0.$$

$(1.46)_{T-1}$ is composed of two parts: The expression in the first brackets describes the marginal change in current loss (composed of instability (w^*_{T-1}) and cost (θC_{T-1})). The expression in the second brackets describes the marginal change in future loss (also composed of instability (EV^*_T) and cost (θEC_T)). Only an expression similar to the first one was involved in problem T, and this makes the difference between T and the other periods exactly as it was in the no-cost case.

Rearranging $(1.46)_{T-1}$, it can be written as:

$$(1.46')_{T-1} \qquad \lambda \cdot \frac{\partial w^*_{T-1}}{\partial \Delta C_{T-1}} + \theta = - \lambda \frac{\partial EV^*_T}{\partial C_{T-1}} - \theta \frac{\partial EC_T}{\partial C_{T-1}} .$$

$(1.46')_{T-1}$ states that to minimize $(1.45)_{T-1}$, it is necessary that current marginal net loss (composed of current marginal destabilization $\left(\lambda \frac{\partial w^*}{\partial \Delta C_{T-1}}\right)$ and current marginal cost (θ)) should be equal to marginal future net benefits (composed of future marginal stabilization $\left(- \lambda \frac{\partial EV^*_T}{\partial C_{T-1}}\right)$ minus marginal future cost $\left(\theta \frac{\partial EC_T}{\partial C_{T-1}}\right)$). Equivalently we can write

$$(1.46'')_{T-1} \qquad \lambda \frac{\partial w^*_{T-1}}{\partial \Delta C_{T-1}} = - \lambda \frac{\partial EV^*_T}{\partial C_{T-1}} - \theta \left(1 + \frac{\partial EC_T}{\partial C_{T-1}}\right)$$

which states that for optimality it is necessary that current marginal destabilization

$\left(\lambda \frac{\partial w^*_{T-1}}{\partial \Delta C_{T-1}}\right)$ should be equal to future marginal stabilization $\left(- \frac{\partial EV^*_T}{\partial C_{T-1}}\right)$

minus marginal cost $(\theta + \theta \frac{\partial EC_T}{\partial C_{T-1}})$. It is the last term which makes the difference between the present case and the no-cost case analyzed before. Recall that in the no-cost case, the unconstrained minimum was always to the right of the point of minimum current instability, i.e., the price after application of the stocks rule was greater or equal to the target price. This was true because when $P_{T-1} = p^*_{T-1}$ it might still be worthwhile to increase stocks which will enable us to reduce future instability. In the present case, however, the cost element might make it unworthy to accumulate that much stocks. It might even be worthwhile to save costs and accumulate less stocks than are needed to equate $P_{T-1} = p^*_{T-1}$ so that the minimum point might be to the left side of minimum current instability, i.e., with the final price lower than the target price.

Formally this can be shown as follows: Looking at $(1.46'')_{T-1}$, it was stated in Proposition 6_T that EV^*_T may have a portion that increases with C_{T-1}, hence $- \frac{\partial EV^*_T}{C_{T-1}}$ may be positive or negative (in the no-cost case $- \partial EV^*_T/\partial C_{T-1}$ was always non-negative . That is, marginal stabilization of future prices can be positive or negative. Even if positive, there is still the cost term on the right hand side, i.e.,

$- \theta(1 + \frac{\partial EC_T}{\partial C_{T-1}})$, which is negative. In balance, the right hand side of

(1.46") might be positive (in which case the minimizing ΔC_{T-1} will be more than is needed for complete current stabilization and the final price (P_{T-1}) will be greater than the target price). However, the right hand side of (1.46") might be negative, when the cost effect is stronger than the future stabilization effect (in which case, the minimizing ΔC_{T-1} will be less than is needed for complete current stabilization and the final price (P_{T-1}) will be lower than the target price).

Graphically, the determination of the unconstrained minimizing stocks change, denoted by $\Delta C^{o}_{T-1}(C_{T-2}, \tilde{P}_{T-1})$ is demonstrated in figures 11 and 12. The curves of w^{*}_{T-1} and of $w_{T-1}/\lambda = w^{*}_{T-1} + \frac{\theta}{\lambda} C_{T-1}$ are depicted in the upper section of figure 11 for the given P_{T-1} and C_{T-2}. The marginal curve of w_T/λ is depicted in the lower section, as in figure 7. In addition, given C_{T-2}, the curves of EV^{*}_{T} and of EV_T/λ are also drawn in the upper section of figure 11. The negative of the marginal curve of EV_T/λ is drawn in the lower section of figure 11. The minimum of the sum $w_{T-1} + EV_T$ is obtained at the point of intersection between the MW_{T-1}/λ curve and the $- MEV_T/\lambda$ curve. In figure 11 this intersection happens to be to the right of the minimum point of w^{*}_{T-1}. In figure 12, which is similar to figure 11, the intersection of the marginal curves is to the left of the minimum point of w^{*}_{T-1}.

Notice that, contrary to period T, the unconstrained minimizing stocks change (ΔC^{o}_{T-1}) is a function of both the market price ($\tilde{P}_{T-1}$) and the beginning stocks (C_{T-2}). The reason for this was discussed in case (a) and can be applied here: Future stabilization depends on current carryover and the beginning stocks affect the marginal stabilization of current stocks change on future prices. However, in the present case there is also a cost factor which is opposite to the stabilization effect.

So far, the unconstrained minimization for period T-1 has been analyzed. As before, $\Delta C^{o}_{T-1}(C_{T-2}, \tilde{P}_{T-1})$, the unconstrained minimizing stocks change

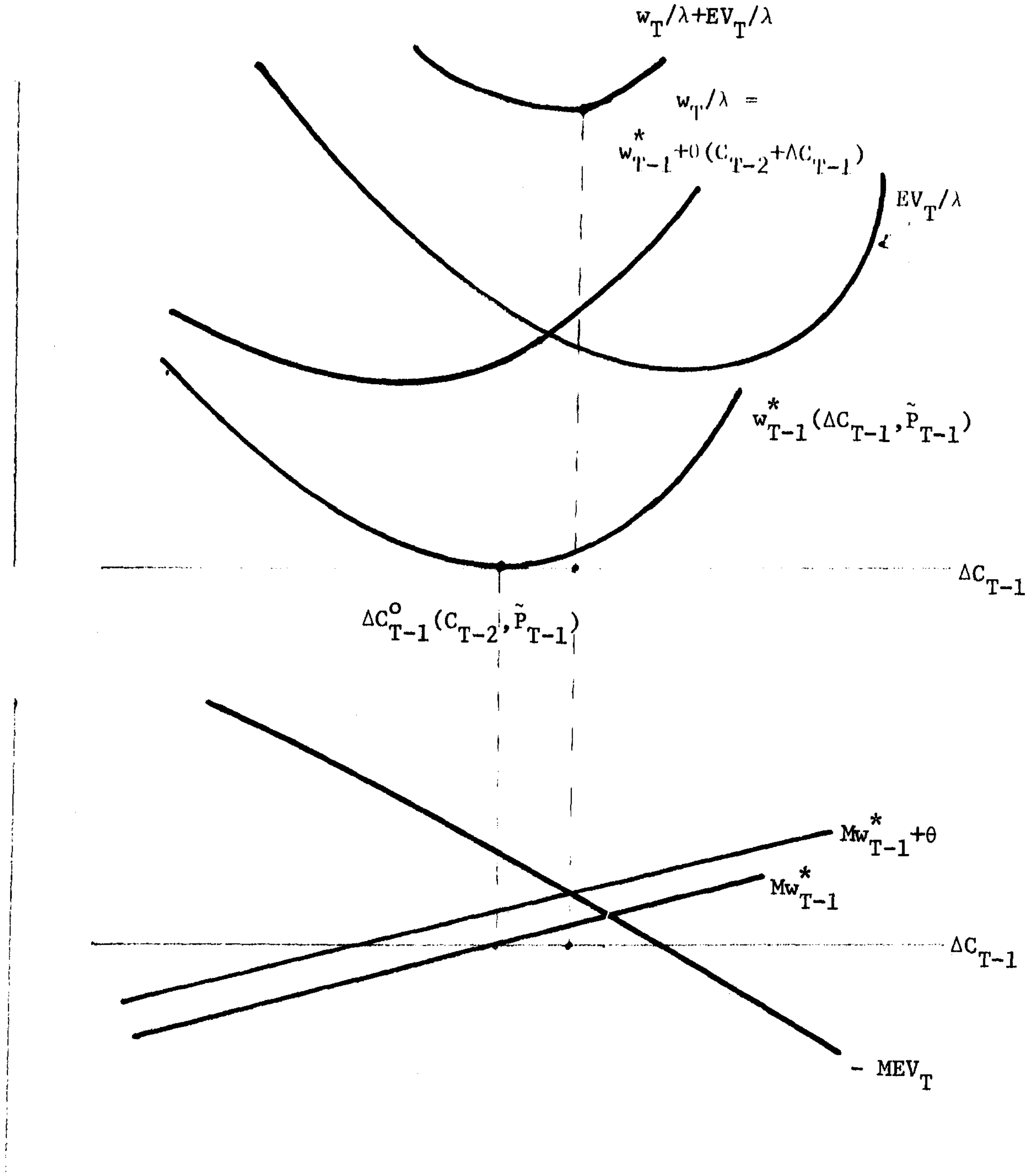

$w_T/\lambda+EV_T/\lambda$
$w_T/\lambda =$
$w^*_{T-1}+\theta(C_{T-2}+\Delta C_{T-1})$
EV_T/λ
$w^*_{T-1}(\Delta C_{T-1},\tilde{P}_{T-1})$
ΔC_{T-1}
$\Delta C^o_{T-1}(C_{T-2},\tilde{P}_{T-1})$
$Mw^*_{T-1}+\theta$
Mw^*_{T-1}
ΔC_{T-1}
$-\ MEV_T$

Figure 11

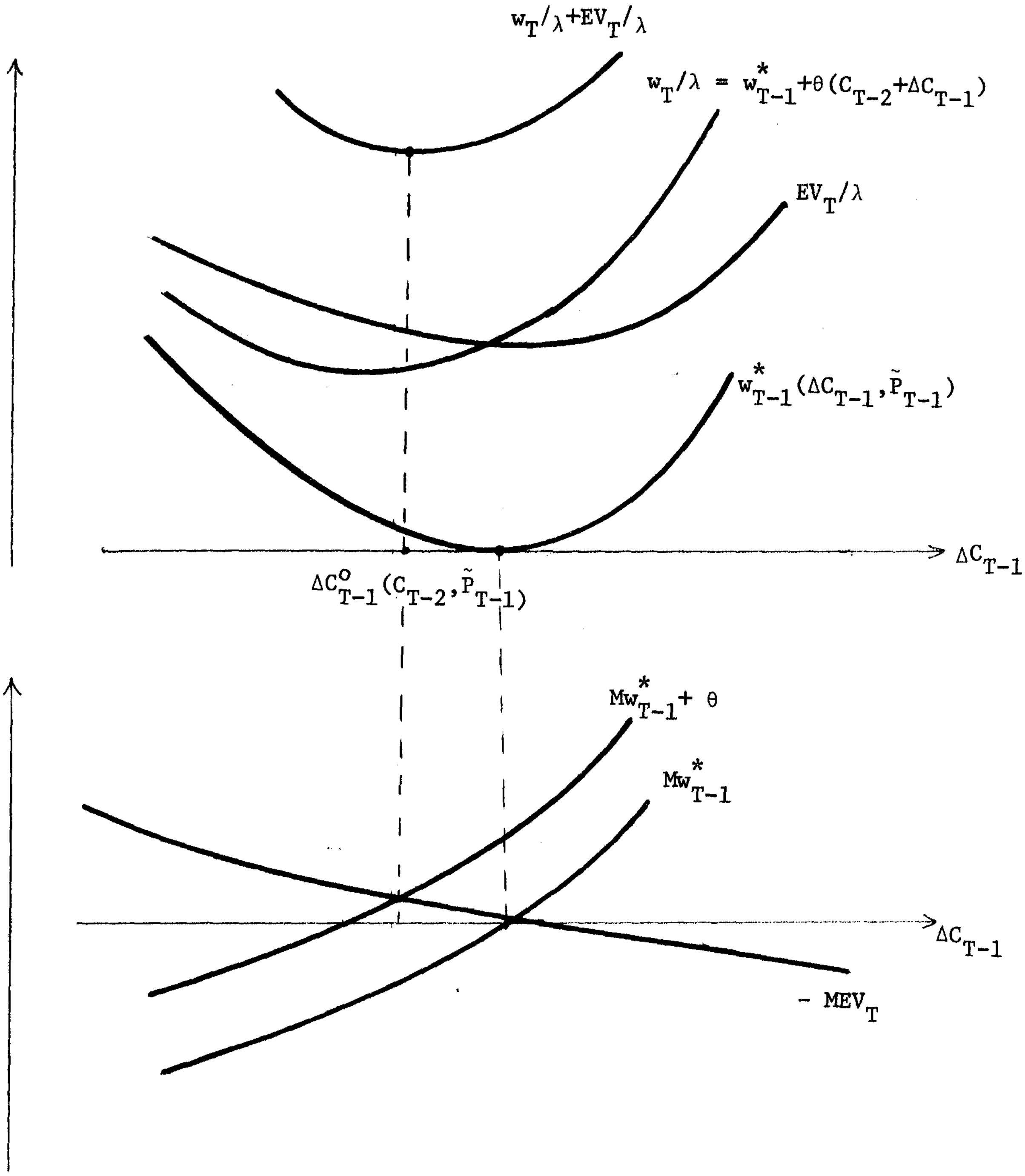

$w_T/\lambda + EV_T/\lambda$
$w_T/\lambda = w^*_{T-1} + \theta(C_{T-2} + \Delta C_{T-1})$
EV_T/λ
$w^*_{T-1}(\Delta C_{T-1}, \tilde{P}_{T-1})$
ΔC_{T-1}
$\Delta C^o_{T-1}(C_{T-2}, \tilde{P}_{T-1})$
$Mw^*_{T-1} + \theta$
Mw^*_{T-1}
ΔC_{T-1}
$- MEV_T$

Figure 12

might not be feasible because of the constraint $(1.44)_{T-1}$. In this case it is optimal to dispose of all stocks. Define $\overline{C}_{T-2}(\tilde{P}_{T-1})$ by

$$\overline{C}_{T-2}(\tilde{P}_{T-1}) \equiv - \Delta C^o_{T-1}(\overline{C}_{T-2}, \tilde{P}_{T-1}),$$

the optimal stocks change is

$$(1.47)_{T-1} \quad \Delta\hat{C}_{T-1}(C_{T-2}, \tilde{P}_{T-1}) = \begin{cases} - C_{T-2} & \text{if } C_{T-2} \leq \overline{C}_{T-2}(\tilde{P}_{T-1}) \\ \\ \Delta C^o_{T-1}(C_{T-2}, \tilde{P}_{T-1}) & \text{if } C_{T-2} \geq \overline{C}_{T-2}(\tilde{P}_{T-1}). \end{cases}$$

We shall not discuss in detail the effect of beginning stocks on the value of the objective under the optimal rule, $V_{T-1}(C_{T-2}, \tilde{P}_{T-1})$. The analysis can be carried out in a way similar to the no-cost case. The main conclusions are summarized in Proposition 6_{T-1}, below. $V_{T-1}(C_{T-2}, \tilde{P}_{T-1})$ can be decomposed into instability and cost components as follows:

$$(1.48)_{T-1} \quad V_{T-1}(C_{T-2}, \tilde{P}_{T-1}) = \lambda\cdot V^*_{T-1}(C_{T-2}, \tilde{P}_{T-1}) + \theta\cdot[\hat{C}_{T-1} + EC_T(\hat{C}_{T-1})]$$

$$= \lambda[w^*_{T-1}(\Delta\hat{C}_{T-1}, \tilde{P}_{T-1}) + EV_T(C_{T-2} + \Delta\hat{C}_{T-1})]$$

$$+ \theta[C_{T-2} + \Delta\hat{C}_{T-1} + EC_T(C_{T-2} + \Delta\hat{C}_{T-1})]$$

where $\Delta\hat{C}_{T-1}(C_{T-2}, \tilde{P}_{T-1})$ is a function of C_{T-2} and $\tilde{P}_{T-1}$.
Similarly, the expected value of V_{T-1}, EV_{T-1}, can be decomposed into instability and cost as follows:

Let us denote by $\overline{EC}_t$ the expectation of the sum of stocks from t on, i.e.,

$$(1.49) \qquad \overline{EC}_t = E\{C_t + C_{t+1} + \ldots, C_T\}$$

Then:

$$(1.50)_{T-1} \quad EV_{T-1}(C_{T-2})$$

$$\equiv \lambda \cdot EV^{*}_{T-1}(C_{T-2}) + \theta \cdot \overline{EC}_{T-1}(C_{T-2})$$

$$\equiv \lambda \cdot E_{\varepsilon_{T-1}}\{w_{T-1}(\Delta\hat{C}_{T-1}, \tilde{P}_{T-1}) + EV_T(C_{T-2} + \Delta\hat{C}_{T-1})\}$$

$$+ \theta \cdot E_{\varepsilon_{T-1}}\{(C_{T-2} + \Delta\hat{C}_{T-1}) + \overline{EC}_T(C_{T-2} + \Delta\hat{C}_{T-1})\}$$

To summarize let us state Propositions 5_{T-1} and 6_{T-1}.

<u>Proposition 5_{T-1}</u> (Optimal stocks change for t=T-1).

1. Generally the optimal stocks change ($\Delta\hat{C}_{T-1}$) is a function of both the market price ($\tilde{P}_{T-1}$) and the beginning stocks (C_{T-2}). However, for any given $\tilde{P}_{T-1}$ there is a $\overline{C}_{T-2}(\tilde{P}_{T-1})$, which is by itself an increasing function of $\tilde{P}_{T-1}$ such that: Whenever $C_{T-2} \leq \overline{C}_{T-2}(\tilde{P}_{T-1})$ the optimal change of stocks is to dispose of all stocks. Whenever $C_{T-2} \geq \overline{C}_{T-2}(\tilde{P}_{T-1})$ the optimal stocks change $\hat{C}_{T-1}(C_{T-2}, \tilde{P}_{T-1})$ decreases with C_{T-2} and $\tilde{P}_{T-1}$. The range of market prices such that $\overline{C}_{T-2}(\tilde{P}_{T-1})$ is positive includes prices above the target price (p^{*}_{T-1}) as well as prices below it (compare to the no-cost case in which this range included only prices above p^{*}_{T-1}).

2. The final price $\hat{P}_{T-1}$ which is implied by the optimal stocks rule may be either higher or lower than the target price.

<u>Proposition 6_{T-1}</u> (Effect of beginning stocks on EV and EV^{*} for t=T-1). The shape of EV_{T-1} and EV^{*}_{T-1} curves depends in general on the probability distributions of $\tilde{P}_{T-1}$ and $\tilde{P}_T$. The instability measure $EV^{*}_{T-1}(C_{T-2})$ is

always positive (i.e., it is never optimal to achieve complete stability in t=T-1, T. $EV_{T-1}(C_{T-2})$ always has an increasing portion. However, for small amounts of C_{T-2} it may be a decreasing function of C_{T-2}.

It should be noted that all periods other than T are qualitatively similar to T-1 so that the analysis of period T-1 can be directly applied to the other ones.

Let us conclude by analyzing the effect of changing λ. For $C_{T-2} < \overline{C}_{T-2}(\tilde{P}_{T-1})$ it is obvious that λ has no effect on $\Delta\hat{C}_{T-1}$ and on V_{T-1}, i.e.,

$$\frac{d\Delta\hat{C}_{T-1}}{d\lambda} = \frac{dV_{T-1}}{d\lambda} = 0$$

In the case that $C_{T-2} > \overline{C}_{T-2}(\tilde{P}_{T-1})$ the analysis is carried out similarly to the analysis in period T:

Differentiating $(1.46)_{T-1}$ gives:

$$(1.51)_{T-1} \quad \left(\frac{\partial w^*_{T-1}}{\partial \Delta C_{T-1}} + \frac{\partial^2 EV_T}{\partial C_{T-1}\partial\lambda}\right) d\lambda + \left(\frac{\partial^2 w^*_{T-1}}{\partial \Delta C^2_{T-1}} + \frac{\partial^2 EV_T}{\partial C^2_{T-1}}\right) d\Delta\hat{C}_{T-1} = 0$$

It can be shown that[11/]

$$\frac{\partial EV_T}{\partial\lambda} = EV^*_T \quad \text{and} \quad \frac{\partial^2 EV_T}{\partial\lambda\partial C_{T-1}} = \frac{\partial EV^*_T}{\partial C_{T-1}}, \quad \text{hence:}$$

[11/] Let us show that $\partial EV_t/\partial\lambda = EV^*_t$:

continued

$$(1.52)_{T-1} \qquad \frac{d\Delta\hat{C}_{T-1}}{d\lambda} = \frac{\partial w^*_{T-1}/\partial\Delta C_{T-1} + \partial EV^*_T/\partial C_{T-1}}{\partial^2 w^*_{T-1}/\partial\Delta C^2_{T-1} + \partial^2 EV_T/\partial C^2_{T-1}}$$

(footnote 11/ continued)

$$\frac{\partial V_t(C_{t-1}, \tilde{P}_t, \lambda)}{\partial\lambda}$$

$$= \frac{\partial}{\partial\lambda} [\lambda \cdot w^*_t(\Delta\hat{C}_t, \tilde{P}_t) + \theta \cdot (C_{t-1} + \Delta\hat{C}_t) + EV_{t+1}(C_{t-1} + \Delta\hat{C}_t, \lambda)]$$

$$= \hat{w}^*_t + [\lambda \frac{\partial w^*_t}{\partial\Delta C_t} + \theta + \frac{\partial EV_{t+1}}{\partial\lambda}] \frac{\partial\Delta\hat{C}_t}{\partial\lambda} + \frac{\partial EV_{t+1}}{\partial\lambda}$$

The second term in the last line vanishes: either the expression in brackets equals zero (if $\Delta\hat{C}$ is an interior solution), or $\Delta\partial\hat{C}_t/\partial\lambda = 0$ (if $\Delta\hat{C}_t$ is a boundary solution $\Delta\hat{C}_t = - C_{t-1}$).

For $t = T$, $\partial EV_{t+1}/\partial\lambda = 0$ by definition. Hence

$$\frac{\partial V_T}{\partial\lambda} = w^*_T \quad \text{and} \quad \frac{\partial EV_T}{\partial\lambda} = EV^*_T$$

For any $t < T$ this can be proved by induction: Assume that $\partial EV_{t+1}/\partial\lambda = EV^*_{t+1}$ and prove for t:

$$\frac{\partial V}{\partial\lambda} = \hat{w}^*_t + \frac{\partial EV_{t+1}}{\partial\lambda} = \hat{w}^*_t + EV^*_{t+1}$$

Hence: $$\frac{\partial EV_t}{\partial\lambda} = E\{\hat{w}^*_t + EV^*_{t+1}\} = EV^*_t \cdot \text{Q.E.D.}$$

From $(1.46)_{T-1}$ it follows that the denominator of $(1.52)_{T-1}$ is negative:

$$\frac{\partial w^*_{T-1}}{\partial \Delta C_{T-1}} + \frac{\partial EV^*_T}{\partial C_{T-1}} = \frac{\theta}{\lambda}\left(1 + \frac{\partial \overline{EC}_T}{\partial C_{T-1}}\right) < 0$$

In addition, the denominator is positive, implied by the second order condition for a minimum. Therefore $(1.52)_{T-1}$ is positive.

The effect of an increase of λ is to reduce the instability index and to increase the mean stocks. This can be proved simply: Suppose that λ^1 and λ^2 are two values of λ such that $\lambda^1 < \lambda^2$. Denote the optimal values of EV^*_{T-1} and $\overline{EC}_{T-1}$ which correspond to λ^1 and λ^2 by EV^{*1}, $\overline{EC}^1$ and EV^{*2}, $\overline{EC}^2$ respectively. The optimal stocks rules corresponding to λ^1 and λ^2 are both feasible. Therefore minimization implies:

$$\lambda^1 EV^{*1} + \theta\overline{EC}^1 \leq \lambda^1 EV^{*2} + \theta\overline{EC}^2$$

and

$$\lambda^2 EV^{*2} + \theta\overline{EC}^2 < \lambda^2 EV^{*1} + \theta\overline{EC}^1,$$

from which it follows that

$$(1.53)_{T-1} \qquad (\lambda^2 - \lambda^1)\cdot(EV^{*2} - EV^{*1}) < 0 \quad \text{or} \quad \frac{\Delta EV^*}{\Delta\lambda} < 0$$

Also:

$$(1.54)_{T-1} \qquad \left(\frac{\theta}{\lambda^1} - \frac{\theta}{\lambda^2}\right)\cdot(\overline{EC}^2 - EC^1) = \frac{\theta}{\lambda^1\lambda^2}(\lambda^2 - \lambda^1)\cdot(\overline{EC}^2 - \overline{EC}^1) > 0$$

$$\text{or} \quad \frac{\Delta\overline{EC}}{\Delta\lambda} > 0$$

Given $\tilde{P}_{T-1}$ and C_{T-2}, it is possible to trace efficient combinations of mean cost $(\theta\overline{EC})$ and instability index (EV^*) by changing λ. There is a substitution between instability and cost. The marginal rate of substitution is $-\lambda$, i.e:[12/]

$$(1.55)_{T-1} \qquad \frac{dEV^{*}_{T-1}}{d\theta\overline{EC}_{T-1}} = -\lambda .$$

[12/] $(1.55)_{T-1}$ can be verified as follows: It is obvious that the values of EV^{*} and $\overline{EC}$ corresponding to a given λ should be on the efficiency frontier and are obtained by minimizing $\lambda EV^{*} + \theta\overline{EC}$ on the frontier. Denote a change of EV^{*} by ΔEV^{*} and the corresponding change of $\overline{EC}$ on the efficiency frontier by $\Delta\overline{EC}(\Delta EV^{*})$, such that:

$$\Delta EV^{*} \cdot \Delta\overline{EC}(\Delta EV^{*}) < 0 .$$

To minimize $\lambda EV^{*} + \theta\overline{EC}$ the following condition should be satisfied at the minimum point: Let ΔEV^{*+} and ΔEV^{*-} be a positive and a negative change in EV^{*} respectively. Then:

$$\lambda \cdot \Delta EV^{*} + \theta \cdot \Delta\overline{EC}(\Delta EV^{*}) > 0 .$$

Substituting ΔEV^{*+} and ΔEV^{*-} for ΔEV^{*} results in:

$$\frac{\theta \cdot \Delta\overline{EC}(\Delta EV^{*+})}{\Delta EV^{*+}} \geq -\lambda \quad \text{and} \quad \frac{\theta \cdot \Delta\overline{EC}(\Delta EV^{*-})}{\Delta EV^{*-}} \leq -\lambda .$$

Assuming differentiability and taking the limit as $\Delta EV^{*} \to 0$ we get

$$\frac{\theta \cdot d\overline{EC}}{dEV^{*}} = -\lambda .$$

$(1.55)_{T-1}$, which is also similar to $(1.52)_T$, is valid for any period t, t+1, ..., T.

In figure 13 the efficiency frontier is drawn. It gives the policy-maker a set of possibilities to choose from according to his evaluation of the importance of stability relative to the cost of achieving it.

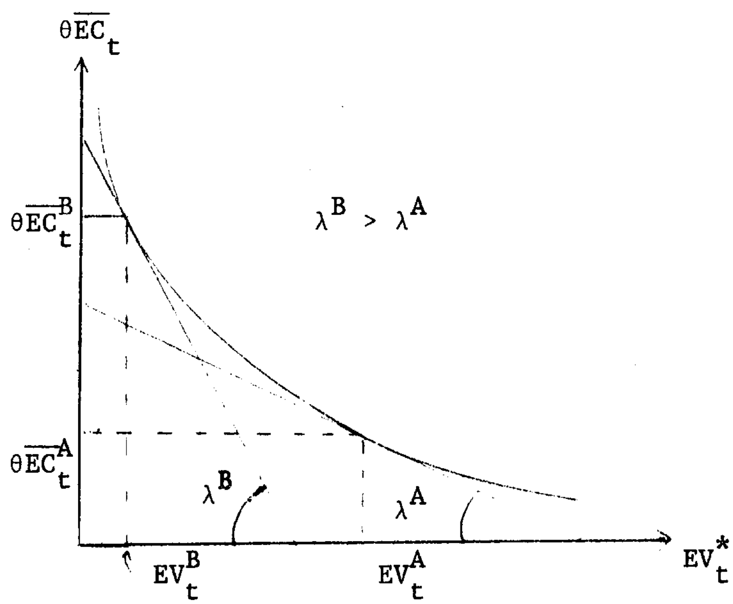


Figure 13

Summary

Based on the market model presented in part I, an optimization model is formulated to be used to find stocks rules which minimize the price instability index. The main part of this section is devoted to an analysis of the stocks rules. First the case in which storage cost plays no role is discussed. For any period t, the optimal rule should depend on two indicators, the market price $(\tilde{P}_t)$ and the level of existing stocks (C_{t-1}). The effect of the market price on the optimal change of stocks is negative.

This is so because current price stabilization activities require that stocks be accumulated in increasing amounts the lower the market price is relative to the target price, and that stocks be released in increasing quantities the higher the market price is relative to the target price. However, price stabilization in the future depends on the quantity of stocks that is carried over. Generally, in the no-cost case, unless there are enough existing stocks to completely stabilize prices throughout the whole planning period, it is worthwhile to accumulate more stocks than would be implied by equating the current price to the target price. The optimal change of stocks depends negatively on the level of existing stocks.

Costs are included in the model by adding a linear cost term to the objective function. The problem is stated to minimize a weighted sum of the price instability index and mean costs. By a change in the weight of the instability index, denoted by λ, a whole set of efficient combinations of instability measure and mean cost is traced. "Efficient" here means to minimize price instability for a given mean storage cost. This gives the policymaker a set of stock policies to choose from that range from no intervention to maximum feasible stabilization (i.e., the no-cost case). The general directions of the effects of market price and existing stocks on the change in stocks do not change in comparison to the no-cost case. However, the greater the relative weight of cost in the objective function (smaller weight of price instability index), the smaller the quantity of stocks that will be accumulated.

2. Computation Procedure

Based on the discussion in the previous section, a program was composed to compute the optimal buffer stock rule for price stabilization.

The program also computes the probability distributions of prices and stocks which result from applying some stock rule. In this section the computational procedure is explained. A detailed flowchart of the computation is given in the appendix.[13/] Subsection 2.1 describes the computation of the stock rule and subsection 2.2 describes the probability computations.

2.1 Computation of the price variability minimization rule. The computation procedure is an approximative one. Any continuous variable (e.g., price, stocks, random disturbance) is approximated by dividing its domain into a series of discrete points. A correspondence between these points and the integers is then defined. For example, suppose that the domain of stocks is from 0 to 1000 (i.e., the probability of being less than 0 or greater than 1000 is zero) and that the approximation is made by using intervals of 100 between any two points, then it is assumed that stocks can obtain values of 0, 100, 200, 111, 1000 and the correspondence between stocks and integer index is defined as follows:

CC(IC)	INDEXC(C)
0	1
100	2
200	3
⋮	⋮
1000	11

[13/] A Fortran program which is based on the flowchart of the appendix is available on request from the author.

where: CC(IC) is the quantity of stocks corresponding to the index IC(IC=1, 2, ...) and INDEXC(C) is the integer index corresponding to a stocks quantity of C. By making the division finer, one can approximate the variable under discussion as close as he wants to, but of course, computation costs increase.

Following the approximation by discrete points, all continuous probability distributions are approximated by discrete probability distributions so that instead of density functions there are probability mass functions.

Before proceeding to the computations let us make a note about notation in this section. The notation is different, even if similar, from the notation used in section 1 and part I. Following the notation of computer programs, no superscripts or subscripts are used. Names of integer variables begin with letters I through N. The name of the first index of an integer variable begins always with M and the last one with N. For example, the index of a period is denoted by IT (corresponding to sub-t in section 1), the first period is denoted by MT, and the last one by NT (corresponding to T in section 1). Some of the notations used in this section with their counterparts in section 1 are summarized in table 1.

Table 1. Notation Used in Section 1 and in Section 2

Variable	Notation in Section 1	Notation in Section 2	Corresponding integer variable		
			Name	First	Last
Current price	P_t	P	IP	1	NP
Lagged price	P_t-1	P1	IP1	1	NP
Market price	$\tilde{P}$	PP	IPP	1	NP
Carryover stocks	C	C	IC	1	NC
Beginning stocks (stocks from previous period)	C_t-1	C1	IC1	1	NC
Target price	p*	PSTAR			
Time index	t=1, 2, ..., T		IT	MT	NT
Random disturbance	ε	E	IE	1	NE
Unit storage cost	θ	THETA			
Discount rate	δ	DELTA			
Marginal substitution of instability for cost	λ	LAMBDA			

The correspondence between the values of a variable and the integers enables us to express some of the functions in the system as functions of integer indices. For example, the stocks rule is a function of the market price and the beginning stocks, $\hat{C}_t(\tilde{P}_t, C_{t-1})$ and can be translated to a function of integer indices, say

$$C(IPP, IC1, IT) \equiv C[P(IPP), CC(IC1), IT] \qquad IC1 = 1, 2, \ldots, NC$$

$$IPP = 1, 2, \ldots, NP$$

where P(IPP) and CC(IC1) are the price and stocks corresponding to the integers IPP and IC1 respectively and IT is the time index.

As mentioned in section 1, the computation procedure is basically a dynamic programming one. Recall from section 1 that the problem is as follows: A planning period is divided into subperiods $t=1, 2, \ldots, T$. In each period the free market price ($\tilde{P}_t$) is determined by the price of the last period (P_{t-1}) and by a stochastic disturbance ε_t which has a known probability distribution (see eq. 1.10 in section 1). The final price of period t, (P_t), is determined by the free market price ($\tilde{P}_t$) and by the change of stocks ($\Delta C_t \equiv C_t - C_{t-1}$). The problem is to find stocks rules $[\hat{C}_t(C_{t-1}, \tilde{P}_t)]$ as functions of the beginning stocks (C_{t-1}) and of the market price ($\tilde{P}_t$), which minimize

$$E\left\{\sum_{t=1}^{T}\left[\left(\frac{P_t-P_t^*}{P_t^*}\right)^2 \cdot \lambda + \theta \cdot C_t\right] \cdot \frac{1}{(1+\delta)^t}\right\} \tag{2.1}$$

where P_t^* is a target price for period t

θ is a unit storage cost per period

λ is a weight, which defines a desired marginal substitution rate between instability and cost

δ is a discount rate

and E stands for the expectation operation.

Following the demand and supply model (see section 1.1 above), the following two functions are defined in the program:

(a) The free market price (PP) at some period IT, as a function of the disturbance (E) and the price of the previous period (P1) (see eq.(1.10)).

This function will be denoted by

<u>PPFEP1</u>(E, P1, IT) .

(b) The final price (P) at some period IT, as a function of the free market price (PP) and the change of stocks (DC) (see eq. 1.11)). This function will be denoted by

<u>PFPPDC</u>(PP, DC, IT) .

The planning period in the present notation is from MT to NT (see table 1). As mentioned in section 1, the computation starts with the last period (NT) of the planning period. For each pair of indices IC1, IPP, corresponding to beginning stocks (C1) and market price (PP), respectively, the program finds the carryover C(IC1, IPP, NT), which minimizes the objective function for period NT (i.e. the instability index for period NT, weighted by LAMBDA, plus carryover cost (THETA times C). Basically, for a given pair (IC1, IPP), the program runs over all the indices IC = 1, 2, ..., NC (which correspond to different values of carryover): For each of them it computes:

(a) The change of stocks

DC = CC(IC) - CC(IC1)

where: CC(I) is the quantity of stocks corresponding to I (I = IC or IC1 respectively)

(b) The price

<u>PFPPDC</u>[P(IPP), DC, NT]

where: P(I) is the price corresponding to the integer index I (I = IPP for the market price PP).

(c) The value of the objective function for period NT, i.e.:

$$\left(\frac{\text{PFPPDC - PSTAR}}{\text{PSTAR}}\right)^2 * \text{ LAMBDA} + \text{C(IC)} * \text{THETA}$$

where PSTAR is the target price.

The program then picks up the level of carryover for which the last expression is minimal. This level of carryover is stored in an array denoted by C(IC1, IPP, NT).

The minimum value of the objective function for period NT, given CC(IC1) and P(IPP), is also stored in an array denoted by V(IC1, IPP).

Recall that the market price (PP) is a function of the lagged price (P1) and the disturbance (E). The next step is to compute the expectation of V over E. This is done as follows: For any given value of the integer index IP1 (IP1 = 1, 2, ..., NP) (corresponding to a lagged price P1) and IC1 (corresponding to a beginning stocks C1), the program runs over all the indices IE and computes:

(a) The market price PP

$$\text{PP} = \underline{\text{PPFEP1}}[\text{E(IE), P(IP1), NT}]$$

where E(IE) is the disturbance corresponding to the integer index IE and P(IP1) is the price corresponding to IP1.

(b) IPP = <u>INDEXP</u>(PP)

where <u>INDEXP</u> is a function which transforms prices into their corresponding integer indices.

(c) V(IC1,IPP)*PROBE(IE)

where PROBE(IE) is the probability of the disturbance E(IE).

Finally the program sums all the last expressions for IE = 1, 2, ..., NE. Obviously, the expected value of V is a function of the beginning stocks C1 and the lagged price P1 (or their corresponding integer indices IC1, IP1) and it is stored in an array denoted by

EV(IC1, IP1), (IC1 = 1, 2, ..., NC. IP1 = 1, 2, ..., NP).

It is possible now to compute the optimal carryover rule for period NT-1 by a similar procedure: Given any pair IC1, IPP (which are now standing for beginning stocks and market price of period NT-1 instead of NT), run over all indices IC = 1, 2, ..., NC (which are now standing for carryover of NT-1) and find the one for which the objective function for the period (NT-1, NT) is minimized. This is done similarly to period NT as follows:

For each IC compute:

(a) The change of stocks

DC = CC(IC) - CC(IC1)

(b) The price

PFPPDC[P(IPP), DC, NT-1]

and its corresponding integer index

IP = <u>INDEXP</u>(PFPPDC)

(c) The value of the objective function for the period NT-1, NT

$$\left(\frac{\text{PFPPDC} - \text{PSTAR}}{\text{PSTAR}}\right)^2 * \text{LAMBDA} + \text{C(IC)} * \text{THETA} + \frac{1}{1 + \text{DELTA}} * \text{EV(IC, IP)}.$$

Then the program picks up the carryover for which the last expression is minimal and stores it in C(IC1, IPP, NT-1). The minimal value V(IC1, IPP) for (NT-1, NT) replaces the V(IC1,IPP) for NT in the same array.
The only difference between NT-1 and NT is that in the latter, the objective function does not include EV as it does for the first.

Next, the EV(IC, IP) for (NT-1, NT) is computed exactly as it was computed for NT and the program is ready to compute the optimal rule for NT-2 which is done exactly as for NT-1; only the time index is changed. Period by period it proceeds from NT down to the first period of the planning period (i.e., MT).

It should be noted that for the computation of the optimal stocks rules the only information which must be carried from period to period of computation is the expected value EV of the last period of computation. Hence there is much saving in memory space. However, the program stores the stocks rules C(IC1, IPP, IT) of all the periods for the computation of probabilities. These computations are described in subsection 2.3. Before going through the probability computations let us describe another stocks rule which was analyzed in addition to the optimal stocks rule described above, namely a bounded price rule.

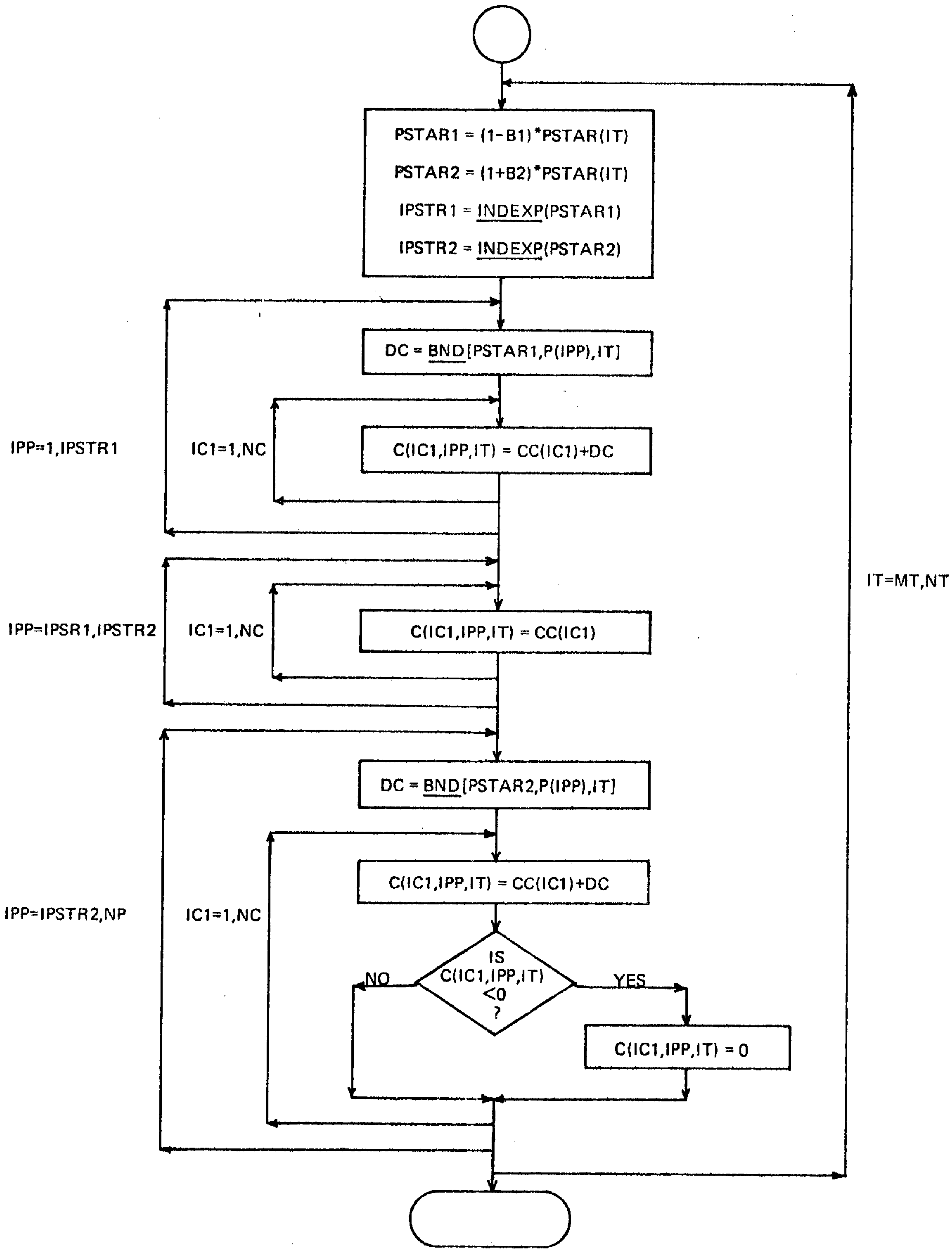
2.2 Bounded Price rule. This rule refers to some proposals which have been proposed for price stabilization and which have a common feature of considering only price signals as indicators for reserve stocks activity.

The level of beginning stocks enters the bounded price rule only in a primitive way, i.e., negative stocks are not feasible, so that the minimal carryover is zero. In particular, a bounded price rule suggested by W. W. Cochrane [1] was examined, so it is worthwhile to describe it in some detail and to relate it to the computational program.

2.2 Bounded price rule. Generally the rule is as follows: Define a range of prices between a lower and an upper boundary. If the price happens to be below the lower boundary the rule is: acquire stocks in the quantity which pushes the price up to the lower boundary. If the price happens to be within the range the rule is: do not sell or buy anything. Finally, if the price happens to be above the upper boundary, the rule is: sell out the quantity of stocks which will push the price down to the lower boundary. If there are not enough stocks to achieve this, then be out of stocks. A flow chart of the bounded price rule is presented in figure 14. In this flow chart the boundaries are defined as percentages of target prices (PSTAR(IT)). B1 and B1 are the percentages of the lower boundary and the upper boundary respectively. PSTAR1 and PSTAR2 are the boundaries and IPSTR1, IPSTR2 their corresponding integer indices. DC is the required change of stocks and BND(PSTARJ,PP,IT) is a function which defines the change of stocks which is needed when the market price is PP, in order to change it to PSTARJ. As before, C(IC1,IPP,IT) is the stocks rule.

2.3. Probability computations of prices and stocks. As mentioned in part I, the most important information for the study of the implications of any stocks rule in the context of price stabilization is embodied in the probability distribution of prices and stocks under the application of the rule. In this subsection the program of probability computation is

Figure 14. A flow chart of the bounded price rule



described. More details can be found in the flow chart of the appendix. The basic formulae were presented in part I (subsection 2.3). The notation described in 2.1 is used here.

To this part of the program, the stocks rule is given exogenously. Actually two rules were examined, namely: (a) the optimal stocks rule, discussed in subsection 2.1, and (b) the bounded price rule, discussed in subsection 2.2. The central computation of this part of the program is the computation of the joint probability distribution of stocks and prices in any period IT. Let us therefore begin with this. Later on, the marginal and cumulative probabilities, as well as some instability indices, will be discussed.

Joint probability distribution of prices and stocks. Recall that according to the approximative procedure (subsection 2.1 above), price may obtain only values of

$$P(IP) \qquad IP = 1, 2, \ldots, NP$$

and stocks may obtain only values of

$$CC(IC) \qquad IC = 1, 2, \ldots, NC \ .$$

The joint probability of stocks and prices in period IT is denoted by PRBCP(IT, IC, IP).

$$PRBCP(IT, IC, IP) \equiv \text{Probability } \{\text{stocks} = CC(IC), \text{ Price} = P(IP)\}$$

Two stochastic elements are involved in the computation of PRBCP(IT, IC, IP), namely:

(a) The joint probabilities of stocks and prices of the previous period (IT-1), i.e.

PRBCP(IT-1, IC1, IP1) IC1 = 1, 2, ..., NC

IP1 = 1, 2, ..., NP

(b) The probabilities of the stochastic disturbance of period IT, which is denoted by

PRBE(IE) IE = 1, 2, ..., NE

Notice that the assumptions of the model imply that (a) and (b) are mutually independent. It follows that the probability of the combination of indices (IC1, IP1, IE) equals the product

PRBCP(IT-1, IC1, IP1) * PRBE(IE).

The following steps are also determined by the same combination of beginning stocks, lagged price and current disturbance:

(a) P(IP1) and E(IE) determine the market price (PP) by the function <u>PPFEP1</u>(E, P1, IT):

PP = PPFEP1(E, P1, IT)

which in turn is transformed to an integer index IPP by

IPP = <u>INDEXP</u>(PP)

where: <u>INDEXP</u>(P) is the function which translates price to the corresponding integer index.

(b) The market price PP (represented by IPP) and the beginning stocks (represented by IC1) determine the quantity of carryover by the stock rule C(IC1, IPP, IT). And this in turn is translated into the integer index (IC) by

IC = <u>INDEXC</u> [C(IC1, IP1, IT)]

where INDEXC is the function which translates quantities of stocks into the integer index.

In addition, the change of stocks DC is computed by

DC = CC(IC) - CC(IC1) ,

which, given the market price (calculated in (a) above), determines the final price by the function PFPPDC and its integer index (IP) by

IP = INDEXP [PFPPDC(PP, DC, IT)] .

In summary, each combination of indices (IC1, IP1, IE) results in a combination of indices (IC, IP) and a probability PRBCP(IT-1, IC1, IP1)*PRBE(IE) attached to it.

However, there might be more than one combination of (IC1, IP1, IE) which results in the <u>same</u> combination of (IC, IP), hence to obtain the joint probability, PRBCP(IT, IC, IP), the program sums all the products

PRBCP(IT-1, IC1, IP1)*PRBE(IE)

corresponding to combinations of (IC1, IP1, IE) which result in the same (IP, IC). Figure 15 summarizes the above description in a flow chart of the program which calculates the joint probabilities.

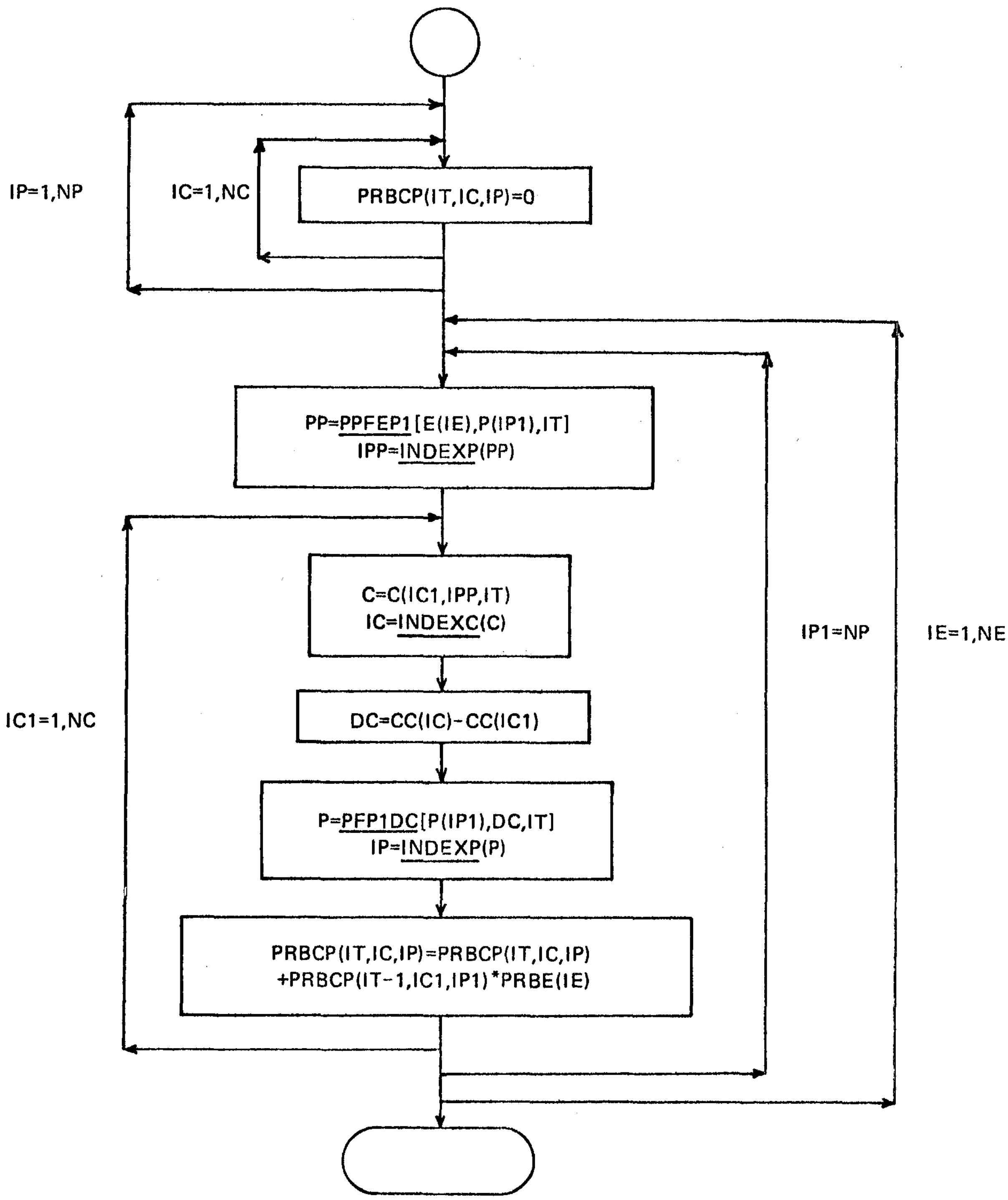


Figure 15. A flow chart of the computations of the joint probabilities of stocks and prices

Using the joint probabilities of stocks and prices, the program proceeds in calculating the marginal probabilities of prices and stocks, their cumulative probabilities and some indicators of magnitudes and instability.

<u>Marginal and cumulative probabilities and some indicators of instability</u>. The marginal probabilities of prices and stocks are calculated by summing over the stocks indices and over the price indices respectively. More specifically, for a certain period IT:

$$PRBP(IP) = \sum_{IC} PRBCP(IT,\ IC,\ IP)$$

$$PRBC(IC) = \sum_{IP} PRBCP(IT,\ IC,\ IP)$$

where PRBP(IP) = Probability {Price = P(IP)}

and PRBC(IC) = Probability {Stocks = CC(IC)}

Using the marginal probabilities, the program then computes the following indicators for each period IT:

(1) The mean of price

$$MEANP(IT) = \sum_{IP} PRBP(IP)*P(IP)$$

(2) The standard deviation of price

$$VARP(IT) \equiv \{\sum_{IP} PRBP(IP)*[P(IP) - MEANP(IT)]^2\}^{1/2}$$

(3) Coefficient of variation of price

$$CVP(IT) = VARP(IT)/MEANP(IT)$$

(4) Index of variability around the target price

$$VIP(IT) = \{\sum_{IP} PRBP(IP)*[P(IP) - PSTAR(IT)]^2\}^{1/2}$$

(5) Coefficient of variation around target price

$$CVTP(IT) = VTP(IT)/PSTAR(IT)$$

(6) Mean of stocks

$$MEANC(IT) = \sum_{IC} PRBC(IC)*CC(IC)$$

(7) Standard deviation of stocks

$$VARC(IT) = \{\sum_{IC} PRBC(IC)*[CC(IC) - MEANC(IT)]^2\}^{1/2}$$

(8) Coefficient of variation of stocks

$$CVC(IT) = VARC(IT)/MEANC(IT)$$

All the above indicators measure the magnitude and variability of price or stocks in a certain period IT. While the standard deviation and the coefficient of variation measure the average absolute and relative deviation around the mean respectively, VIP and CVTP measure the average absolute and relative deviation of prices around the target price respectively.

In addition the program computes for each IT cumulative indicators for the whole period from MT through IT. These are:

(9) Discounted instability index

$$DIIN = \sum_{I=MT}^{IT} CVTP(I)^2 * 1/(1+DELTA)^{(IT-MT)}$$

(10) Instability index (not discounted)

$$IIN = \sum_{I=MT}^{IT} CVT(I)^2$$

(11) Average coefficient of variation around target price

$$ACVTP = (IIN)^{1/2}/(IT-MT+1)$$

This is a measure of the average deviation around the target price for the whole period MT through IT.

(12) Discounted mean of stocks

$$DMEANC = \sum_{I=MT}^{IT} MEANC(I)*1/(1+DELTA)^{(IT-MT+1)}$$

(13) Average (non-discounted) mean of stocks

$$AMEANC = \sum_{I=MT}^{IT} MEANC(I)/(IT-MT+1)$$

Finally the program computes the cumulative probabilities of prices and of stocks by

$$PRBP(IP) = \sum_{I=1}^{IP} PRBP(I)$$

$$PRBC(IC) = \sum_{I=1}^{IC} PRBC(I)$$

where PRBP(IP) now stands for Probability $\{Price \leq P(IP)\}$ and PRBC(IC) now stands for Probability $\{Stocks \leq CC(IC)\}$.

3. Application to Grain Reserve Problem

The model and the computation procedure described in the previous two sections were applied to two cases, U.S. wheat and world grains, to demonstrate the use of the model to obtain a set of efficient stock policies for price stabilization and to evaluate the expected implications of different stock policies. However, it should be noted that the empirical work in the present research is very preliminary. No econometric work has been done to estimate the various parameters of the assumed models. Those were evaluated by judgment, using information from other studies. Therefore, the numerical results should be considered merely as illustrative examples. The U.S. wheat model is presented first in some detail and then the world grain model.

3.1 U.S. wheat model. The order of presentation is as follows: First, the general assumptions of the model are presented. Second, the different stocks rules are discussed. Finally, the probability distributions of prices and stocks under the different policies are analyzed together with the various indicators which are based on these distributions.

General assumptions

Supply is composed of an acreage equation and a yield equation. The acreage equation is[14/]

$$\text{PLW} = 49.10 + 4.37\ \text{PWH-1} \tag{3.1}$$

[14/] In deriving the acreage equation we followed Hoffman [2]. We reestimated his equation using the same data but deflated the price variables by the GNP deflator in order to eliminate the effect of general inflation. The equation which was estimated was

$$\text{PLW} = 4367\ \text{PWH-1} - 11695\ \text{EVW} + 864\ \text{RNC} - 20043$$

(continued)

where PLW is planted area of wheat (million acres) and PWH-1 is wheat price, lagged one year ($/bushel).

The yield equation is

$$YW = (32.75 + EPS)\cdot(1.033)^{(t-75)} \tag{3.2}$$

where YW is yield of wheat per planted area (bushel/acre)

EPS is a random variable, assumed to be normally distributed with zero mean and standard deviation 3.46 (bushel/acre).

From (3.1) and (3.2) the supply equation is

$$X_t = (49.10 + 4.37P_{t-1})(32.75 + E_s)(1.033)^{(t-75)} \tag{3.3}$$

The demand has two components, domestic and export.[15/] The domestic demand equation is

$$WLD = 1190 - 83.3P\cdot(1.01)^{(t-75)} \tag{3.4}$$

(footnote 14/ continued)

where PLW is the planted area of wheat

PWH-1 is wheat price, lagged one year, deflated by GNP deflator (in $/bushel)

EVW is effective voluntary rate which is a variable representing government policy (EVW was also deflated by the GNP deflator)

RNC is a range condition index.

For an explanation of the last two variables see Hoffman [2]. Assuming EVW=0 and RNC=80, the equation is transformed to (3.1). The yield equation is based on fitting a logarithmic trend line to a time series of yields. The standard deviation of EPS was estimated by the deviations from this line.

[15/] The demand equations are based on equations used in a simulation study by Sharples and Walker [3].

The export demand equation is

$$\text{(3.4')} \qquad WEX = (1765 - 161.7P + EPS)(1.03)^{(t-75)}$$

where WLD is domestic demand for wheat (million bushels)

P is the price of wheat (\$/bushel)

WEX is wheat export and

EPS is a random variable, assumed to be normally distributed with a zero mean and a standard deviation of 255 million bushels.

Total demand is the sum of (3.4) and (3.4').

$$\text{(3.5)} \qquad \begin{aligned} Y_t = {} & [1190(1.01)^{(t-75)} + 1765(1.03)^{(t-75)}] \\ & - [83.3(1.01)^{(t-75)} + 161.7(1.03)^{(t-75)}]P_t \\ & + E_d(1.03)^{(t-75)}. \end{aligned}$$

In summary, the supply and demand equations (3.3) and (3.5) are a linear version of the general model assumed in section 1 (equations (1.2) and (1.5)).

The planning period is defined from 1975 to 1985. The target prices are assumed to be equal to the "long-run" equilibrium mean prices, i.e., prices such that $P_t = P_{t-1}$ and $E_{st} = E_{dt} = 0$.

Beginning stocks in 1975 are assumed to be zero and the initial lagged price is assumed to be \$4.25 per bushel. These initial values are used in the probability computations.

It is assumed that annual storage cost (Θ) is \$0.2/bushel and that the discount rate (δ) is 0.05.

In the present experiment, the program calculated the following stocks policies: a bounded price rule (BPR) with a range of $\pm$ 10 percent around the target price (see subsection 2.2), and a set of price variability minimization rules (PVM) with different values of λ.

<u>Stock rules</u>

<u>Bounded price rule (BPR)</u>. The rule was described in subsection 2.2. Generally, a range of prices is defined in which there is no intervention. Whenever the price is below the lower boundary of the range, a quantity of stocks is purchased that raises the price to that boundary. Whenever the price is above the upper boundary of the range, stocks are sold to reduce the price as much as possible (if there are enough stocks) toward that boundary. In the present experiment the boundaries were defined to be plus or minus 10 percent of the target price.

The bounded price rule in 1975 is demonstrated graphically by the dashed curves (denoted by BPR) in figures 16, 17, and 18, in which change of stocks (ΔC) is measured along the vertical axis and market price ($\tilde{P}$) along the horizontal one. The lower and upper boundaries are denoted by B_1 and B_2 respectively. In figure 16 the BPR curve is drawn under the assumption that there are always enough stocks to sell out when needed ($\tilde{P} > B_2$). In figure 17 the BPR curve is drawn under the assumption that the existing stocks are zero. It is the same curve as in figure 16 but truncated at $\Delta C = 0$. Similarly in figure 18 the BPR curve is drawn assuming that the beginning stocks are 500 million bushels. This curve is also identical to the BPR curve in figure 16 but truncated at $\Delta C = -500$. This demonstrates the fact mentioned previously, that the existing stocks affect the stocks rule only in limiting the feasible negative change of stocks (i.e., selling), otherwise the BPR depends on price only. This is in contrast to the price

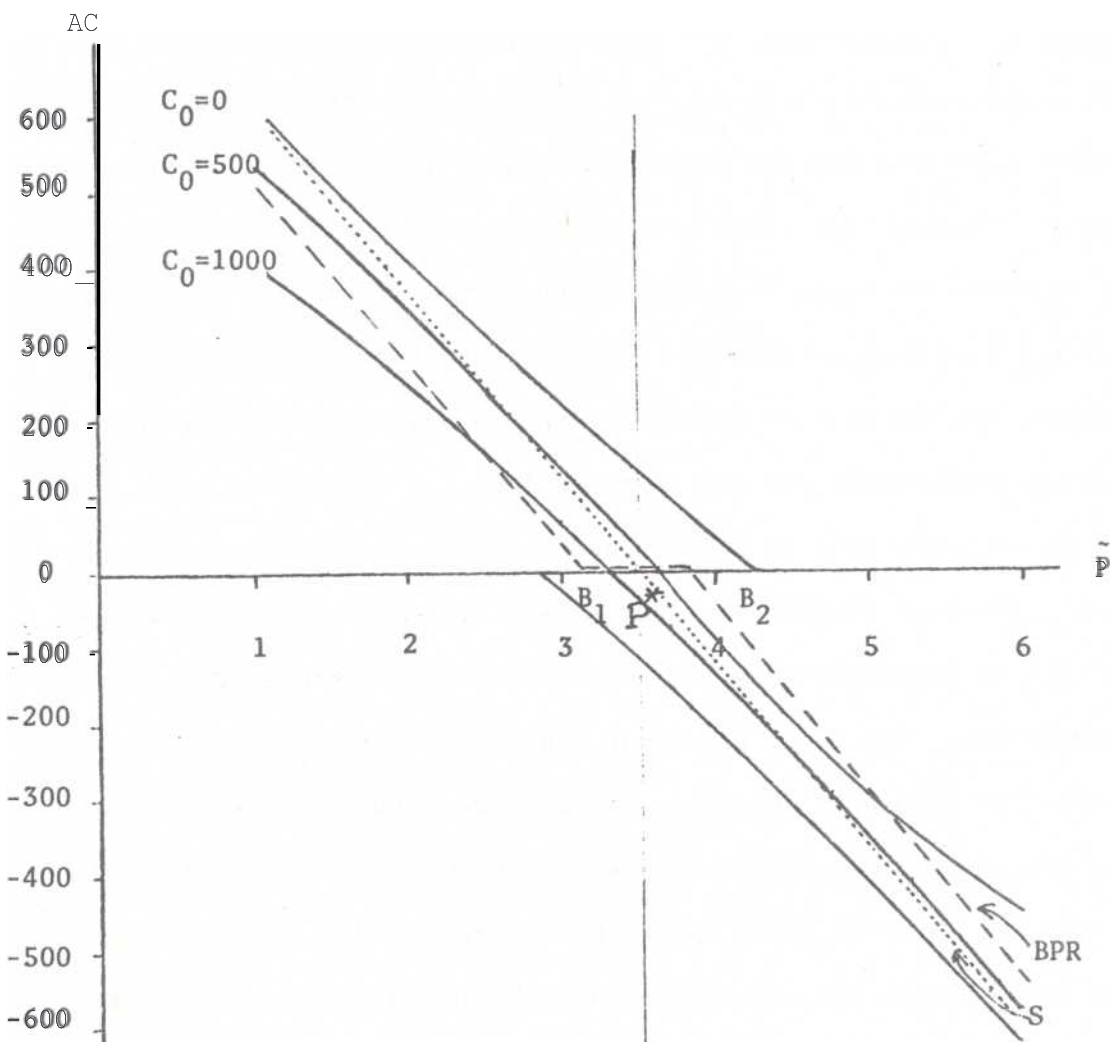


Figure 16. U.S. Wheat Model
PVM rule in 1975 with 0=0
different beginning stocks.

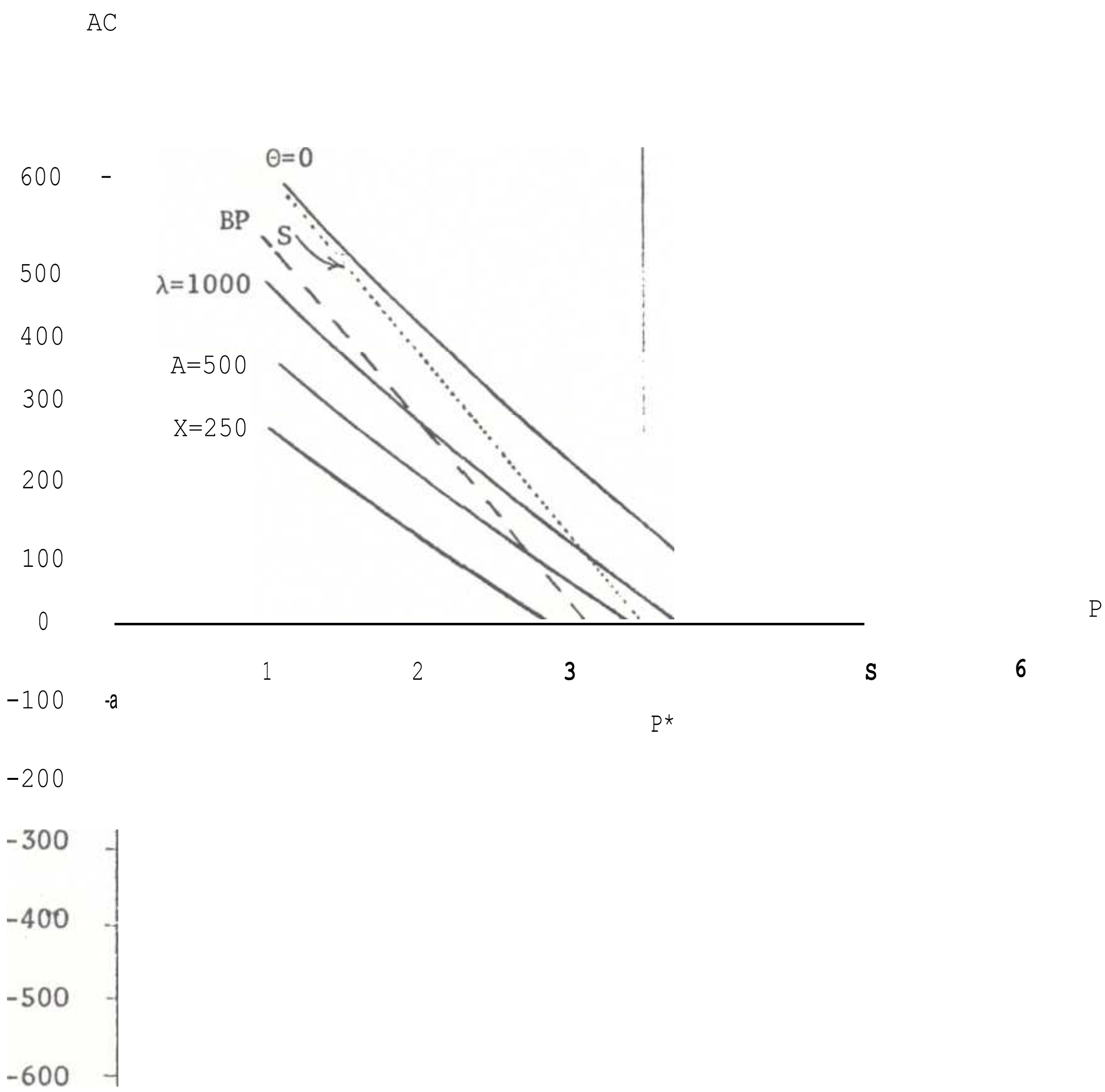


Figure 17.

U.S. Wheat Model
PVM rule in 1975
with zero beginning stocks
under different values of X.

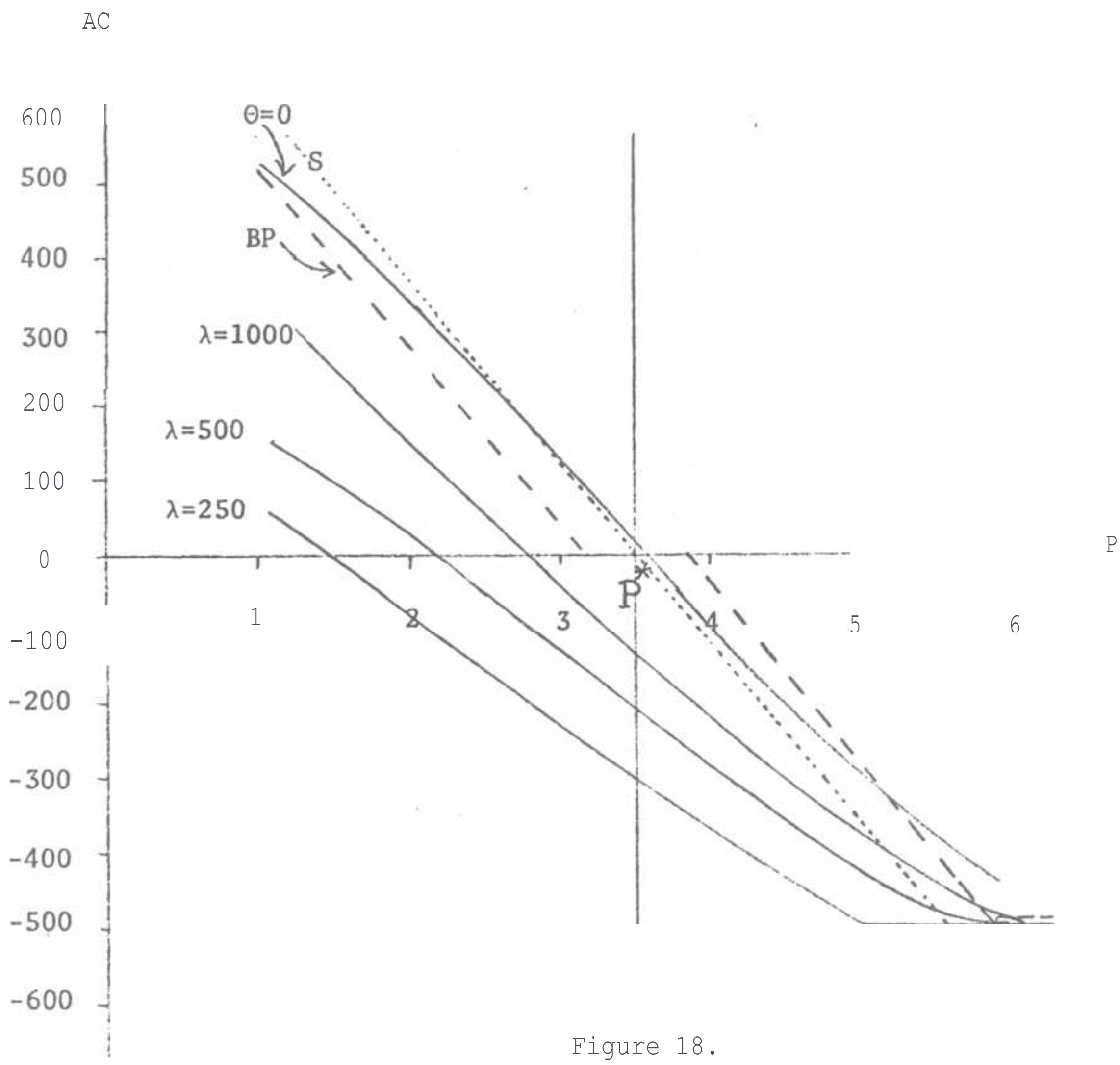


Figure 18.
U.S. Wheat Model
PVM rule in 1975
with beginning stocks of 500 million bushels
under different values of X.

variability minimization rule (PVM) and depends on both price and quantity.

Price variability minimization rule (PVM). The PVM computation program was applied to the U.S. wheat model with several alternative values of λ (rate of substitution of cost for stability). Recall that by changing λ, an efficiency frontier is traced along which mean storage cost increases as price instability decreases. Experiments were made using two extreme cases and four intermediate cases. In one extreme case it was assumed that $\Theta = 0$, i.e., that there is no cost at all. This program gives the minimal feasible degree of instability that can be obtained if one does not care about the cost of stocks.[16/] In the other extreme case $\lambda = 0$. In this case one does not care about instability and the result is, of course, no intervention, or the "free market" situation. In the intermediate cases, $\lambda = 125, 250, 500,$ and 1000. For the given λ and Θ, the PVM rule was computed for each year of the planning period. On request, the computer can print out the rule for selected years in the form of a table. In the table, columns correspond to different values of market prices ($\tilde{P}_t$) and lines correspond to different values of beginning stocks (C_{t-1}) (of the specific year). Each box of the table, corresponding to a pair ($\tilde{P}_t$, C_{t-1}), contains three numbers: the change of stocks (ΔC_t), the final stocks (C_t), and the resulting price (P_t). The behavior of PVM for 1975 is illustrated graphically in figures 16, 17, and 18. The curves in these figures are free hand fittings of data from the computer print-out tables. In figure 16 the extreme case of PVM ($\Theta = 0$) (when there is no cost) is drawn. Change of stocks (ΔC) is measured along the vertical axis and market price ($\tilde{P}$) along the horizontal one. The curves describe the change of stocks (ΔC_{75}) as a function of the market price ($\tilde{P}_{75}$), given alternative

[16/] The value of λ does not matter when $\Theta = 0$.

levels of beginning stocks ($C_{74} = 0$, $C_{74} = 500$, $C_{74} = 1000$, and $C_{74} = 1400$ million bushels). It can be seen easily that as the beginning stocks increase the change of stocks decreases for any given market price. This is compatible with the theoretical conclusion in section 1, in which it is shown that when existing stocks are relatively small, it is desirable to accumulate more in order to reduce future instability. For comparison, the BPR curve is drawn in the same figure by the dashed curve. The dotted curve denoted by S indicates the change of stocks that is needed to equate the price to the target price. It can be seen that when stocks are relatively small it pays to accumulate more stocks than are needed to completely stabilize current price (i.e., to equate P_{15} to p^*_{15}). However, there is a ceiling on the amount of stocks,[17/] so in some situations a positive change of stocks might be desirable, but not feasible, for price stabilization. It follows that when the quantities of existing stocks are large enough, positive accumulation will not have a stabilizing effect on future prices but will have an opposite effect, because it might be impossible to buy stocks in enough quantity in case the market price is below the target price. Thus, in figure 16, those parts of the curves corresponding to high levels of beginning stocks and low market price are below the current stability curve.

The effect of λ on the PVM rule is demonstrated in figures 17 and 18 for beginning stocks of zero and 500 million bushels respectively. The upper curve in each of the figures describes the PVM rule in the case of no cost ($\Theta = 0$), and the other curves describe the rule for $\lambda = 1000$, $\lambda = 500$, and $\lambda = 250$. In addition, the bounded price rule BPR curve is

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[17/] The ceiling on the amount of stocks is defined implicitly in the computation procedure by the values that are assigned to UC and NC (see section 2).

drawn and also the complete current price stabilization curve (curve S). As would be expected, the smaller λ is, the smaller is the change of stocks for any given market price. This is compatible with the reasoning stated in section 1. Smaller λ means that relatively more weight is given in the objective function to the cost that has to be paid for stabilization, hence less stocks are accumulated.

Probability distributions of prices and stocks and summarizing indicators. Using the assumptions of the model, the computer program calculates the probability distributions of prices and stocks in each year of the planning period under any specific stocks rule, given the initial values of stocks and lagged price. Based on these distributions, some summarizing indices are computed. Detailed results of these computations are not reported here. Instead, the probability distribution of prices for selected years and some indices computed from the distributions are presented graphically. But first the magnitudes of stocks under the different stocks rules discussed above are shown. The initial values are zero beginning stocks (i.e., $C_{74} = 0$) and a lagged price of \$4.25 per bushel (i.e., $P_{74} = 4.25$). The mean accumulation of stocks through time under the different stock rules is demonstrated graphically in figure 19. It is clear that the highest rate of accumulation is associated with the PVM rule with no cost (Θ=0), and accumulation is reduced when λ is reduced. Mean accumulation under the bounded price rule (BPR) (within a range of $\pm$ 10 percent) is between the PVM with Θ=0 and the PVM with λ=1000. It is clear that, with certain probability, stocks may be greater or less than the mean. The standard deviation is an indicator of the dispersion of the stocks probability distribution. The standard deviations of stocks under

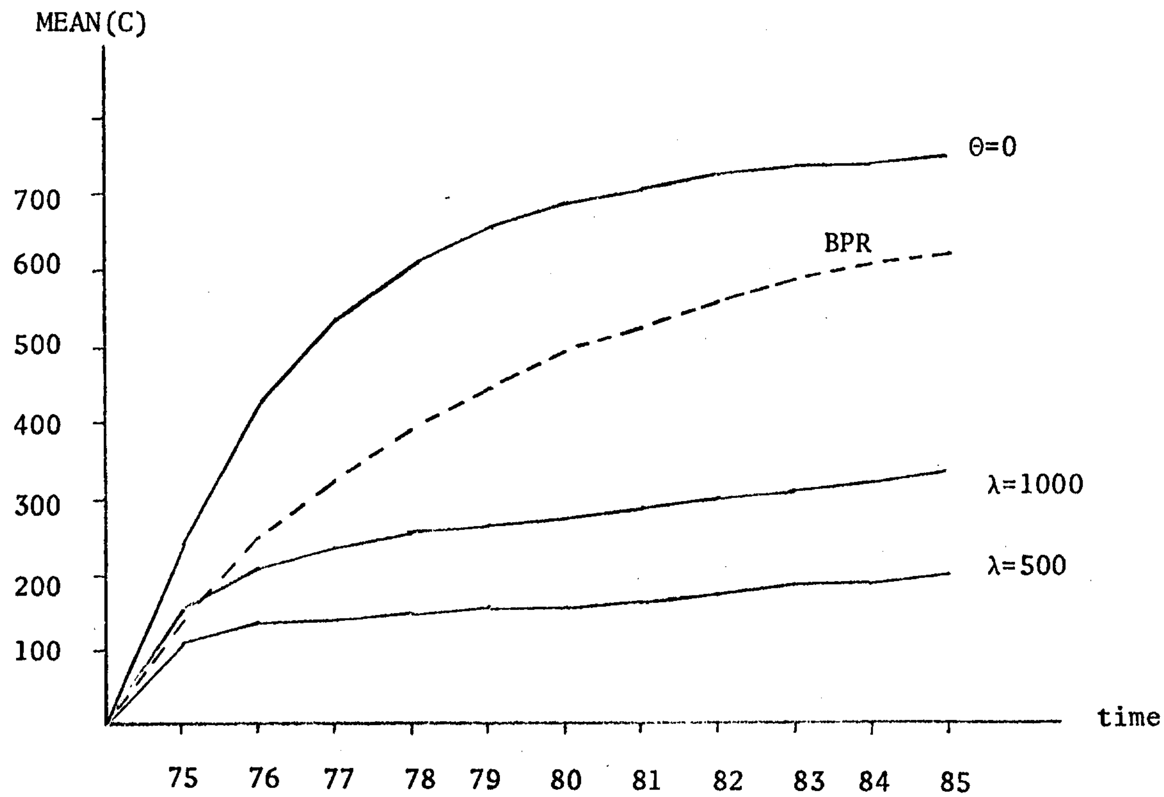


Figure 19.

U.S. model: mean stocks accumulation

the various stock policies are given in table 2 for the years 1975, 1980, and 1985. In general, as the mean of stocks increases through time the standard deviation also increases in any given stock polciy. Also, in a particular year, the higher λ is, the greater are the mean of stocks and their standard deviation. However, the greater λ is, the smaller the coefficient of variation of stocks (i.e., the standard deviation divided by the mean).

Another interesting feature of the probability distribution of stocks is the probability of being out of stocks, i.e., Prob $\{C=0\}$, which is indicated in table 2 for 1975, 1980, and 1985. In general, as stocks accumulate through time, the probability of zero stocks decreases under a given stock policy. In the no-cost PVM case, for example, this probability is 28% in 1975, 6% in 1980, and 5% in 1985. In the BPR case this probability is 52% in 1975, 20% in 1980, and 17% in 1985. When the different rules in a particular year are compared, it is clear that the smaller λ is, the higher is the probability of zero stocks. In 1975 the probability of being out of stocks under the PVM rule with $\Theta=0$, $\lambda=1000$, $\lambda=500$, $\lambda=250$, and $\lambda=125$ are 28%, 40%, 44%, 56%, and 75% respectively. These probabilities in 1980 are 6%, 26%, 41%, 56%, and 76% and in 1985 they are 5%, 24%, 35%, 51%, and 70%.

Let us now turn to the probability distribution of prices. As mentioned in section 2, the computer program calculates the cumulative probability distributions of prices in each year of the planning period following an approximation procedure. The results are printed in the form of tables of cumulative probabilities. Here we present free hand curves fitted to these probabilities for selected years and stock policies. The

Table 2. U.S. Wheat Model: Summary Indicators for 1975, 1980 and 1985

	Year	Free market	Bounded price rule $\pm$ 10% of the target price	Price variability minimization rule				
				λ=125 Θ=.2	λ=250 Θ=.2	λ=500 Θ=.2	λ=1000 Θ=.2	Θ=0
Mean price ($/bushel)	1975	3.12	3.74	3.24	3.41	3.59	3.76	4.12
	1980	3.23	3.33	3.23	3.21	3.23	3.22	3.28
	1985	2.99	3.02	3.00	3.03	3.01	3.00	2.98
Coefficient of variation around the target price CVTP (in %)	1975	44.1	27.5	40.1	34.9	31.1	28.9	29.1
	1980	51.2	20.2	45.0	37.9	31.2	25.5	17.7
	1985	56.1	21.6	50.1	41.3	34.2	27.8	20.6
Mean stocks (million bushels)	1975	-	146	28	71	116	157	246
	1980	-	494	30	82	163	277	689
	1985	-	621	29	106	202	334	751
Standard deviation of stocks (million bushels)	1975	-	187	53	96	134	166	221
	1980	-	417	57	113	178	252	397
	1985	-	472	47	133	205	288	411
Probability of being out of stocks (in %)	1975	-	52.1	75.1	56.2	43.8	39.8	28.4
	1980	-	20.1	75.6	56.2	41.1	26.1	6.1
	1985	-	17.2	69.6	51.3	35.3	23.5	4.9

cumulative probability curves of prices in 1975, 1980, and 1985 are drawn in figures 20, 21, and 22 respectively. In each of these figures the curve for the free market (i.e., with no intervention) is denoted by F, the curves for the PVM are denoted by their corresponding λ or Θ, and the curve for the bounded price rule with a range of $\pm$ 10 percent is the dashed curve denoted by BP. As mentioned above, these probabilities are conditioned on the initial values in 1975 of zero beginning stocks and a lagged price of $4.25 per bushel.

In 1975 beginning stocks are zero with probability one so it is not possible by any stock rule to reduce prices that are above the target price. However, prices that are below the target price can be increased by buying stocks, which is also desirable for future contingencies. In the PVM no-cost case, the probability of prices below the target price is eliminated in comparison to the free market case (figure 20). There is accumulation in the no-cost case even for some prices above the target price; therefore for these prices the cumulative probability curve of the no-cost case is still below that of the free market. However, for prices that are high enough, these two curves coincide in 1975 because of the fact that prices cannot be reduced. For PVM cases with cost there is less stock accumulation than in the no-cost case; hence the cumulation probability curves of the first ones are between the curve of the no-cost case and the free market curve. Also they coincide with the free market curve before the no-cost case. The PBR cumulative curve in 1975 coincides with the free market for all prices above the lower boundary. The probability of prices below that boundary is, of course, zero. The performance of the various policies in 1975 can be measured by the price instability

Figure 20.

U.S. Wheat Model cumulative probability distribution of price in 1975 under various stocks rules.

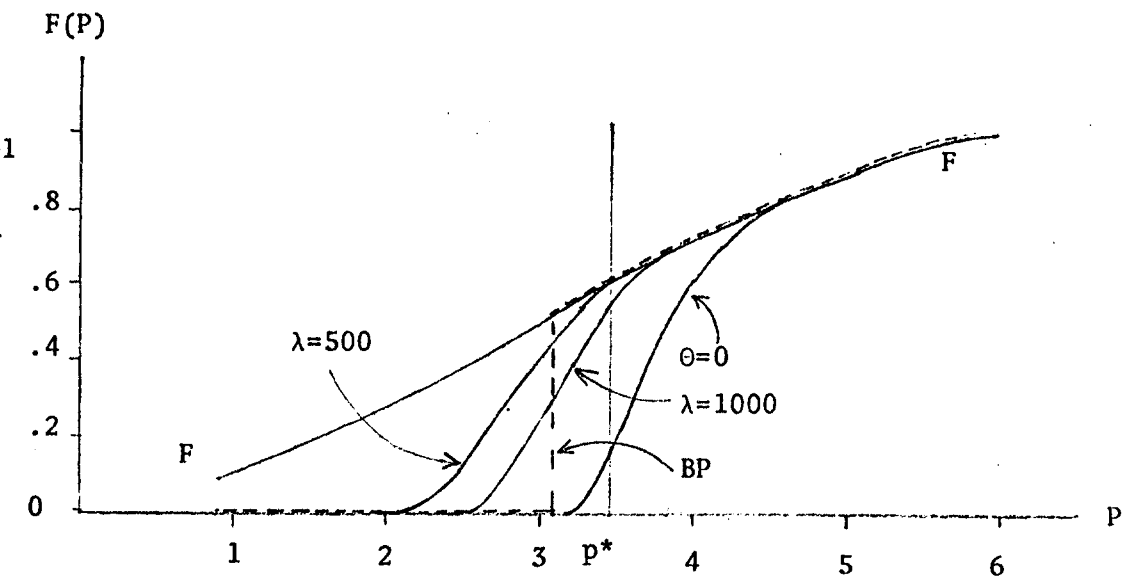


Figure 19

Figure 21.

U.S. Wheat Model cumulative probability distribution of price in 1980 under various stocks rules

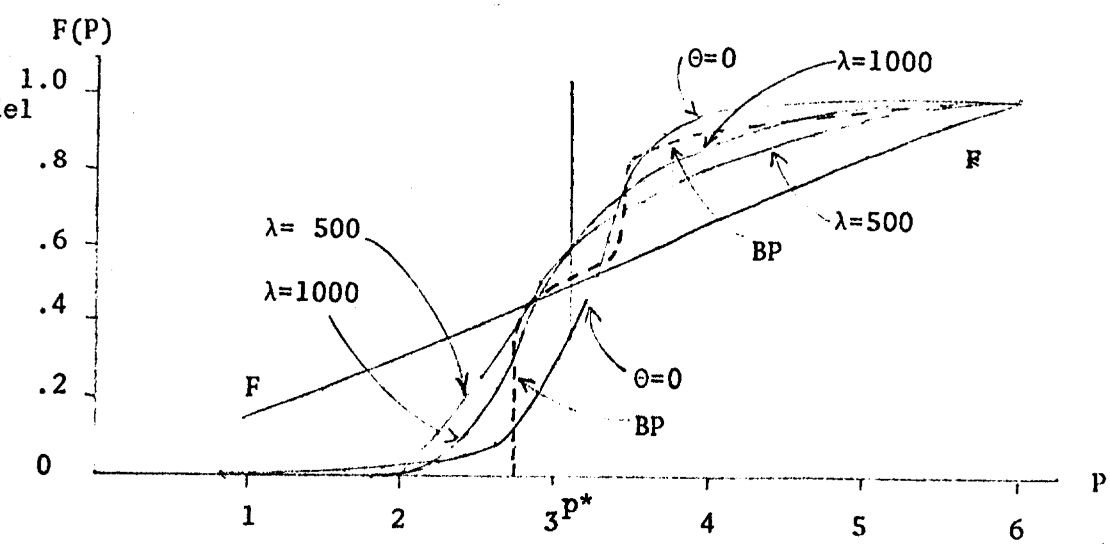
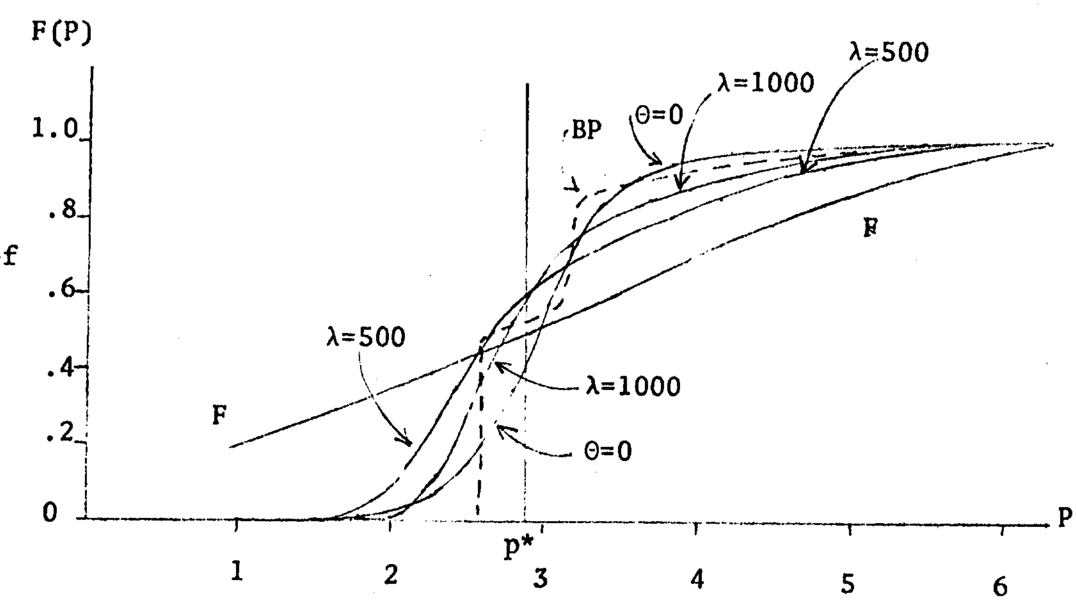


Figure 22.

U.S. Wheat Model cumulative probability distribution of price in 1985 under various stocks rules.



index for that year, which is calculated on the basis of the probability distributions for that year. As can be seen in table 2, the coefficient of variation around the target price (CVTP) in 1975 is 47% in the free market, 29% in the PVM with Θ=0, and 29%, 31%, 35%, and 40% in the PVM with λ=1000, λ=500, λ=250, and λ=125. The CVTP in the BPR is 27%.

Contrary to 1975 figures, the probability of positive stocks is not zero in 1980 and 1985. Therefore it is possible also to reduce the price when being above the target price. It is expected that prices will be more concentrated in probability around the target price in comparison to the free market case (figures 21 and 22). The greatest probability concentration around the target price is achieved in the no-cost PVM case. As λ decreases, the cumulative probability curve is closer to the free market one. It is interesting to look at the bounded price curves (figures 21 and 22). According to the stock rule, all prices below the lower boundary are eliminated, so the probability of price being less than the lower boundary is zero. However, the possibility of eliminating prices which are above the upper boundary depends on the availability of adequate stocks, which is not certain. Hence, the cumulative probability of prices above the upper boundary is not one but less than that.

On the average, the degree of price instability is measured by the coefficients of variation around the target price, CVTP, which are given for 1980 and 1985 in table 2. In 1985 the CVTP in the free market situation is 51%; it is 18% in the no-cost PVM case; and it is 26%, 31%, 38%, and 45% in the PVM cases with λ=1000, 500, 250, and 125 respectively. The CVTP in the BPR in 1980 is 20%. The figures for 1985 are 56% in the free market; 21% in the no-cost PVM case; 28%, 34%, 41%, and 50% when

λ is 1000, 500, 250, and 125 respectively; and 22% in the BPR case.

So far we have considered the performance of the various stocks policies in specific years. Let us now look at the performance over the whole planning period. This can be measured by the instability part and the cost part of the objective function. Recall that the latter is to minimize

$$\lambda \cdot \text{Mean} \left\{ \sum_{t=1975}^{1985} \left(\frac{P_t - p_t^*}{p_t^*} \right)^2 \frac{1}{(1+\delta)^t} \right\} + \Theta \text{ Mean} \left\{ \sum_{t=1975}^{1985} C_t \frac{1}{(1+\delta)^t} \right\}$$

$$= \lambda \text{DII} + \Theta \text{DMEANC}$$

where DII stands for the discounted instability and DMEANC stands for the discounted mean stocks (see section 2), i.e.

$$\text{DII} = \text{Mean} \left\{ \sum_t \left(\frac{P_t - p_t^*}{p_t^*} \right)^2 \frac{1}{(1+\delta)^t} \right\}$$

$$\text{DMEANC} = \text{Mean} \left\{ \sum_t C_t \frac{1}{(1+\delta)^t} \right\}$$

Table 3 gives the values of DII and DMEANC for the different stocks rules. According to the analysis of section 1 it can be expected that as λ increases, mean stocks increase and instability decreases. In fact, by changing λ, a whole set of efficient combinations of instability (as measured by DII) and stocks (as measured by DMEAN) is traced. This efficiency frontier is drawn in figure 23, in which storage cost (ΘDMEANC) is measured along the vertical axis and price instability (DFF) along the horizontal one. The slope of this curve at some point equals the value of λ corresponding to that point. The highest degree of instability

Table 3. U.S. Wheat Model: Summary Indicators for the Period 1975-1985

	Free market	Bounded price rule 10% of the target price	Price variability minimization rule				
			λ=125 Θ=.2	λ=250 Θ=.2	λ=500 Θ=.2	λ=1000 Θ	Θ=0
Discounted instability index of prices DII	2.128	.407	1.666	1.197	.821	.563	.333
Discounted mean stocks DMEANC	-	3548	239	679	1317	2181	4967
Average coefficients of variation around target price. ACVTP (in %)	50.9	21.7	45.0	38.2	31.6	26.0	19.3
Annual average mean stocks (million bushels per year)	-	450	29	86	166	277	631

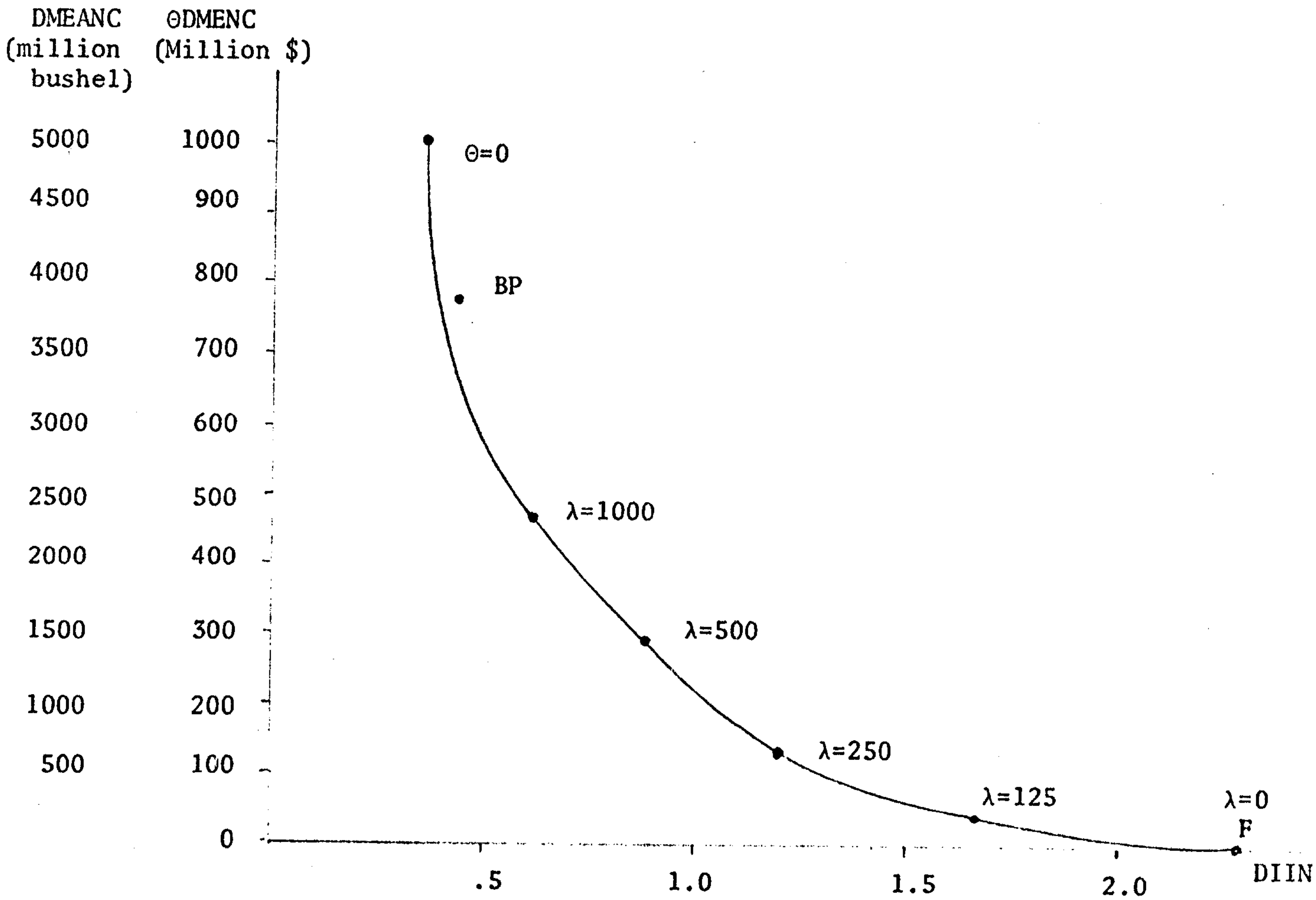


Figure 23.

U.S. Wheat Model

PVM Rule

Discounted Instability Index DII vs.

Discounted Mean Stocks

1975 - 1985

is achieved at the point corresponding to the no-cost case (Θ=0). The slope of the efficiency frontier at this point is infinite. On the other extreme, with no intervention, the cost is zero, the instability is maximal along the efficiency frontier, and the slope at this point is 0. The performance of the bounded price rule with a range of $\pm$ 10 percent over the whole period is also given in table 3 and is plotted in figure 23. It can be seen that point BPR in this figure is relatively close to the efficiency frontier. This means that in this experiment the bounded price rule is almost efficient; however it has the advantage of being a simple rule that can be easily explained to a nonprofessional.

Another measure of performance that is based on non-discounted values can be defined by the average coefficient of variations around the target prices through the whole planning period. This measure, which is denoted by ACVTP, is defined by

$$ACVTP = \left(\prod_{t=1}^{T} CVTP_t\right)^{1/T}$$

That is, ACVTP is the geometric average of the coefficients of variation around the target prices of the various years. Similarly, a nondiscounted average mean stock was calculated and denoted by AMEANC.

The ACVTP and AMEANC corresponding to the different stock policies are reported in table 3 and plotted in figure 24. Stated in this form, the free market price instability over the whole period is 51 percent.

Under the cost PVM rule, price instability is 19 percent with average annual mean stocks of 620 million bushels. Other PVM cases are between the two extreme cases. The BPR with a range of $\pm$ 10 percent results in a price instability of 22 percent and an annual mean stock of 270 million

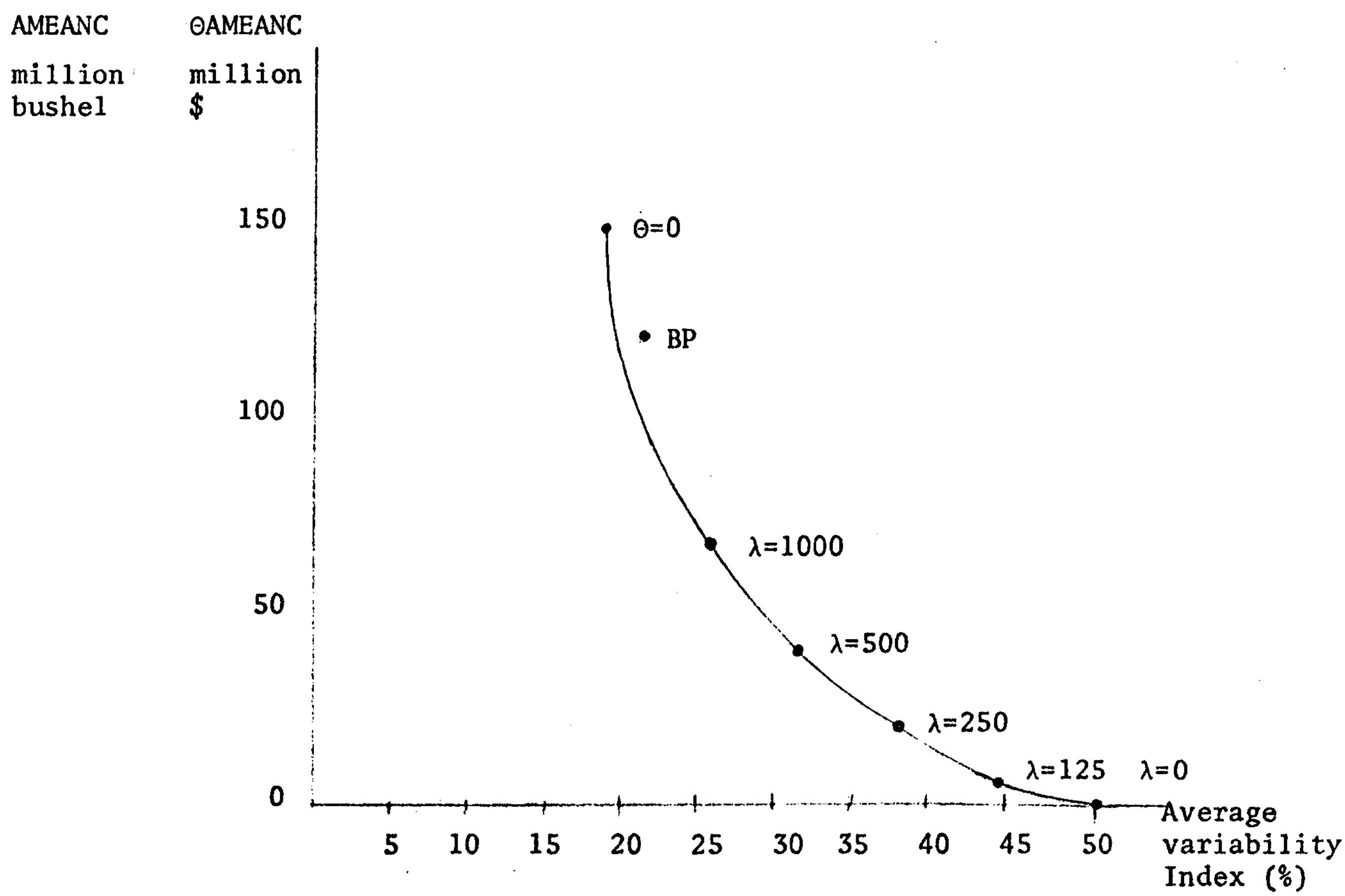


Figure 24.

U.S. Wheat Model

PVM Rule

Average variability around target price (ACVTP) vs.

Average annual mean stocks AMEANC

1975 - 1985

bushels. This concludes the discussion on the experiment with the U.S. wheat model. The world all grains model will be discussed only briefly.

3.2 World grains model. The model is a synthetic one in which world all cereals are assumed to constitute a composite commodity which is traded in a single market. It is assumed that supply does not depend on price, that mean production grows through time at a constant proportional rate, and that production is normally distributed. The supply equation is:[18/]

$$X_t = (1308 + \varepsilon_s)(1 + .029)^{(t-75)} \tag{3.10}$$

where X_t is the quantity produced in year t (in million tons)

ε_s is normally distributed with zero mean and a standard deviation of 40 million tons.

Demand is assumed to be nonstochastic, linearly dependent on price, and grows at the same rate as the supply, i.e., at 2.9 percent a year. The demand equation is:[19/]

$$Y_t = (1439 - 1.31P_t)(1 + .029)^{(t-75)} \tag{3.11}$$

where Y_t is the quantity demanded (in million tons).

These assumptions imply that the mean equilibrium price of the

---

[18/] The rate of growth was estimated by fitting a logarithmic trend line to a time series of world grain production in the years 1950-1973. The deviations from the trend line were used to estimate the standard deviation.

[19/] The demand equation was synthesized such that the mean equilibrium price would be 100 and that the mean equilibrium quantity would be equal to mean production (see the supply equation) and that the price elasticity at this point is -.1.

aggregated commodity is 100 for all t and this was also defined to be the target price. Beginning stocks in 1975 are assumed to be zero. Storage cost is Θ = \$7.5 per ton and the discount rate is $\delta = .05$.

The stock policies tested were:

1. Bounded price rule (BPR) with the following alternative ranges: $\pm$ 10 percent of the target price; $\pm$ 20 percent of the target price; and - 5 percent, + 10 percent of the target price.
2. Price variability minimization rule (PVM) with the following alternatives: no cost (i.e., Θ=0) and λ = 2,500, 5,000, 10,000, 20,000, 40,000.

Summary of results

As in the U.S. wheat model, the computation program computes the cumulative probabilities of prices and stocks for any specific stock rule for each year of the planning period. In addition, it calculates summary indicators which measure price instability and magnitudes of stocks. The various indices have been explained above. The summary results of the world grain model for selected years are reported in table 4, which is similar to table 2. The performance of the PVM rules over the whole planning period under alternative values of λ is summarized in table 5, which is similar to table 3 of the U.S. wheat model.

4. Concluding Remarks and Recommendations for Further Research

The subject of this study is grain price stabilization by means of a stock policy. The controversy about the desirability of price stability and who gains or loses from stabilization is not investigated here. It is taken for granted that price stabilization is desirable. The objective of the research was to establish an analytical framework within which the

Table 4. World Grains Model: Summary Indicators for 1975, 1980 and 1985

	Year	Free market	Bounded price rule			Price variability minimization rule					
			±10% of the target price	±20% of the target price	+10%, -5% of the target price	λ=2500 θ=7.5	λ=5000 θ=7.5	λ=10000 θ=7.5	λ=20000 θ=7.5	λ=40000 θ=7.5	θ=0
Mean price	1975	100	107	104	109	102	106	109	111	113	115
	1980	100	103	102	105	100	101	103	104	106	107
	1985	100	103	102	104	100	101	103	104	105	105
Coefficient of variation around target price CVTP (in %)	1975	27.4	20.4	22.5	19.7	24.0	21.2	20.0	19.6	19.7	20.3
	1980	27.4	15.4	19.9	13.3	24.2	20.5	17.0	14.6	13.4	12.6
	1985	27.4	14.2	19.1	11.8	24.9	21.1	17.5	14.9	13.5	12.6
Mean stocks (million tons)	1975	-	9	5	12	3	6	9	12	14	16
	1980	-	38	24	51	3	9	16	25	33	46
	1985	-	57	39	73	2.5	9	18	30	42	61
Standard deviation of stocks (million tons)	1975	-	16	12	18	6	10	13	14	16	17
	1980	-	40	29	46	7	13	19	27	33	42
	1985	-	50	40	53	6	13	22	33	43	55
Probability of being out of stocks (in %)	1975	-	65.2	76.1	59.3	74.4	59.3	49.0	40.7	38.7	34.8
	1980	-	29.8	36.5	22.8	76.8	54.2	37.5	26.1	21.0	14.4
	1985	-	22.9	27.5	16.1	80.6	58.4	39.2	27.5	19.9	13.0

Table 5. World Grains Model: Summary Indicators for the Period 1975-1985

	Price variability minimization rule					
	$\lambda=2500$ $\Theta=7.5$	$\lambda=5000$ $\Theta=7.5$	$\lambda=10000$ $\Theta=7.5$	$\lambda=20000$ $\Theta=7.5$	$\lambda=40000$ $\Theta=7.5$	$\Theta=0$
Discounted instability index, DII	.486	.351	.255	.202	.180	.171
Discounted mean stocks DMEANC (million tons)	24	68	125	190	249	342
Average coefficient of variability around target price ACVTP (in %)	24.2	20.6	17.5	15.3	14.3	13.7
Annual average mean stocks (million tons)	3	8	15	24	31	43

implications of a stock policy for price stabilization can be analyzed. For this purpose a price instability index over a period of time was suggested. A simple economic model of a single storable commodity was assumed, where both the demand and supply can be stochastic with known probability distribution. The imposition of any stock rule results in a probability distribution of prices and stocks and a computation program was composed which calculates these distributions. The program also computes some summary indicators of price instability and magnitudes of stocks. In addition, the problem of an efficient stock policy in the space of price instability and storage cost was analyzed and a procedure to obtain a set of efficient stock policies was developed. This may give the policymaker a set of policies to choose from. In addition, the set of efficient stock policies can be used as a reference to which any other suggested policy (e.g., a bounded price rule) can be compared. The use of the procedure was demonstrated by application to two models: U.S. wheat and world grains. These experiments were made mainly for illustrative purposes and the econometric basis of the various parameters assumed is rather poor. However, the numerical results may serve as indicators to the order of magnitudes which might be involved in price stabilization schemes. In conclusion, the present study is a preliminary one and further research is needed. The following points should be regarded as recommended directions for further research and as criticism of the present study.

1. Structure of the model. At present the model is of a single commodity. However, cross effects of substitutes are probably important in the determination of price. Thus, prices of wheat, corn, rice, and

other grains are not independent but are jointly determined. Aggregation to a composite product might be misleading for two reasons: First, different grains are not perfect substitutes on the demand side and their prices do not necessarily move proportionally. Second, even if they were substitutes on the demand side, they are not so from the supply side. In particular, the seasons of harvest are different and are important (the season for corn is different from that for wheat; varieties of wheat have different seasons and of course the northern and southern halves of the world have different seasons). In summary, a simultaneous model might be more compatible than a single commodity model.

The present model is a simple "cobweb" model in which supply depends on a one-period lagged price, and demand depends on current price. However, a more sophisticated expectation process might prove to be better and should be tested (e.g., some form of distributed lags).

2. Probability distributions. The form of the probability distributions of the various random variables in the model, and the magnitudes of their parameters play an important role in a problem of price stabilization and stocks. Normal distribution is convenient but it may not be a sufficient approximation. For example, it is argued that the probability distribution of yield is not symmetric. In addition, if the model should be extended to a simultaneous model of different grains, their joint probability distribution should be investigated.

3. Timing of decisions. In the present formulation decisions are made once a year. However, information flows continuously throughout the year and should be used to revise previous decisions. In particular, harvest times of the main crops can be appropriate dates for decisions within the year.

4. Improving the efficiency of computations. The approximative procedure used in the present study is a very simple one but it seems to be inefficient. The probability distributions are approximated by discretionalization and used directly in the computations. However, the dimensionality of the computations is already big even in the simple cases illustrated above and it will increase steeply with the addition of new elements such as more commodities and joint distributions. Probably the solution might be to introduce simulation of the distributions rather than use the distribution directly. Thus, an improvement of the computation procedure, in general, is needed in order to handle problems that are more complex than the one discussed in this research.

APPENDIX

A FLOW CHART OF THE COMPUTATIONAL PROGRAM

A description of the computations procedure was given in section 2. In this appendix a flow chart of computations is presented. It is based on a FORTRAN program which was used in this study.[1] Some details of the program have been omitted and slight changes have been introduced in the flow chart for convenience.

Notes and explanations of the main symbols are forthcoming in the order in which they appear in the flow chart. Underlined names indicate functions. A flow chart of the functions is presented at the end. The numbers in circles ◯ are related to the numbered connectors in the flow chart.

① ⟶ ② Defining the approximation to prices, stocks and stochastic disturbance.

NP = number of integer indices of price

PO = value of price corresponding to the first price index

UP = interval between any two points of price

NC = number of integer indices of stocks

CO = quantity of stocks corresponding to the first stocks index

UC = interval between any two points of stocks

The first do-loop defines a correspondence from integer indices to prices, P(IP). The inverse correspondence (i.e., from price to indices)

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[1] Printout of a FORTRAN program is available on request.

is defined by the function INDEXP(P) (all functions are described at the end). The second do-loop defines a correspondence from integer indices to stocks, CC(IC). The inverse correspondence is defined by the function INDEXC(IC) (functions are described at the end).

EO = number of standard deviations (+ or -) which define the range of the stochastic disturbance (E)

UE = the interval between any two points of the stochastic disturbance (expressed in standard deviations)

NE = number of integer indices of the stochastic disturbance

The last part of the program before (2) defines an approximation to the standardized normal distribution by a discrete probability function.

PRBE(IE) = the probability that the random disturbance will obtain the value E(IE)

The correspondence E(IE) from the integer indices IE to values of E is defined in the first row in the large box before (2).

(2) ⟶ (3)

MT = first period of the planning period

NT = last period of the planning period

NMT = total number of periods

The do-loop in K=1, 2, ..., NMT defines a series of target prices PSTAR(KT) KT=1, 2, ..., NMT. The function TARGET(IT) must be determined and set by the user of the program. (An example is given in the description of the function at the end.)

LAMBDA = λ, the desired marginal rate of substitution of instability for cost

THETA = θ, unit storage cost per period

DELTA = δ, discount rate

After reading DELTA, the discount coefficient $1/1+\delta)$ replaces the value of δ in the same variable.

(3) ⟶ (4) Calculation of the optimal stocks rules.

IC1 = integer index of beginning stocks (C1), i.e., stocks of previous period

IPP = integer index of free market price (PP)

DC = ΔC, the change of stocks

IC = integer index of current stocks

<u>PFPPD</u>(PP,DC,IT) = a function which calculates the final price of time IT, when the market price is PP and the change of stocks is DC (see description of functions at the end)

C(IC1,IPP,IT) = the optimal carryover, which minimizes the objective function for beginning stocks CC(IC1) and market price P(IPP)

PRICE(IC1,IPP) = final price after applying the optimal rule, when the beginning stocks are CC(IC1) and the market price is P(IPP)

V(IC1,IPP) = the minimum value of the objective function, given CC(IC1) and P(IPP)

(4) ⟶ (5) Calculation of the expectation of V(IC1,IPP)

LAG = an index which indicates whether there is a lagged price effect in the model (LAG = 0 if NOT, LAG = 1 if YES)

<u>PPFEP1</u>(E,P1,IT) = a function which calculates the free market price (PP) at time IT, when the lagged price is P1 and the disturbance is E (see description of functions at the end)

EV(IC1,IP1) = the expected value of V(IC1,IPP) when beginning stocks are CC(IC1) and lagged price is P(IP1)

EV(IC1) = similar to EV(IC1,IP1), but for the case of no lagged price

Note that when there is no lagged price effect, EV depends only on the beginning stocks (C1) and much computation time can be saved by using the right-hand side of (4) $\longrightarrow$ (5).

(5) $\longrightarrow$ (6) Preparation for probability computation.

C10 = beginning stocks of the first period (MT)

P10 = lagged price of the first period

PRBCP(J,IC,IP) = joint probability of stocks CC(IC) and price P(IP)

PRBCP(J1,IC1,IP1) = joint probability of stocks CC(IC1) and price P(IP1) of previous period

Note: When there is no lagged price effect, it is not necessary to calculate the joint probabilities of stocks and price. However the space of PRBCP is used for the (marginal) probabilities of stocks and of prices.

PRBCP(·,IC,1) is used for the probability of stocks and

PRBCP(·,IP,2) is used for the probability of prices.

Note also that for the calculation of probabilities of period IT it is not necessary to remember all the probabilities which have been computed for previous periods but only the ones of IT-1. The use of the indices J and J1 enables one to use a space for only two periods, namely: the current one and the previous one.

J = an index which indicates that the joint probability PRBCP is that of the current period of calculations. It may obtain values of 1 or 2:

$$J = \begin{cases} 1 \text{ if } K = 1, 3, 5 \ldots \\ 2 \text{ if } K = 2, 4, 6 \ldots \end{cases}$$ where K is the number of the current calculation period.

J1 = an index which indicates that the joint probability PRBCP is that of the previous period. It always obtains the opposite value of J:

$$J1 = \begin{cases} 2 \text{ if } J = 1 \\ 1 \text{ if } J = 2 \end{cases}$$

7 ⟶ 8 Calculation of the joint probabilities when there is a lagged price effect.

7 ⟶ 9 Calculation of the marginal probabilities of stocks and of prices in the no-lagged price case. There is no need to calculate the joint probabilities because in this case only the probability of stocks affects the probabilities of price and of stocks of the next period.

8 ⟶ 11 Claculations of marginal probabilities of prices and of stocks in the case of lagged price effect.

PRBP(IP1) = marginal probability of P(IP1)

PRBC(IC1) = marginal probability of CC(IC1)

9 ⟶ 11 Transformation of the marginal probabilities of stocks and of prices to new variables in the case of no-lagged price. (The marginal probabilities have been calculated in 7 ⟶ 9 .)

11 ⟶ 12 Calculation of various indicators and of cumulative probabilities of prices and stocks. The cumulative probabilities replace the marginal one in the corresponding variables (PRBP and PRBC).

MEANC = mean of stocks

VARC = variance of stocks (Later, the standard deviation of stocks replaces the variance in the same variable.)

CVC = coefficient of variation of stocks

MEANP = mean of price

VARP = variance of price (Later, the standard deviation of price replaces the variance in the same variable.)

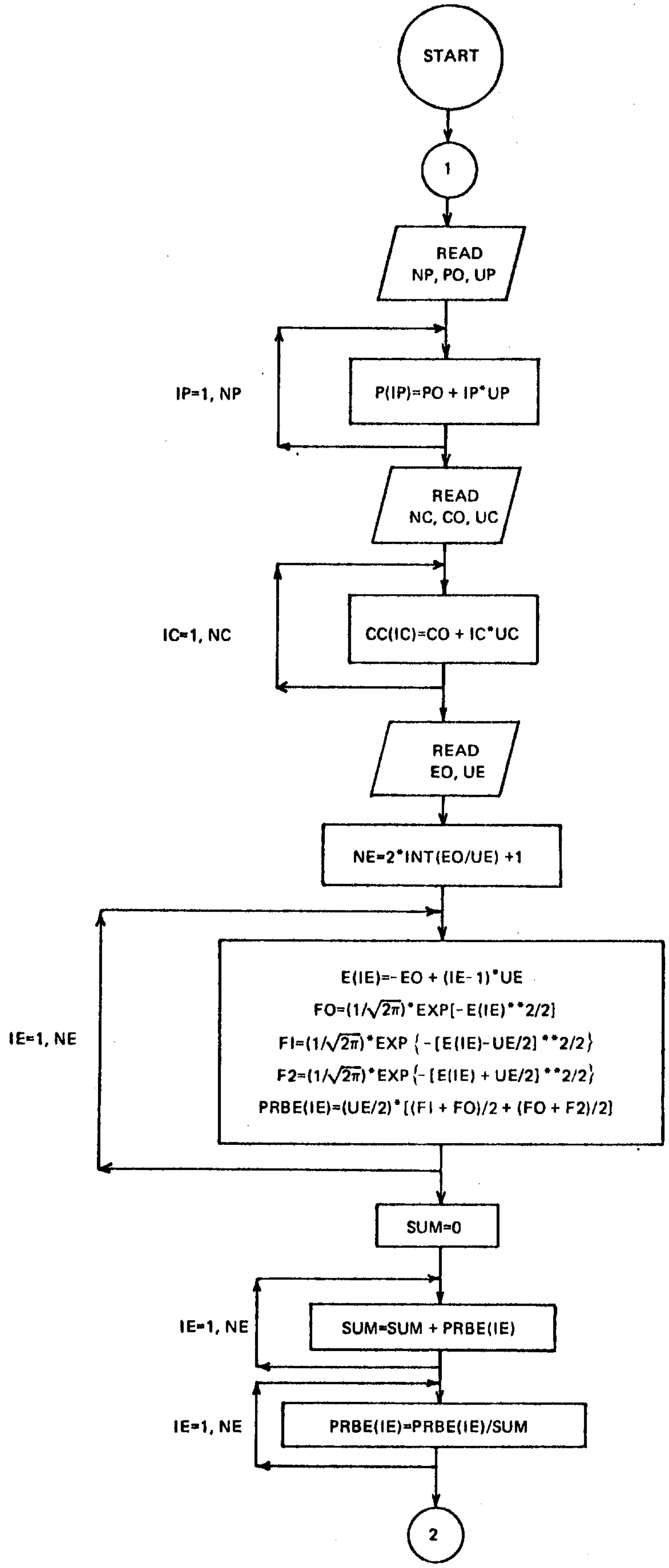
IIN = accumulated price instability index

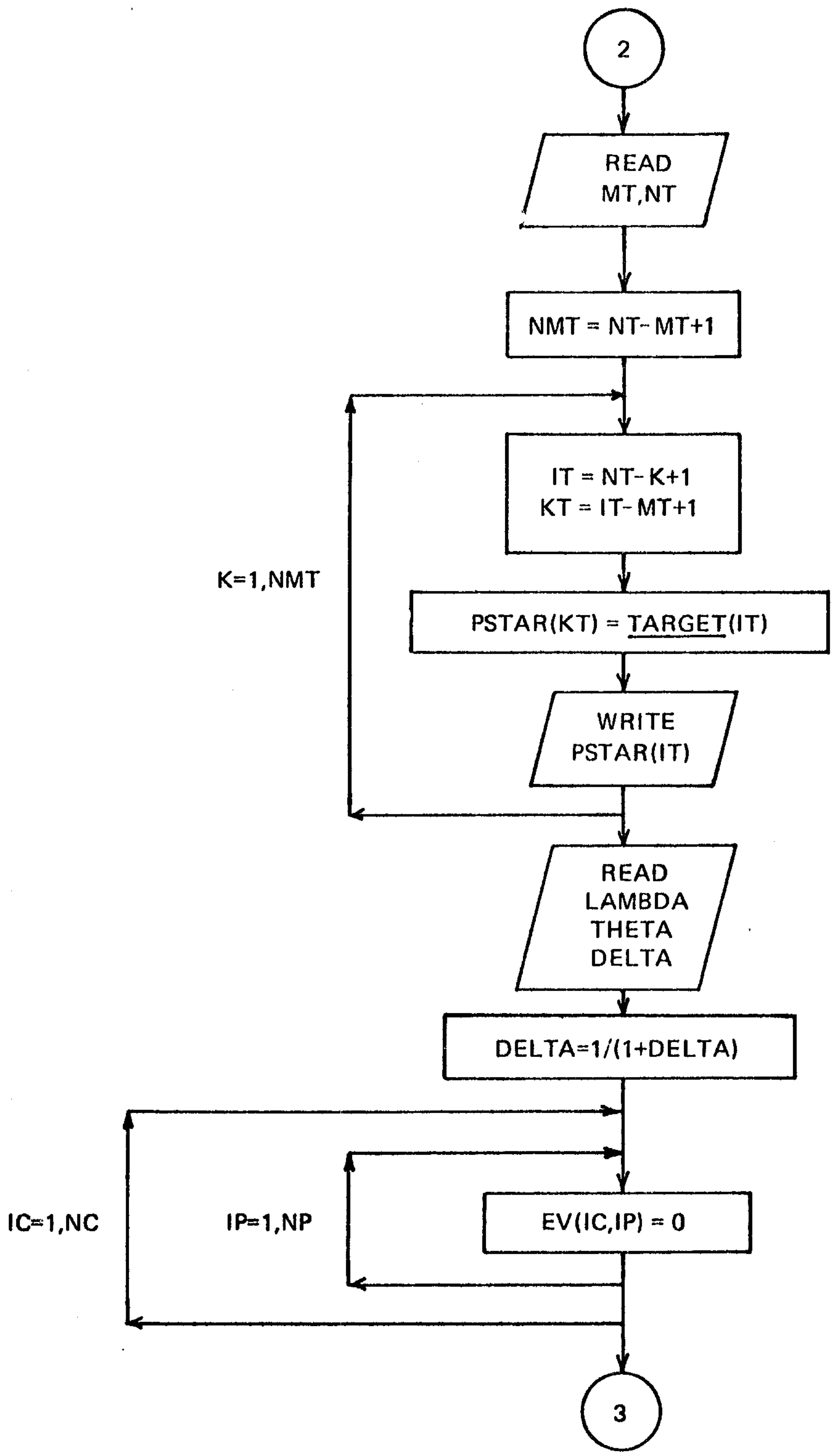
DIIN = discounted accumulated price instability index

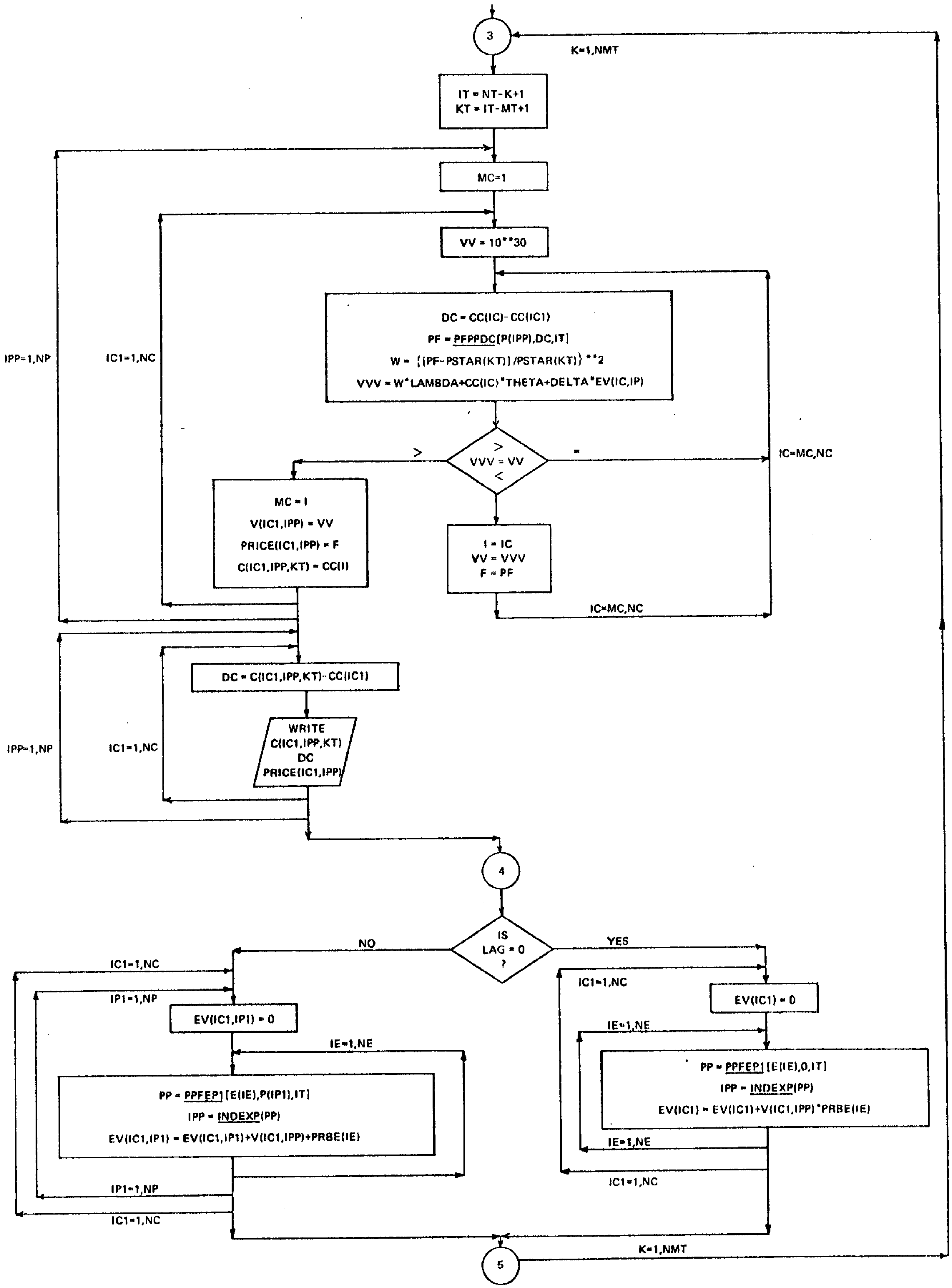
ACVTP = average coefficient of variation of prices around the target prices

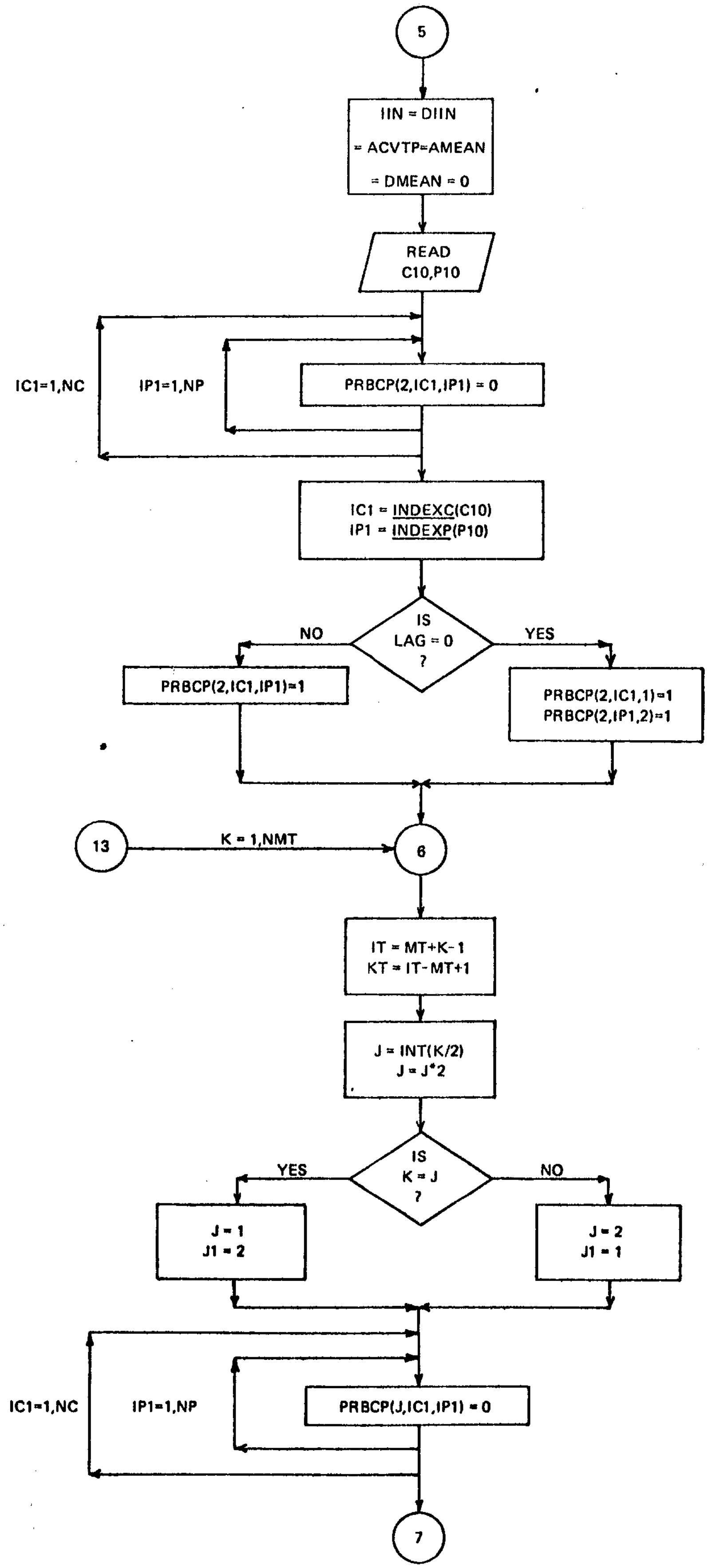
DMEANC = discounted accumulated mean of stocks

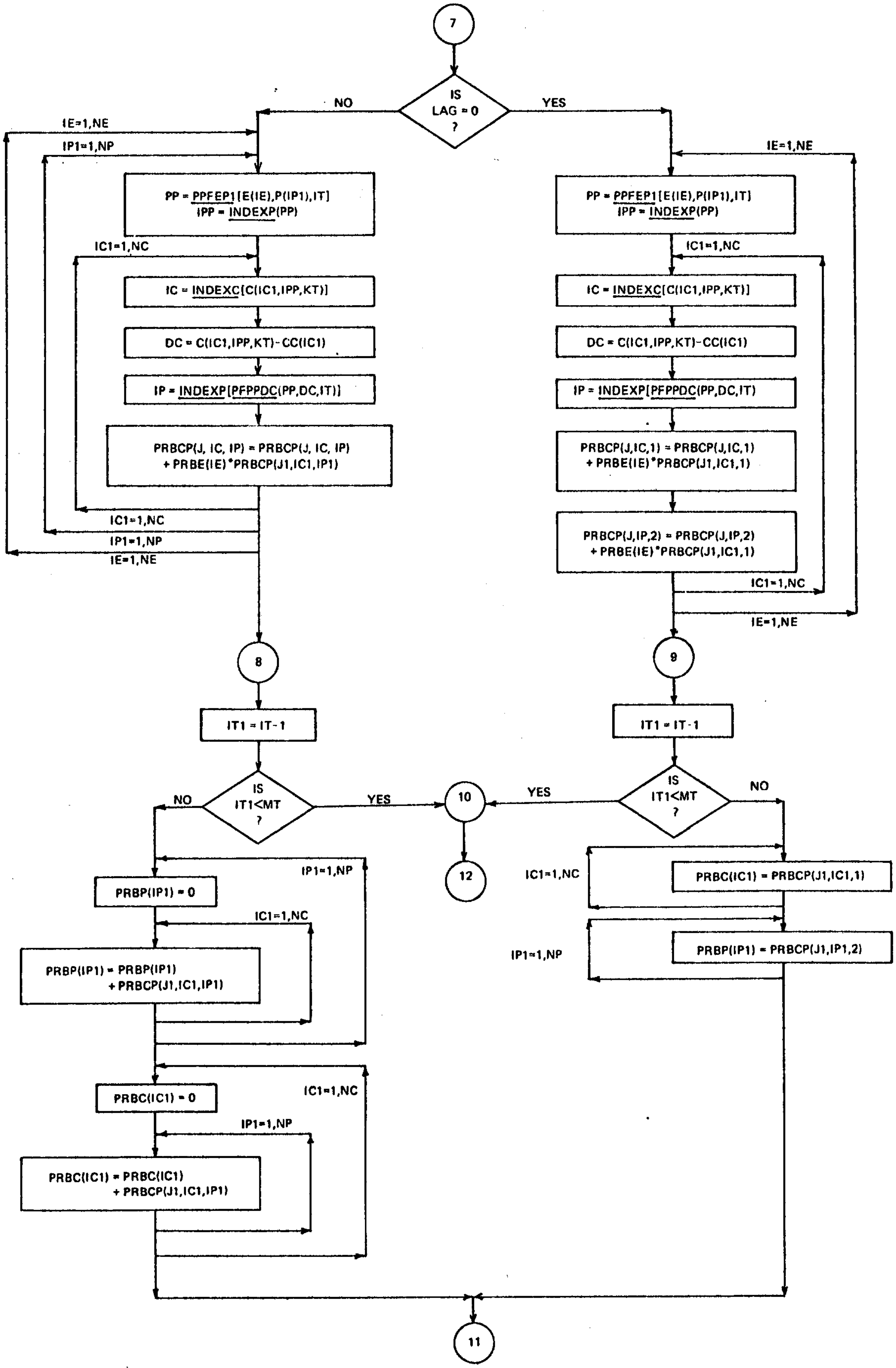
AMEANC = average mean of stocks

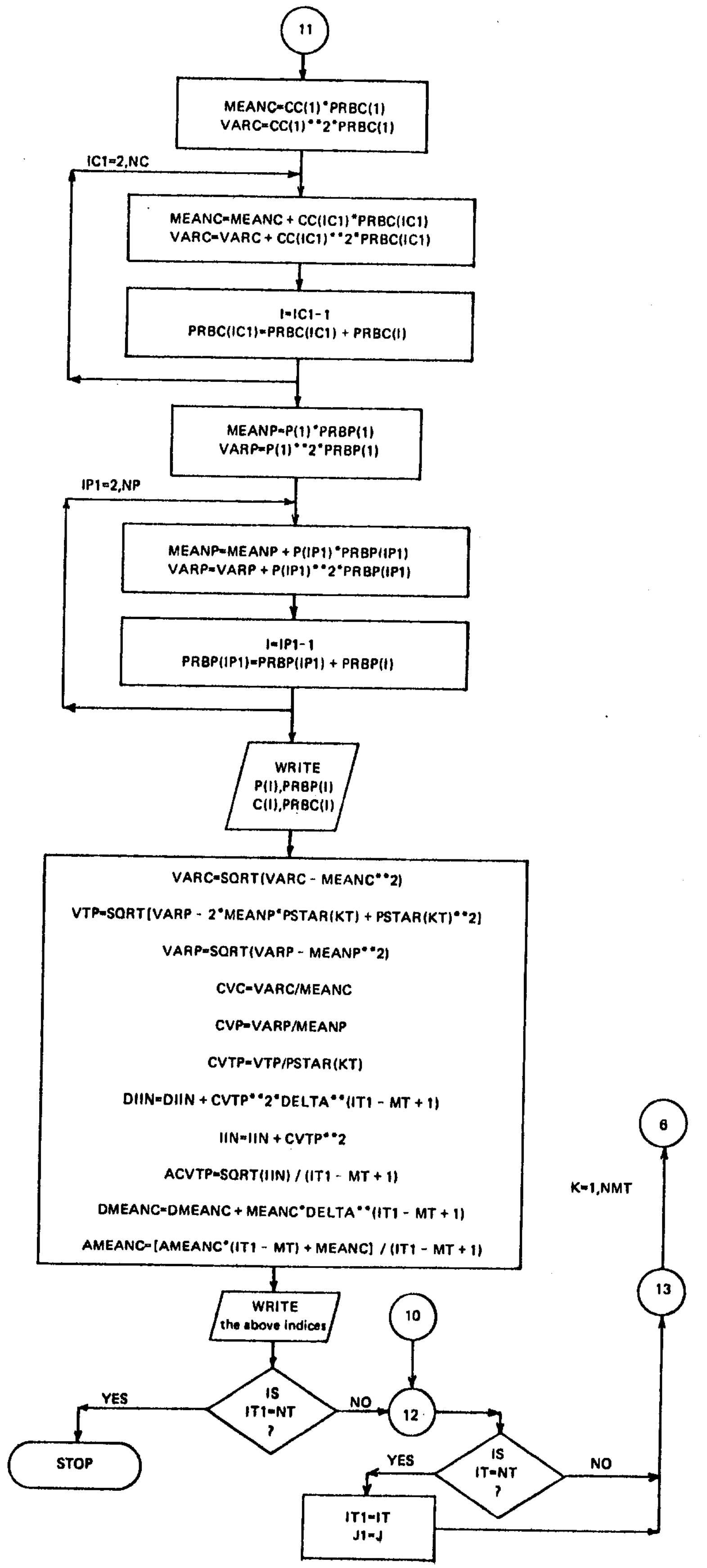

START
1
READ
NP, PO, UP
IP=1, NP
P(IP)=PO + IP*UP
READ
NC, CO, UC
IC=1, NC
CC(IC)=CO + IC*UC
READ
EO, UE
NE=2*INT(EO/UE) +1
IE=1, NE
E(IE)=- EO + (IE- 1)*UE
FO=(1/√2π)*EXP[- E(IE)**2/2]
FI=(1/√2π)*EXP {- [E(IE)- UE/2]**2/2}
F2=(1/√2π)*EXP {- [E(IE) + UE/2]**2/2}
PRBE(IE)=(UE/2)*[(FI + FO)/2 + (FO + F2)/2]
SUM=0
IE=1, NE
SUM=SUM + PRBE(IE)
IE=1, NE
PRBE(IE)=PRBE(IE)/SUM
2


2
READ
MT,NT
NMT = NT-MT+1
IT = NT-K+1
KT = IT-MT+1
K=1,NMT
PSTAR(KT) = TARGET(IT)
WRITE
PSTAR(IT)
READ
LAMBDA
THETA
DELTA
DELTA=1/(1+DELTA)
IC=1,NC
IP=1,NP
EV(IC,IP) = 0
3


3
K=1,NMT
IT = NT-K+1
KT = IT-MT+1
MC=1
VV = 10**30
DC = CC(IC)-CC(IC1)
PF = PFPPDC(P(IPP),DC,IT)
W = {(PF-PSTAR(KT))/PSTAR(KT)}**2
VVV = W*LAMBDA+CC(IC)*THETA+DELTA*EV(IC,IP)
IPP=1,NP
IC1=1,NC
>
VVV = VV
<
=
IC=MC,NC
MC = I
V(IC1,IPP) = VV
PRICE(IC1,IPP) = F
C(IC1,IPP,KT) = CC(I)
I = IC
VV = VVV
F = PF
IC=MC,NC
DC = C(IC1,IPP,KT)-CC(IC1)
WRITE
C(IC1,IPP,KT)
DC
PRICE(IC1,IPP)
IPP=1,NP
IC1=1,NC
4
IS
LAG = 0
?
NO
YES
IC1=1,NC
IP1=1,NP
EV(IC1,IP1) = 0
IE=1,NE
PP = PPFEP1(E(IE),P(IP1),IT)
IPP = INDEXP(PP)
EV(IC1,IP1) = EV(IC1,IP1)+V(IC1,IPP)+PRBE(IE)
IP1=1,NP
IC1=1,NC
IC1=1,NC
EV(IC1) = 0
IE=1,NE
PP = PPFEP1(E(IE),0,IT)
IPP = INDEXP(PP)
EV(IC1) = EV(IC1)+V(IC1,IPP)*PRBE(IE)
IE=1,NE
IC1=1,NC
K=1,NMT
5


5
IIN = DIIN
= ACVTP=AMEAN
= DMEAN = 0
READ
C10,P10
IC1=1,NC
IP1=1,NP
PRBCP(2,IC1,IP1) = 0
IC1 = INDEXC(C10)
IP1 = INDEXP(P10)
IS
LAG = 0
?
NO
YES
PRBCP(2,IC1,IP1)=1
PRBCP(2,IC1,1)=1
PRBCP(2,IP1,2)=1
13
K = 1,NMT
6
IT = MT+K-1
KT = IT-MT+1
J = INT(K/2)
J = J*2
IS
K = J
?
YES
NO
J = 1
J1 = 2
J = 2
J1 = 1
IC1=1,NC
IP1=1,NP
PRBCP(J,IC1,IP1) = 0
7


7
IS LAG = 0 ?
NO
YES
IE=1,NE
IP1=1,NP
PP = PPFEP1[E(IE),P(IP1),IT]
IPP = INDEXP(PP)
IC1=1,NC
IC = INDEXC[C(IC1,IPP,KT)]
DC = C(IC1,IPP,KT)-CC(IC1)
IP = INDEXP[PFPPDC(PP,DC,IT)]
PRBCP(J, IC, IP) = PRBCP(J, IC, IP) + PRBE(IE)*PRBCP(J1,IC1,IP1)
PRBCP(J,IC,1) = PRBCP(J,IC,1) + PRBE(IE)*PRBCP(J1,IC1,1)
PRBCP(J,IP,2) = PRBCP(J,IP,2) + PRBE(IE)*PRBCP(J1,IC1,1)
8
9
IT1 = IT-1
IS IT1<MT ?
10
12
PRBP(IP1) = 0
PRBP(IP1) = PRBP(IP1) + PRBCP(J1,IC1,IP1)
PRBC(IC1) = 0
PRBC(IC1) = PRBC(IC1) + PRBCP(J1,IC1,IP1)
PRBC(IC1) = PRBCP(J1,IC1,1)
PRBP(IP1) = PRBCP(J1,IP1,2)
11


11
MEANC=CC(1)*PRBC(1)
VARC=CC(1)**2*PRBC(1)
IC1=2,NC
MEANC=MEANC + CC(IC1)*PRBC(IC1)
VARC=VARC + CC(IC1)**2*PRBC(IC1)
I=IC1-1
PRBC(IC1)=PRBC(IC1) + PRBC(I)
MEANP=P(1)*PRBP(1)
VARP=P(1)**2*PRBP(1)
IP1=2,NP
MEANP=MEANP + P(IP1)*PRBP(IP1)
VARP=VARP + P(IP1)**2*PRBP(IP1)
I=IP1-1
PRBP(IP1)=PRBP(IP1) + PRBP(I)
WRITE
P(I),PRBP(I)
C(I),PRBC(I)
VARC=SQRT(VARC - MEANC**2)
VTP=SQRT(VARP - 2*MEANP*PSTAR(KT) + PSTAR(KT)**2)
VARP=SQRT(VARP - MEANP**2)
CVC=VARC/MEANC
CVP=VARP/MEANP
CVTP=VTP/PSTAR(KT)
DIIN=DIIN + CVTP**2*DELTA**(IT1 - MT + 1)
IIN=IIN + CVTP**2
ACVTP=SQRT(IIN) / (IT1 - MT + 1)
DMEANC=DMEANC + MEANC*DELTA**(IT1 - MT + 1)
AMEANC=[AMEANC*(IT1 - MT) + MEANC] / (IT1 - MT + 1)
WRITE
the above indices
IS
IT1=NT
?
YES
STOP
NO
10
12
IS
IT=NT
?
YES
IT1=IT
J1=J
NO
13
K=1,NMT
6

Functions.

INDEX P(P), INDEX (C)

These functions define the correspondence from prices and stocks to integer indices.

TARGET, PF PPDC, PPF EP

These three functions should be supplied by the user of the program according to the specification of the demand and supply model. The functions which are presented here are examples which are based on the following simple linear model. We denote by small letters parameters, the numerical values of which are inserted in the program.

Demand function: $Y_{IT} = (a_o - a_1 P_{IT} + E_d)(1 + g_d)^{(IT-MT)}$

Supply function: $X_{IT} = (b_o + b_1 P_{IT-1} + E_s)(1 + g_s)^{(IT-MT)}$

E_d and E_s are distributed normally with zero mean and standard deviations s_s and s_d respectively.

TARGET(IT)

This function defines a series of target prices. In the present example the target prices are defined as the long-run equilibrium prices (i.e., $P_{IT} = P_{IT-1}$) .

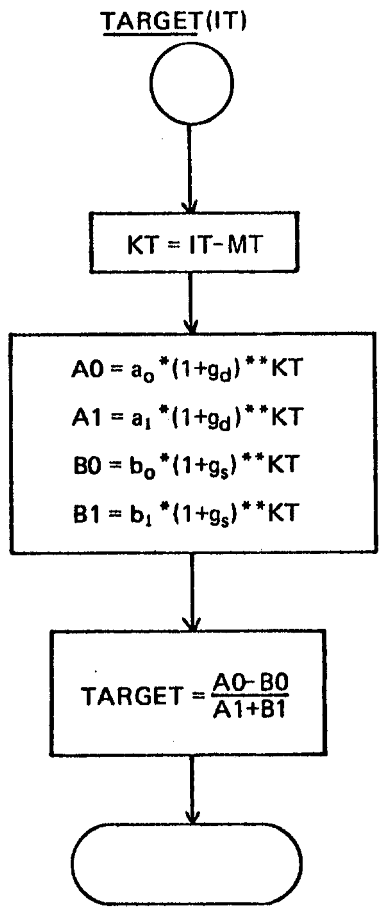
PFPPDC(PP,DC,IT)

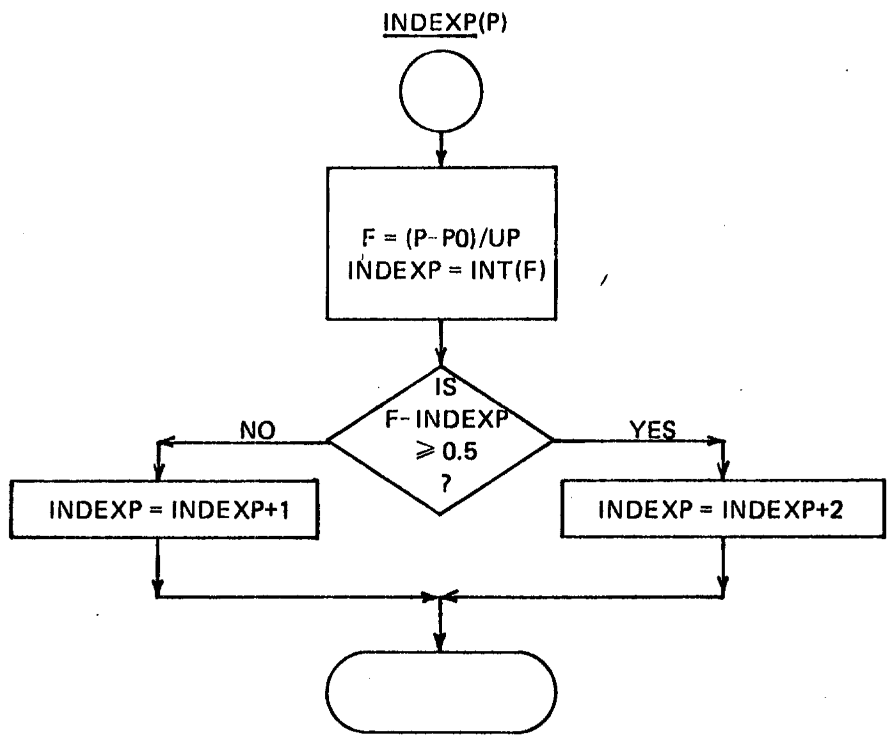
This function defines the relation between the free market price (PP) in period IT, the change of stocks (DC) and the final price (PFPPDC).

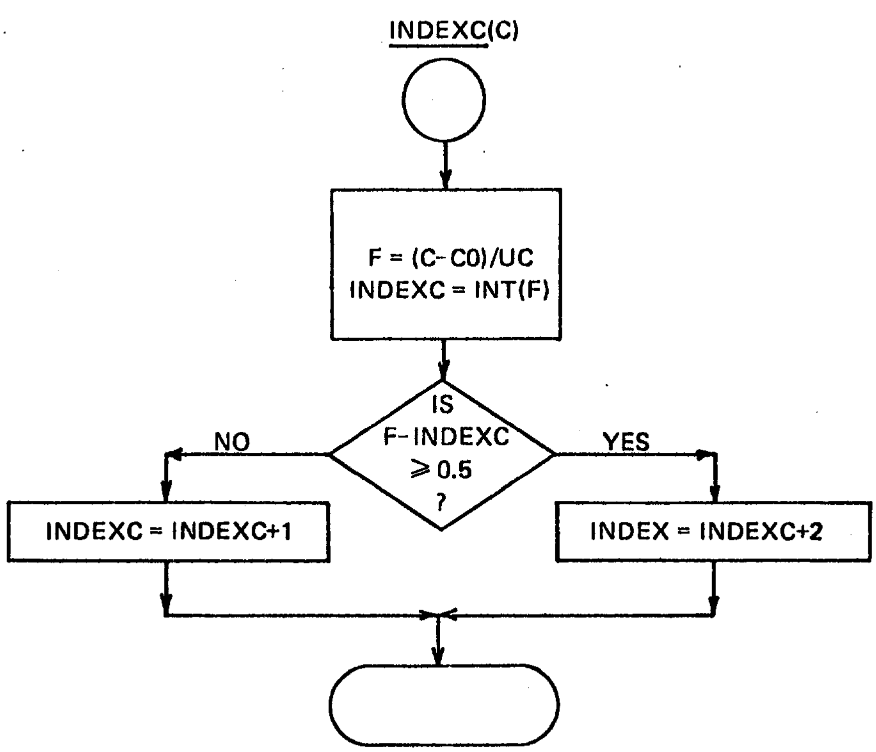
<u>PPFEP1</u>(E,P1,IT)

This function defines the determination of the free market price (PPFEP1) at period IT, by the lagged price (P1), and the composed disturbance (E). SD and SS are the standard deviations of the demand and supply, respectively, at period IT.

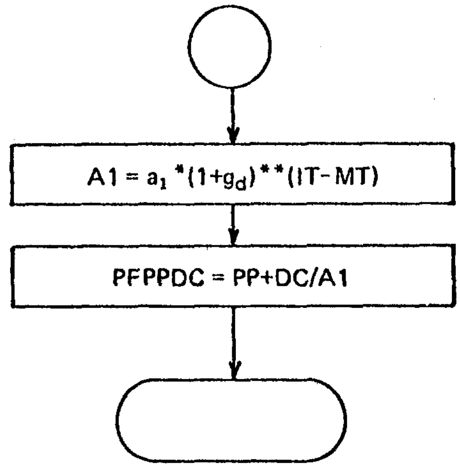
S = SD + SS is the composed standard deviation at time IT.


TARGET(IT)
KT = IT-MT
$A0 = a_o * (1+g_d) ** KT$
$A1 = a_1 * (1+g_d) ** KT$
$B0 = b_o * (1+g_s) ** KT$
$B1 = b_1 * (1+g_s) ** KT$
$TARGET = \frac{A0-B0}{A1+B1}$

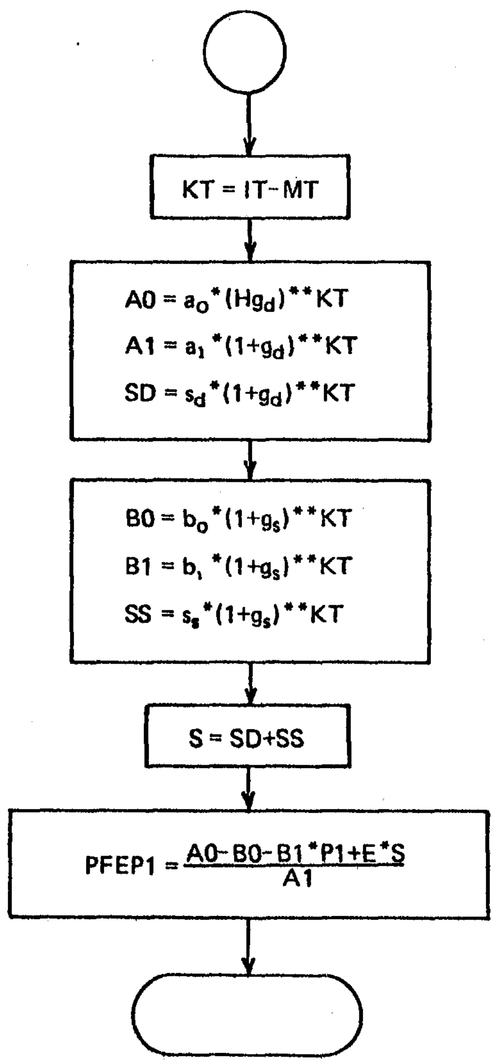

INDEXP(P)
F = (P-P0)/UP
INDEXP = INT(F)
IS
F-INDEXP
⩾ 0.5
?
NO
YES
INDEXP = INDEXP+1
INDEXP = INDEXP+2


INDEXC(C)
F = (C-C0)/UC
INDEXC = INT(F)
IS
F-INDEXC
⩾ 0.5
?
NO
YES
INDEXC = INDEXC+1
INDEX = INDEXC+2

PFPPDC(PP,DC,IT)



PPFEP1(E,P1,IT)



Part II

REFERENCES

[1] Cochrane, W. W. Feast or Famine: The Uncertain World of Food and Agriculture and Its Policy Implications for the United States. Report No. 136, National Planning Association, Washington, D. C. 1974.

[2] Hoffman, R. G. "Wheat-Regional Supply Analysis." The Wheat Situation, WS-225. Economic Research Service, U. S. Department of Agriculture. August 1973.

[3] Sharples, J. A. and Walker, R. L. Analysis of Wheat Loan Rates and Target Prices Using a Wheat Reserve Stocks Simulation Model. Research Status Report No. 2, Reserve Stocks of Grain. Commodity Economics Division Working Paper. Economic Research Service, U. S. Department of Agriculture, Washington, D. C. May 1975.