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SPLINE FUNCTIONS: THEIR USE IN ESTIMATING NON-REVERSIBLE RESPONSE

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In this paper specification of non-reversibilities in estimable polynomial relations is considered in terms of spline functions. Wolffram, Houck, and Trail *et al*. have recently suggested techniques to estimate non-reversibility in linear supply response functions. Houck proposed an alternative to Wolffram's technique of modeling complete non-reversibility. Traill *et al*. posited that complete non-reversibility was an inappropriate specification of the nature of supply response asymmetry. They offered a partial non-reversibility specification and estimation procedure, the Modified Wolffram. This paper does not discuss or present any of the theoretical underpinnings for non-reversible response, rather the focus here is exclusively on illustrating the use of spline functions for modeling this phenomenon.

The objectives of this paper are: (1) to introduce the concept of spline functions; and (2) to account for complete and partial non-reversibility with spline functions. This paper illustrates this approach using a supply response example. The sections in this paper correspond to the above objectives: the first section introduces splines functions; the second section concerns the specification of complete non-reversibility; the third section discusses partial non-reversibility; and the final section concludes the paper.

Spline Functions

Spline functions enable modeling of changing parameter structure in regimes when the switching point between regimes is known. Each regime is characterized by a n-th degree polynomial and continuity up to the (n-1)-th derivative can be required at the switching points. Figure 1 illustrates the response structure for a first degree polynomial function (i.e., linear) when the response is known or assumed to change at switching points P_1 and P_2.

Spline functions have recently been introduced to the econometrics literature by, for example, Poirier, Buse and Lim, and Suits et al. Poirier describes spline functions and provides several applications. Buse and Lim prove that spline functions are a special case of restricted least squares. Suits _et al_. demonstrate that by imposing continuity restrictions spline functions can be fitted by ordinary regression methods.

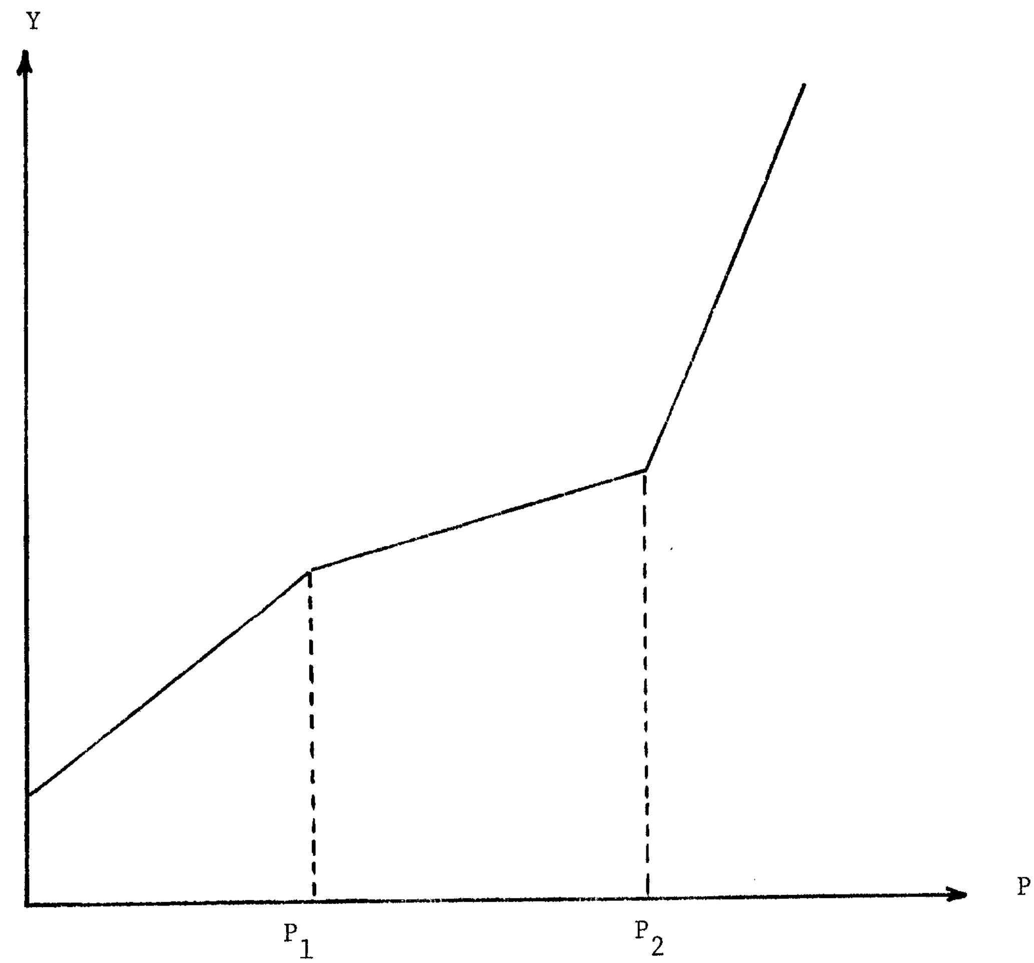
The following time series model with three different parameter regimes for a j^{th} degree polynomial is used to demonstrate the continuity restrictions of spline functions.

$$(1) \qquad Y_t = [a_o + \sum_j b_j P_t^j]\, D_{t_o}^{t_1} + [a_1 + \sum_j (b_j + c_j) P_t^j]\, D_{t_1}^{t_2} + [a_2 + \sum_j (b_j + c_j + d_j) P_t^j]\, D_{t_2}^{T} + \varepsilon_t,\ j = 1, 2, 3,$$

where $t = t_o..t_1..t_2..T$; the switching points occur at t_1 and t_2; $D_{t_i}^{t_{i+1}}$ is equal to one when $t_i \leq t < t_{i+1}$ and zero elsewhere; for all j, c_j is the change in response on P_t^j when $t_1 \leq t < t_2$, and d_j is the response change on P_t^j from $(b_j + c_j)$ when $t_2 \leq t \leq T$.

A spline function is continuous at each of the switching points. For a cubic spline this requires that:

Figure 1. Structural Change at Switching Points P_1 and P_2 in a Relation Between Y and P.



(2') $$a_1 + \sum_{j=1}^{3} (b_j + c_j)P^j_{t_1} = a_o + \sum_{j=1}^{3} b_j P^j_{t_1} \text{ , or}$$

(2) $$a_1 = a_o - \sum_j c_j P^j_{t_1} \text{ , and}$$

(3) $$a_2 = a_1 - \sum_j d_j P^j_{t_2} \text{ , for } j = 1, 2, 3.$$

Continuity of the first derivative at the switching points can be imposed on the above cubic function and this requires that:

(4') $$\sum_j jb_j P^{j-1}_{t_1} = \sum_j j(b_j + c_j)\ P^{j-1}_{t_1}\text{, or}$$

(4) $$c_1 = -2c_2 P_{t_1} - 3c_3 P^2_{t_1} \text{ , and}$$

(5) $$d_1 = -2d_2 P_{t_2} - 3d_3 P^2_{t_2} \text{ .}$$

The restrictions for continuity of the second derivative are:

(6) $$c_2 = -3c_3 P_{t_1}\text{, and}$$

(7) $$d_2 = -3d_3 P_{t_2}.$$

Imposing restrictions (2, 3, 4, 5, 6, 7) on equation (1) produces the following estimable equation: (See Suits _et al_. for a slightly different approach to obtain this result.)

(8) $$Y_t = \alpha_o + b_1 P_t + b_2 P^2_t + b_3 P^3_t - c_3(P_t - P_{t_1})^3\ [D^{t_2}_{t_1} + D^T_{t_2}]$$
$$- d_3(P_t - P_{t_2})^3\ [D^T_{t_2}] + \varepsilon_t.$$

The continuity restrictions of the spline functions at switching points enable the estimable equation to be a function of six variables (including the constant) compared to twelve variables in the original specification. If the original specification of equation (1) was quadratic, then $c_3(\cdot)^3$ and $d_3(\cdot)^3$ would be replaced by $c_2(\cdot)^2$ and $d_2(\cdot)^2$. This can be shown by accounting for the continuity restrictions. A similar correspondence applies to a linear specification of the model.

Specification of Complete Non-reversibility

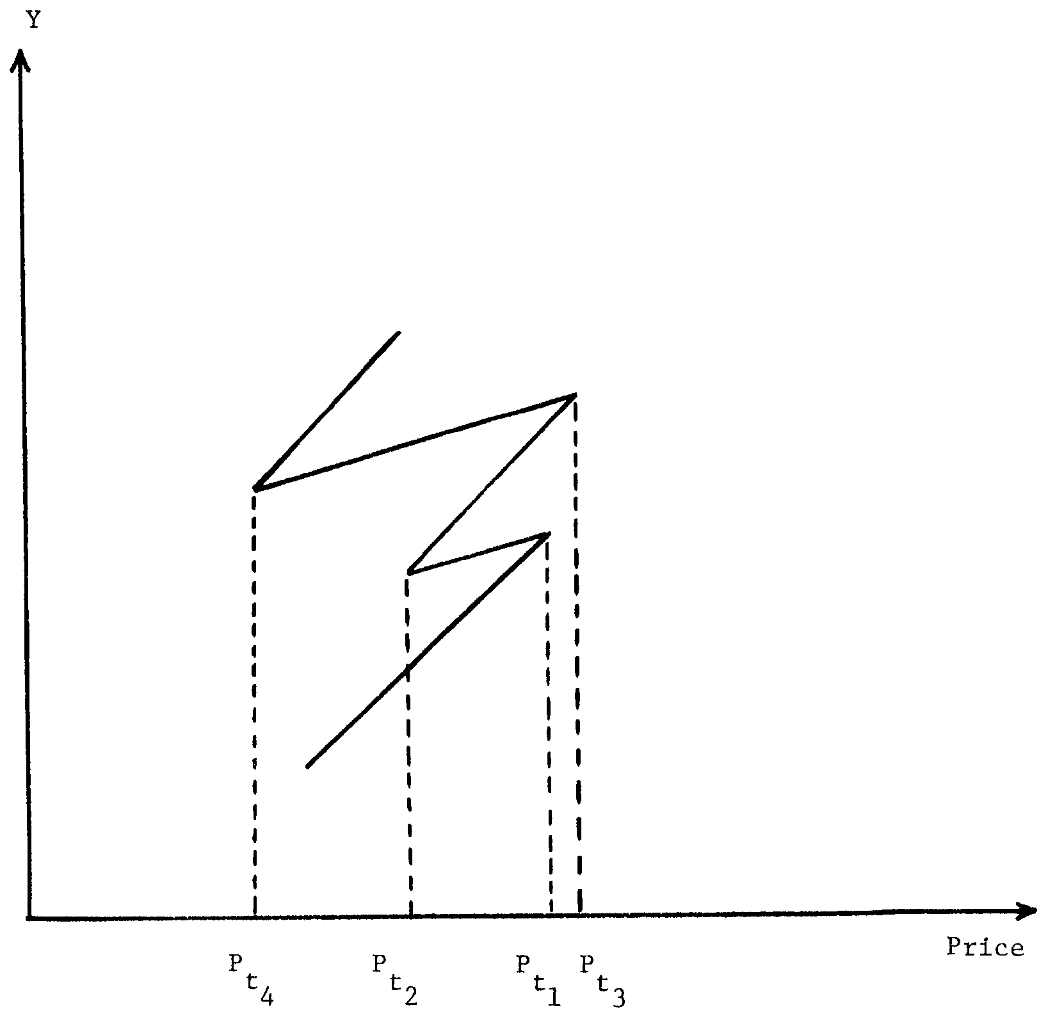
Specifying non-reversible response with spline functions requires that all of the continuity restrictions on the derivatives at the switching points can not be imposed. In the following specification of complete non-reversible response in price (P), a switching point is defined as a point where movements in price reverse direction. In figure 2 P_{t_1}, P_{t_2}, P_{t_3}, and P_{t_4} are the prices at switching points for a complete non-reversible specification in a linear model.

For expositional purposes, consider the following model of non-reversible response in a n-th degree polynomial function.

$$(9) \qquad Y_t = [a_o + \sum_j b_j P_t^j] D_{t_o}^{t_1} + [a_1 + \sum_j (b_j + c_j) P_t^j] D_{t_1}^{t_2} + [a_2 + \sum_j b_j P_t^j] D_{t_2}^{t_3} + [a_3 + \sum_j (b_j + c_j + d_j) P_t^j] D_{t_3}^{t_4} + [a_4 + \sum_j b_j P_t^j] D_{t_4}^{T} + \varepsilon_t, j = 1 \ldots n,$$

where $t = t_o..t_1..t_2..t_3..t_4..T$; switching points are at t_1, t_2, t_3, t_4; $D_{t_i}^{t_{i+1}}$ is equal to one when $t_1 < t \leq t_{i+1}$ and zero elsewhere;

Figure 2. Complete Non-reversible Response to Price



$D_{t_o}^{t_1}$, $D_{t_2}^{t_3}$, and $D_{t_4}^{T}$ designate regimes when P is increasing; $D_{t_1}^{t_2}$ and $D_{t_3}^{t_4}$ reflect P decreasing regimes ($P_t \leq P_{t-1}$); for all j, c_j represents the change in price response in $D_{t_1}^{t_2}$, d_j represents the change in response in $D_{t_3}^{t_4}$ relative to $D_{t_1}^{t_2}$; and if price response is hypothesized not to change in this period, then $d_j = 0$.

The continuity restrictions on equation (9) are

$$(10') \qquad a_1 + \sum_j (b_j + c_j)\, P_{t_1}^j = a_o + \sum_j b_j P_{t_1}^j \text{ , or}$$

$$(10) \qquad a_1 = a_o - \sum_{j=1} c_j P_{t_1}^j \text{ ,}$$

$$(11) \qquad a_2 = a_1 + \sum_{j=1} c_j P_{t_2}^j \text{ ,}$$

$$(12) \qquad a_3 = a_2 - \sum_{j=1} (c_j + d_j)\, P_{t_3}^j \text{ , and}$$

$$(13) \qquad a_4 = a_3 + \sum_{j=1} (c_j + d_j) P_{t_4}^j \text{ .}$$

Imposing these continuity restrictions on equation (9) yields

$$(14') \qquad Y_t = a_o + \sum_j b_j P_t^j + \left[\sum_j c_j\, (P_t^j - P_{t_1}^j)\right] D_{t_1}^{t_2} + \left[\sum_j c_j\, (P_{t_2}^j - P_{t_1}^j)\right]$$

$$D_{t_2}^{t_3} + \left[\sum_j c_j [(P_{t_2}^j - P_{t_1}^j) + (P_t^j - P_{t_3}^j)] + \sum_j d_j (P_t^j - P_{t_3}^j)\right]$$

$$D_{t_3}^{t_4} + \left[\sum_j c_j [(P_{t_2}^j - P_{t_1}^j) + (P_{t_4}^j - P_{t_3}^j)] + \sum_j d_j\, (P_{t_4}^j - P_{t_3}^j)\right]$$

$$D_{t_4}^{T} + \varepsilon_t, \text{ for } j = 1...n.$$

Equation (14') can be simplified by noting that

$$(P^j_t - P^j_{t_i})\, D^{t_{i+1}}_{t_i} \equiv \sum_{k = t_i+1}^{t \leq t_{i+1}} (P^j_k - P^j_{k-1}) \quad \text{, where}$$

$\sum(\cdot)_-$ indicates summation of negative values. Also in $D^{t_3}_{t_2}$ and $D^{T}_{t_4}$

$$(P^j_{t_{i+1}} - P^j_{t_i})\, D^{t_{i+2}}_{t_{i+1}} \equiv \sum_{k = t_i+1}^{t_{i+2}} (P^j_k - P^j_{k-1})_- \text{ since } P_k > P_{k-1} \text{ in } D^{t_{i+2}}_{t_{i+1}}.$$

Hence, the following is an estimable form of complete non-reversible price response in a polynomial function.

$$(14) \qquad Y_t = a_o + \sum_j b_j P^j_t + \sum_j c_j \left[\sum_{k = t_o+1}^{t \leq T} [P^j_k - P^j_{k-1}]_- \right] + \sum_j d_j \left[\sum_{k = t_3+1}^{t \leq T} [P^j_k - P^j_{k-1}]_- \right] + \varepsilon_t, \text{ for } j = 1 \ldots n.$$

Equation (14) can be used to estimate complete non-reversibility for a n-th degree polynomial function without imposing any further continuity restrictions. For example, the case of a first degree polynomial (i.e., a linear model) with $d_1 = 0$ (response to falling price is hypothesized the same throughout the observation period) results in the following estimable equation:

$$(15) \qquad Y_t = a_o + b_1 P_t + c_1 \left[\sum_{k=t_o+1}^{t} (P_k - P_{k-1})_- \right] + \varepsilon_t$$

where b_1 is the response to rising price and $b_1 + c_1$ is the response to falling price. Regardless of the number of price increasing and price decreasing regimes, equation (15) specified a complete non-reversible relationship between price (P) and output (Y).

Hypotheses concerning complete non-reversibilities can easily be accepted or rejected by simple F tests on the c_j parameters in equation (14) and (15). For example the null hypotheses that price response is reversible can be tested by restricting $c_j = d_j = 0$ for all j. These results indicate that complete non-reversibility can be specified by spline functions and estimated by simple estimation procedures.

For illustrative purposes equation (15) is applied to the non-reversible data supplied by Wolffram. This data is reproduced in the first two columns of table 1. The third column in table 1 contains the data associated with the spline variable in equation (15). The estimated equation using ordinary least squares is

$$Y_t = -30.0 + 5.0P_t - 2.0\left(\sum_{k=t_o+1}^{t} (P_k - P_{k-1})\_\right).$$

The coefficients are as specified by Wolffram and the $R^2 = 1.0$. These results indicate that price response is 5.0 when price increases and is (5-2) = 3 when price decreases.

Continuity restrictions can be imposed on the derivatives at switching points for second degree and higher order polynomial functions. For example, the continuity restriction on the first derivative at the switching points for a quadratic function is (using equation (9) when $d_j = 0$):

$$c_1 = -2c_2 P_{t_i} \text{ , or } c_2 = -\frac{c_1}{2P_{t_i}}$$

where P_{t_i} refers to price at each respective switching point. In this case the estimable equation is:

Table 1. Wolffram's Data for Complete Non-Reversibility and the Associated Spline Variable

Q_t	P_t	$\left[\sum_{k=t_o+1}^{t} (P_k - P_{k-1})^-\right]$
20	10	0
35	13	0
29	11	-2
44	14	-2
59	17	-2
44	12	-7
35	9	-10
70	16	-10
90	20	-10
84	18	-12

$$(16) \qquad Y_t = a_o + b_1P_t + b_2P_t^2 + c_1 \left\{ \sum_{k=t_o+1}^{t} (P_k - P_{k-1})_- - \left[\left(\frac{P_t^2 - P_{t_1}^2}{P_{t_1}}\right) D_{t_1}^{t_2} + \left(\frac{P_{t_2}^2 - P_{t_1}^2}{P_{t_2}}\right) D_{t_2}^{t_3} + \left(\frac{P_{t_2}^2 - P_{t_1}^2}{P_{t_3}}\right) D_{t_3}^{t_4} + \left(\frac{P_t^2 - P_{t_3}^2}{P_{t_3}}\right) D_{t_3}^{t_4} + \left(\frac{P_{t_2}^2 - P_{t_1}^2}{P_{t_4}}\right) D_{t_4}^{T} + \left(\frac{P_{t_4}^2 - P_{t_3}^2}{P_{t_4}}\right) D_{t_4}^{T} \right] \right\} + \varepsilon_t .$$

In this equation only four parameters (a_o, b_1, b_2, c_1) are estimated. If the continuity restriction were not imposed the c_2 would also be estimated as in equation (14). Equation (16) can be simplified or rearranged in a variety of ways for computation in regression packages.

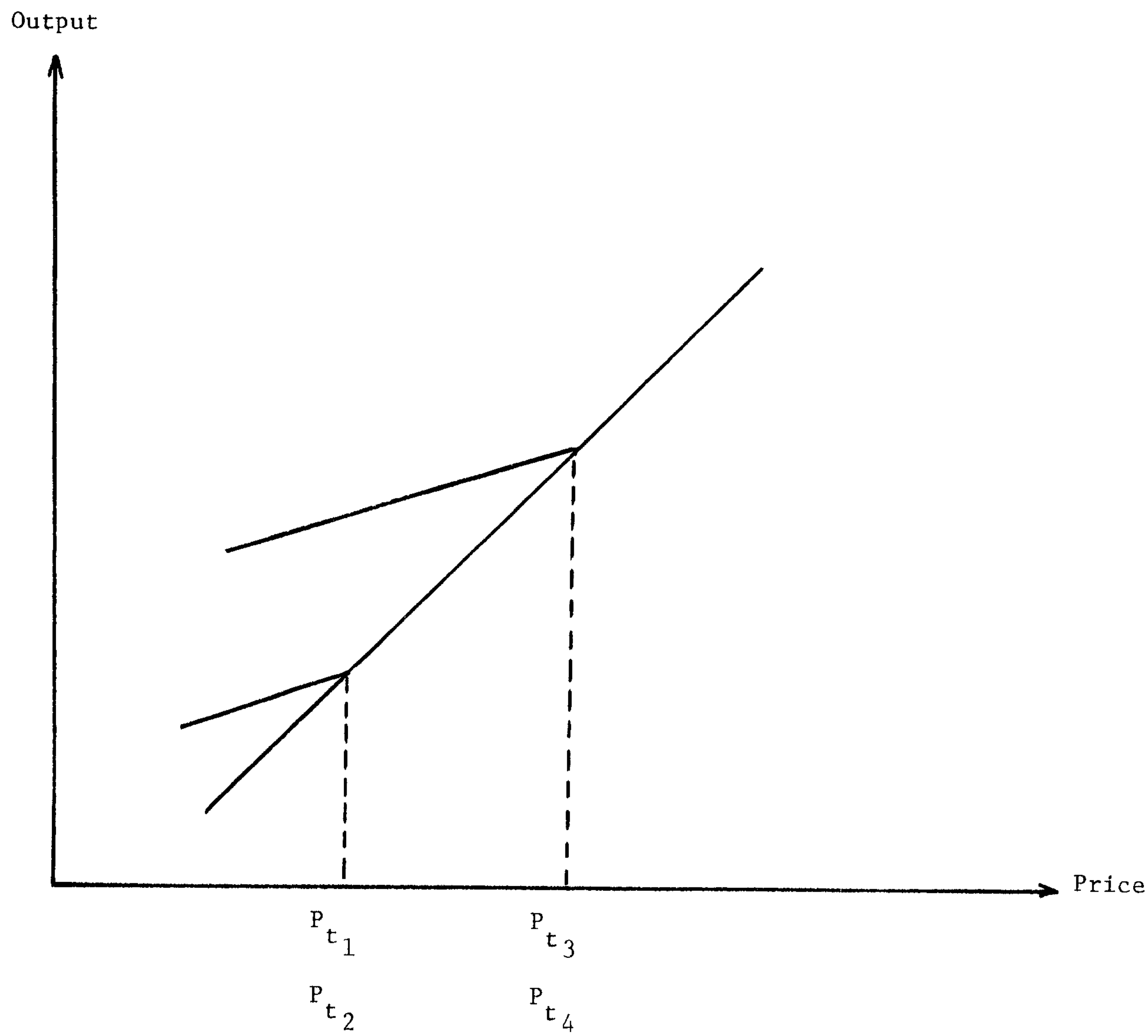
Specification of Partial Non-reversibility

Trail et al. suggested that non-reversibility on the supply side should be considered in a partial context. Partial non-reversibility is defined here as price response being less (or more) elastic whenever price is below a prior maximum price (P_{tm}). This specification is illustrated in figure 3. The P_{tm}'s in this figure are P_{t_1}, P_{t_2}, P_{t_3}, and P_{t_4}, and are also defined as switching points. Equation (9) can be modified to account for partial non-reversibility in the following way for a n-th degree polynomial:

$$(17) \qquad Y_t = [a_o + \sum_j b_jP_t^j]\, D_{t_o}^{t_1} + [a_1 + \sum_j (b_j + c_j)P_t^j]\, D_{t_1}^{t_2} + [a_o + \sum_j b_jP_t^j]\, D_{t_2}^{t_3} + [a_2 + \sum_j (b_j + C_j)\, P_t^j]\, D_{t_3}^{t_4} + [a_o + \sum_j b_jP_t^j]\, D_{t_4}^{T} + \varepsilon_t, \; j = 1, n,$$

where $t = t_o \ldots t_1 \ldots t_2 \ldots t_3 \ldots t_4 \ldots T$; switching points occur at t_1, t_2, t_3,

Figure 3. Partial Non-reversible Response to Price



and t_4; $D_{t_i}^{t_{i+1}}$ are as defined before; $D_{t_o}^{t_1}$, $D_{t_2}^{t_3}$. and $D_{t_4}^{T}$ designate regimes where current price is equal to the maximum price ($P_t = P_{tm}$ and $t = tm$); $D_{t_1}^{t_2}$ and $D_{t_3}^{t_4}$ reflect the regimes when current price is less than the maximum price ($P_t \leq P_{tm}$, $t > tm$); c_j reflects the change in price response for the regimes when $P_t < P_{tm}$; and note that the intercept is the same for all regimes when $P_t = P_{tm}$ and $t = tm$.

The requirements for continuity of the function at switching points are:

$$a_1 = a_o - \sum_j c_j \, P_{t_1}^j = a_o - \sum_j c_j \, P_{t_2}^j \text{ , since } P_{t_1} = P_{t_2} = P_{t_m} \text{ in } D_{t_1}^{t_2} \text{ ,} \tag{18}$$

and

$$a_2 = a_o - \sum_j c_j \, P_{t_3}^j = a_o - \sum_j c_j \, P_{t_4}^j \text{, since } P_{t_3} = P_{t_4} = P_{tm} \text{ in } D_{t_3}^{t_4} \text{ .} \tag{19}$$

The continuity restrictions in equations (18) and (19) simplifies equation (17) to the following:

$$Y_t = a_o + \sum_j b_j \, P_t^j + [\sum_j c_j \, (P_t^j - P_{t_1}^j)] \, D_{t_1}^{t_2} + [\sum_j c_j \, (P_t^j - P_{t_3}^j)] \, D_{t_3}^{t_4} + \varepsilon_t . \tag{20}$$

This specification indicates that the spline variable is $(P_t^j - P_{tm}^j)$ when $P_t \leq P_{tm}$ for $t \geq tm$, and zero (0) when $P_t = P_{tm}$ for $t = tm$.

For a linear response function equation (20) simplifies to

$$Y_t = a_o + b_1 P_t + c_1 (P_t - P_{tm})^{t \geq tm} + \varepsilon_t , \tag{21}$$

where $b_1 + c_1$ is the response to price when current price is below the maximum past price.

Equation (21) was applied to the Traill et al. data which reflected partial non-reversibility. This price and quantity data is reported in the first two columns of table 2. The third column contains the data reflecting the spline formulation. The calculated response function is

$$Y_t = 2.0 + 2.0P_t - 1.0(P_t - P_{tm})^{t>tm},$$

which is the same as the Traill et al. equation except for a different value of the intercept because of the differences in specification. These results indicate that the response to current maximum prices is 2.0 and that the response to price is (2.0 - 1.0) = 1.0 when price is below the maximum price.

As with complete non-reversibility, continuity restrictions can be imposed on the derivatives of second and higher degree polynomials. This is not pursued here since it is a direct extension of the methodology used in the previous section.

Concluding Comments

This paper used spline functions to specify for estimation partial and complete non-reversibilities in polynomial relations. In both cases the spline variable is easily computed and any hypotheses concerning non-reversibilities is simply tested by standard statistical procedures. This spline function approach to modeling non-reversibilities is very flexible and can be readily extended to account for other changes in response when the switching points between regimes are known or hypothesized to exist. The analysis in this paper did not include other explanatory variables and these variables can be directly included in the model. If the response to these additional variables is hypothesized to change

Table 2. Traill *et al.*'s Data for Partial Non-reversibility and the Associated Spline Variable.

Y_t	P_t	$(P_t - P_{tm})^{t>tm}$
20	9	0
30	14	0
23	7	-7
20	4	-10
27	11	-3
25	9	-5
30	14	0
34	16	0
29	11	-5
38	18	0

at prespecified switching points then spline variables can be similarly defined to account for this change in response. Poirier discusses the case of interaction between variables associated with changing response. A final comment: Splines are only defined within the estimation range, and assumptions on parameter structure are required for extrapolation.

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