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LEAST SQUARES ESTIMATION OF DISTRIBUTED LAG MODELS: RELATIONSHIPS BETWEEN ACTUAL AND FIRST DIFFERENCE EQUATIONS

By

Abraham Subotnik and James P. Houck

Department of Agricultural Economics

University of Minnesota
Institute of Agriculture
St. Paul, Minnesota 55108

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RELATIONSHIPS BETWEEN ACTUAL AND FIRST DIFFERENCE EQUATIONS

Abraham Subotnik
James P. Houck
University of Minnesota

Distributed lag models of economic behavior have won an important place in applied econometric research. The two models most often used in agricultural economics research were brought to the profession's attention by Nerlove [4]. Following the terminology in Goldberger's text, these two models can be called the "expectations model" and the "rigidity model." [1, p. 275]. The former involves hypotheses about the way producers or consumers form their expectations of prices, incomes, or other variables on which they base economic decisions. The latter model involves hypotheses about the way producers or consumers adjust over time toward "desired" or long-run output or consumption.

The purpose of this paper is to focus on the rigidity model and illustrate some relationships between the typical use of the model and a version of it involving first differences in the dependent variable. These relationships will be extended to least squares estimation of rigidity models. A useful correspondence between least squares estimates of the typical model and the first difference model will be demonstrated.^{1/}

^{1/} After the completion of this paper, the authors discovered a similar demonstration of these properties in a bulletin by George Ladd [3, p. 836]. The discussion in this paper is more extensive than the terse proof provided by Ladd and may be a useful adjunct to that work.

Since the relationships to be discussed here can have particular usefulness in the development of inventory or stock equations, let a simple inventory equation be

$$(1) \quad y_t^* = \alpha_0 + \alpha_1 x_t + e_t$$

where

y_t^* = desired level of stock at the end of time period t

x_t = an exogenous variable (or set of variables) at time t affecting the stock level

e_t = an unobserved, mean-zero, random disturbance with finite variance.

The parameters α_0 and α_1 are unknown, and generally y_t^* cannot be observed. The rigidity model specifies the process of adjustment of stock levels between time periods as

$$(2) \quad y_t = y_{t-1} + \gamma(y_t^* - y_{t-1})$$

where γ is the familiar "adjustment coefficient" hypothesized to lie between 0 and +1. The usual procedure to obtain an estimable equation is to insert equation (1) into (2) and clear algebraically. This gives

$$(3) \quad y_t = \gamma\alpha_0 + \gamma\alpha_1 x_t + (1-\gamma)y_{t-1} + \gamma e_t$$

with sufficient observations on y and x_1 , the parameters α_0 , α_1 , and γ can be estimated with least squares.

Now let

$$(4) \quad \Delta y_t = y_t - y_{t-1}$$

In this formulation, Δy_t is the change in stock during period t .

The change in stocks may be specified as an important dependent variable in economic models [see for example, 2]. Manipulate equation (2) algebraically by bringing y_{t-1} to the right side. Then using (4)

$$(5) \quad y_t = \gamma \alpha_0 + \gamma \alpha_1 x_t - \gamma y_{t-1} + \gamma e_t$$

This equation expresses the change in stocks in period t as a linear function of x_t and y_{t-1} . Equation (5) also can be estimated by least squares. Whether equation (5) or equation (3) is selected for estimation normally would depend upon the purpose of the investigation and whether or not the stock equation is part of a larger system.

Imagine, for example, that the inventory equation is part of a larger model which includes a market-clearing identity to insure that consumption in a given period is equal to production plus inventory change. In this case, an equation, like (5), with stock change specified might be less cumbersome to deal with in the model than a stock level equation like (3).

Next it will be shown that, given a series of observations on y and x , least squares estimates of α_0 , α_1 and γ are identical no matter whether equation (3) or equation (5) is specified. Employing standard matrix notation let the regression model indicated by equation (3) be written as

$$(6) \quad y = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + u$$

where

y = level of stocks; a vector of dimension $T \times 1$

X_1 = a row vector of 1's and the independent variable; a matrix of dimension $T \times 2$

X_2 = lagged value of y ; a vector of dimension $T \times 1$

$$\beta_1 = \begin{bmatrix} \gamma \alpha_0 \\ \gamma \alpha_1 \end{bmatrix}$$

$$\beta_2 = [1 - \gamma]$$

u = disturbance term; a vector of dimension $T \times 1$ with a typical element being γe_t

The least squares estimates of the β 's are well-known as

$$(7) \quad \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} X_1' X_1 & X_1' X_2 \\ X_2' X_1 & X_2' X_2 \end{bmatrix}^{-1} \begin{bmatrix} X_1' y \\ X_2' y \end{bmatrix}$$

Now define the following variable

$$(8) \quad z = y - X_2$$

Notice that $z = \Delta y$

Then the regression model of equation (5) becomes

$$(9) \quad z = \begin{bmatrix} X_1 & X_2 \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} + u$$

where

$$\pi_1 = \begin{bmatrix} \gamma \alpha_0 \\ \gamma \alpha_1 \end{bmatrix}$$

$$\pi_2 = [-\gamma]$$

and X_1 , X_2 , and u are the same as in equation (6)

The least squares estimates of (9) are

$$\begin{bmatrix} \hat{\pi}_1 \\ \hat{\pi}_2 \end{bmatrix} = \begin{bmatrix} X_1'X_1 & X_1'X_2 \\ X_2'X_1 & X_2'X_2 \end{bmatrix}^{-1} \begin{bmatrix} X_1' \\ X_2' \end{bmatrix} z$$

Then using equation (8)

$$(10) \begin{bmatrix} \hat{\pi}_1 \\ \hat{\pi}_2 \end{bmatrix} = \begin{bmatrix} X_1'X_1 & X_1'X_2 \\ X_2'X_1 & X_2'X_2 \end{bmatrix}^{-1} \begin{bmatrix} X_1' \\ X_2' \end{bmatrix} y - \begin{bmatrix} X_1'X_1 & X_1'X_2 \\ X_2'X_1 & X_2'X_2 \end{bmatrix}^{-1} \begin{bmatrix} X_1'X_2 \\ X_2'X_2 \end{bmatrix}$$

Notice that the first expression in equation (10) reduces directly to the least squares estimates of $\hat{\beta}_1$ and $\hat{\beta}_2$. The second expression can be viewed as an inverse matrix multiplied by part of its original matrix. As such, it reduces to a part of an identity matrix -- in this case, the final column vector of an identity matrix.

Therefore

$$(11) \begin{bmatrix} \hat{\pi}_1 \\ \hat{\pi}_2 \end{bmatrix} = \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 - 1 \end{bmatrix}$$

Both $\hat{\pi}_1$ and $\hat{\beta}_1$ are identical estimates of the parameter vector $[\gamma\alpha_0 \quad \gamma\alpha_1]'$. Similarly $\hat{\pi}_2$ is an estimate of $-\gamma$, and $\hat{\beta}_2$ is an estimate of $(1 - \gamma)$. Since from equation (11) $\hat{\pi}_2 = \hat{\beta}_2 - 1$, both provide identical estimates of γ . ^{2/}

^{2/} This particular property of the least squares method is not confined to the economic illustration which has been employed here. It holds any time the dependent variable is altered by the subtraction (or addition) of one or more of the specified independent variables.

This direct correspondence between the estimated coefficients of these two models extends to the estimated standard errors of the regression coefficients. The usual expression of the standard error of the regression coefficients, $\hat{\beta}$, is

$$S_{\hat{\beta}} = \left[\frac{\hat{u}'\hat{u}}{\text{d.f.}} \cdot (X'X)^{-1} \right]^{\frac{1}{2}}$$

where \hat{u} is the calculated residual vector based on $\hat{\beta}$, X is the matrix $[X_1, X_2]$, and d.f. indicates degrees of freedom. Similarly for the estimated coefficients, $\hat{\pi}$, the expression is

$$S_{\hat{\pi}} = \left[\frac{\tilde{u}'\tilde{u}}{\text{d.f.}} \cdot (X'X)^{-1} \right]^{\frac{1}{2}}$$

where all is the same except that \tilde{u} is the calculated residual vector based on $\hat{\pi}$. If it can be shown that $\hat{u}'\hat{u}$ is equal to $\tilde{u}'\tilde{u}$ then $S_{\hat{\beta}} = S_{\hat{\pi}}$.

First look at $\hat{u}'\hat{u}$. Observing that $\hat{u} = y - X\hat{\beta}$ and using a familiar relationship from least square methodology [1, p. 159].

$$(12) \quad \hat{u}'\hat{u} = y'y - y'X\hat{\beta}$$

Similarly

$$(13) \quad \tilde{u}'\tilde{u} = z'z - z'X\hat{\pi}$$

Recall that $\hat{\pi} = \hat{\beta} - K$ where K is a column vector of zero's except for the last element which is in this case 1, see equation (11). Then

$$(14) \quad \begin{aligned} \tilde{u}'\tilde{u} &= z' \left[z' - X(\hat{\beta} - K) \right] \\ &= z \left[z - X\hat{\beta} + X_2 \right] \end{aligned}$$

Substituting for z from equation (8)

$$\begin{aligned}
 \tilde{u}'\tilde{u} &= (y - x_2)' [y - x_2 - x\hat{\beta} + x_2] \\
 &= (y'y - y'x\hat{\beta}) - x_2'(y - x\hat{\beta}) \\
 &= \hat{u}'\hat{u} - x_2'(y - x\hat{\beta}) \\
 (15) \quad &= u'u - x_2'y + x_2'x\hat{\beta}
 \end{aligned}$$

Because x_2 is a column vector partition of X and since $\hat{\beta} = (X'X)^{-1} X'y$ the final expression in equation (15), namely $x_2'x\hat{\beta}$, collapses to $x_2'y$. Thus

$$\tilde{u}'\tilde{u} = \hat{u}'\hat{u}$$

and the standard errors of the estimated coefficients in each model are identical. This is appropriate since the error in estimating from equation (7) and the error in estimating β_2 from equation (10) both stem from the error attached to estimating δ (recall that $\beta_2 = 1 - \delta$ and $\pi_2 = -\delta$).

In this illustration either the actual level of stocks or the change in stocks can be used as the dependent variable in a least squares estimation process. Parameter and standard error estimates based on one specification are identical with those based on the other specification and can be used interchangeably.

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