A Discrete Model of Replenishable Resource Management Under Uncertainty

J. Barry Smith

Department of Economics
Institute of Applied Economic Research
Concordia University
Montreal, Canada

Abstract This paper examines issues in the management of replenishable resources under uncertainty. The stochastic resource dynamics are given by the discrete-time counterpart of the classic logistic growth model. The use of discrete-time stochastic dynamics allows for a more general characterization of growth uncertainty than is possible with continuous-time models.

Given a general specification of the resource management problem, necessary and sufficient conditions for the optimal management policy are derived. Many important properties of the management policy are derived and comparisons are made with the deterministic counterpart policy. An example serves to illustrate many of the results of the analysis.

Introduction

Recently, papers by Bourguignon (1974), Merton (1975), Bismut (1975), Brock and Mirman (1972), and Mirman and Zilcha (1975) have addressed issues in dynamic capital theory under uncer-

Marine Resource Economics, Volume 1, Number 3
0738–1360/85/010283–00$02.00/0

tainty. At the same time, Reed (1974), Lewis (1975), Ludwig (1977), and Smith (1977) have examined related questions that arise in the management of replenishable natural resources. In the natural resource literature, Reed and Ludwig focus attention on "bang-bang" control policies in continuous time. Lewis presents the control problem in terms of a finite state Markov process and uses numeric techniques to provide illustrative solutions. Finally, Smith examines a class of continuous-time stochastic resource problems that do not have bang-bang solutions.

The goal of all the papers noted above has been to generalize the policy results obtained from deterministic models. Perhaps the most important reason why such generalizations are required lies in the very stringent requirements of deterministic management policies. In particular, all present as well as future resource states must be known with certainty. Without this information, the social planner would never be able to implement the optimal policy that guarantees to keep the resource on the (generally) unique stable arm path to the long-run equilibrium steady state. Within a deterministic setting, any perturbation caused by random growth or imperfect observation will in general result in disaster, in that continued application of the deterministic policy will virtually guarantee that the appropriate equilibrium state is not achieved.

In this paper the problem of characterizing the optimal resourçe management policy when the growth process is random is examined within a discrete model. The use of a discrete model is important in that it allows for more general characterization of uncertainty than continuous models and also involves simpler analytic techniques. The search for optimal policies in continuous-time models involves the introduction of difficult mathematics. Also, for the most part, the random growth models must be defined in terms of normal and Poisson continuous processes.

The paper begins with the derivation of the difference equation associated with logistic growth. The assumption that the parameters of the model are random variables serves to characterize the resource as a stochastic process defined by a stochastic difference equation. In the absence of harvesting, the resource model is studied in order to determine stochastic steady-state properties.

In the next section management is introduced under the assumption of perfect state observability. Necessary and sufficient conditions for a policy to be optimal are derived. In addition, general properties of the optimal policy are discussed. The section ends with an example that is solved in closed form and that illustrates the discussion in the section. It is also possible to compare the optimal policy to the optimal policies from counterpart deterministic models.

The paper concludes with a discussion of useful topics for future inquiry.

Resource Dynamics

Discrete Deterministic Growth

Since replenishable resources grow continuously and eventually at a rate less than exponential, many authors have argued that in the absence of harvesting, the logistic curve provides a reasonable representation of the time path of biomass of many such resources. The logistic curve is defined by

$$N_t = \frac{\lambda N_0}{\epsilon N_0 + (\lambda - \epsilon N_0)e^{-\lambda t}} \qquad \lambda, \epsilon, t \geq 0 \tag{1}$$

where N_s is biomass at time s and λ and ϵ are respectively, birth and death parameters. Equation (1) is the solution of the nonlinear differential equation:

$$dN_t/dt = \lambda N_t - \epsilon N_t^2; \; N_{t=0} = N_0 \tag{2}$$

As can be seen in Figure 1, the time path of N_t is smooth and asymptotic to $N_\infty = \lambda/\epsilon$.

In order to derive the discrete counterpart implied by Equation (1), we begin by considering some discrete interval $[t + h, t]$.

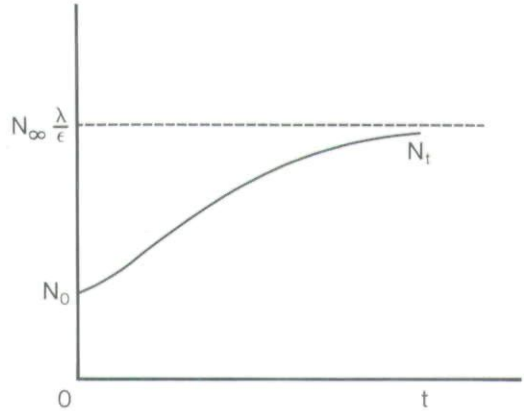

FIGURE 1. The classic logistic curve.

Taking the first difference of Equation (1), it is possible to derive the difference equation satisfied by N_t as:

$$N_{t+h} - N_t = (\lambda N_t - \epsilon N_{t+h} N_t) \cdot a(\lambda h) \cdot h \tag{3}$$

where $N_{t=0} = N_0$, $t \geq 0$, and $a(\lambda h) = (e^{\lambda h} - 1)/\lambda h$. If we divide Equation (3) by h it is quickly verified that, as $h \to 0$, Equation (3) limits to Equation (2). Equation (3) can be usefully rewritten in the form

$$N_{t+1} = \frac{(1 + \bar{\lambda}) N_t}{1 + \bar{\epsilon} N_t} \tag{4}$$

where $\bar{\lambda} = \lambda a(\lambda)$, $\bar{\epsilon} = \epsilon a(\lambda)$, $N_{t=0} = N_0$, and h is taken as 1. The graph of Equation (4) is presented in Figure 2 for particular values of λ and ϵ. Note that if $\lambda > 0$ this is sufficient to guarantee positive net growth for small stock sizes in the absence of harvesting. The time path of N_t from Equation (4) can be graphically represented as a sequence of points, all of which would lie on the smooth logistic curve shown in Figure 1.

It should be noted that the difference equation

$$N_{t+h} - N_t = \lambda N_t - \epsilon N_t^2 \tag{5}$$

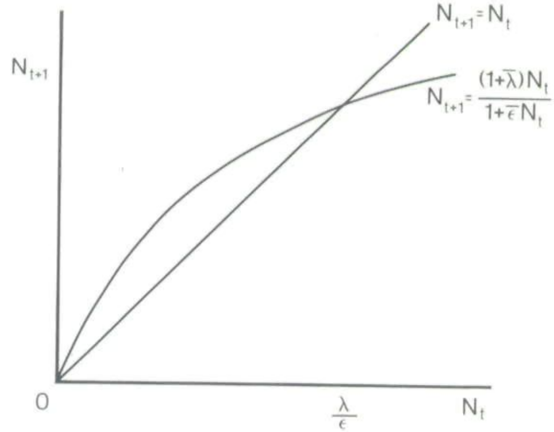

FIGURE 2. Discrete equivalent of the logistic model.

where $t \geq 0$ and $N_{t=0} = N_0$, which "looks" as if it corresponds to Equation (2) and hence (1), is not in general consistent with (1). In fact, Equation (5) admits a variety of solutions. Independent of ϵ, if $\lambda \geq 2$ then the resource will "bounce" around the steady-state equilibrium of $N_\infty = \lambda/\epsilon$ according to the solution

$$N_t = \frac{\lambda + 2}{2\epsilon} + (-1)^t \cdot \frac{(\lambda + 2)(\lambda - 2)^{1/2}}{4\epsilon^2} \qquad (6)$$

Similarly, it is straightforward to show that if $1 < \lambda < 2$, then N_t will approach λ/ϵ with damped oscillations. Finally, for $0 < \lambda < 1$, the approach to λ/ϵ is nonoscillatory. In all of these cases, the graph of N_t does not coincide with the logistic curve.

Finally, it is worth observing that Equation (4) does not represent a rederivation of the classic Beverton-Holt model. This is apparent by comparing the derivation here with the derivation and parameter restrictions given by Clark (1976, p. 218).

In the rest of this paper we will restrict our analysis to the case where the resource dynamics are generated by Equation (4). Although Equation (5) has some interesting properties, it is difficult to extend the deterministic analytic results past those presented above. As might be imagined, it is even more difficult to use Equation (5) to obtain insights into replenishable resource growth and management under uncertainty.

Discrete Stochastic Growth

Formally, the extension of the discrete dynamic analysis to uncertainty is accomplished by considering λ and ϵ in Equations (1) and (4) as random variables with (potentially) different realizations at successive points in time. Thus $\{N_t,\ t \in I^+\}$ becomes a stochastic process defined by the now stochastic difference equation:

$$N_{t+1} = \frac{(1 + \bar{\lambda}_t)N_t}{1 + \bar{\epsilon}_t N_t} \qquad (7)$$

where $\bar{\lambda}_t = \lambda_t a(\lambda_t)$, $\bar{\epsilon}_t = \epsilon_t a(\lambda_t)$, and $\text{Prob}(N_{t=0} = N_0) = 1$. In Equation (7), $\{\lambda_t,\ \epsilon_t\}$ represent random variables that have realizations at the beginning of each period t. For the purposes of this paper, we will focus attention upon situations where all possible realizations of $\{\lambda_t,\ \epsilon_t\}$ are constrained to be strictly positive and bounded at any point in time and further where the process generating realizations of $\{\lambda_t,\ \epsilon_t\}$ is stationary. This latter assumption implies a regularity in natural activity since it implies that the underlying forces generating natural activity are time-invariant.

Properties of the Discrete Stochastic Growth Model

An examination of the properties of the model described by Equation (7) is most easily undertaken in terms of the inverse stock process $\{N_t^{-1},\ t \in I^+\}$. The assumptions for $\{\lambda_t,\ \epsilon_t\}$ introduced above guarantee that $N_t \gg 0$ for all finite t and therefore that N_t^{-1} is well defined. It may well be argued that the assumptions guaranteeing that $N_t \gg 0$ are overly restrictive in that they guarantee that the resource will never become extinct in finite time because of natural phenomena. This point is granted; however, it is noted that the model permits situations where the resource can be very small for an arbitrarily long period of time.

In addition, in the management models that we consider later in this paper, resource extinction is a possible outcome.

Cross-multiplying in Equation (7) and subsequently dividing through by the product $N_{t+1} \cdot N_t$ yields

$$\frac{1}{N_t} + \bar{\epsilon}_t = \frac{1 + \bar{\lambda}_t}{N_{t+1}} \tag{8}$$

If we define $z_t = N_t^{-1}$, Equation (8) leads to the linear stochastic difference equation:

$$z_{t+1} = \alpha_t z_t + \beta_t \tag{9}$$

where $\alpha_t = e^{-\lambda_t}$, $\beta_t = (\epsilon_t/\lambda_t)(1 - \epsilon^{-\lambda_t}) \geqslant 0$, and $\text{Prob}(z_0 = N_0^{-1}) = 1$. It will be noted that $0 \leqslant \alpha_t \leqslant 1$ for all t as a result of previous assumptions.

Steady-State Properties of the Process $\{\beta_t,\ t \in I^+\}$. Equation (9) could provide the description of a discrete deterministic process if (for example) we replaced $\{\lambda_t, \epsilon_t\}$ by time-invariant nonrandom constants $\{\lambda, \epsilon\}$. In such a situation $\{\alpha_t, \beta_t\}$ would also be positive constants $\{\alpha, \beta\}$ and the deterministic process would asymptotically approach the steady-state value $z_\infty = \beta/(1 - \alpha) = \epsilon/\lambda$. Alternatively, N_t would approach $N_\infty = \lambda/\epsilon$. The deterministic steady state is seen to be independent of the initial resource size N_0 and time.

There is a stochastic counterpart to the deterministic steady state. A stochastic steady state is said to arise if the sequence of the random variable z_t defined by Equation (9) converges to a random variable z_∞, the distribution of which is independent of the initial stock size and time. The remainder of this section will be devoted to demonstrating that under quite general circumstances the stochastic process defined by Equation (9) will converge towards a stochastic steady state.

We begin by developing an expression for the limiting random variable z_∞. In particular,

$$z_\infty = \lim_{t \to \infty} z_t = \lim_{t \to \infty} \left[z_0 \prod_{i=0}^{t-1} \alpha_i + \sum_{i=0}^{t-1} \beta_i \prod_{j=i+1}^{t-1} \alpha_j \right] \tag{10}$$

where we follow the convention that

$$\prod_{i=\delta_1}^{\delta_2} = 1$$

whenever $\delta_1 > \delta_2$. The first term in brackets on the right hand side of Equation (10) vanishes with probability 1 in the limit since by assumption, $0 \leqslant \alpha_i \leqslant 1$. With this result we conclude that the limiting random variable is independent of the initial condition z_0. Thus Equation (10) can be rewritten

$$z_\infty = \lim_{t \to \infty} \left[\sum_{i=0}^{t-1} \beta_i \prod_{j=i+1}^{t-1} \alpha_j \right] \tag{11}$$

The expression on the right-hand side of Equation (11) can be analyzed within a measure theoretic setting. To this end we define Ω as a one-dimensional space with typical element w. There is no important loss in generality (for our purposes) in considering Ω as a closed interval in R. We assume that Ω is time-invariant and therefore that $\Omega_t = \Omega_{t+1} = \Omega$. We next define U as a system of subsets of Ω forming a σ-algebra in Ω. In particular, we will take U as the system of all right-half open intervals in Ω or the Borel sets of Ω. Finally, we define P as a measure on U with the normalization requirement that $P(\Omega) = 1$. Hence, we have the probability space (Ω, U, P). We extend this to define the infinite product probability space:

$$(\Omega_\infty, U_\infty, P_\infty) = (\otimes_{i=0}^{\infty} \Omega, \otimes_{i=0}^{\infty} U, \otimes_{i=0}^{\infty} P) \tag{12}$$

The existence of such a space is well documented (see, for example, Bauer [1972], p. 168). For our purposes, we define the real random variables $\{\lambda_t, \epsilon_t\}$ as bounded measurable mappings from Ω_t to R. Since $\Omega_t = \Omega$ for all t, it follows that the random variables which we consider are stationary. In particular, the random variables $\{\lambda_t, \epsilon_t\}$ are defined by

$$\lambda_t : \Omega \rightarrow [\boldsymbol{\lambda}, \bar{\boldsymbol{\lambda}}] \subset R^+ \tag{13a}$$

$$\epsilon_t : \Omega \rightarrow [\boldsymbol{\epsilon}, \bar{\boldsymbol{\epsilon}}] \subset R^+ \tag{13b}$$

This implies that $\{\alpha_t, \beta_t\}$ are stationary random variables with $0 \ll \alpha_t \ll 1$ and $\beta_t \gg 0$ for all t.

The proof of the existence of a limiting random variable z_∞ requires only that one note that under the conditions defined above, the right-hand side of Equation (11) represents the convergent sum of a sequence of bounded functions, all of which are defined on $(\Omega_\infty, U_\infty, P_\infty)$. The existence of z_∞, along with the strict positivity implied by the assumptions introduced above, guarantees the existence of N_∞.

Some Examples. The model described by Equation (9) is quite rich with respect to possible characteristics of the stochastic steady state. In what follows we examine two examples where closed-form solutions are possible.

The first example demonstrates a situation in which the limiting distribution is degenerate in that the limiting "random variable" is a nonrandom constant. To examine this case we begin by rewriting Equation (9) to stress the functional dependence upon $\{\lambda_t, \epsilon_t\}$. In particular,

$$z_{t+1} = e^{-\lambda_t} z_t + \epsilon_t (1 - e^{-\lambda_t})/\lambda_t \qquad \text{Prob}(z_0 = N_0{}^{-1}) = 1 \tag{14}$$

We next define w_t as an element at time t of the stationary set defined by $\Omega = [\underline{w}, \overline{w}] \subset R^+$. The random variables are defined as linear functions from $\Omega \rightarrow R$. In particular,

$$\lambda_t = \lambda w_t \qquad \lambda \text{ constant and greater than zero} \tag{15a}$$

$$\epsilon_t = \epsilon w_t \qquad \epsilon \text{ constant and greater than zero} \tag{15b}$$

These definitions imply that a good outcome (large w_t) yields a large birth parameter λ_t and a large death parameter ϵ_t. Hence if conditions are especially good for growth, they are also good for the growth of predators and therefore the death parameter of the resource is large. Introducing Equation (15) into (14) yields

$$z_{t+1} = e^{-\lambda w_t} z_t + k(1 - e^{-\lambda w_t}) \qquad k = \epsilon/\lambda \tag{16}$$

Dividing through by k and substituting for the new random variable defined by $y_t = z_t/k - 1$ yields

$$y_{t+1} = e^{-\lambda w_t} y_t \qquad \text{Prob}[y_0 = kN_0^{-1} - 1] = 1 \tag{17}$$

With probability 1, the limiting random variable associated with the process defined by Equation (17) is degenerate and its distribution function has all of its mass at $y_\infty = 0$. From this we conclude that in the limit the natural resource is nonrandom with a steady-state value equal to λ/ϵ.

The second example is interesting in that it demonstrates a variety of characterizations of the steady-state distribution function of the limiting random variable. In particular, one result that emerges is that the distribution function can be constant over a range of values of the limiting stock random variable. The implication of this result is that there are intervals nested in the range of the limiting random variable that will never be observed. To examine this and other cases we begin with Equation (14) and the assumption that λ_t is nonrandom and time-invariant at the value λ. Uncertainty enters the model with the assumption that ϵ_t has the following stationary point binomial distribution:

$$\begin{aligned} \epsilon_t &= \bar{\epsilon} \ \text{Prob} = p \qquad 0 < \epsilon < \bar{\epsilon} \\ &= \epsilon \ \text{Prob} = 1 - p \end{aligned}$$

Using these assumptions and introducing the change of variables

$$z_t = h_t \cdot \frac{(1 - e^{-\lambda})(\bar{\epsilon} - \epsilon) + \epsilon}{\lambda}$$

leads to the following model:

$$h_{t+1} = bh_t + \tilde{\epsilon}_t \tag{18}$$

where

$$\begin{aligned} \tilde{\epsilon}_t &= 1 \qquad \text{Prob} = p \\ &= 0 \qquad \text{Prob} = 1 - p \end{aligned}$$

and

$$b = e^{-\lambda} \in (0, 1)$$

From the results of the general proof introduced above, there is no loss in generality in assuming $h_0 = 0$. Proceeding, we can write the expression for the limiting random variable h_∞ as

$$h_\infty = \sum_{t=0}^{\infty} b^t \tilde{\epsilon}_t \tag{19}$$

The limiting random variable is bounded from above by $1/(1 - b)$ and from below by 0. In addition, it can be shown that for all $b \in (0, 1)$ the possible realizations of h_∞ form a nondenumerable set within these bounds. To see this it is sufficient to note that different realizations of h_∞ correspond to permutations of the infinite set $\{\tilde{\epsilon}_0, \tilde{\epsilon}_1 \cdots\}$. Since $\tilde{\epsilon}_t$ can take one of two possible values, the number of possible permutations is 2^K where K is the first infinite integer. Clearly, the set of realizations is not denumerable.

The set of possible realizations of h_∞ may not be connected. In particular, if $b < 0.5$ there will be intervals contained in the range $[0, 1/(1 - b)]$ that will never be occupied. If, however, $b > 0.5$ the range of the limiting random variable will be $[0, 1/(1 - b)]$ and every interval will have positive probability.

The proof of the assertions for $b < 0.5$ proceeds in the following way. Consider any number $b/(1 - b) + \delta < 1$ with $\delta >$

0. In order for this to be a possible realization of h_∞ it is necessary that $\bar{\epsilon}_0 = 0$ (otherwise $h_\infty > 1$). If we assume that $\bar{\epsilon}_t = 1$ for all $t \geq 1$, the infinite sum has the value $b/(1 - b)$, which is less than the number introduced above. Hence, when $b < 0.5$ the set of possible realizations of h_∞ is not connected. Indeed, for any such $b < 0.5$ there will be an infinite number of gaps in the range of h_∞. This result demonstrates a property which is similar to that obtained in the study of Cantor sets.

To prove the assertions for the case where $b \geq 0.5$, we show that there is always a permutation of $\{\bar{\epsilon}_0, \bar{\epsilon}_1, \ldots\}$ which will generate any y in the range $[0, 1/(1 - b)]$. Without loss of generality we take $\bar{\epsilon}_0 = 0$ and examine only the range $[0, b/(1 - b)]$. Let y take on a value in $[0, b/(1 - b)]$ and consider the sequence $\{x_1, x_2, \ldots\}$ where the elements are related by

$$x_{n+1} = x_n + b^{n+1}\bar{\epsilon}_{n+1} \tag{20}$$

We show that the sequence of x's will converge to y if the set $\{\bar{\epsilon}_1, \bar{\epsilon}_2, \ldots\}$ solves the following problem at each point in time:

$$\begin{aligned} &\max \bar{\epsilon}_i \qquad \text{subject to } x_i \leq y \qquad i = 1, 2, \ldots \\ &\qquad\qquad \text{subject to } \bar{\epsilon}_i = 0 \text{ or } 1 \end{aligned}$$

After the first stage, $y - x_1 \leq b/(1 - b) - b = b^2/(1 - b)$. After the nth stage, $y_n - x_n \leq b^n/(1 - b)$. Thus, lim $n \to \infty$, $y - x_n = 0$. Of course, the sequence of x's may converge in a finite number of steps. In addition, there may be several sequences that converge to y if $b > 0.5$, and the algorithm defined above will yield only one.

As a last point to this example, we illustrate a closed form solution to the problem of describing the limiting density function for h_∞ when $b = 0.5$. We begin by noting that the right-hand side of Equation (19) can be thought of as an infinite sum of independent point binomial random variables x_s where

$$\begin{aligned} x_s &= b^s \qquad \text{Prob} = p \qquad s = 0, 1, \ldots \\ &= 0 \qquad \text{Prob} = 1 - p \end{aligned} \tag{21}$$

The characteristic function for x_s is $[pe^{i\tau b} + (1 - p)]$, where $i = \sqrt{-1}$. Since the x_s values are independent, the characteristic function for h_∞ is given by

$$\phi_{h\infty}(\tau) = \prod_{s=0}^{\infty} [pe^{i\tau b^s} + (1 - p)] \tag{22}$$

Although Equation (22) is formidable to analyze, when $b = p = 0.5$ it reduces to:

$$\phi_{h\infty}(\tau) = \frac{e^{i2\tau} - 1}{i2\tau} \tag{23}$$

Equation (23) represents the characteristic function of a random variable uniformly distributed over the interval [0, 2]. Thus h_∞ is uniformly distributed with density

$$\begin{aligned} f(h_\infty) &= 0.5 \qquad h_\infty \in [0, 2] \\ &= 0 \qquad \text{otherwise} \end{aligned} \tag{24}$$

It follows that the density of N_∞ will be inverse uniform in the range

$$\left[\frac{\lambda}{(1 - e^{-\lambda})(\bar{\epsilon} - \epsilon) + \epsilon}, \frac{\lambda}{\epsilon}\right]$$

For large values of λ the range approaches $[\lambda/\bar{\epsilon}, \lambda/\epsilon]$. In the case where uncertainty is not large [i.e., $(\bar{\epsilon} - \epsilon)$ is small], the probability mass will be tightly packed around λ/ϵ.

Optimal Resource Management Policies Under Uncertainty

Statement of the Problem

In this section we state the resource management problem posed to the central manager under uncertainty. In the section that follows we derive a set of conditions that characterize the optimal extraction policy. After demonstrating many properties of the solution we conclude with an example that can be solved in closed form for the optimal management policy. The properties of this policy are studied in some detail.

The management problem is expressed in the following way. In the absence of harvesting, the resource is assumed to grow according to the stochastic difference equation given by (7). The introduction of extraction implies that if the amount C_t is harvested at time t then, residually, the amount $X_t = N_t - C_t$ becomes the stock that grows according to Equation (7). The sequence of harvests $\{C_t, t \in I^+\}$ is chosen to maximize the expected present discounted value of benefits from harvesting where extraction benefits are measured by the utility associated with the extraction at any point in time.

Before turning to the mathematical statement of the problem, two points should be noted. First, it is assumed that the social manager can perfectly observe all resource states and that the distribution of the random variables $\{\lambda_t, \epsilon_t\}$ is stationary and known to the planner. Second, although the foregoing analysis ignored this point, it is possible to include extraction costs into the model in a limited sense. In particular, if extraction costs represent a constant proportion of the extracted resource and are effectively payable in units of the resource then all of the results generated in the sections that follow will continue to hold. In such a situation, benefits are defined over net extraction.

Formally, the programming problem faced by the social manager can be written

$$\max_{\{C_0, C_1 \ldots\}} E\left[\sum_{t=0}^{\infty} \delta^t U(C_t)\right] \tag{25}$$

subject to

$$N_{t+1} = \frac{(1 + \bar{\lambda}_t)(N_t - C_t)}{1 + \bar{\epsilon}_t(N_t - C_t)} \qquad N_0 \text{ given}$$

subject to

$$N_t, C_t \geq 0$$

In (25), E is the expectations operator defined over infinite vectors $\{C_0, \ldots\}$ and δ is the constant discount factor satisfying $0 < \delta < 1$. The twice continuously differentiable utility function in (25) is assumed to satisfy the following conditions:

$$\begin{aligned} \lim_{x \to 0} U'(x) &= \infty \\ \lim_{x \to \infty} U'(x) &= 0 \\ U''(x) &< 0 \qquad \text{for all } x \\ \lim_{x \to 0} U(x) &= -\infty \end{aligned} \tag{26}$$

The last condition, while not necessary for the optimality arguments that follow (indeed, it provides some complications) introduces a useful conservation motive into the decision making process. Finally, we continue to assume that λ_t and ϵ_t are strictly positive and bounded for each t by $[\boldsymbol{\lambda}, \overline{\boldsymbol{\lambda}}]$ and $[\boldsymbol{\epsilon}, \overline{\boldsymbol{\epsilon}}]$. Given the nonnegativity constraint, this implies that the resource growth process given in Equation (25) is strictly concave in net stock $(N_t - C_t)$ and N_{n+1} is strictly bounded by $[0, \overline{\boldsymbol{\lambda}}/\boldsymbol{\epsilon}]$.

General Characteristics of the Solution to Equation (25)

The Optimal Policy Function. The solution to Equation (25) involves the determination of an optimal policy function $C_t = f(N_t, t)$ that is feasible and that relates the optimal extraction at time t to the size of the stock at time t and calendar time. However, given the way in which the problem is posed in Equation (25), the optimal policy function will be independent of explicit calendar time and can be written $C_t = f(N_t)$.

The proof of this assertion lies in first noting that the utility function is explicitly independent of time and the planning horizon is infinite. These points, combined with the facts that the processes generating $\{\lambda_t, \epsilon_t\}$ are stationary and the discount factor is constant, imply that the variable t simply indexes the sequence of states and that a transformation of this index would not affect the decision making process.

Existence of a Feasible Policy. The set of feasible policies is not denumerable. For example, any policy of the form $C_t = kN_t$ where k is a positive constant satisfying $0 < k < 1$ is feasible in that the growth process will yield $C_t, N_t > 0$ for all finite t. See, for example, Mirman and Zilcha (1975).

Finiteness of the Solution. Define $V(N_0)$ as the optimized value of the objective function in Equation (25). This will be termed the *optimal value function*. It is possible to demonstrate that as long as N_0 is positive and finite, $V(N_0)$ will be finite.

To prove the finiteness assertion, it is sufficient to demonstrate that there is a feasible solution to Equation (25) such that $V(N_0)$ is finite. In particular, choose the feasible policy $C_t = kN_t$ where k is sufficiently small such that $(1 + \boldsymbol{\lambda})(1 - k) \gg 1$. The preceding proofs, which demonstrate that N_t approaches a limiting random variable defined over a strictly positive interval, continue to hold. This guarantees that C_t will be strictly bounded from below by a positive constant. In addition, the boundedness of C_t from above is guaranteed by the fact that N_t is bounded by $\bar{\boldsymbol{\lambda}}/\boldsymbol{\epsilon}$. Hence $U(C_t)$ can take values only in a finite range and the infinite sum $E \sum_{t=0}^{\infty} \delta^t U(C_t)$ must converge. Since this sum converges for a feasible policy it must also converge for the optimal policy and $V(N_0)$ is finite.

This result is very strong. It guarantees that even though $\lim_{x \to 0} U(x) = -\infty$ and though it may be optimal to *asymptotically* extinguish the resource, it will always be the case that $\lim_{t \to \infty} E[\delta^t U(C_t)] = 0$ when C_t is chosen according to the optimal policy function.

$V(N_0)$ Is an Increasing Concave and Differentiable Function of N_0. The fact that $V(N_0)$ is an increasing function of N_0 follows immediately from the assumption of everywhere positive marginal utilities. The following argument serves to demonstrate that $V(N_0)$ is differentiable. The discussion that follows is directly

patterned after the proof of a similar result in Mirman and Zilcha (1975).

Let $\{C_t\}$ represent an optimal consumption sequence starting from initial stock size N_0. Let ΔN_0 be some small positive amount of stock and $\{\hat{C}_t\}$ be a feasible consumption policy defined by $\hat{C}_0 = C_0 + \Delta N_0$ and $\hat{C}_t = C_t$ for $t = 1, 2, \ldots$. The expression for the difference in the optimal value function is

$$V(N_0 + \Delta N_0) - V(N_0) \geq E \sum_{t=0}^{\infty} \delta^t U(\hat{C}_t) - E \sum_{t=0}^{\infty} \delta^t U(C_t) \quad (27)$$

where the inequality follows from the fact that $\{\hat{C}_t\}$ may not be optimal. By the definition of $\{\hat{C}_t\}$, Equation (27) can be rewritten

$$\begin{aligned} V(N_0 + \Delta N_0) - V(N_0) &\geq U(C_0 + \Delta N_0) - U(C_0) \\ &= U'(C_0)\Delta N_0 + o(\Delta N_0) \quad (28) \end{aligned}$$

The last equality follows from the differentiability of U. Dividing though Equation (28) by ΔN_0 and taking the limit as $\Delta N_0 \to 0$ yields

$$V'(N_0) \geq U'(C_0) = U'[f(N_0)] \quad (29)$$

where $f(\ )$ is the optimal policy function. Since an identical argument can be used to show that for a small stock *reduction* (ΔN_0), $V'(N_0) \leq U'[f(N_0)]$, we conclude that $V(N_0)$ is differentiable, with the derivative given by

$$V'(N_0) = U'[f(N_0)] \quad (30)$$

Finally, the concavity of $V(N_0)$ can be established in the following way. Let $\{C_t^1\}$ denote an optimal policy starting from stock size N_0^1 and $\{C_t^2\}$ denote the optimal policy from N_0^2. If $0 < \alpha < 1$ then $\{\alpha C_t^1\} + \{(1 - \alpha)C_t^2\}$ is a feasible policy starting from $\alpha N_0^1 + (1 - \alpha)N_0^2$. However, since the feasible policy so described may not be optimal, it follows that

$$V(\alpha N_0^1 + (1 - \alpha)N_0^2) \geq E \sum_{t=0}^{\infty} \delta^t U(\alpha C_t^1 + (1 - \alpha)C_t^2) \quad (31)$$

Using the fact that U is concave, the right-hand side of Equation (31) can be rewritten to yield

$$\begin{aligned} V[\alpha N_0^1 + (1 - \alpha)N_0^2] \\ \geq \alpha E \sum \delta^t U(C_t^1) + (1 - \alpha)E \sum \delta^t U(C_t^2) \\ = \alpha V(N_0^1) + (1 - \alpha)V(N_0^2) \quad (32) \end{aligned}$$

This establishes the concavity of $V(N_0)$.

The Optimal Policy Function $C_t = f(N_t)$ is Continuous and Increasing with $0 = f(0)$. The fact that $0 = f(0)$ follows immediately from the non-negativity constraints. The fact that $f(\ )$ must be increasing in N_t follows from the result proved above—that $V'(N)$ exists and is positive and hence that there is always a feasible increasing policy function that dominates as constant or decreasing policy function. Continuity of $f(\ )$ is established in the following way.

Define $X(N_t) = N_t - f(N_t)$. By the feasibility constraints and the finiteness of V it follows that $X(\ )$ must be an increasing function of N_t. Next, define the upper and lower limits of X and C at N_0 by

$$C_0^- = \lim_{N \uparrow N_0} f(N) \leq \lim_{N \downarrow N_0} f(N) = C_0^+$$
$$X_0^- = \lim_{N \uparrow N_0} X(N) \leq \lim_{N \downarrow N_0} X(N) = X_0^+$$

The inequalities follow from the fact that both X and f are increasing in N_0. By definition,

$$C_0^- + X_0^- = N_0$$
$$C_0^+ + X_0^+ = N_0$$

Subtracting these two equations we obtain

$$(C_0^- - C_0^+) + (X_0^- - X_0^+) = 0$$

Since X and f are increasing functions it follows that the only way for equality to hold is if $C_0^- = C_0^+$ and $X_0^- = X_0^+$. Thus, continuity of the optimal policy function is established.

Identification of the Optimal Policy

Having described many of the characteristics of the optimal policy, we now turn to the development of the necessary and sufficient conditions for a policy to be optimal.

Using a Dynamic Programming argument, the optimal value function can be written

$$V(N_0) = \max_{C_0} \left[U(C_0) + \delta EV \left(\frac{(1 + \bar{\lambda})(N_0 - C_0)}{1 + \bar{\epsilon}(N_0 - C_0)} \right) \right] \quad (33)$$

Since $V(\ )$ is differentiable and concave, the term in square brackets is a concave differentiable function of C_0 in the range $0 \leq C_0 \leq N_0$. Indeed, since we know that $V(\ )$ is finite and $\lim_{x \to 0} U(x) = -\infty$, we can conclude that the C_0 which maximizes the term in square brackets must lie in the open interval $(0, N_0)$. Finally, since the term in square brackets is a concave differentiable function, the choice of C_0 will be unique.

Differentiating in Equation (33) we obtain

$$U'(C_0) = \delta E \left[V'(N_1) \cdot \frac{(1 + \bar{\lambda}_0)}{[1 + \bar{\epsilon}(N_0 - C_0)]^2} \right] \quad (34)$$

Now, if we recognize that Equation (34) must be true for any time period, we can drop the time subscript from C_0. Further, we can replace every occurrence of C in Equation (34) with $f(N)$, the optimal policy function. Finally, rewriting $V'(\ )$ according to Equation (27), we arrive at the following statement of the necessary condition:

$$U'[f(N)] = \delta E \left\{ U' \left[f \left(\frac{(1 + \bar{\lambda})[N - f(N)]}{1 + \bar{\epsilon}[N - f(N)]} \right) \right] \cdot \frac{(1 + \bar{\lambda})}{\{1 + \bar{\epsilon}[N - f(N)]\}^2} \right\} \tag{35}$$

Equation (35) is a functional equation which, in principle, can be used to solve for the optimal policy function $f(N)$. It can be interpreted as requiring that the optimal policy should leave no opportunity for gain from an intertemporal redistribution of resources arising from an intertemporal reallocation of consumption.

Satisfaction of Equation (35) by a feasible policy function is also a sufficient condition for the feasible policy function to be the optimal policy function. In order to demonstrate sufficiency we show that the value of discounted benefits from the optimal policy sequence $\{C_t\}$ exceeds the value of discounted benefits from any other feasible policy sequence $\{\hat{C}_t\}$ given N_0. Since the optimal value function is known to be finite (see section above, "Finiteness of the Solution"), we assume that $\{\hat{C}_t\}$ yields a finite value for discounted benefits. Alternative feasible policies where the value of benefits may diverge to negative infinity are of no interest.

In terms of the above discussion, the establishment of sufficiency requires that

$$E \sum_{t=0}^{\infty} \delta^t [U(C_t) - U(\hat{C}_t)] \geq 0 \tag{36}$$

Since $U(\ )$ is concave it satisfies the following inequality for any t:

$$U(C_t) - U(\hat{C}_t) \geq U'(C_t)(C_t - \hat{C}_t) \tag{37}$$

Thus in order to demonstrate the validity of Equation (36) it is sufficient to demonstrate that

$$E \sum_{t=0}^{\infty} \delta^t U'(C_t)(C_t - \hat{C}_t) \geq 0 \tag{38}$$

In order to evaluate Equation (38) we require information on the difference $C_t - \hat{C}_t$. Inverting the constraint in Equation (25) we obtain

$$N_t - C_t = \frac{N_{t+1}}{1 + \bar{\lambda}_t - \bar{\epsilon}_t N_{t+1}} \geq 0 \tag{39a}$$

$$\hat{N}_t - \hat{C}_t = \frac{\hat{N}_{t+1}}{1 + \bar{\lambda}_t - \bar{\epsilon}_t \hat{N}_{t+1}} \geq 0 \tag{39b}$$

Combining Equations (39*a*) and (39*b*) we obtain

$$C_t - \hat{C}_t = N_t - \hat{N}_t - \frac{N_{t+1}}{1 + \bar{\lambda}_t - \bar{\epsilon}_t N_{t+1}} - \frac{\hat{N}_{t+1}}{1 + \bar{\lambda}_t - \bar{\epsilon}_t \hat{N}_{t+1}} \tag{40}$$

Notice that $-N_{t+1}/(1 + \bar{\lambda}_t - \bar{\epsilon}_t N_{t+1})$ is a concave function of N_{t+1} in the range of possible values of N_{t+1}. In addition, the sufficiency condition would still be satisfied if the inequality in (38) was replaced by an expression which, for each t, was less than $C_t - \hat{C}_t$. In particular, using concavity, Equation (40) can be rewritten as an inequality:

$$C_t - \hat{C}_t \geq N_t - \hat{N}_t - \frac{(N_{t+1} - \hat{N}_{t+1})(1 + \bar{\lambda}_t)}{[1 + \bar{\lambda}_t - \bar{\epsilon}_t N_{t+1}]^2} \tag{41}$$

Substituting the right-hand side of Equation (41) into (38), the sufficiency condition reduces to

$$E \sum_{t=0}^{\infty} \delta^t U'(C_t) \left[N_t - \hat{N}_t - \frac{(N_{t+1} - \hat{N}_{t+1})(1 + \bar{\lambda}_t)}{(1 + \bar{\lambda}_t - \bar{\epsilon}_t N_{t+1})^2} \right] \geq 0 \tag{42}$$

Since $N_0 = \hat{N}_0$, Equation (42) can be rewritten

$$E \sum_{t=1}^{\infty} \delta^{t-1} \frac{(N_t - \hat{N}_t)(1 + \bar{\lambda}_{t-1})}{(1 + \bar{\lambda}_{t-1} - \bar{\epsilon}_{t-1} N_t)^2} \times \left[\delta U'(C_t) \frac{(1 + \bar{\lambda}_{t-1} - \bar{\epsilon}_{t-1} N_t)^2}{(1 + \bar{\lambda}_{t-1})} - U'(C_{t-1}) \right] \geq 0 \tag{43}$$

Next, we use the growth constraint to substitute for N_t in the term in square brackets in Equation (43) to obtain

$$E \sum_{t=1}^{\infty} \delta^{t-1} \frac{(N_t - \hat{N}_t)(1 + \bar{\lambda}_{t-1})}{(1 + \bar{\lambda}_{t-1} - \bar{\epsilon}_{t-1} N_t)^2} \times \left[\frac{\delta U'(C_t)(1 + \bar{\lambda}_{t-1})}{[1 + \bar{\epsilon}_{t-1}(N_{t-1} - C_{t-1})]^2} - U'(C_{t-1}) \right] \geq 0 \tag{44}$$

Let us denote the term in large brackets in Equation (44) by J_t. Depending on the realizations of the random variables, J_t can be positive, negative, or zero, as can $N_t - \hat{N}_t$. However, we can rewrite the sufficiency condition as

$$E \sum_{t=1}^{\infty} \delta^{t-1} \left[\frac{\operatorname{sgn}(J_t)(N_t - \hat{N}_t)(1 + \bar{\lambda}_{t-1})}{(1 + \bar{\lambda}_{t-1} - \bar{\epsilon}_{t-1} N_t)^2} \right] [\operatorname{sgn}(J_t) \cdot J_t] \geq 0 \tag{45}$$

where $\operatorname{sgn}(J_t) = -1$ whenever $J_t < 0$, 1 whenever $J_t > 0$, and 0 whenever $J_t = 0$. The term in small brackets in Equation (45) is now nonnegative for all t. The sufficiency requirement is made no less stringent if we replace the term in large brackets in Equation (45) by a term that is always smaller. The following two replacements are made conditional on the sign of J_t:

$$H_t^+ = N_{t-1} - C_{t-1} - (\hat{N}_{t-1} - \hat{C}_{t-1}) \leq \frac{(N_t - \hat{N}_t)(1 + \bar{\lambda}_{t-1})}{(1 + \bar{\lambda}_{t-1} - \bar{\epsilon}_{t-1}N_t)^2} \qquad (46a)$$

whenever $J_t \geq 0$, and

$$H_t^- = -(N_{t-1} - C_{t-1})[1 + \bar{\epsilon}(N_{t-1} - C_{t-1})] \leq \frac{(\hat{N}_t - N_t)(1 + \bar{\lambda}_{t-1})}{(1 + \bar{\lambda}_{t-1} - \bar{\epsilon}_{t-1}N_t)^2} \qquad (46b)$$

whenever $J_t \leq 0$. Using the replacements defined in Equation (46) and the definition of $\text{sgn}(J_t)$, the sufficiency condition can be expressed as

$$E \sum_{t=1}^{\infty} \delta^{t-1} \left[\frac{\text{sgn}(J_t)(H_t^+ - H_t^-) + H_t^+ - H_t^-}{2} \right] \cdot \text{sgn}(J_t) \cdot J_t \geq 0 \qquad (47a)$$

Since it is always true that $E[f(x)g(x, y)] = E\{f(x)E[g(x, y) \mid x]\}$, and further, since H_t^+ and H_t^- are functions only of N_{t-1} (because C_{t-1} is a function of N_{t-1}), Equation (47*a*) can be rewritten

$$E \sum_{t=1}^{\infty} \delta^{t-1} \left[\frac{\text{sgn}(J_t)(H_t^+ - H_t^-) + H_t^+ - H_t^-}{2} \right] \cdot \text{sgn}(J_t)E[J_t \mid N_{t-1}] \geq 0 \qquad (47b)$$

Since the second expectation in Equation (47*b*) is zero from the necessary condition, sufficiency is established. It should be stressed that the sgn( ) operator in no way affects the expectations.

An Example

As an example of the foregoing analysis, we consider the case where benefits are evaluated according to the isoelastic utility function:

$$U(C_t) = -A/C_t \qquad A > 0 \qquad (48)$$

Substituting Equation (45) and its derivative into Equation (35) and solving the resulting functional equation yields the optimal policy function:

$$C_t = f(N_t) = (1 - k)N_t \qquad k = E[\delta/(1 + \bar{\lambda})]^{1/2} \qquad (49)$$

Some interesting properties of the optimal extraction policy are worth noting. In the first place, it will be noted that the extraction decision is independent of the random death variable ϵ_t. Secondly, the optimal amount of extraction decreases as the variance in the growth parameter λ_t increases. Finally, if we take a deterministic model as one with parameters given by the means of $\{\bar{\lambda}_t, \bar{\epsilon}_t\}$ then, for any stock size, deterministic extraction will be greater than extraction under uncertainty. This suggests that the certainty equivalent of this stochastic model does not correspond to the mean of the stochastic model.

To establish the second property it is sufficient to prove that $E[1/(1 + \bar{\lambda}_t)]$ is an increasing function of the second central moment (u_2) of λ_t. Dropping the time subscript and expanding yields

$$E[1/(1 + \bar{\lambda})] = E[\exp(-\lambda)] = 1 - u_1 + u_2/2 - u_3/3! \ldots \qquad (50)$$

As required, the right-hand side is an increasing function of u_2.

To establish the third property it is sufficient to note that the functional equation holds in the special case of certainty. Thus, defining $\bar{\lambda}_m$ as the mean of $\bar{\lambda}$, the optimal deterministic management policy is

$$C_t = (1 - k_m)N_t \qquad k_m = [\delta/(1 + \lambda_m)]^{1/2} \qquad (51)$$

By Jensen's Inequality, $k > k_m$ and the result is established.

Directions for Future Research

Rather than summarizing the results and conclusions that have been reached in the course of this paper, we will conclude by discussing directions for future research.

In the first place, it is important to try to extend the analysis to consider the case where desired and actual extraction may differ. I have made a first pass at this problem and considered the case where a given amount of extraction may appear as a greater amount in terms of its effects on the growth process. This situation would arise if, for example, the actual harvest in fisheries contained proportionately too many females or if the actual harvesting activities damaged unextracted members of the species or feeding and breeding grounds in a probabilistic fashion. The harder problem of introducing an explicit production technology with random marginal products and the like remains to be solved.

Second, it would be useful if the model could be used to gain some insight into all of the above cases when decisions are decentralized and one is searching for an extraction policy that maximizes the expected present discounted value of profits.

Third, it would be interesting to use the models introduced above and those that have been suggested to evaluate past policy towards extraction. The linear properties of the inverse growth process suggest that existing econometric techniques (for example, random coefficient models) could be adapted to obtain estimates of the distribution of the parameters that govern the growth processes of fish species, forests, and other replenishable resources. These estimates could provide a valuable contribution to public policy decisions.

Finally, many results remain to be derived from the present models. For example, what are the certainty equivalence relationships contained in these models? How are risk and the discount rate traded off in an optimal policy? The answers to these questions will provide potentially useful input for policy decisions.

References

Bauer, H. 1972. *Probability theory and elements of measure theory*. New York: Holt, Rinehart, and Winston.

Bismut, J.-M. 1975. Growth and optimal intertemporal allocation of risks. *J. Eco. Theory* 10: 239–257.

Bourguignon, F. 1974. A particular class of continuous-time stochastic growth models. *J. Econ. Theory* 9: 141–158.

Brock, W. A., and L. J. Mirman. 1972. Optimal economic growth and uncertainty: the discounted case. *J. Econ. Theory* 4: 479–513.

Clark, C. W. 1976. *Mathematical bioeconomics: the optimal management of renewable resources*. Toronto: Wiley.

Lewis, T. R. 1975. Optimal resource management under conditions of uncertainty: the case of an ocean fishery. Social Science Working Paper 104. Pasadena: California Institute of Technology.

Ludwig, D. 1977. Optimal harvesting of a randomly fluctuating resource, I: Application of perturbation methods. Technical Report 77-11. Vancouver: Institute of Applied Mathematics and Statistics, University of British Columbia.

Merton, R. C. 1975. An asympotic theory of growth under uncertainty. *Rev. Econ. Stud*. 42: 375–393.

Mirman, L. J., and I. Zilcha. 1975. On optimal growth under uncertainty. *J. Econ. Theory* 11: 329–339.

Reed, W. J. 1974. A stochastic model of a renewable animal resource. *Math. Biosci*. 22: 313–337.

Smith, J. B. 1977. An analysis of optimal replenishable resource management under uncertainty. Ph.D. dissertation. London, Ont.: University of Western Ontario.