Incorporating Risk Preferences into Real Options Models

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Abstract

This paper develops a framework to link the expected utility analysis to real options models in order to capture the joint effects of risk aversion and irreversibility associated with real investments. It aims at modifying the theory of investment under uncertainty by incorporating decision makers' risk preferences and allows explicitly analyzing the impacts of risk aversion, uncertainty and irreversibility on decisions such as investment and resource allocations. It addresses the shortcomings of the commonly used expected utility and investment under uncertainty models by generalizing the theory of irreversible investment under uncertainty by allowing for risk-averse investors. We found that uncertainty, irreversibility and risk aversion are important determinants of the optimal timing of irreversible decisions. Ignoring risk preferences in real options models would lead to over or underestimation of magnitude of investments.

Key Word: expected utility, investment under uncertainty, irreversibility, real options, risk aversion.

JEL Classification: D81, G1

1. Introduction

The assumption that a decision maker maximizes expected utility has been a frequently employed model specification. The contributions of Arrow and Pratt led to a large number of mainly theoretical papers concerning economic decisions involving risk and uncertainty. The expected utility framework has been widely used to examine various economic and social problems in economics and in agricultural economics. There now exits a substantial set of definitions, theorems, and empirical procedures available to those applying this paradigm (see Meyer (2002)). The expected utility framework incorporates uncertainty and risk preferences of decision makers and has been the foundation of many economic analyses. A special case of the expected utility framework derived with the exponential utility function is the mean-variance framework, which was popularized by Tobin (1958) for portfolio allocation decisions. The mean-variance framework has been much used as a basis of various empirical problems such as investment in new technologies and resource management decisions in analyzing the impacts of risk aversion and uncertainty.

Recent studies in economics and finance have introduced the use of real options models. These theoretical models to capital investment decisions of firms depend on the financial options and have been recently popularized by Dixit and Pindyck (1994) in modeling irreversible investment decisions. These models stress the irreversibility of most investment decisions and the ongoing uncertainty of the economic environment in which those decisions are made. This new approach recognizes the option value of waiting for better information. The theory of irreversible investment has been much used in both the empirical research and the theoretical literature in analyzing risk-neutral decision makers' investment decisions (Brennan and Schwartz, 1985; Dixit, 1989; McDonald and Siegel, 1985; Majd and Pindyck, 1987; Myers and

Majd, 1990; Pindyck, 1988; Dixit and Pindyck, 1994; Trigeorgis, 1996). One characteristic of the theory of irreversible investment under uncertainty is that it explicitly incorporates the value of waiting or cost of commitment in analyzing decisions that are at least partially irreversible. This theory gives decision makers an opportunity to delay irreversible decisions such as investment in new technologies to learn more about market and economic conditions before making irreversible decisions. However, until now this theory has only been developed in the cases of risk neutrality using dynamic programming and risk aversion using contingent claims analysis. Contingent claims analysis incorporates risk (i.e., market price risk) through using risk adjusted rate of return instead of discount rate in the analysis. Although this procedure adjusts the rate of return required according to the variability of return in the market[1], it does not explicitly take into account the decision makers' subjective degree of risk aversion. Individuals are often faced with a variety of flexibilities (for instance health coverage options) which can not be valued properly without taking into account subjective degree of risk aversion. The market price of risk does not capture this effect.

Many investment decisions such as whether to expand the capacity of their current operations or exit the industry involve sunk costs of investment and uncertainty about prices, demand, or cost. Most firms have the opportunity to delay the investment decisions to learn more about prices, costs, and other market conditions before making decisions that are at least partially irreversible. Firms currently operating also have the option to choose risk-free investment alternatives such as long-term saving accounts or government bonds to diversify their portfolio. Although such investment alternatives are considered to be risk free and reduce the variability of

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[1] The capital assets pricing model is usually used to determine the risk-adjusted rate of return as: $m = r + fsd_{pm}$, where r is the risk free interest rate, d_{pm} is the correlation between the asset and the market, f is the market price or risk and s is the variability of returns of the asset.

income, firms' decisions to enroll in such programs are at least partially irreversible. An example of such decisions is the farmer participation in the conservation reserve program in the U.S. Instead of operating their risky farming operations, farmers have the option to voluntarily retire their land for 10-15 years to receive annual rental payments[2]. However, they face a decision whether to continue to operate their risky operations or participate in an irreversible program that provides risk-free returns over a fixed period of time. If farmers participate in the program, they can reduce the variability of their income. Although the participation is likely to reduce the variability of returns, it involves an irreversible decision. It is therefore important to consider both risk aversion and irreversibility of the decision in modeling such decisions.

An important issue is to determine which of the theoretical frameworks reviewed above is appropriate in modeling decisions such as investment decisions of firms that are facing a decision to invest in an irreversible program or continue to operate their risky operations. Real options models and expected utility framework are useful in modeling decision-making under uncertainty for a wide variety of problems. The theory of irreversible investment under uncertainty has been criticized because it does not allow decision makers' subjective degree of risk aversion to be explicitly incorporated in decision-making process. Specifically, this theory does not take into account the impacts of reductions in variability of the firm's portfolio on investment decisions. The expected utility framework, on the other hand, allows decision makers' risk preferences and therefore the risk premium to be explicitly incorporated into the firms' decision making. However, the expected utility theory does not consider the importance of irreversibility of investment decisions and ability to delay irreversible decisions.

[2] Another example is to decide whether to invest in the risk free long-term government certificate of deposit or invest in the stock market.

The purpose of this paper is to develop a framework that allows the incorporation of decision makers' risk preferences into real options (investment under uncertainty) models. The framework developed in this paper aims at modifying the theory of irreversible investment under uncertainty by incorporating decision makers' risk preferences into real options models. Additionally, it allows explicitly analyzing the impacts of risk aversion, uncertainty and irreversibility on decisions such as investment and resource allocations. It, therefore, addresses the shortcomings of the commonly used expected utility and investment under uncertainty models. This paper contributes to the literature on investment under uncertainty by generalizing the theory of irreversible investment under uncertainty by allowing for risk-averse investors. The results indicate that uncertainty, irreversibility and risk aversion all play an important role in determining the optimal timing of irreversible decisions. Ignoring risk preferences in real options models would lead to over or underestimation of magnitude of investments.

2. Theoretical Model

To illustrate the impact of risk aversion in real options model, we develop a simple model of decision-making under uncertainty and irreversibility. We consider a decision maker who must decide whether to continue to operate the current operation of a firm or invest in a riskless portfolio. The decision maker operating the firm faces with various sources of uncertainty such as demand, price, or weather. Therefore, operating the firm is a risky business. We denote the expected present value of the returns from the current operation at time t as $E(R_t)$. Let the variance of the returns be represented by $Var(R_t)$. The decision maker has the option of investing in a risk-free portfolio with the present value of returns V_t at time t. It is for simplicity assumed that the investment in the risk-free alternative is completely irreversible[3]. Let the discount rate be represented by r. Note that the threshold returns required from the risk free portfolio to shut down the current operation of the firm under risk neutrality is equal to the expected returns from the current operation, $E(R_t)$. We now derive the firm's optimal decision rule with an expected utility model and with a real options model. Then, we introduce an alternative model that combines these two models to address the shortcomings of these two models.

Decision with Expected Utility Framework

We first examine the firm's optimal investment strategy under uncertainty and risk aversion using a utility function. A risk-averse decision maker maximizes the expected utility of wealth, $EU(R)$. To determine the optimal investment strategy, we derive the certainty equivalent wealth from the expected utility. We first define the risk premium as the amount of money that an individual is willingly to pay to avoid uncertainty of income and get the expected value of the income for sure. The risk premium (P) can be derived from $U(E(R)-P)=EU(R)$. Using a second-order Taylor series approximation, the risk premium as: $P=\frac{-Var(R)}{2}\frac{U_{WW}(E(R))}{U_W(E(R))}$,

where $l=-\frac{U_{WW}(E(R))}{U_W(E(R))}$ is called the coefficient of absolute risk aversion evaluated at the mean wealth $\overline{W}$. The certainty equivalent level of the wealth then can be written as:

$$R_{CE}=E(R_t)-\frac{l}{2}Var(R_t). \quad (1)$$

[3] Real world investment decisions are much more complex than the case considered here. Many investment decisions are at least partially irreversible and firms could have several alternatives to invest. To focus on the impact of risk preferences on irreversible investment decisions, we make several simplifying assumptions in the model.

The threshold returns in which the decision maker would be indifferent between choosing to operate the risky operation of the firm and investing in the riskless portfolio can be obtained from (1) as: $V_t^* \equiv E(R_t) - \frac{l}{2} Var(R_t)$. Thus, the decision maker would invest in the risk-free portfolio at time t if the return of the risk-free portfolio were greater than the threshold return, i.e., $V_t \geq E(R_t) - \frac{l}{2} Var(R_t)$. This indicates that there is a tradeoff between the expected return and reductions in the variability of the return. The decision maker is willingly to reduce the expected income by investing in the risk-free portfolio in order to reduce the variability of the returns. The amount that the decision maker would be willingly to pay to receive the expected income for sure is equal to the risk premium.

Decision under Uncertainty and Irreversibility

The firm's investment strategy under uncertainty and irreversibility is modeled using two alternative approaches: dynamic programming techniques and contingent claims analysis, as in Dixit and Pindyck (1994). We assume that R is stochastic and evolves according to the following geometric Brownian motion processes represented by:

$$dR = aRdt + sRdz \,. \tag{2}$$

The variance of R_t, $Var(R_t)$, can be obtained from (2) as: $Var(R_t) = R_0^2 e^{2at}\left(e^{s^2 t} - 1\right)$.

The decision problem at each time t is to maximize the net returns from the investment by choosing an optimal time t as:

$$F(R) = \max_t E\left(V_t - R_t\right)e^{-rt} \,. \tag{3}$$

We can obtain two alternative solutions to the firm's investment decision. First, dynamic optimization techniques are used to derive the optimal investment rule. The Bellman equation is $rF(R)dt = E[F(R)]$. Using Ito's Lemma to expand the right-hand side of this expression, $F(R)$

can be shown to satisfy the following differential equation $0.5\left(s^{2}R^{2}F_{RR}\right)+aRF_{R}-rF=0$, where F_R and F_{RR} are the derivatives of $F(R)$. We solve this differential equation with respect to the following boundary conditions: $F(0)=0$, $F(R)=V-R$, and $F_R(R)=-1$. Solving the differential equation subject to the boundary conditions reveals that the threshold return to be received at which it is optimal to invest in the risk-free portfolio is given by[4]:

$$V_t^{*}=\left(\frac{b_2-1}{b_2}\right)E(R_t) \tag{4}$$

where $b_2<0$ is the smaller root $0.5s^{2}b\left(b-1\right)-ab-r=0$. Note that $\left(\frac{b_2-1}{b_2}\right)>1$ since $b_2<0$. This decision rule requires the decision maker to invest in the riskless portfolio if the expected present value of the return from the riskless portfolio (V_t) is greater than the expected return from the risky operation ($E(R_t)$) by a factor $\left(\frac{b_2-1}{b_2}\right)>1$. This is because the model of the investment under uncertainty incorporates the value of waiting or cost of the commitment in the investment decisions, requiring the firm to demand a premium to account the value of waiting. The extent to which uncertainty and irreversibility of the investment affect the decision-making depends on the magnitude of the multiple $\left(\frac{b_2-1}{b_2}\right)$. This factor increases with an increase in s and/or a decrease in a.

Second, we use contingent claims analysis to incorporate risk using risk-adjusted rate of return (m) instead of the exogenously given discount rate in the analysis (as in Dixit and

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[4] The model developed here can be generalized to the case where both V and R are stochastic and follow as geometric Brownian motion. In that case, $b_2<0$ is the smaller root of $0.5\left(s_V^{2}-2gs_Vs_R+s_R^{2}\right)b\left(b-1\right)+\left(a_V-a_R\right)b-\left(r-a_R\right)=0$, where the parameter g represents the covariance between changes in R and V.

Pindyck (1994)). The solution to the above problem is similar to the dynamic programming, with $b_2 < 0$ being the smaller root $0.5s^2 b(b-1) - (r - m + a)b - r = 0$, where r is the risk free rate of return. If the discount rate is equal to the risk free rate of return, the contingent claims solution to the investment decision is equal to the dynamic programming solution. Thus, option price theory incorporates the market risk into the model by using risk adjusted rate of return instead of the discount rate. However, this model does not incorporate subjective degree of risk aversion.

This decision rule derived using the real options model explicitly incorporates the value of waiting into the firm's investment decisions. However, it does not explicitly take into account the decision maker's subjective degree of risk aversion in determining whether to invest as well as the trade-off between the expected returns and the variability of returns because it assumes risk neutrality. The decision rule under uncertainty and irreversibility presented above is the opposite of that with the expected utility framework. The threshold return required to invest in the risk-free alternative under the option value framework is higher than that under uncertainty and risk aversion of individuals.

Decision under Uncertainty, Irreversibility, and Risk Aversion

We now explicitly incorporate the degree of risk aversion, uncertainty about the returns and irreversibility of the investment into the firm's decision making by combining the two alternative models presented above. The decision problem of the firm is to maximize the certainty equivalent returns from the investment in the irreversible risk-free portfolio by choosing an optimal time t as:

$$F(R) = \max_t E\left(V_t - R_t - \frac{l}{2} Var(R_t) \right) e^{-rt} . \tag{5}$$

Although this model is similar to (3), it allows incorporating the subjective degree of risk aversion and extends that model beyond risk neutrality using utility functions. Until now, dynamic programming has only been applied to the problem of irreversibility under the assumption of risk neutrality or under the market price risk. In this paper we consider the economically relevant problem faced by risk-averse investors who contemplate an irreversible investment in an asset. Use of the dynamic programming methods described above reveals that the threshold returns to be received at which it is optimal to invest is given by:

$$V_t^* = \left(\frac{b_2 - 1}{b_2}\right)\left(E(R_t) - \frac{l}{2}Var(R_t)\right). \quad (6)$$

The threshold return in (6) incorporates the impacts of both risk aversion and the value of waiting on the investment decisions. An increase in risk aversion and/or variability of the return increases the risk premium and therefore decreases the threshold return required to invest in the risk-free portfolio given in (6). This therefore encourages the investment in the risk-free portfolio by risk-averse firms. On the other hand, an increase in the value of waiting (i.e., $\left(\frac{b_2 - 1}{b_2}\right)$) increases the threshold return and therefore discourages the investment in the risk-free portfolio. Thus, there is a trade-off between the value of waiting and reductions in the variability of return in the portfolio. Equation (6) takes into account both these effects in evaluating the firm's investment decisions.

The threshold return under both risk aversion and irreversibility of the investment given in (6) is greater than that under only risk aversion. The threshold in (6) is also greater than that under certainty and risk-neutrality (R_t), if $-E(R_t) > b_2\left(\frac{b_2 - 1}{b_2}\right)\frac{l}{2}Var(R_t)$. These results indicate that the investment rule under both risk aversion and irreversibility of the decision is different than that under only risk aversion or under only certainty and risk-neutrality. These results also imply that ignoring risk preferences in developing real option options models can lead to over or underestimation of magnitude of investments. These results indicate how incorporating subjective degree of risk aversion changes the nature of the optimal investment rule. It is therefore important to incorporate the subjective degree of risk aversion in modeling irreversible investment decisions.

3. Numerical Example

We now examine the extent to which risk aversion, uncertainty and irreversibility affect investment decisions of active firms by providing a numerical example. In this example, we consider a firm that is currently operating a widget factory and decides whether to continue to operate the widget factory or invest in a risk-free portfolio. To keep matters as simple as possible, we assume that the factory produces one widget per year forever with zero operating cost. Currently, the expected net present value of investment over the cost of investment is assumed to be \$200, but next year the price will change thus the net preset value of the investment could change. It is assumed that the firm has an option to invest in a risk-free alternative that pays \$185 (the net present value of investment). However, this investment decision is assumed to be irreversible[5]. We assume that the risk-free rate of interest is 10%. It is simplicity assumed that the utility function is represented by a negative exponential function: $U = -e^{-lR}$, where l is the risk aversion coefficient. Given that R_t is normally distributed, certainty equivalent level of income can be written as: $R_{CE} = E(R_t) - \frac{l}{2}Var(R_t)$. The risk aversion coefficient l is assumed to be 0.015. The parameters used in the numerical simulation are presented in Table 1.

Table 2 presents the firm's alternative investment strategies under various models

[5] It is, however, possible to consider a more realistic scenario in which the investment decision is partially irreversible. In that case, the impact of uncertainty and irreversibility on investment would be less than the case of complete irreversibility.

presented above. Under certainty and risk neutrality, the firm does not invest in the risk-free portfolio because the expected net preset value of operating the widget factory ($200) is greater than the expected net present value of the earnings from the investment in the risk-free asset ($185). Thus, the firm's optimal decision is to continue to operate the widget factory under the assumption of certainty and risk neutrality.

We then calculate the critical values at which is optimal to invest in the risk-free asset under uncertainty and risk aversion for various values of $Var(R)$. The critical values at which it is optimal to invest presented in Table 2 indicate that a risk-averse firm would invest in the risk-free asset in most of the cases examined here. The reason is that the certainty equivalent level of the returns takes into account the risk premium and therefore the required threshold returns are much lower than those under certainty and risk-neutrality. This is because a risk-averse firm would accept less return in order to reduce the variability of the returns and therefore is likely to invest more in the risk-free asset.

Under the real option model, the critical values at which it is optimal to invest in the risk-free portfolio are higher than the expected net preset value from the investment in the risk-free portfolio ($185). Thus, the firm would delay the investment decision in the risk-free portfolio. Instead, the firm continues to operate the widget factory because it takes into account the irreversibility of the investment decision and the value of waiting. On the other hand, under both the risk aversion and irreversibility of the investment, we take into account the risk preferences of the decision maker as well as the tradeoff between the expected return and the variance of return and the impact of value of waiting in the decision-making. In this case, as the variance of the return of the widget factory increases, the critical value at which it is optimal to invest in the riskless portfolio decreases significantly and therefore the firm decides to invest in the riskless portfolio. This framework allows the risk-averse firm to take into account the tradeoff between the reductions in the variability of the firm's returns and the irreversibility of the alternative investment option. Thus, the framework developed in this paper incorporates the two important characteristics of the commonly employed expected utility and investment under uncertainty models. These results emphasize the importance of incorporating risk preferences in real options models.

4. Conclusions

Two important economic models, the expected utility framework and investment under uncertainty, have been widely used to examine various economic and social problems in economics and in agricultural economics involving uncertainty and irreversibility. In this paper, we consider the appropriateness of these models in modeling active firms' irreversible decisions such as investment and portfolio allocations. We develop an alternative model that addresses the shortcomings of the expected utility and real options models. The model developed in this paper combines the two important characteristics of these theoretical models, risk preferences of decision makers and the irreversibility of investment decisions. The paper makes contributions to the literature by generalizing the theory of irreversible investment under uncertainty by allowing for risk-averse investors and by showing how incorporating subjective degree of risk aversion changes the nature of the optimal investment rule.

The results indicate that uncertainty, irreversibility and risk aversion all play significant role in determining the optimal timing of irreversible decisions such as investment and resource allocations. Under the expected utility framework, a risk-averse firm would invest in the risk-free asset in most of the cases examined in the numerical examples because the certainty equivalent level of the returns takes into account the risk premium. Under the real option framework, the critical values at which it is optimal to invest in the risk-free portfolio are higher than the expected net preset value of the alternative investment because this framework takes into account the value of waiting. The model developed here takes into account the tradeoff between the return and variance of return and the impact of value of waiting in the decision-making. As the variance of the return of the widget factory increases the firm decides to invest more in the riskless portfolio to reduce the variability of income. These results underline the importance of incorporating the degree of risk aversion into real options models. Our results show that ignoring risk preferences in real options models can lead to significant over or underestimation of magnitude of investments.

The model developed in this paper can be helpful in analyzing risk-averse decision makers' irreversible decisions in economics and finance. Further research in this area is needed to incorporate many important features of the real world investment decisions under uncertainty. Empirical applications of the model developed in this paper are also needed to determine the extent to which risk aversion and irreversibility of the investment impact the investment decisions of firms in various industries.

Table 1. Parameters Used in the Numerical Example

Example #	a	s	$\boldsymbol{Var(R)}$	$\left(\frac{b_2 - \mathbf{1}}{b_2}\right)$
1	0.05	0.15	1005.9	1.19
2	0.05	0.30	4163.1	1.65
3	0.05	0.45	9922.7	2.29
4	0.05	0.60	19156.1	3.14

Table 2. Critical Values at Which It is Optimal to Invest in the Riskless Portfolio

Example #	Certainty and Risk Neutrality	Uncertainty and Risk Aversion	Real Option		Real Option With Risk Aversion (Dynamic Programming)
			Dynamic Programming	Contingent Claims Analysis*	
1	200.0	192.5	238.7	223.8	229.7
2	200.0	168.8	329.3	309.3	277.9
3	200.0	125.6	458.9	487.1	288.2
4	200.0	56.5	628.2	771.8	176.9

* $m = 0.10$, $r = 0.05$.

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