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An overview of steady-state distributions resulting from firm dynamics models

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An overview of steady-state distributions resulting from firm dynamics models

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Abstract
Empirical firm size distributions are the cumulated result of underlying firm dynamics involving entry of new firms and growth, decline, and exits of incumbent firms. In this paper we give an overview of firm size distributions that result as steady states from models differing in the way these firm dynamics are modelled. In the process we (i) derive common results and explain seemingly contradictory results, (ii) propose new functional forms to describe firm size distributions, (iii) give insight in the interrelationships between the distributions in terms of underlying firm dynamics, (iv) give possible firm dynamical interpretations of the parameters of the distributions, and (v) analyse to which extent the steady-state approach is able to explain the shape of firm size distributions that are encountered in practice.

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1. Introduction

Empirical firm size distributions are the cumulated result of underlying firm dynamics involving entry of new firms and growth, decline, and exits of incumbent firms. In this paper we give an overview of firm size distributions that result as steady states from models differing in the way these firm dynamics are modelled.

What is the use of such an overview? First, it facilitates researchers in deciding which steady-state distribution to use dependent on the situation with respect to entry, exit, growth, and decline of firms in an industry. Second, it gives information about what future changes in the firm size distribution one could expect when there are changes in the underlying firm dynamics. Alternatively, historical changes in the firm size distribution over time or differences between industries might be explained out of changes or differences in underlying firm dynamics. Third, it gives insight into what kind of firm dynamics may be underlying specific firm size distributions. Hence, possible interpretations of the parameters of the size distribution in terms of firm dynamics are provided. Furthermore, relationships between different firm size distributions become clear in terms of underlying firm dynamics. Fourth, it gives possible candidates which firm size distributions to use for fitting purposes. Fifth, one gets an impression to what extent it is possible to explain the various shapes of firm size distributions that we encounter in practice by the steady-state approach. Sixth – because the precise relationship between firm dynamics and the resulting firm size distribution becomes clear – determinants of firm dynamics can be translated into determinants of firm size distributions and vice versa.

What is the contribution of this paper to the literature? First, it catalogues in one place all analytic steady-state distributions that have so far been derived in the literature. Till now these were scattered over the literature. Among others, we bring together the results of Gibrat (1931), Kalecki (1945), Simon (1955, 1960), Steindl (1965), Ijiri and Simon (1977), Levy and Solomon (1996), Sutton (1997), Gabaix (1999), and Malcai et al. (1999).
Second, by confronting results from many different sources we were able to compare them, derive common results, and explain seemingly contradictory results. In a few instances, this led to the discovery of flaws in the existing literature.
Third, to improve the value of this overview we sometimes reinterpreted existing models, elaborated more upon them, or gave the firm size distributions the right labels. For example, this paper introduces the *Waring* distribution and the *extended Katz* distribution (already present but unnamed in the work of Simon) into this area of research.
Fourth, by making this overview we discovered some gaps in the literature, which we closed by developing some extensions to existing models. In particular, we extended the Simon model to involve (i) a general deviation from Gibrat's law and (ii) size-dependent exits.
Fifth, the latest empirical insights concerning the shape of firm size distributions are shortly reviewed (noteworthy those of Sutton (1997) and Axtell (2001) and references therein) and it is analysed to which extent the reviewed models are able to explain these.

The remainder of this paper is structured as follows. The various models leading to steady-state distributions appear to differ with respect to the precise steady-state concept used, the way the steady state is imposed or the steady-state distribution is calculated. These theoretical aspects are discussed in section 2. The various models also differ with respect to the assumptions made. Section 3 discusses the plausibility and usefulness of these assumptions in the light of theoretical and empirical considerations. Section 4 is the heart of the paper. It tabulates all the steady-state firm size distributions and shows how these depend on the underlying assumptions regarding firm dynamics. It is indicated where these distributions are calculated in the literature and it is discussed how results from different models compare to each other. Section 5 summarizes the general tendencies resulting from the steady-state approach and shows to what extent this approach is able to explain the shape of firm size distributions that we meet in practice. Finally, section 6 concludes.

2. Theory

This paper reviews models in which firms are assumed to grow, decline or exit with probabilities that may be a function of current firm size. Optionally new firms may enter at a certain rate or as a fraction of aggregate industry growth. Such firm dynamics models are said to have a steady state if the firm size distribution that results from these conditions evolves to a state in which it does not change any more.
Not all models evolve to a steady state. Furthermore, models may differ with respect to the precise steady-state concept used, the way the steady state is imposed or the steady-state distribution is calculated. In this section some typical models are discussed that differ in these respects from each other. Successively, we will discuss the models of Gibrat (1931), Kalecki (1945), Simon (1955, 1960), Steindl (1965) and Levy and Solomon (1996).[1] The main characteristics of these models are summarized in table 1.

Gibrat (1931)

The most elementary firm dynamics model is that of Gibrat (1931). He considers a fixed number of firms, each facing the same distribution of growth rates, independent of their size. This is referred to as the Gibrat assumption in the literature. Gibrat shows that in his model the logarithm of firm size follows an unrestricted random walk. Accordingly, he finds that the firm size distribution approaches a lognormal distribution with ever increasing mean and variance when time goes on.[2] Hence, the model has no real steady state: the parameters of the firm size distribution keep changing. In the end, when the variance of the lognormal distribution is infinite, firm size is undetermined: the probability that a firm's size is in a certain interval approaches zero for *any* size interval.

The Gibrat model has no steady state. It is in need for a stability condition to restrict the random walk of the (logarithm of the) firm's size. In the models below we will encounter three alternative stability devices to impose a steady state.

1. The average growth rate of firms is negative correlated with size.
2. There is a constant stream of small new firms at the minimum firm size.
3. Firms cannot decline below a certain minimum firm size.

[1] In section 4 results of more models are discussed. However, those extra models do not differ from the ones discussed here with respect to the issues here at hand.

[2] Proof, based on Scherer (1980, p. 147). Let s_0 denote the initial size of a firm at time 0 and let ε_t denote the random growth multiplier in time interval t. Then the size of a firm at time t is $s_t = s_0\, \varepsilon_1\, \varepsilon_2 \,..\, \varepsilon_t$, that is: the cumulative product of its initial size times a string of t random growth multipliers. Taking logarithms we obtain: $\ln s_t = \ln s_0 + \ln \varepsilon_1 + \ln \varepsilon_2 + .. + \ln \varepsilon_t$. Hence, the logarithm of firm size follows an unrestricted random walk. By the central limit theorem the distribution of the sum of t random variables is asymptotically normal with mean $t\mu$ and variance $t\sigma^2$, where μ and σ^2 denote mean and variance of the random variables $\ln \varepsilon_t$. By definition, if log size is distributed normally, then the size itself is distributed lognormally.

All these three devices restrict in a different way the random walk result that in the end firm size is undetermined. The first device accomplishes this by giving up the Gibrat assumption. Growth is impeded: on average, growth rates are smaller for larger firms so that small firm sizes become more probable than large firm sizes. The second device introduces a constant stream of small new firms at a certain minimum firm size. As a result, there will be relatively more small firms than large firms in the steady state. Finally, the third device introduces a minimum firm size below which firms cannot decline. Hence, there will be relatively many firms at or somewhat above the minimum firm size. These are the firms that would have decreased below the minimum firm size if there had not been a lower bound.

Kalecki (1945)

The Gibrat model has no steady state because the variance of the lognormal distribution keeps increasing. Hence, to impose a steady state, Kalecki (1945) proposes a new model in which it is demanded that the variance of the firm size distribution remains constant. He deduces that this condition can be satisfied if there is a negative correlation between the average growth rates of firms and their size.[3]

Simon (1955)

Simon (1955) introduces a constant stream of new small firms to accomplish a steady-state solution. In particular, he considers a growing industry where at each time interval one size unit is added to the industry.[4] Each new unit could be either a new firm of size 1 (with probability p) or a unit of growth of an incumbent firm (with probability 1-p). If the unit of growth accrues to an incumbent firm, the probability that it accrues to a particular firm is proportional to the firm's size. Hence, the Gibrat assumption holds. Furthermore, the model considers no decline and/or exits of firms.

As a result, the model describes a *growing* industry: industry size and the number of firms grow. Obviously, in the steady state – having by definition a constant firm size distribution - the number of firms of any size grows at the same speed in this model.

The above described growth process determines a relationship between the probability density before and after the adding of a size unit to the industry. Furthermore, there is the steady-state condition that the probability density should remain unaffected by the adding of the size unit once the steady state has set in. Simon (1955) shows that both relationships together determine the steady-state distribution. It is a discrete distribution function because of the discrete size unit approach of Simon.

Simon (1955, alternative)

In the same paper Simon (1955) introduces a variant of his model that is interesting because of the differing formulation of the steady-state condition. The model is exactly the same except that now the size of the industry and the number of firms is kept constant by introducing size-independent firm exits that occur at the same frequency as firm entries. Hence, the model describes a *stationary* industry. Furthermore, the model leads to the same steady-state distribution as before because - obviously - the introduction of size-independent exits cannot change the shape of a steady-state size distribution.

Because the industry has been made stationary in this model, the steady-state condition can be formulated differently. Now, it demands that in the steady state the number of firms of any size – say s – should be constant. Hence, it reads:

[3] See also the discussion of the Kalecki model in Steindl (1965, pp. 32-33).

[4] Originally, Simon's model has been stated in terms of a word frequency model. We have translated it here in terms of a firm dynamics model.

“inflow because of growth from s-1 to s” = “outflow because of growth from s to s+1”

+ “outflow because of exits of size s”

By assumption, the inflow because of growth from s-1 to s is proportional to the density function at s-1. On the other hand, the outflow because of growth from s to s+1 and the outflow because of exits are both proportional to the density function at s. Hence, the steady state condition prescribes a recurrent relationship between f(s-1) and f(s) for all sizes s. This determines the density function completely because density functions are normalised to 1. See appendix 1 for a specific example.

Simon (1960)

Simon (1960) introduces another way to make his original model describe a stationary industry. Each time a size unit is added to the industry, also a size unit is taken away from an incumbent firm. Hence, either an incumbent firm declines with one size unit or an incumbent firm exits – if its size was only one size unit.

This model describes a stationary industry: industry size and the number of firms are constant, once the steady state has set in. However, the process that causes the industry to be stationary is fundamentally different than in Simon (1955, alternative) because of the introduction of firm decline instead of size-independent exits. As a result, the steady-state condition and the resulting steady-state distribution are also different. Because the industry is stationary, for any size *s* the number of firms should be constant. A sufficient condition for this to hold is the following steady-state condition:

“inflow because of growth from s-1 to s” = “outflow because of decline from s to s-1”

Note the difference with respect to the steady-state condition of Simon (1955, alternative). First, the inflow of firms from s-1 to s is not compensated by the outflow of firms of size s due to growth to larger sizes but by the outflow due to the *decline* of firms of size s to smaller sizes. Second, there is no outflow of firms due to direct exits (except for firms of size 1).

As before, the density function can be derived from the steady-state condition. For, by assumption, the inflow because of growth from s-1 to s is proportional to the density function at s-1, while the outflow because of decline from s to s-1 is proportional to the density function at s. Hence, the steady state condition prescribes a recurrent relationship between f(s-1) and f(s) for all sizes s. Normalization of the density function does the rest. See appendix 1 for a specific example.

Steindl (1965)

Steindl (1965, pp. 45-63) generalizes the Simon (1955) model by introducing the possibility of firm decline and firm exits (equivalent to a firm declining to size 0), while still considering an industry with a growing number of firms. As a result, Steindl has to calculate the steady-state distribution in a totally different way as compared to Simon. Steindl starts by calculating the distribution of a cohort of firms of the same age. Subsequently, he calculates how firm ages are distributed in the steady state. Finally, he mixes these two distributions to arrive at the proper steady-state firm size distribution. Just as the distribution functions of Simon it is discrete due to the discrete approach of Steindl.

Levy and Solomon (1996)

Levy and Solomon (1996) start with the Gibrat model: a fixed number of firms, each facing the same (arbitrary) distribution of growth rates. However, they add to the model a minimum firm size below which firms can't decline.[5] In particular, all firms face the same (arbitrary!) distribution of growth rates in this model. At each time interval a firm draws a growth rate from this distribution.[6] If, however, a firm draws a negative growth rate such that it would decline below the minimum firm size, it is assigned the minimum firm size instead.

Because the model assumes an arbitrary distribution of growth rates of individual firms, it is possible that on average individual firms grow. Obviously, in this case the mean of the firm size distribution of all firms together will grow forever, so that there is no steady state. To get rid of this trivial result and to focus on the *shape* of the steady-state distribution instead, Levy and Solomon shift their attention to *normalised* firm sizes: firm sizes divided by the average firm size of the industry.[7] Hence, the search in this model is for the steady-state distribution of normalised firm sizes.

The steady-state distribution is derived along the same lines as in Simon (1955).[8] First, the above described growth process determines a relationship between the probability density before and after the time interval in which firms grow. Second, there is the steady-state condition that the probability density should be the same before and after a time interval once the steady state has set in. Levy and Solomon (1996) show that both relationships together determine the steady-state distribution uniquely. It is continuous due to the continuous approach taken by Levy and Solomon.

3. Plausibility of assumptions

The models of the next section for which steady-state distribution are derived, start from different assumptions. In this section the plausibility and usefulness of these assumptions are discussed in the light of theoretical and empirical considerations.

Minimum firm size

In most models of the next section a minimum firm size is introduced. This can be justified by the existence of a minimum efficient size, an idea that is well established in economic theory. Moreover, if one measures firm size as the number of persons that work in the firm, there is a natural minimum size of one person (see e.g. Axtell, 2001).

The role that the minimum firm size plays differs between models. First, it may serve as the starting size of new firms. Second, it may determine the exit of a firm: if a firm would decline below the minimum firm size it exits. Third, it may only serve as a lower bound to sizes: firms cannot decline below it but remain in business at the minimum size if they hit upon it.

[5] This is (arguably) the simplest and most elegant way to prevent firms from becoming too small and hence to guarantee a steady-state solution. However, it is just one of the possibilities. Another possibility would have been that of Kesten (1973). See also Gabaix (1999, pp. 750-751 and pp. 761-762) and references therein.

[6] While in the models of Simon and Steindl growth in a time interval is restricted to one size unit up or down, the Levy-Solomon model is more elegant in this respect: an arbitrary size growth up or down is permitted in one time interval.

[7] The original Gibrat model cannot produce a steady-state solution by the same trick. For, in the Gibrat model the distribution of normalised sizes has still a variance that increases without limit when time goes on.

[8] Mathematics is totally different though because of the more general continuous approach of Levy and Solomon.

Assumptions concerning entries and exits

In many models simplifying assumptions are made such as: no entry of firms, no exit of firms or even no entries and exits at all. This seems to diminish the practical relevance of these models to almost zero because in reality firms *do* enter and exit. However, one should realize that in this paper model results are only dependent on the *net sum* of entry and exit. This increases the practical relevance of these models a great deal. For example, results of a model with no entry and exit at all are also relevant for a situation in which entry and exit cancel each other out. Or, to give another example, results of a model with no exits and only entry at the minimum size are also relevant for a situation in which there are exits at the minimum size that are more than compensated by entries at the minimum size.

In the next section we will review quite a few models starting from the most basic assumption one could think of: a fixed number of firms. Because of its simplicity this assumption is a natural starting point of analysis. Furthermore – as we remarked above - models starting from this assumption have also practical relevance for understanding industries in which entries and exits cancel each other out approximately.[9]

Apart from models with a fixed number of firms, in the next section also models with a positive net entry rate of firms are reviewed.[10] In most of these models net entry takes only place at the minimum firm size. This seems a reasonable approximation of reality because in practice most firms start small. There is one model that is more general in this respect. In this model firms enter in different sizes that are geometrically distributed. Hence, also in this model there are more entries of small firms than of large firms. In those models that explicitly model the exits of firms, in most cases exit is modelled as firms declining below the minimum size. Hence, firms exit only at the minimum size in these models. There is one more general model in this respect. In this model firm exits are proportional to $1+x/s$ with x a positive parameter and s denoting firm size. Both these ways of modelling do justice to the stylised fact noted by Sutton (1997, p. 46) – based on an overview of the empirical literature - that small firms exit more often than large firms.

Assumptions concerning growth and decline of incumbent firms

Various models of the next section adopt the Gibrat assumption: each firm faces the same distribution of growth rates, independent of its size. The theoretic argument in favour of this assumption is that - from a certain size onwards - firms exhibit constant returns to scale and thus have the same growth chance (see, e.g., Ijiri and Simon, 1977, pp. 3-11, 140-142). Besides, there is a large empirical literature about the validity of this assumption. See e.g. the overview of Sutton (1997, pp. 43-47) and references therein. On the basis of this empirical literature it seems fair to conclude that – in

[9] This is also the reason why the model of Levy and Solomon – a fixed number of firms with a minimum firm size below which firms cannot decline – has more practical importance than one maybe would think at first sight. For, the model can reinterpreted in the sense that firms exit when they decline below the minimum size but are replaced by new firms at the same time so that the number of firms remains the same.

[10] Obviously, no models with a negative net entry rate are reviewed. Such models have no steady state because in the end there will be no firms left in such models.

agreement with the theoretic argument - *especially for larger firms* the Gibrat assumption may be not a bad first approximation.[11]

However, especially if smaller firms are taken into account, it is an empirical fact that, on average, growth rates are decreasing in size (Sutton, 1997, p. 46). To do justice to this stylised fact there are models in the next section in which a firm's average growth rate is proportional to 1+*g/s* where *g* is a positive parameter and *s* denotes firm size.[12] This way of modelling simulates exactly the features we find in practice for firms in most cases:

- The average growth rate of a firm is gradually decreasing in firm size.
- Smaller firms have, on average, a significantly larger growth rate than larger firms.
- Large firms share a similar average growth rate because the term *g/s* becomes negligible for large firms. Hence, for larger firms the Gibrat assumption approximately holds.

There are also models in the next section in which growth rates are proportional to s^{-g} (*g* a positive parameter, *s* denoting size). In these models growth rates decrease in size, just as in the previous models. However, contrary to the previous models, the speed at which growth rates decrease remains the same with climbing size so that the Gibrat assumption does not hold approximately between large firms either. Hence, this is the case of *impeded growth*. It is relevant for describing *plants* instead of firms because for plants there are natural limits to growth (see, e.g., Steindl, 1965, p. 32).

The case in which growth rates are inversely proportional to size is a special case of impeded growth. According to Sutton (1997, pp. 48-52) this special case if of relevance for firms as well. He claims (based on empirical evidence) that the steady-state distribution that follows from this growth assumption provides a good description of the *least unequal* distribution that we are likely to find in practice at the 4/5-digit SIC level or higher.

Some models of the next section adopt the Gibrat assumption, modified with the condition that there is a minimum size below which firms can't decline.[13] Although sometimes it may be indeed the case in practice that firms hitting upon the minimum size may linger there instead of exiting, on the whole this does not seem a very realistic assumption. Hence, we prefer the reinterpretation of this model already noted above: firms that decline below the minimum size actually exit and are replaced by new firms of the minimum size.

[11] Sutton (1997) summarizes the empirical literature till 1980 by quoting Scherer (1980): "at least for the United States, empirical studies suggest that assuming growth rates uncorrelated with size is not a bad first approximation". Sutton goes on by reviewing the empirical literature from the eighties, notably the studies of Hall (1987), Evans (1987a, 1987b), and Dunne et al. (1988, 1989). Sutton notes that all these studies suggest that growth rates decrease with size. However, Sutton fails to note that Evans (1987a) remarks that for larger firms the deviation from the Gibrat assumption is moderate. Furthermore, the results of Dunne et al. are derived only for plants, not for firms. Hence, we conclude that *for larger firms* the Gibrat assumption tends to hold approximately. See for collaborate evidence Simon and Bonini (1958) and references therein.

[12] In some of these models growth and decline are modelled separately: growth is proportional to 1+g/s and decline is proportional to 1+d/s.

[13] Hence, especially for smaller firms (which have a higher chance of hitting upon the minimum size) the Gibrat assumption is violated in this model. Average growth will be higher and the variance in growth will be smaller for them.

Finally, there are models in the next section in which – for simplicity - only growth and no decline of firms is modelled. Obviously, these models are only relevant for describing *growing* industries.[14]

Evaluation

Although across the various models there is a large variety in assumptions, of which most can be justified by theoretic or empirical arguments, it cannot be denied that most assumptions are rather basic and elementary. Of course, this is necessary in order to preserve analytic tractability. In the light of this, three remarks are in order.

First, we cannot expect the results of these models necessarily relevant for small and very specific industries, such as narrowly defined product markets. In such industries very specific mechanisms may be prevalent that cannot be captured by quite general regularities. See Sutton (1997, p. 47) and references therein. However, it is to be expected that if we consider broader industries, e.g. 4/5-digit SIC level or higher, very specific mechanisms cancel each other out so that the models become more relevant.

Second, the models in this paper focus on the steady state. No attention is paid to the transitional state before the steady state sets in nor to the time that it costs before the steady state is expected to set in. Hence, these models are not fit to explain the firm size distributions for specific young industries with phenomena such as shake-outs. See Sutton (1997, p. 47-48) and references therein.

Third, one should bear in mind that a model starting from some stylised assumptions can produce valuable results, although these stylised assumptions are never met in practice precisely. For example, in physics the gas laws for ideal gases produce useful results, while in practice gases never are ideal.[15] In the same manner, the Gibrat assumption may produce useful results for a sample of large firms, while it is not satisfied fully in the sample. Hence, the question is not whether a stylised assumption is precisely met in practice but whether or not it can serve as a (first) good approximation.

4. Steady-state distributions

In this section we describe which steady-state firm size distributions result from different assumptions regarding the underlying firm dynamics. Table 2a gives results for a fixed number of firms. Table 2b gives also results for a fixed number of firms but now also offsetting entries and exits at the minimum firm size are modelled explicitly. Table 2c gives results for a growing number of firms. Main characteristics of the steady-state distributions that are found are presented in table 3. For ease of reference, appendix 2 catalogues the symbols used. We will discuss tables 2a-c successively.

Fixed number of firms

Consider the Levy-Solomon model already described in section 2: a fixed number of firms growing according to the Gibrat assumption, modified with the condition that there is a minimum size below which firms can't decline. Malcai et al. (1999, pp. 1300-1301) show that this model leads in the steady state to a continuous *Pareto* distribution, of which the parameter is dependent on the number of firms *N* and the minimum-

[14] Obviously, in practice some firms also decline in growing industries. Nevertheless, models with only growing firms have practical relevance if we assume that underneath there is a (not modelled) process of firm growth and decline that cancel each other out. Hence, we interpret these models as models in which only the *net* growth is modelled.

[15] We have borrowed this example from Ijiri and Simon (1977, pp. 4-5, 109-116) who elaborate on the argument much more.

size parameter c, defined as the fraction of minimum and average firm size.[16] See table 2a, 1st row.

The model contains two interesting special cases that deserve attention.

If we remove the condition that firms do not decline below a minimum size (that is, if we set c=0) the Gibrat assumption holds perfectly and we arrive at the model of Gibrat (1931). As already shown in section 2 in this case the firm size distribution approaches a *lognormal* distribution with parameters ς, σ^2 going to infinity. See table 2a, 2nd row. For large values of σ^2 the lognormal distribution (with a parabolic density function on a log-log scale) resembles quite well the Pareto distribution with a parameter equal to 0 (straight line on a log-log scale) for firm sizes that are not too small or large (Montroll and Shlesinger, 1982, or Sornette and Cont, 1997, p. 432). In fact, in the limit when $\sigma^2 = \propto$ the two distributions coincide. This explains why we find a *Pareto* distribution with a parameter equal to 0 when setting c=0 in the Levy-Solomon model (see Malcai et al. (1999, p. 1301).[17]

We get another special case of the model if we take the minimum firm size relatively small (but not *too* small) as compared to average firm size. In this case we get in the steady state a Pareto distribution with a parameter approximately equal to 1. See table 2a, 3rd row.

Gabaix (1999, pp. 756-757) introduces deviations from the Gibrat assumption in the model. He finds that the parameter of the Pareto distributions will become smaller than 1 in some size domain, if in that domain average growth and/or the variance in growth is larger. See table 2a, 4th row.

Finally, Kalecki (1945) also analyses a model with a fixed number of firms. Instead of introducing a minimum firm size, he introduces a negative correlation between growth and firm size. He shows that this leads in the steady state to a *lognormal* distribution.[18] See table 2a, 5th row.

Fixed number of firms; entries and exits at minimum size

Consider a continuous stream of new firms entering at the minimum size, which is precisely offset by a continuous stream of exiting firms at the minimum size so that the number of firms is constant. The fraction of industry growth due to new firms is constant and labelled *p*. Moreover, the average growth rate of incumbents is proportional to 1+g/s (g>=0), while the average decline rate is proportional to 1+d/s (d>=0). This is the model of Simon (1960), for which he derives the steady-state distribution. Since 1975 this distribution is called the *extended Katz* distribution.[19] See table 2b, 1st row. Note that its parameters depend on the entry parameter *p*, and the growth and decline parameters *g* and *d*.

[16] Gabaix (1999) also analyses the model. When deriving the expression for the parameter ρ of the Pareto distribution from the normalization condition (p. 750), he overlooks the fact that the normalized distribution function is bounded from above. As a result, he gets $\rho = 1/(1-c)$, which in fact is only valid if the number of firms would be infinite (see Malcai et al, 1999, p. 1301). The same flaw is present in the heuristic proof of Gabaix (1999, p. 744).

[17] When setting c=0 Gabaix (1999) finds ρ=1because he starts from $\rho = 1/(1-c)$. Because this expression is only valid for an infinite number of firms (see the previous note on the subject), Gabaix's result is false for any finite number of firms. His conclusion that Gibrat's law in itself is a sufficient condition to get a Pareto distribution with ρ=1 is therefore false. Also an extra condition with respect to the minimum firm size (to be discussed next in the main text) is needed.

[18] See also the discussion of the model in Steindl (1965, pp. 32-33).

[19] Proof and results are somewhat sketchy in Simon (1960). Hence, we provide more details in appendix 1 of this paper. These are necessary to derive some of the results for the special and limiting cases to be discussed next in the main text.

The model contains a number of interesting special or limiting cases (see also figure 1). If decline is independent of size (that is: d=0), we find in the steady state the *negative binomial* distribution. See table 2b, 2nd row. If also the average growth rate is proportional to 1+1/*s* (that is: g=1) the distribution becomes *geometric*. See table 2b, 3rd row. Both these results are derived easily by substituting the parameter values into the extended Katz distribution.
Ijiri and Simon (1977, p. 54 and p. 61) mention also three special/limiting cases. They derive that for d=g (that is: both the growth and decline rate are on average proportional to 1+g/s), for d=g=0 (the Gibrat assumption), and for d=0/g→∞ (decline independent of size while the average growth rate is proportional to 1/s) the steady state is characterized by a *generalization of the logarithmic* distribution, the *logarithmic* distribution, and the *Poisson* distribution, respectively. See table 2b, rows 4-6. This can be verified by substituting the parameter values in the extended Katz distribution.[20]
If the entry parameter *p* approaches zero, the model approaches the situation of a fixed number of firms. In the model with the Gibrat assumption (*d*=*g*=0) the logarithmic distribution approaches then the Pareto distribution with parameter 0.[21] This is as expected: see table 2a, 2nd row.
If, on average, the decline rate of incumbents decreases substantially faster with size than the growth rate (that is: if d >> g), the model has no steady state. The highest value of *d* for which there exists a steady state, is reached for *d* = (1+*g*)/(1-*p*). We can prove that in this case the steady-state distribution boils down to the *Waring* distribution.[22] See table 2b, 7th row. If we set g=0 in this latter model, we get the case for which the average growth rate is independent of size, while the average decline rate decreases – mainly for smaller values of firm size – according to 1+ [1/(1-p)]/s. It follows (by substituting g=0 in the above derived Waring distribution) that in this case the steady state is characterized by the *Yule* distribution. See table 2b, 8th row.

Growing number of firms

Consider a constant stream of new firms entering at the minimum size s_{min}. The fraction of industry growth due to these new firms is constant and labelled *p*. Entries may be (partly) offset by exits as long as the exit chance is independent of firm size. Furthermore, growth rates of incumbent firms deviate in an unspecified way from the Gibrat assumption: average growth rates are proportional to g(s) where g(s) is an unspecified function and s denotes firm size. Finally, firms do not decline. It is noted that these assumptions imply that (i) average firm size is equal to s_{min}/p and that (ii) the net entry rate of new firms is larger than the average growth rate of the incumbent firms.[23] In appendix 1 we show that this model leads in the steady state to the *generalization of the Waring* distribution presented in table 2c, cell (1,1).

[20] Only the Poisson distribution does not follow straightforwardly. To get it from d=0/g→∞, first substitute d=0 to arrive at the negative binomial distribution (see above in the main text). Second, substitute: λ_{nb} = 1- exp(-λ_p/g). Third, use $\lim_{g\to\infty} g\lambda_{nb} = \lambda_p$ to find the Poisson distribution.

[21] Proof. If *p* approaches 0, the parameter λ of the logarithmic distribution approaches 1. See table 2b, 5th row. Hence, the logarithmic distribution approaches 1/*s* (see table 3).

[22] From the expression of the extended Katz distribution (see table 3) it is clear that for λ=1 the distribution boils down to the Waring distribution (take δ=γ+ρ). Hence, we know the mean of the distribution: E*s*=d/(d-g-1) (from table 3 on the Waring distribution). Then, from the expression λ=(1-p)(Es+d)/(Es+g) (see appendix 1 on the extended Katz distribution) it follows that d=(1+g)/(1-p).

[23] Proof. Let N denote the number of firms, S the size of the industry, S_{inc} the size of the incumbent firms together. By definition we have: s_{min} dN = p dS at all times. Then, we must have in the steady state: s_{min} N = p S. Hence, average firm size S/N is equal to s_{min}/p. Furthermore, we have by definition: dS_{inc} = (1-p) dS. It follows that (1/N) dN/dt = 1/(1-p) (1/S) dS_{inc}/dt > (1/S) dS_{inc}/dt.

The model contains a number of interesting special or limiting cases, which deserve attention.[24] See also figure 1.

By setting g(*s*) = 1+*g/s* with *g* a positive parameter we get in the steady state the *Waring* distribution. See table 2c, cell (2,1). This model was briefly analysed by Ijiri and Simon (1977, p. 38) as a sensitivity check. However, they do not interpret assumptions, do not derive the probability density fully, and are not aware that the Waring distribution that they derive is called that way.

If the average growth rate of incumbents is inversely proportional to firm size we arrive at the "boundary" model of Sutton (1997). He derives that the steady state is characterized by the *geometric* distribution. See table 2c, cell (3,1). This is easily verified by setting g(*s*)=1/*s* in the generalization of the Waring distribution derived above.

If the average growth rate of incumbents is set independent of firm size – that is, if the Gibrat assumption holds - we arrive at the model of Simon (1955). He shows that it leads in the steady state to the *Yule* distribution. See table 2c, cell (4,1). This result is easily verified by setting g(s)=1.

If in the latter model the entry parameter *p* approaches zero, the model seems to approach the model of Gibrat: a fixed number of firms growing according to the Gibrat assumption. At the same time the steady state distribution approaches the Pareto distribution with parameter 1.[25] This is unexpected, because characteristic for the Gibrat model is a Pareto distribution with parameter 0 instead of 1 (cf. table 2a, 2nd row). The result can be understood if one realizes that – however small parameter *p* gets – the entry rate of new firms remains larger than the average growth rate of incumbent firms (we showed this in a previous note). Hence, the Gibrat model cannot really be reached by taking the limit of *p* to zero. Note also that when approaching the limit situation *p*=0 convergence of the model to the steady state distribution becomes increasingly troublesome (see Krugman, 1996, pp. 96-97).

Three generalizations of special cases

There are three generalizations of special cases of the above model (that are not included in the original model) worth discussing. See also figure 1.

First, in the model in which small firms grow faster than large firms (g(*s*)=1+*g/s*) we introduce *size-dependent exits*. More specific, the chance of exiting is taken proportional to 1+x/s with *x* a positive parameter. (We get back to size-independent exits or no exits by setting x=0.). In appendix 1 we show that the model leads in the steady state to the particular *generalized hypergeometric* distribution that is shown in table 2c, cell (2,3). Its parameters are dependent on the entry parameter *p*, the growth parameter *g*, and the exit parameter *x*.

Second, in the Simon model in which average growth rates are independent of size (g(s)=1) we introduce *size-dependent entries*. More specific, sizes of entering firms are geometrically distributed with parameter *n*. (We get back to entries solely at the minimum size by setting *n*=0.). The resulting model is that of Ijiri and Simon (1977, pp. 78-81) although they have a different interpretation of the model. They show that in the steady state the model leads to the *Waring* distribution (although they are unaware of

[24] Ijiri and Simon (1977, pp. 76-78) discuss a special case of this model in which the growth rate between incumbent firms is distributed according to a negative binomial distribution. They show that it leads to a Yule distribution with an incomplete beta function in the steady state.

[25] Proof. If *p* approaches 0 the parameter of the Yule distribution approaches 1. See table 2c, cell (4,1). The Yule distribution with parameter 1 is equivalent to the discrete Pareto distribution with parameter 1 as one can verify easily from table 3.

its name), where the parameters are dependent on the entry parameters *p* and *n*. See table 2c, cell (4,4). [26]

Third, in the Simon model in which average growth rates are independent of size (g(s)=1) we introduce the possibility of size-independent *decline* together with *exits* when firms decline below the minimum size. The resulting model is that of Steindl (1965, pp. 45-73). He shows that the tail of the resulting steady-state distribution follows a *Pareto* distribution of which the parameter is dependent on the net entry rate of new firms *e* and the average growth rate of incumbent firms *f*. See table 2c, cell (4,2).[27]

In the Steindl model the parameter of the Pareto distribution is above 1 as long as *e*>*f*. This is consistent with the findings in the Simon model. For, as we have shown, in the Simon model *e*>*f* holds by implication, while we find in the steady state a Yule distribution (that has the same tail behaviour as the Pareto distribution) with a parameter above 1.

Because of the more general set-up of the Steindl model as compared to the Simon model, it is possible to let the model approach the Gibrat model by letting the net entry parameter *e* approach zero. The steady state distribution approaches the Pareto distribution with parameter 0 in this case (see table 2c, cell (4,2)), as we would expect (cf. table 2a, 2nd row).

Gabaix (1999, pp. 751-752) introduces the entry of new firms into the model of Levy-Solomon (with a fixed number of firms) that we discussed earlier. In this way he arrives at a model of which the main characteristics are the same as those of the Steindl model. Not surprisingly therefore he finds that the tail of the steady-state distribution is Pareto with a parameter larger than 1 as long as the entry rate of new firms is larger than the average growth rate of incumbent firms.[28]

5. Discussion and evaluation

The overview given in the previous section is necessarily rather technical because of the complicated mathematics involved. Hence, we begin the section by summarizing in a non-technical way the general tendencies that emerge from the overview. Second, we evaluate how well the steady-state approach is able to explain the shape of firm size distributions that we encounter in practice. Third, this paper reviews quite a few functional forms that can be used to describe and/or fit empirical firm size distributions. We conclude by reviewing other functional forms that are sometimes used for this purpose.

General tendencies

Point of departure is the well-known model of Gibrat (1931): a sample of a fixed number of firms growing according to the so-called Gibrat assumption, that is: they each

[26] Steindl (1965, pp 58-61) also introduces size-dependent entries in his model (which is to be discussed next in the main text). In accordance with the findings of Ijiri and Simon he finds that the tail behaviour of the steady-state distribution is not changed by the introduction of size-dependent entries.

[27] Blank and Solomon (2000) – clearly unaware of the model of Steindl – analyse a model that is very similar to that of Steindl. Indeed, they find (not analytically but only by simulation) in the steady state a Pareto distribution.

[28] For the case that the entry rate of new firms is smaller than the growth rate of incumbent firms Gabaix (1999, pp. 751-752, 762-763) finds that the tail of the distribution is Pareto distributed with parameter 1. This *is* surprising because the Steindl model predicts a Pareto distribution with a parameter *between* 0 and 1. Noting this discrepancy we have studied the Gabaix proof in detail and indeed discovered a flaw. In particular, Gabaix overlooks in his proof the fact that for each age cohort of new firms minimum firm size (that is strictly related to the average firm size of the age cohort) increases beyond all bounds for very large values of time *t*.

face the same growth distribution independent of their size. In this model, after some time a *lognormal* firm size distribution[29] emerges. However, this distribution is only transitional. In the end it will break down and firm size becomes undetermined.
Shifting our attention temporarily to plants instead of firms, we introduce the case of impeded growth, that is: the Gibrat assumption is abandoned and replaced by average growth rates negatively correlated with plant size. Kalecki (1945) shows that under these circumstances the *lognormal* is not only transitional but remains permanent in the steady state.
Shifting our attention back to firms, we introduce a minimum firm size below which firms can't operate: if firms happen to decline below it they exit and are replaced by new firms. Then a *Pareto* distribution[30] emerges (Levy and Solomon, 1996). The larger the minimum firm size is with respect to average firm size, the thinner the tail of the distribution will be. Its parameter is near 1 as long as the minimum firm size is small (but not *too* small) with respect to average firm size.
When on top of the minimum firm size a constant stream of small new firms is introduced, the picture remains more or less the same. Dependent on the precise specifications a *Yule* distribution[31] (Simon, 1955, 1960) or a *Pareto* distribution (Steindl, 1965, Gabaix, 1999) emerges. The more small firms enter, the thinner the tail of the distribution. Its parameter is near 1 as long as the growth due to new firms is small compared to aggregate industry growth.
It is a stylised empirical fact that for a sample of firms that include small firms the average growth rate is decreasing with size, although this effect will be moderate for larger sizes. If we introduce this feature, the steady-state distribution changes to a *Waring* distribution (this paper based on Simon, 1955). It is more concave for smaller firm sizes but still has for larger sizes the same tail behaviour as the Pareto distribution.
In the limit, if we reject the Gibrat assumption altogether by taking the average growth rates of firms inversely proportional to firm size, we get the boundary model of Sutton (1997). He claims that the resulting *geometric* distribution[32] provides a good description of the *least unequal* firm size distribution that we are likely to find in practice at the 4/5-digit SIC level or higher.
Most models exhibit new entries only at the minimum size. Starting from a model exhibiting the Yule distribution in the steady state, the introduction of size-dependent entries changes it into a *Waring* distribution (this paper, based on Ijiri and Simon, 1977). Hence, size-dependent entries make the steady-state distribution more concave for smaller sizes, while the tail behaviour of the distribution is not altered. This is in accordance with the findings of Steindl (1965).
Finally, Gabaix (1999) shows that if the average growth and/or the variance in growth is larger in some size domain, in that domain the distribution will be relatively less steep downwards.

[29] The density function of the lognormal is concave parabolic on a log-log scale. See table 3.

[30] The density function of the Pareto distribution is a straight downwardly sloped line on a log-log scale. See table 3.

[31] The Yule distribution exhibits the same tail behaviour as the Pareto distribution. See table 3.

[32] The density function of the geometric distribution is exponentially decreasing on a log-log scale. See table 3.

Explaining the shape of firm size distributions in practice

In order to evaluate how well the steady-state approach is able to explain the shape of the firm size distributions that we encounter in practice, we give a – very brief – overview of the empirical literature. In particular, we describe how the probability density looks like on a log-log scale.

1 Especially for large number of firms, the density is - for a very long size range – well described by a straight line with a downward slope of approximately -2. That is: a *Pareto distribution* with a parameter near 1 is found. Only for very small and very large sizes there is a noteworthy deviation from this line. See e.g. Axtell (2001) and references therein.
2 Often - on closer inspection - the above-mentioned straight line is in fact somewhat concave. See e.g. Ijiri and Simon (1977).
3 It is also reported (see e.g., Gibrat (1931), Hart and Prais (1956), Hall (1987, p. 584) and Stanley et al. (1995)) that the empirical density can be described quite well by a *lognormal* density function.[33] That is: a (concave) parabolic density function is found. Note that this is in line with the previous remark.
4 For smaller industries it can appear that a Pareto distribution is found with a parameter substantially different from 1, or even that neither the Pareto nor the lognormal describes the empirical density function satisfactorily. See, e.g., Quandt (1966) and Silberman (1967). Hence, Sutton (1997, p. 52) concludes that probably there is no general density function that describes all empirical densities well.

From the general tendencies summarized at the beginning of this section it follows that these empirical regularities might be explained as follows.

First, the basic shape of firm size distributions in practice – the Pareto distribution with a parameter near 1 – can be explained by adopting the Gibrat assumption together with the introduction of a minimum firm size below which firms can't operate. This result is valid whether or not a stream of small new firms is present.

Second, various alternative explanations are offered to explain the often observed concavity of firm size distributions in practice. For example, this might be due to (i) small firms growing faster than large firms on average, (ii) small firms having a higher variance of growth rates, or (iii) firms entering with different sizes instead of only at the minimum size.

Third, the appearance of a lognormal firm size distribution can be due to impeded growth (average growth rate negatively correlated with firm size). Alternatively, it may be just the transitional state of the Gibrat model: a sample of firms all facing the same growth distribution independent of their size.

Fourth, the appearance of firm size distributions with a downward slope substantially steeper than a Pareto distribution with a parameter near 1 might be due to (i) a substantial entry rate of small new firms, and/or (ii) a relatively high minimum firm size. Alternatively, the appearance of firm size distributions with a downward slope less steep than a Pareto distribution with a parameter near 1 might be due to (i) small firms having on average a higher growth rate or a higher variance in their growth, and/or (ii) a particular small minimum firm size with respect to average firm size.

[33] Sornette and Cont (1997, p. 432) note that the lognormal distribution can be mistaken for an apparent Pareto distribution with a parameter that is slowly varying with the range on which firm sizes are measured. This can be understood quite easily by realizing that if the parabolic shape of the lognormal is stretched it can be approximated by successive straight lines. This may complicate matters when trying to decide empirically whether the Pareto or the lognormal distribution describes the empirical density function best.

From the above we conclude that the steady-state approach offers many plausible (sometimes alternative) explanations of the various shapes of firm size distributions that we encounter in practice. However, the following caveats are in order. First, the steady-state approach cannot give all answers with respect to explaining the shape of firm size distributions encountered in practice, because of the obvious fact that not all industries are in a steady state. Second, as we suggested at the end of section 3, due to the rather basic and elementary assumptions adopted, the approach is more appropriate for broader industries, e.g. 4/5-digit SIC level or higher. For smaller industries other approaches seem more appropriate to explain the shape of the firm size distribution.

Firm size distributions without firm dynamical basis

This paper reviews quite a few functional forms that can be used to describe and/or fit empirical firm size distributions. In practice, still other functional forms are sometimes used. Noteworthy are the generalized beta distributions of the first and second kind and the many special and limiting cases of these distributions, for example: the beta distributions of the first/second kind, the (generalized) gamma distribution, the Singh-Maddala distribution, the lognormal distribution, the Weibull distribution, the Fisk distribution, and the exponential distribution. See McDonald (1984).

However - as far as we know - no firm dynamics models have been developed yet that lead to these distributions – with the exception of course of the lognormal distribution. Hence, at present, the parameters of these distributions cannot be related explicitly to underlying firm dynamics. This is in contrast with the parameters of the firm size distributions reviewed in this paper, which can be directly related to underlying firm dynamics: see section 4.[34]

6. Conclusion

This paper provides an up-to-date overview of analytic steady-state firm size distributions that can be derived from assumptions concerning underlying firm dynamics. By bringing together results from many different sources we were able to sort out common results and explain seemingly contradictory results. Furthermore, we gave an impression of how well at present the steady-state approach is able to explain the shape of the firm size distribution that we encounter in practice.

The overview can be used in many different ways. To name a few, it could facilitate researchers in deciding which steady-state distribution to use dependent on the situation with respect to entry, exit, growth, and decline of firms in an industry. Second, it could provide information about what changes in the firm size distribution one might expect when there are changes in the underlying firm dynamics. Third, it gives insight into what kind of firm dynamics may be underlying specific firm size distributions. Hence, it provides possible interpretations of the parameters of the size distribution in terms of firm dynamics.

Appendix 1. Derivations

Derivation of generalization of Waring distribution

Assume (i) a constant stream of new firms of size 1 (the fraction of growth due to new firms is labelled p), (ii) exits independent of firm size, (iii) the average growth rate of in-

[34] Because the same steady-state firm size distribution can be derived from different model assumptions, there are alternative firm dynamical interpretations of the same distribution parameter, though. See section 4.

cumbent firms of size *s* proportional to an unspecified function g(s), and (iv) no decline of firms. Assume also - without loss of generality - a stationary industry. Hence, the steady-state condition of Simon (1955, alternative), see section 2 on theory, is applicable:

"inflow because of growth from s-1 to s" = "outflow because of growth from s to s+1"

+ "outflow because of exits of size s"

This condition can be written as:

$$(1-p)\frac{(s-1)g(s-1)}{Esg(s)}f(s-1)=(1-p)\frac{sg(s)}{Esg(s)}f(s)+pf(s)$$

where: f(s) denotes the density function of firm sizes and E() the expectation. The reasoning behind this is as follows.

- The inflow term on the left should be proportional to (s-1) g(s-1) by assumption and to the number of firms of size (s-1) and hence to f(s-1). The factor (1-p) follows from the fact that the growth from all incumbent firms together should add up to 1-p.
- The expression for first term on the right is explained in the same way.
- The exit term on the right should be proportional to the number of firms of size *s* and hence to f(s). The factor *p* follows from the fact that the exits of all incumbent firms together should add up to the fraction of growth due to new firms, *p*.

Rearranging the steady state condition gives the following relationship between f(s) and f(s-1):

$$f(s)/f(s-1)=\frac{(s-1)g(s-1)}{sg(s)+\rho}$$

where the auxiliary parameter ρ is defined as:

$$\rho\equiv\frac{p}{1-p}Esg(s)$$

It follows that the steady state density can be expressed as:

$$f(s)=A\;\frac{\prod_{t=1}^{s-1}tg(t)}{\prod_{t=1}^{s}(tg(t)+\rho)}$$

where *A* is a normalizing constant. It is calculated most easily by using the steady-state condition for firms of size 1:

$$p=(1-p)\frac{g(1)}{Esg(s)}f(1)+p\,f(1)$$

where we have used that the term on the left "inflow because of growth from size 0 to size 1" should be equal to the fraction of growth due to new firms: *p*. Substituting f(1) and using the definition of ρ gives: A=ρ.

Derivation of particular hypergeometric distribution

With respect to the previous model two assumptions are changed: (i) exits are proportional to 1+x/s and (ii) the average growth rate of firms of size *s* is proportional to 1+g/s with *g* a nonnegative parameter. Then the steady state condition changes into:

$$(1-p)\frac{s-1+g}{Es+g}f(s-1)=(1-p)\frac{s+g}{Es+g}f(s)+p\frac{1+x/s}{1+xE(1/s)}f(s)$$

Note that the last term on the right has now an extra factor because exits are assumed to be proportional to 1+x/s. Rearranging the steady state condition gives the following relationship between f(s) and f(s-1):

$$f(s)/f(s-1)=\frac{s-1+g}{s+g+\rho(1+x/s)}$$

where the auxiliary parameter ρ is defined as:

$$\rho=\frac{p}{1-p}\frac{(g+E(s))}{(1+xE(\frac{1}{s}))}$$

It follows that the steady state density can be expressed as:

$$f(s)=A(\rho,g,x)\frac{(1+g)_{s-1}}{\prod_{t=1}^{s}\left[t+g+\rho\,(1+x/t)\right]}$$

where *A* is a normalizing constant of which the value can be found as a function of the parameters ρ, g, and x by normalizing the density function to unity.

Derivation of extended Katz distribution

With respect to the previous model two assumptions are changed: (i) the average decline rate of firms of size *s* is proportional to 1+d/s (with *d* a nonnegative parameter) and (ii) exits are equivalent to declining firms of the minimum size 1. Because in this model decline is included, while exits occur only at the minimum size the steady state condition changes into (see section 2 on theory, Simon (1960)):

"inflow because of growth from s-1 to s" = "outflow because of decline from s to s-1"

This condition boils down to:

$$(1-p)\frac{s-1+g}{Es+g}f(s-1)=\frac{s+d}{Es+d}f(s)$$

The term on the left is already familiar from the previous model. The reasoning behind the term on the right is as follows. It should be proportional to *s*+*d* by assumption and to the number of firms of size *s* and hence to f(s). The term on the right is normalized in such a way that the decline of all incumbent firms together is equivalent to the growth of all incumbent firms plus the growth due to new firms.
Rearranging the steady state condition gives the following relationship between f(s) and f(s-1):

$$f(s)/f(s-1)=\lambda\,\frac{s-1+g}{s+d}$$

where the auxiliary parameter λ is defined as:

$$\lambda \equiv (1-p)\frac{Es+d}{Es+g}$$

It follows that the steady state density can be expressed as:

$$f(s) = A(\lambda,\gamma,\delta)\frac{(1+g)_{s-1}}{(1+d)_s}\lambda^s$$

where *A* is a normalizing constant, of which the value can be found as a function of the parameters λ, g, and d by normalizing the density function to unity.
The steady-state condition for firms of size 1 boils down to:

$$p = \frac{1+d}{Es+d}f(1)$$

where we have used that the term on the left "inflow because of growth from size 0 to size 1" should be equal to the fraction of growth due to new firms: *p*. We can use this expression to get an expression for the auxiliary parameter λ in terms of the original parameters p, d, and g, and the normalization constant A:

$$\lambda = 1 - p\left(1+\frac{g-d}{A}\right)$$

Appendix 2. Explanation of symbols

Symbols denoting parameters of firm dynamics models (Roman characters):

c	ratio between minimum and average firm size
d	decline parameter
e	entry rate of new firms
f	growth rate of incumbent firms
g	growth parameter
g(*s*)	growth function, also function in generalization of Waring distribution
n	distribution parameter of new firms
N	number of firms
p	fraction of growth due to new firms
x	exit parameter

Symbols denoting parameters of derived distributions (Greek characters):

γ	parameter of (generalization of) Waring, particular generalized hypergeometric, generalization of logarithmic, negative binomial, extended Katz
δ	parameter of extended Katz
ζ	parameter of lognormal
λ	parameter of geometric, Poisson, (generalization of) logarithmic, negative binomial, extended Katz
ρ	parameter of Pareto, Yule, (generalization of) Waring, particular generalized hypergeometric
σ	parameter of lognormal
χ	parameter of particular generalized hypergeometric

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Table 1. Characteristics of typical models

paper	*stability device*	*steady-state concept*	*approach*
Gibrat (1931)	no	no steady state	continuous
Kalecki (1945)	impeded growth	only *shape* of distribution is stable in steady state (mean of distribution may change)	continuous
Simon (1955)	small new firms	distribution is stable in steady state; number of firms grows	discrete
Simon (1955, alternative)	small new firms	distribution is stable in steady state; fixed number of firms	discrete
Simon (1960)	small new firms	distribution is stable in steady state; fixed number of firms	discrete
Steindl (1965)	small new firms	distribution is stable in steady state; number of firms grows	discrete
Levy and Solomon (1996)	minimum firm size	only *shape* of distribution is stable in steady state (mean of distribution may change)	continuous

Table 2a. Steady-state distributions for a fixed number of firms.

Growth type		*Distribution*
distribution of growth rate independent of size; however, firms do not decline below some minimum firm size[a]: $s_{min} = c\ s_{av}$	$c \in (0,1)$	continuous Pareto[b] (ρ); $\rho > 0$
	$c = 0$ (the Gibrat assumption)	no real steady state: lognormal (ζ, σ); $\zeta, \sigma \to \infty$ = continuous Pareto (ρ); $\rho \to 0$
	c small[c]	continuous Pareto (ρ); $\rho \approx 1$
	c small[c]; average growth and/or variance in growth is larger in some size domain	in size domain: continuous Pareto (ρ); $\rho < 1$
average growth rate ~ s^{-g}		lognormal (ζ, σ)

a. s_{av} denotes average firm size.

b. Parameter ρ implicitly defined by: $N = \frac{\rho - 1}{\rho}\left[\frac{(c/N)^{\rho} - 1}{(c/N)^{\rho} - (c/N)}\right]$, where *N* denotes the number of firms.

c. From the expression in note *b* it follows that in order to get ρ approximately equal to 1 – say in the interval (0.8, 1.2) - parameter *c* should be in the interval (0.04, 0.20) for *N*=1000 or $c \in (0.01, 0.18)$ for *N*=100.000. Hence, *c* should be small but not *too* small (especially for smaller values of *N*).

Table 2b. Steady-state distributions for a fixed number of firms; entries/exits at minimum size.[a]

Growth type		*Distribution*
decline rate ~ 1+d/s with $d \in (0, \frac{1+g}{1-p})$	growth rate ~ 1+g/s with g>0	extended Katz[b] (λ, γ, δ) $\lambda = 1 - p\left(1 + \frac{g-d}{A}\right) \in (0,1)$ $\gamma = g > 0$ $\delta = d \in (0, \frac{1+g}{1-p})$
decline rate independent of size	growth rate ~ 1+g/s with g>0	negative binomial (λ, γ) $\lambda = 1 - p^{\frac{1}{1+g}} \in (0,1)$ $\gamma = g > 0$
	growth rate ~ 1+1/s	geometric (λ) $\lambda = 1 - \sqrt{p} \in (0,1)$
	growth rate ~ 1/s	Poisson (λ) $\lambda = -\ln(p) > 0$
	growth rate independent of size (the Gibrat assumption)	logarithmic (λ) $\lambda = 1 - p \in (0,1)$
decline rate ~ 1+g/s with g>0	growth rate ~ 1+g/s with g>0	generalization of logarithmic (λ, γ) $\lambda = 1 - p \in (0,1)$ $\gamma = g > 0$
decline rate ~ $1 + \frac{(1+g)}{(1-p)} \frac{1}{s}$	growth rate ~ 1+g/s with g>0	Waring (ρ, γ) $\rho = \frac{1+gp}{1-p} > 1$ $\gamma = g > 0$
decline rate ~ $1 + \frac{1}{(1-p)s}$	growth rate independent of size	Yule (ρ) $\rho = \frac{1}{1-p} > 1$

a. The fraction of growth due to new firms is labelled $p \in (0,1)$.

b. The expression for λ is only implicit. For, the normalization constant A of the extended Katz distribution is dependent on its parameters: λ, γ, δ. Because A is not known in analytic form, the shown expression is the best we can offer.

Table 2c. Steady-state distributions for a growing number of firms.[a] [b]

horizontal axis: different entry/exit types:	entry at minimum size			entry size geometrically distributed with parameter n $\in$(0,1); no exits or size-independent exits
vertical axis: different growth types:	no exits or size-independent exits	exits at minimum size: net entry rate *e*	exits proportional to 1+x/s with x>0	
growth rate ~ g(s)	generalization of Waring[c] (ρ, g(s)) $\rho = \frac{p}{1-p} E(sg(s))$ g(s) = g(s)			
growth rate ~ 1+g/s with g>0	Waring (ρ, γ) $\rho = \frac{1+gp}{1-p} > 1$ γ = g >0		particular generalized hypergeometric[d] (ρ, γ, χ) $\rho = \frac{p}{1-p} \frac{(g+E(s))}{(1+xE(\frac{1}{s}))}$ γ = g >0 χ = x >0	
growth rate ~ 1/s	geometric (λ) λ = 1-p $\in$(0,1)			
growth rate independent of size (the Gibrat assumption)	Yule (ρ) $\rho = \frac{1}{1-p} > 1$	tail: Pareto[e] (ρ) ρ = *e*/*f* > 0		Waring (ρ, γ) $\rho = \frac{1}{1-p} > 1$ $\gamma = \frac{1}{(1-p)} \frac{n}{(1-n)} > 0$

a. The cells give the steady-state distribution resulting from the combined assumptions on the horizontal and vertical axes. A blank cell means that the steady-state distribution has not been calculated in the literature.

b. The fraction of growth due to new firms is labelled p $\in$(0,1).

c. The expression for ρ is only implicit. For, E(sg(s)) – the expectation of sg(s) - will be dependent on the characteristics of the generalization of the Waring distribution: parameter ρ and function g(s). Because E(sg(s)) can only be calculated once the function g(s) has been specified, the shown expression is the best we can offer.

d. See - mutatis mutandis - note c.

e. Parameter *f* (>0) denotes the average growth rate of incumbent firms. If *f*<0 we get: ρ =1–*e*/*f* > 1. If *f*=0 no Pareto distribution occurs.

Table 3. Characteristics of distributions

name and parameters[a]	probability density function f(s)[b, c]	mean[d, e]	tail behaviour on log-log scale of pdf	additional information[f]
lognormal (ζ, σ), continuous	$\frac{1}{(s-1)\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2}\frac{[\ln(s-1)-\zeta]^2}{\sigma^2}\right)$	$1+\exp(\zeta+\frac{1}{2}\sigma^2)$	parabolic downwards	Jo94 (207-209, 211-212)
continuous Pareto (ρ)	$\rho\, s^{-(1+\rho)}$	$\frac{\rho}{\rho-1}$	straight line with slope –(1+ρ)	Jo94 (574)
discrete Pareto[g] (ρ)	$s^{-\rho}-(s+1)^{-\rho}$	ζ(ρ)	straight line with slope –(1+ρ)	IS77 (75-76)
generalization of Waring (ρ, g(s))	$\rho\,\frac{\prod_{t=1}^{s-1} tg(t)}{\prod_{t=1}^{s}(tg(t)+\rho)}$	-[h]	dependent on specification of g(s)	
Waring (ρ, γ)	$\rho\frac{(1+\gamma)_{s-1}}{(1+\gamma+\rho)_s}$	$\frac{\rho+\gamma}{\rho-1}$	straight line with slope –(1+ρ)	Jo92 (278-279)
Yule (ρ)	$\rho\frac{(s-1)!}{(1+\rho)_s}$	$\frac{\rho}{\rho-1}$	straight line with slope –(1+ρ)	Jo92 (275)
particular generalized hypergeometric (ρ, γ, χ)	$A(\rho,\gamma,\chi)\frac{(1+\gamma)_{s-1}}{\prod_{t=1}^{s}[t+\gamma+\rho\,(1+\chi/t)]}$		straight line with slope –(1+ρ)	Jo92 (84-91)
extended Katz (λ, γ, δ)	$A(\lambda,\gamma,\delta)\frac{(1+\gamma)_{s-1}}{(1+\delta)_s}\lambda^s$		dependent on value of parameters	Jo92 (79-80)
negative binomial (λ, γ)	$\frac{1}{(1-\lambda)^{-\gamma}-1}\frac{(\gamma)_s}{s!}\lambda^s$	$\frac{\gamma\lambda}{(1-\lambda)(1-[1-\lambda]^\gamma)}$	exponentially decreasing	Jo92 (225)
generalization of logarithmic (λ, γ)	$A(\lambda,\gamma)\frac{\lambda^s}{s+\gamma}$		exponentially decreasing	
logarithmic (λ)	$\frac{1}{-\ln(1-\lambda)}\frac{\lambda^s}{s}$	$\frac{1}{-\ln(1-\lambda)}\frac{\lambda}{1-\lambda}$	exponentially decreasing	Jo92 (285)
geometric (λ)	$(1-\lambda)\,\lambda^{s-1}$	$\frac{1}{1-\lambda}$	exponentially decreasing	Jo92 (201)
Poisson (λ)	$\frac{1}{(\exp(\lambda)-1)}\frac{\lambda^s}{s!}$	$\frac{\lambda}{1-\exp(-\lambda)}$	decreases faster than exponentially	Jo92 (181-182)

a. All distributions are discrete (s=1,2,3 …) except for the first two distributions, which are continuous.

b. To keep expressions simple and comparable all density functions are defined (and normalized) on the interval (1,∞). This means that in some cases density functions are truncated with respect to the familiar form in the literature. Generalization to intervals with an arbitrary minimum value is straightforward.

c. $(x)_s$ denotes an ascending factorial. It is defined by: $(x)_s$ = x (x+1) (x+2) … (x+s-1) for s= 1, 2, … and $(x)_0$=1. Hence, $(1)_s$ = s!.

d. Means are calculated with the presented density functions, defined on the interval (1,∞). If the cell is blank the mean is (to the best of our knowledge) not known.

e. ζ() denotes the Riemann zeta function. It is defined by: $\zeta(\rho)=\sum_{s=1}^{\infty}s^{-\rho}$. See Johnson et al. (1992, pp. 29-30).

f. Jo92 refers to Johnson et al. (1992); Jo94 refers to Johnson et al. (1994); IS77 refers to Ijiri and Simon (1977). To make distributions in this paper comparable to each other, parametrization in this paper may differ somewhat from parametrization in these standard works. If the cell is blank we have not found additional information about the distribution in the literature.

g. There is another variant of a discrete Pareto distribution with slightly different properties. See Johnson et al. (1992, pp. 465-471).

h. Although we do not know the mean of the distribution in terms of ρ and g(s), we do know it in terms of the underlying firm dynamics parameters. For, as was shown in the main text, average firm size is equal to s_{min}/p, so that (when setting s_{min}=1) the mean of the distribution must be equal to 1/p.

Figure 1. Tree of discrete distributions