



Via Po, 53 – 10124 Torino (Italy)
Tel. (+39) 011 6704917 - Fax (+39) 011 6703895
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Interactive and common knowledge in the state-space model

Ivan Moscati

Dipartimento di Economia "S. Cagnetti de Martiis"

Centro di Studi sulla Storia e i Metodi dell'Economia Politica
"Claudio Napoleoni"
(CESMEP)

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Università di Torino

IVAN MOSCATI *

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ABSTRACT: This paper deals with the prevailing formal model for knowledge in contemporary economics, namely the state-space model introduced by Robert Aumann in 1976. In particular, the paper addresses the following question arising in this formalism: in order to state that an event is interactively or commonly known among a group of agents, do we need to assume that each of them knows how the information is imparted to the others? Aumann answered in the negative, but his arguments apply only to canonical, i.e., completely specified state spaces, while in most applications the state space is not canonical. This paper addresses the same question along original lines, demonstrating that the answer is negative for both canonical and not-canonical state spaces. Further, it shows that this result ensues from two counterintuitive properties held by knowledge in the state-space model, namely Substitutivity and Monotonicity.

KEYWORDS: Knowledge; Interactive knowledge; Common knowledge; State-space model; Substitutivity; Monotonicity; Robert Aumann

JEL CLASSIFICATION: B40; C70; D80; D82; D83

* Bocconi University, Department of Economics. Via Roentgen 1, 20136 Milano, Italy. E-mail: ivan.moscatti@unibocconi.it. I am grateful to Robert Aumann, Pierpaolo Battigalli, Giacomo Bonanno, Paolo Colla, Marco Dardi, Alfredo Di Tillio, Vittorioemanuele Ferrante, Francesco Guala for useful discussions and suggestions. I thank the participants in a Choice Group Seminar at the London School of Economics, and in a seminar at Queen Mary, University of London, for helpful comments on an earlier draft. I also thank Bocconi for financial support and the Centre for Philosophy of Natural and Social Science at LSE for its hospitality during part of the work on this paper. Any errors are mine.

1 INTRODUCTION

Models of economic theory are peopled by agents who take actions on the basis of their knowledge and beliefs about the world, and about each other's knowledge and beliefs. The prevailing formal model for knowledge in contemporary mainstream economics was introduced by Robert Aumann in a seminal paper published in 1976.¹ Aumann's basic idea is that an agent knows an event if, in every state of the world the agent considers possible, that event occurs. This idea is formalized in a set-theoretic setting where the knowledge of an agent becomes an operator K mapping subsets of the space of the states of the world Ω into other subsets of Ω . Aumann's model of knowledge and the generalizations of his model have been variously labeled as the event-based approach, possibility correspondence model, semantic formalism, knowledge space, Aumann structures, and state-space model of knowledge. The latter name will be adopted here.

The state-space model makes it possible not only to represent what each agent knows about the world, but also what each agent knows about what other agents know about the world. This kind of knowledge – knowledge of what others know – is called *interactive knowledge*. In strategic environments interactive knowledge has important consequences on the actions agents take. Consider Ann and Bob, who both observe a certain event E . For instance, E may be Ann's effort in a principal-agent game or Ann's planned output in a duopoly game. If Ann is uncertain whether Bob knows E she may choose a low effort ("Maybe Bob won't find out") or a low output ("I don't know what Bob's costs are or what he knows about my costs, so it's better to keep my output low"). In contrast, if Ann knows that Bob knows E , she will probably choose a high effort ("I'd better work hard, or he'll fire me") or a high output ("Well, I'm a Stakelberg leader, and Bob will adapt").

Given its strategic importance, it is fundamental to understand clearly how interactive knowledge works in the state-space model. At an intuitive level, it seems that interactive knowledge of an event requires the additional assumption that agents know how information is imparted to the others: Ann needs to know how information is imparted to Bob in order to know that he knows E , otherwise Ann would have no clue about what Bob knows. In fact, it turns out that in the state-space formalism interactive knowledge requires no additional assumption about the knowledge of other agents' informational structure. The first contribution of the present paper is to clarify what are the formal features of the knowledge operator K provoking this counterintuitive behavior of interactive knowledge in the state-space model.

In effect there are multiple levels of interactive knowledge. Level 1 is the one discussed above: it is about what each agent knows about what other agents know about the world. Level

¹ In philosophy the formal analysis of knowledge dates back to Hintikka (1962).

2 is about what each agent knows about what other agents know about her/his knowledge of the world. The staircase of levels of interactive knowledge escalates in the predictable way. A specific kind of interactive knowledge is *common knowledge*. An event is said to be common knowledge among a group of agents if all know it, all know that all know it, and so on *ad infinitum*.²

Besides interactive knowledge of level 1, interactive knowledge of higher levels and common knowledge are also of great consequence in strategic environments. For instance, consider interactive knowledge of level 2: if Bob knows that Ann knows that he is able to observe her effort, Bob may think that Ann's commitment to the firm is not sincere, and decide to fire her even if she works hard. As regards common knowledge, some elements of the game are typically assumed to be commonly known among the players, and this assumption has a key role in equilibrium analysis. More precisely, in games of complete information, the set of players, the set of strategies, and the payoff functions are assumed to be common knowledge among the players. In games of incomplete information, players usually have prior probability distributions about the unknown variables, and such distributions are typically taken to be common knowledge. Furthermore, some important game-theoretic solution concepts require that each player is rational, and that the rationality of the players is common knowledge among them.³ Finally, common knowledge of posterior probabilities is essential for so-called "agreeing to disagree" results, and common knowledge of willingness to trade for no-trade theorems.⁴

When higher levels of interactive knowledge or common knowledge are involved, the question about the knowledge of other agents' informational structure comes out again, at higher levels. Level 2 of interactive knowledge of an event raises a question about level 1 of interactive knowledge: to say that Bob knows that Ann knows that he knows her effort, does Bob need to know that Ann knows how the information is imparted to him? More generally, if we consider level n of interactive knowledge, level $(n-1)$ of interactive knowledge of the agents' informational structure seems to be involved, so that when common knowledge is at issue the question becomes: to state that a certain event E is common knowledge among a group of agents, do we need to assume that the way information is imparted to them is itself common knowledge? Again, even if the intuitive answer is in the affirmative, it turns out that in the state-space model common knowledge of an event requires no additional assumption about the

² To circumvent the infinitely recursive nature of this definition of common knowledge, a number of alternative characterizations of it have been proposed. On them, see Geanakoplos (1992, 1994) as well as Vanderschraaf and Sillari (2005). However, these alternative characterizations play no role in the current contribution.

³ More on this in Brandenburger (1992, 2007); Dekel and Gul (1997); Battigalli and Bonanno (1999).

⁴ The seminal paper for "agreeing to disagree" results is, again, Aumann (1976); for no-trade theorems it is Milgrom and Stokey (1982).

agents' knowledge of the way information is imparted to them. The second contribution of the paper is to show that the counterintuitive behavior of common knowledge in the state-space model originates from the same formal features of the knowledge operator K that provoke the counterintuitive behavior of interactive knowledge.

The puzzles surrounding interactive and common knowledge in the state-space model have already been discussed by Aumann, but in a way that does not appear completely satisfactory.⁵ The main problem with Aumann's arguments is that they affect only so-called *canonical state spaces*, that is, state spaces that are completely specified. However most applications employ a reduced state-space that is not canonical, so Aumann's arguments do not apply. Whereas Aumann's case is based on the notion of state of the world, the present paper addresses the topic along different lines. Its basic insight is a methodological distinction between the intuitive and philosophical understanding of knowledge on the one hand, and knowledge as modeled in the state-space model through the operator K on the other. In effect, K possesses a number of properties that are at odds with both commonsense and the philosophical analysis of knowledge, and the counterintuitive behavior of interactive and common knowledge in the state-space model can be explained by two of these properties, namely Substitutivity and Monotonicity. Substitutivity says that, if two events E and F collect the same states of the world, when the agent knows E she also knows F . Although Substitutivity has attracted little attention among economists, it turns out to be not only a demanding property of K but also one that is intrinsic to any set-theoretic knowledge operator, so that it appears difficult to get rid of. Monotonicity states that, if event E is a subset of event F , when the agent knows E she also knows F . Monotonicity is stronger than Substitutivity (the former implies the latter), and its unrealistic character has been thoroughly examined in the literature. Unlike Substitutivity, however, Monotonicity can be easily eliminated through minor modifications in the definition of the operator K .

The paper shows that, when interactive and common knowledge are at issue, in some cases Substitutivity alone suffices to make superfluous any additional assumption about the agents' knowledge of the way information is imparted to other agents. Moreover, whenever Substitutivity alone does not suffice, Monotonicity does. These results hold for both canonical and non-canonical state spaces, so that the present contribution may be seen as a completion of Aumann's analysis.

⁵ See in particular Aumann (1976, p. 1237; 1987, p. 9; 1999, pp. 272–3, 276–8; 2005, pp. 92–4). Other contributions to the discussion are Gilboa (1988); Brandenburger and Dekel (1993); Margalit and Yaari (1996); Hart, Heifetz and Samet (1996); Dekel and Gul (1997); Heifetz and Samet (1998); Heifetz (1999); Fagin, Geanakoplos, Halpern and Vardi (1999); Aumann and Heifetz (2002, Appendix); Cubitt and Sugden (2003, Appendix 2); Sent (2006).

Some final specifications on scope and intended audience of the current contribution are in order. First, in the philosophical discussion, one of the characteristics that distinguishes knowledge from belief is that knowledge is assumed to be truthful while belief can be false. In fact, knowledge is traditionally defined by philosophers as “justified true belief”.⁶ In the state-space model, on the contrary, nothing prevents knowledge from being false, so that in fact the present paper covers not only knowledge but also belief.

Second, an important subset of state-space models is *partitional* models. Since these have a number of nice properties (among other things, in them knowledge is always truthful), much of the literature focuses on them. Since neither Substitutivity nor Monotonicity depends on the conditions that make the state space partitional, the arguments made in the present paper hold for both partitional and non-partitional state-space models.

Third, in the state-space model of knowledge, agents consider possible certain states of the world in Ω , and impossible the other states, but they are not endowed with probability distributions that represent their beliefs about Ω . If we first add to the model a probability distribution for each agent, then introduce a belief operator B that identifies the probability assigned by an agent to any given event, and finally redefine knowledge as “belief with probability 1”, we obtain a different model that is variously labeled as probabilistic belief space, probabilistic structure or Harsanyi type space. There are a number of analogies between the state-space model and the probabilistic belief space, and in particular the issue about interactive and common knowledge arising in the former has an analog in the latter. However, the answers to the issue diverge in the two formalisms. This is mainly due to the circumstance that in probabilistic belief spaces the probability measures defining the belief operator B endow it with certain continuity properties that the knowledge operator K fails to have. Now, the present paper deals only with interactive and common knowledge in the state-space model, and does not examine the analogous issue in probabilistic belief spaces.⁷

Fourth, the state-space model of knowledge employs set-theoretic tools that are familiar to economists. There is another model of knowledge, mainly elaborated by logicians and philosophers, that employs the language and tools of logics and has been variously called the logic-based approach, the syntactic formalism, Kripke structure or simply epistemic logic.⁸ The parallels between the state-space model and the logic-based approach have been explored by

⁶ For an introduction to the definition of knowledge as “justified true belief”, and the refinements of this definition as a consequence of the so-called Gettier problem, see Steup (2006).

⁷ On probabilistic belief spaces and their relationships to the state-space model, see Mertens and Zamir (1985); Monderer and Samet (1989); Brandenburger and Dekel (1993); Heifetz and Samet (1998, 1999a, 1999b); Battigalli and Bonanno (1999); Fagin, Geanakoplos, Halpern and Vardi (1999); Meier (2005); Mariotti, Meier and Piccione (2005).

⁸ For a comprehensive presentation of the logic-based approach see Fagin, Halpern, Moses and Vardi (1995).

Michael Bacharach (1985) and Aumann himself (1989, 1999), among others. The logic-based formalism proved useful for understanding the properties of the knowledge operator K and has other nice features, but its language remains unfamiliar to many economists. Therefore, the focus of the present paper is on the state-space model, and the questions about interactive and common knowledge are tackled and answered within this model.

Finally, the paper is addressed to all scholars dealing with formal models of knowledge and interested in the notion of common knowledge. In particular, economists may be glad to be reassured that no additional assumption is surreptitiously introduced into their models when an event is said to be interactively or commonly known among a group of agents. However, they may be surprised that this depends on reasons other than those put forward by the standard view moulded by Aumann's arguments. Furthermore, they may be concerned that assumptions about interactive and common knowledge are dispensable thanks to properties of the operator K that neither commonsense nor philosophy judges plausible. As regards philosophers, they may consider the state-space model of knowledge and its internal riddles as "an economist thing". However, philosophers in the analytic tradition are familiar with formal models of knowledge, and common knowledge has become a major topic of research for them.⁹ Therefore, the internal puzzles of the state-space model (especially those involving common knowledge) and the solution to those puzzles suggested here may be of interest for philosophers too. The paper is organized as follows. Section 2 reviews the state-space model of knowledge. Section 3 illustrates through an example the puzzles surrounding interactive knowledge in the state-space model. Section 4 discusses Aumann's solution to the puzzle. Section 5 examines Substitutivity and Monotonicity. Section 6 shows that Substitutivity and Monotonicity are sufficient to clarify the counterintuitive behavior of interactive knowledge in the state-space model. Section 7 does the same for common knowledge. Section 8 sums up the paper.

2 THE STATE-SPACE MODEL OF KNOWLEDGE

Consider a set Ω whose generic element is ω , and a correspondence $P : \Omega \rightarrow 2^\Omega \setminus \{\emptyset\}$ that associates to each element $\omega \in \Omega$ a set $P(\omega)$ of elements of Ω (2^Ω is the set of all subsets of Ω). Based on P , define an operator $K : 2^\Omega \rightarrow 2^\Omega$ as follows: for every $E \subseteq \Omega$, $K(E) = \{\omega \in \Omega : P(\omega) \subseteq E\}$.¹⁰

The interpretation of the above set-theoretic structure is the following. Ω is the set of the

⁹ For an introduction to the philosophical research on common knowledge see Vanderschraaf and Sillari (2005) and the references cited there.

¹⁰ This review of the state-space model of knowledge is based on Osborne and Rubinstein (1994, Chapter 5); Dekel and Gul (1997); Battigalli and Bonanno (1999); Samuleson (2004).

possible states of the world. A state $\omega \in \Omega$ specifies all *epistemic* and *non-epistemic* aspects of the world that are relevant to the situation. The non-epistemic aspects of the world are those that do not involve the agents' knowledge, that is, aspects such as "it rains" or "agent i has transitive preferences". In the literature, to indicate the whole of the non-epistemic aspects of the world the term *nature* is often used. The epistemic aspects of the world are those concerning the agents' knowledge about nature and about each other's knowledge, e.g., aspects such as "agent i knows that it rains" or "agent j knows that agent i knows that it rains".

Only one state of the world is the true one, but the agent may be uncertain about which one. This uncertainty is modeled by a correspondence P , which associates to each state ω the set of states that the agent regards as *possible* at ω . This is why P is called a *possibility correspondence*.¹¹ The possibility correspondence of an agent expresses formally the way information is imparted to her. Notice however that possibility correspondences are just a tool that the external, omniscient model-maker employs to encode and represent the agents' epistemic states, not something that they are aware of.

A subset $E \subseteq \Omega$ is called an *event*, and can be thought of as the collection of all states that share a certain feature. For instance, the event "it rains" collects all states $\omega \in \Omega$ characterized by rain. Note that, if $P(\omega) \subseteq E$, in all states the agent regards as possible in ω , the event E occurs. The operator K is interpreted as a knowledge operator: if $\omega \in K(E)$, then at ω the agent knows that the event E occurs, and this is because in every state the agent regards as possible in ω – that is, in $P(\omega)$ – the event E occurs. Observe that $K(E)$ is itself an event, the event "the agent knows E ". As such, $K(E)$ may become the object of further knowledge or uncertainty for another agent.

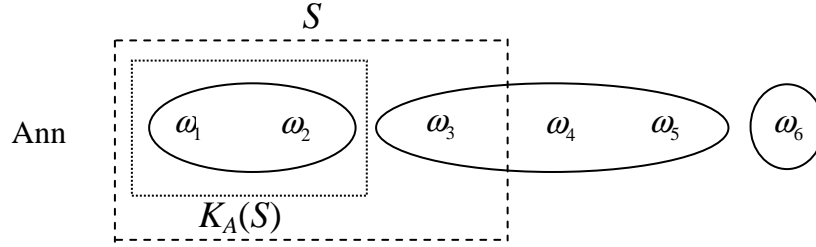
As an illustration of the state-space model of knowledge, suppose that Ann is interested in a variable v that can take values from 1 to 6, like a die, and that each state of world is completely characterized by the value taken in it by v . This means that each state of world is completely characterized by its non-epistemic, or natural, aspects. Under these assumptions, there are six possible states of the world: $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}$. P_A , the possibility correspondence of Ann, is as follows: $P_A(\omega_1) = P_A(\omega_2) = \{\omega_1, \omega_2\}$, $P_A(\omega_3) = P_A(\omega_4) = P_A(\omega_5) = \{\omega_3, \omega_4, \omega_5\}$, $P_A(\omega_6) = \{\omega_6\}$. So if $v = 1$, Ann considers possible both $v = 1$ and $v = 2$; if $v = 3$, Ann is uncertain whether $v = 3$, $v = 4$ or $v = 5$, and so on.

Let us now consider the event S " v is not greater than 3". S occurs at states ω_1 , ω_2 and

¹¹ If P satisfies the following two properties: (i) for every $\omega \in \Omega$, $\omega \in P(\omega)$ and (ii) if $\omega' \in P(\omega)$, $P(\omega') = P(\omega)$, then the state-space model is partitional. In particular, property (i) entails that knowledge is truthful.

$\omega_3 : S = \{\omega_1, \omega_2, \omega_3\}$. In which states of the world does Ann know S ? Since only in ω_1 and ω_2 $P_A(\omega) \subseteq S$, Ann knows S only in these two states: $K_A(S) = \{\omega_1, \omega_2\}$. Note that $K_A(S)$ is itself an event: the event that Ann knows that $v \leq 3$.

All this has an intuitive graphical representation. In Figure 1 below, the ovals stand for the sets $P_A(\omega)$ representing Ann's knowledge and uncertainty about the true state of the world, whereas the rectangles stand for events:

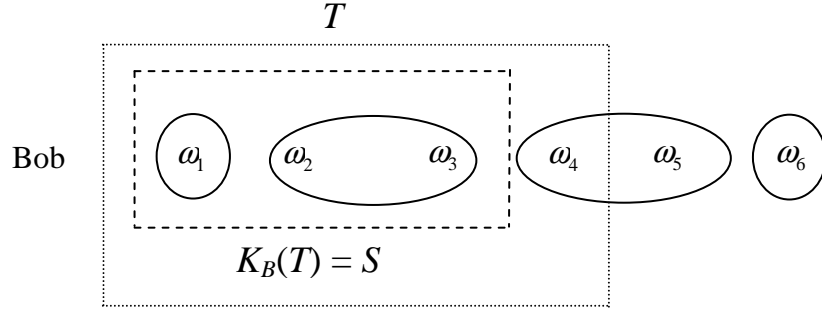


For future reference, notice again that if the true state is ω_1 Ann knows $S : \omega_1 \in K_A(S)$.

3 INTERACTIVE KNOWLEDGE: THE PUZZLE

The state-space formalism can also be used to model interactive knowledge. The simplest setting with two agents – Ann and Bob – will be considered here since this makes the discussion simpler without loss of generality. In this setting, P_i and K_i , with $i \in \{A, B\}$, are the possibility correspondence and the knowledge operator of Ann and Bob, respectively.

Assume that Bob's possibility correspondence is as follows: $P_B(\omega_1) = \{\omega_1\}$, $P_B(\omega_2) = P_B(\omega_3) = \{\omega_2, \omega_3\}$, $P_B(\omega_4) = P_B(\omega_5) = \{\omega_4, \omega_5\}$, $P_B(\omega_6) = \{\omega_6\}$. Consider now the event T “ $v \leq 4$ ” that occurs at states $\omega_1, \omega_2, \omega_3$ and ω_4 : $T = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. It is easy to show that the states of the world where Bob knows that $v \leq 4$ are ω_1, ω_2 and ω_3 : $K_B(T) = \{\omega_1, \omega_2, \omega_3\}$. $K_B(T)$ is itself an event, and in our example it happens that the event S , “ $v \leq 3$ ”, and the event $K_B(T)$, “Bob knows that $v \leq 4$ ”, occur exactly in the same states of the world: $K_B(T) = S$. This situation is represented in Figure 2:



Hence, if the true state is ω_1 Bob knows T : $\omega_1 \in K_B(T)$. At this point, interactive knowledge enters the scene. We can ask whether at ω_1 Ann knows that Bob knows that $v \leq 4$. Since $K_B(T)$ is itself an event, in the state-space formalism the question can be restated as follows: does $\omega_1 \in K_A(K_B(T))$?

From an intuitive viewpoint, the answer is that it depends on what Ann knows about the way information is imparted to Bob. If Ann knows that in ω_1 Bob is certain that $v = 1$, and that in ω_2 Bob regards as possible both $v = 2$ and $v = 3$, then in ω_1 Ann can reason as follows: “I don’t know whether the true state is ω_1 or ω_2 , but I’m sure that in both states Bob knows that $v \leq 4$ “. Therefore, Ann does indeed know that Bob knows T . On the contrary, if Ann does not know how information is imparted to Bob in ω_1 and ω_2 , neither does she know what Bob knows in these two states, and so cannot conclude that Bob knows T . In other words, the intuitive answer is that we do need to make some additional assumption about Ann’s knowledge of Bob’s informational structure to state that at ω_1 she knows that Bob knows T .

However, consider the following, formalist-oriented objection to this intuitive answer. It was established that at ω_1 Ann knows S (i.e., $\omega_1 \in K_A(S)$), and that the set of states where Bob knows T coincides with S (i.e., $K_B(T) = S$). But if $\omega_1 \in K_A(S)$ and $K_B(T) = S$, it is also the case that $\omega_1 \in K_A(K_B(T))$, that is, in fact at ω_1 Ann knows that Bob knows that $v \leq 4$. And this is independent of any additional assumption about Ann’s knowledge of the way information is imparted to Bob.

Still, from the intuitive viewpoint there is an obvious reply to the formalist objection: if Ann does not know how information is imparted to Bob, she is not aware that $K_B(T) = S$, so that she cannot go from $K_A(S)$ to $K_A(K_B(T))$. In other words, from Ann’s subjective viewpoint, S and $K_B(T)$ are different events. To say that, for Ann, S is subjectively equivalent to $K_B(T)$, the additional assumption that Ann knows how information is imparted to Bob in ω_1 and ω_2 , is indeed necessary.

Which stance is correct, the intuitive or the formalist one?

4 AUMANN'S SOLUTION TO THE PUZZLE

Since in the state-space model the way information is imparted to agent i is formally represented by his possibility correspondence P_i , one may think that our Ann-Bob puzzle reduces to the question whether at ω_1 Ann knows Bob's possibility correspondence P_B , and that this question could be easily answered by checking whether $\omega_1 \in K_A(P_B)$. The problem with this idea is that the knowledge operator K applies to sets, not to possibility correspondences. Therefore, the very expression "knowledge of possibility correspondences" has no formal counterpart in the state-space model, and the expression $K_A(P_B)$ is meaningless in it. This is not just a technical issue. At a methodological and more substantial level, the point is that possibility correspondences exist for the model-maker, not for the agents in the model. As observed in Section 2, possibility correspondences are in fact just a tool that the modeler employs to encode and represent the agents' epistemic states, not something that they are aware of or even know. Therefore, the idea of solving the Ann-Bob puzzle by framing it in the terms of Ann's knowledge of Bob's possibility correspondence risks mixing up the viewpoint of the modeler with that of the agents, and hence could be misleading.

Since 1976 Aumann has proposed a different solution to the puzzle, which is based on the very notion of state of the world and goes as follows.¹² If the model is well specified, a state of the world should be a complete description of every epistemic and non-epistemic aspect of the world that is relevant to the situation. Therefore, a state of the world should contain also a description of the manner in which information is distributed among the agents when this is relevant to the situation. In our Ann-Bob example, assume for instance that at state ω_1 Ann is uncertain about what Bob may know. Ann could think: "If $v=1$, there are two alternatives: either Bob knows that $v=1$ or he wrongly believes that $v=6$. And I do not know which alternative is the true one". But if this is the case, our Ann-Bob model, where $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}$ and each state of the world is characterized only by non-epistemic aspects, is ill-specified. In the correct model, in fact, the state ω_1 should be split into two states: ω_1' where $v=1$ and Bob is certain that $v=1$, and ω_1'' where $v=1$ and Bob wrongly believes that $v=6$. Accordingly, the state space Ω should be expanded and Ann's informational structure should be such that she cannot distinguish between ω_1' and ω_1'' .

More generally, if in a state of the world agent i is uncertain about the way information

¹² See in particular Aumann (1976, p. 1237; 1987, p. 9).

is imparted to agent j , then that state should be broken into different states and Ω should be expanded until the point where all uncertainty of agent i about the informational structure of j is eliminated. Therefore, in the correct and complete state space Ω , which is also called *canonical*, each agent knows by construction how the information is imparted to the other agents. But this knowledge “is not an assumption, but a ‘theorem’, a tautology; it is implicit in the model itself” (Aumann, 1987, p. 9), that is, it is the outcome of the way the canonical state-space model, as an economist’s tool to represent appropriately both the nature and the agents’ epistemic states, is built up.

There are two problems with Aumann’s solution to the Ann-Bob puzzle. The first and minor one, already pointed out by Aumann himself and others, is that in some situations the construction of the canonical Ω is precluded because no number of state splits is sufficiently large to exhaust all interactive uncertainty of the agents. In particular, this happens when no number of sentences is sufficiently large to describe the agents’ interactive uncertainty. However, these situations appear to be quite peculiar so that they do not affect Aumann’s solution to the puzzle in a significant way.¹³

The second problem has received little attention in the literature but in my opinion is the major one. Aumann’s solution to the Ann-Bob puzzle requires that the interaction among the agents is modeled by using the canonical state-space. However most applications employ a *reduced* state-space, in which the states describe only the non-epistemic aspects of the situation at issue, and therefore are in fact just *states of nature* rather than states of the world. Reduced models are used because as soon as one attempts to split the states of nature in order to eliminate interactive uncertainty and construct the canonical Ω , the state-space formalism loses its simplicity and tractability, and becomes a cumbersome machinery. In effect, also our formal representation of the Ann-Bob interaction is a reduced model: the six states $\omega_1 - \omega_6$ represent only the different values the die can take, and the possibility correspondences of Ann and Bob express their uncertainty about these values alone, not about the other’s epistemic states.

If the agents have no kind of interactive uncertainty about the epistemic states of the others, then the reduced model is also canonical, and Aumann’s arguments work fine. However, when the model is not canonical Aumann’s case does not apply and the puzzle returns: to say that at state of *nature* ω_1 Ann knows that Bob knows that $v \leq 4$, is any additional assumption about Ann’s knowledge of the way information is imparted to Bob necessary? The answer put forward in the present paper is in the negative: the properties of Substitutivity and Monotonic-

¹³ More on the cases where the construction of the canonical Ω is problematical in Aumann (1989, 1999); Hart, Heifetz and Samet (1996); Heifetz and Samet (1998); Heifetz (1999); Fagin, Geanakoplos, Halpern and Vardi (1999); Aumann and Heifetz (2002, Appendix).

ity that the knowledge operator K holds by construction in the state-space model, make any additional assumption dispensable, and this not only in canonical state-space models, but also in reduced ones. This conclusion is in accord with Aumann's, and the present contribution may in fact be seen as a completion of his case when non-canonical state spaces are involved. Let us now examine in more detail Substitutivity and Monotonicity.

5 SUBSTITUTIVITY AND MONOTONICITY

The state-space model of knowledge makes knowledge easy to handle in economic models, and captures certain features of the intuitive and philosophical understanding of knowledge. In effect, the idea that we know a fact when this fact takes place in any situation we consider possible sounds sensible. On the other hand, the definition of knowledge through K implies some properties of knowledge that appear too demanding from the intuitive and philosophical viewpoint, and have been discussed in the philosophical, economic and artificial intelligence literature under the banner of the *logical omniscience problem*. The present paper focuses on two properties of K : Monotonicity, which has already attracted considerable attention among economists, and Substitutivity, which on the contrary has been rather neglected by the profession.¹⁴

5.1 Substitutivity

Substitutivity states that, if two events collect exactly the same states of the world, when the agent knows one event she also knows the other. Formally:

Substitutivity: if $E = F$, then $K(E) = K(F)$.

Although Substitutivity may appear a quite natural property of knowledge, a brief aside on the philosophical notions of *extension* and *intension* will show that it is not.¹⁵

Arguably since Medieval discussions about the status of universals, philosophers have recognized that there is a difference between what a linguistic expression designates and what it means. What a linguistic expression *designates* consists of a set of things to which the expression applies, and has been labeled as *denotation* by John Stuart Mill (1843), *reference* by Gottlob Frege (1892), and *extension* by Rudolf Carnap (1947). Carnap's terminology has be-

¹⁴ On the logical omniscience problem, see the references cited in note 10, as well as Dekel, Lipman and Rustichini (1998) and Fagin, Halpern, Moses and Vardi (1995, Chapter 9). On Substitutivity in particular, see Bacharach (1986) and Vilks (1995, 2007). In the logic-based approach Substitutivity is usually called the Equivalence Rule. Lismont and Mongin (1994, 2003), as well as Ferrante (1996), have introduced logical models of knowledge that are based on so-called Montague-Scott or neighbourhood semantics, and where, at least to a certain extent, Monotonicity is replaced with the weaker Equivalence Rule.

¹⁵ This aside is largely based on Bealer (1998); Christmas (1998); Fitting (2007).

come standard in contemporary philosophy and will be adopted here. So, for instance, the extension of the term “computer” is the set of existing computers. What a linguistic expression *means* is the notion or idea conveyed by the expression, and has been called *connotation* by Mill, *sense* by Frege, and *intension* by Carnap. The intension of “computer” is the idea of an electronic machine that can store, retrieve, and process data.

Two expressions can have the same extension but different intensions. Frege proposed the example of the morning star, which is the star that can be seen at sunrise, and the evening star, the star that appears at sunset. The morning star and the evening star have different intensions but the same extension, since both designate the planet Venus. Other expressions with different intensions but equal extension are “51” and “ 17×3 ”, or “equilateral triangle” and “equiangular triangle”.

In certain contexts, extensional equality is sufficient to apply the so-called *principle of substitutivity*, according to which equals can be substituted by equals in any statement without modifying the truth-value of the statement. Contexts where substitution of equals requires only extensional equality are called *extensional contexts*. Classical logic, mathematics and standard set theory, that is, Zermelo-Fraenkel set theory, are typical instances of extensional contexts. Contexts in which intension also matters, and in which extensional equality alone does not warrant the principle of substitutivity, are called *intensional contexts*. Typical examples of intensional contexts are statements involving verbs of propositional attitude such as “believes”, “wants”, “knows”. For instance, even if Ann knows that the morning star is Venus, she may not know that the evening star is also Venus. Even if Bob knows that the triangle in front of him is equilateral he may not know that it is also equiangular.

One could think that the failure of the substitutivity principle in these two examples is due to the fact that the extensional equality among the expressions involved is only accidental, that is, non necessary: equilateral and equiangular triangles coincide in Euclidean geometry but may differ in some non-Euclidean system. Similarly, the evening star and the morning star are the same in the actual astronomical universe, but may be different in another possible universe. In effect, the principle of substitutivity can fail even when extensional equality is necessary, that is, holds in every imaginable universe. For instance, even if it is always the case that $17 \times 3 = 51$, Carl may know that 17×3 is not prime but not know that 51 is not prime. Logical systems developed for intensional contexts are called *intensional logics*. Even if a number of intensional logics have been proposed in the last 60 years, none of them has gained general acceptance.¹⁶

¹⁶ The main systems of intensional logic are those proposed by Carnap (1947); Church (1951); Montague (1960, 1970); Gallin (1975); Zalta (1988).

Going back to the state-space model, here we do not find linguistic expressions but subsets of Ω called events. However, we have seen that events are typically interpreted as set-theoretic images of linguistic expressions like “it rains”, “ v is not greater than 3”, or “Bob knows that v is not greater than 4”. According to this interpretation, the extension of an event is the set of states of Ω constituting the event, whereas its intension is identified with the intension of the linguistic expression represented by the event, e.g., the intension of “it rains”.

Now, the problem with Substitutivity as a property of the knowledge operator K is that it says that extensional equality ($E = F$) is sufficient to apply the substitutivity principle and deduce that $K(E) = K(F)$: if Ann knows that the morning star is Venus, she must also know that the evening star is Venus. This means that in the state-space model, the contexts involving knowledge are purely extensional, and that the operator K misses the intensional dimension that both philosophy and commonsense recognize in actual knowledge.

In particular, in contexts involving interactive knowledge the extensional nature of K has an even more striking consequence: it entails that an agent may know what an other agent knows, even if the former has no clue about the way information is imparted to the latter. In fact, if a generic event E and event $K_i(F)$ concerning agent i 's knowledge are extensionally equal, Substitutivity applies with no need of additional assumptions, so that if agent j knows E she also knows that agent i knows F . We will see this happening in the Ann-Bob puzzle.

Note that Substitutivity draws from the axioms of Zermelo-Fraenkel set theory and the circumstance that K operates on sets, rather than from the specific way K is defined in the state-space model, that is, as $K(E) = \{\omega \in \Omega : P(\omega) \subseteq E\}$. In fact, as far as K has sets as its domain, and set E is equal to set F , it must be that $K(E) = K(F)$, and this independently of the proposed definition of K .¹⁷ Therefore, Substitutivity turns out to be a fundamental property of any set-theoretic knowledge operator, that is, a property that cannot be removed by modifications of the standard K . This also means that any set-theoretic knowledge operator tacitly endows the agent with epistemic capabilities that appear problematic in the economic and philosophical interpretations of the state-space model.

5.2 Monotonicity

Monotonicity states that, if event E is a subset of event F , when an agent knows E she also knows F . Formally:

¹⁷ Notice that Substitutivity is also independent of the two properties of P that make the state-space partitional and were mentioned in note 11.

Monotonicity: if $E \subseteq F$, then $K(E) \subseteq K(F)$.¹⁸

Clearly, when Monotonicity holds, so does Substitutivity. Monotonicity is usually interpreted as stating that the agent knows the implications of what she knows. This means that if the agent knows the axioms of a mathematical system, she also knows all the theorems that are valid in the system, and this appears at odds with ordinary intuitions about knowledge and the logical abilities of human beings. Here a slightly different interpretation of Monotonicity is suggested, which proves helpful in clarifying the counterintuitive behavior of interactive and common knowledge in the state-space model.

According to the usual interpretation, Monotonicity seems to deal with the deductive capacities of the agent, so that it enters the scene only when the agent knows something and remains silent otherwise. If the agent does not know the axioms of the system, Monotonicity has nothing to say about what theorems she knows. However, Monotonicity is much more pervasive. In fact, at any state ω the agent knows and cannot avoid knowing the event $P(\omega)$, that is, the event collecting all the states she regards as possible at ω . By Monotonicity, she also knows and cannot avoid knowing all the events that include $P(\omega)$, i.e., all the events that are supersets of $P(\omega)$. Therefore, Monotonicity enters the scene at any ω , and implies that there is always a number of events that the agent knows and cannot avoid knowing, namely $P(\omega)$ and its supersets. In a sense, at ω $P(\omega)$ and its supersets make themselves manifest to the agent.

In the economic literature, this epiphanic character of K has been noticed (and exploited for a number of results) with reference to a particular class of events called *self-evident events* or *truisms*.¹⁹ An event E is said to be self-evident if, for every $\omega \in E$, $P(\omega) \subseteq E$. Therefore, if E is a self-evident event and $\omega \in E$, then it is also the case that $\omega \in K(E)$, i.e. $E \subseteq K(E)$. In words, whenever a self-evident event occurs the agent knows and cannot avoid knowing it. The interpretation of Monotonicity proposed here highlights that the epiphanic character of K is not restricted to self-evident events, since in any state ω there is a number of events that are immediately and necessarily known by the agent, namely $P(\omega)$ and its supersets.

Note that among the events that make themselves manifest to the agent, there may also be events concerning the knowledge of other agents. Since Monotonicity implies Substitutivity,

¹⁸ To see why Monotonicity holds when $K(E) = \{\omega \in \Omega : P(\omega) \subseteq E\}$, note that if $\omega \in K(E)$ then $P(\omega) \subseteq E$. If $E \subseteq F$, it is also the case that $P(\omega) \subseteq F$, and hence $\omega \in K(F)$. Notice that Monotonicity, like Substitutivity, does not depend on the two properties of P that make the state-space partitional and were mentioned in note 11.

¹⁹ See e.g. Milgrom (1981), Geanakoplos (1992, 1994) and Binmore and Brandenburger (1989).

under Monotonicity, the epistemic capabilities of the agent already entailed by Substitutivity cannot become weaker. In effect, under Monotonicity these capabilities become even stronger: agent j will know that agent i knows event F not only when $K_i(E)$ and $E = K_i(F)$, but also whenever $E \subseteq K_i(F)$.

From a philosophical viewpoint it can be argued that certain events related to sensations (e.g. “I see this object as white”) or thoughts (e.g. the Cartesian “I am thinking” or the analytical truth “A is A”) are immediately and necessarily known, and that any knowledge ultimately relies on this kind of event. However, in most real-world circumstances knowledge refers to states of affairs that do not make themselves manifest, and this certainly holds for mental states of other individuals. Therefore, even in the interpretation proposed here Monotonicity appears an unrealistic property of K . Moreover, to Monotonicity apply all criticisms of the extensional nature of K discussed in relation to Substitutivity. It can be added that Dekel, Lipman and Rustichini (1998) have also shown that Monotonicity is incompatible with our intuitions about a feature of actual knowledge that is relevant for economic analysis, namely that an agent may be unaware of some possible events.

Unlike Substitutivity, however, Monotonicity can be easily ruled out by slightly modifying the standard definition of the knowledge operator K . Consider for instance a mapping $X : \Omega \rightarrow 2^{2^\Omega}$, that associates to each state ω a collection of subsets of Ω . X may be interpreted as a “comprehension correspondence” that associates to each ω the events that the agent is able to figure out in ω . $K(E)$ may then be defined as follows: $K(E) = \{\omega \in \Omega : P(\omega) \subseteq E \text{ \& } E \in X(\omega)\}$, whereby $P(\omega)$ is the customary possibility correspondence.²⁰ The interpretation of this modified knowledge operator is that knowing an event requires not only that the event occurs in every state the agent regards as possible, but also that the agent can figure out the event at issue. For instance, if at ω_1 Bob is not able to figure out the meaning of “odd number”, although in all states he regards as possible at ω_1 the value of the die is odd, Bob does not know that it is. So, if $O = \{\omega_1, \omega_3, \omega_5\}$ is the event “ v is an odd number”, $O \notin X_B(\omega_1)$, so that $\omega_1 \notin K_B(O)$, although $P_B(\omega_1) \subseteq O$. Whereas Substitutivity holds also for this modified knowledge operator, Monotonicity does not: $E \subseteq F$ and $K(E)$ no longer imply $K(F)$, since it may be that $E \in X(\omega)$ but $F \notin X(\omega)$.

6 INTERACTIVE KNOWLEDGE

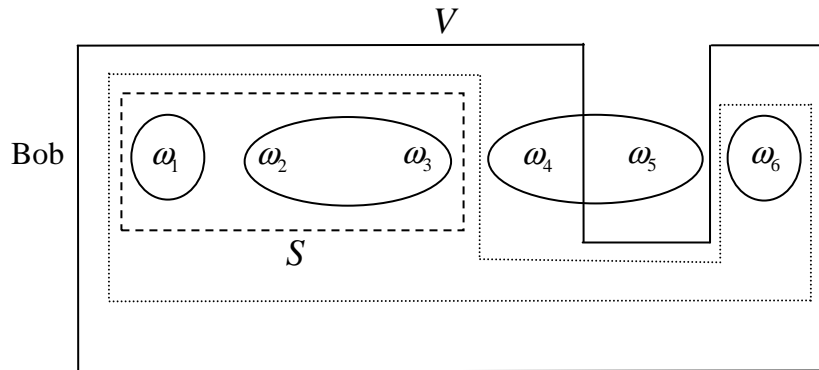
Let us now return to the Ann-Bob puzzle: is any additional assumption about Ann’s knowledge

²⁰ This definition of K is largely inspired by Fagin, Halpern, Moses and Vardi (1995, Chapter 9).

of Bob’s informational structure necessary for Ann to know that Bob knows that $v \leq 4$, i.e., the event $K_B(T)$, at ω_1 ? The intuitive answer was “Yes”: even if at ω_1 Ann knows S , i.e., that $v \leq 3$, and $K_B(T) = S$, if Ann does not have any clue about the way information is imparted to Bob, she is not aware that $K_B(T) = S$, so that she cannot go from knowing S to knowing $K_B(T)$.

The analysis of Substitutivity put forward in Section 5.1 makes clear that this answer is erroneous. The error derives from interpreting the operator K on the basis of the commonsensical and philosophical understanding of knowledge, according to which intension matters. In fact, for both commonsense and philosophy even if S and $K_B(T)$ are extensionally equal, their intensional difference (“ $v \leq 3$ ” is intensionally different from “Bob knows that $v \leq 4$ ”) does not allow Ann to jump from $K_A(S)$ to $K_A(K_B(T))$. However, K is not an exact copy of actual knowledge, and in particular K obliterates the intensional dimension of knowledge. Therefore, the extensional equality of S and $K_B(T)$ is indeed sufficient to apply Substitutivity and go from $K_A(S)$ to $K_A(K_B(T))$, and this without any additional assumption about Ann’s knowledge of Bob’s informational structure.

To this line of reasoning one may object that the Ann-Bob puzzle and its solution refer to a particular case, namely the one where $K_B(T) = S$ and Substitutivity applies. In effect, in general Substitutivity does not suffice, and the stronger Monotonicity is needed. For instance, consider event V “ $v \neq 5$ ” that occurs at all states except ω_5 : $V = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_6\}$. It is easy to show that $K_B(V) = \{\omega_1, \omega_2, \omega_3, \omega_6\}$, so that $K_B(V) \neq S$ and Substitutivity is ruled out. This situation is represented in Figure 3 below:



Here at ω_1 Bob knows that $v \neq 5$. Moreover, since $P_A(\omega_1) = \{\omega_1, \omega_2\} \subseteq \{\omega_1, \omega_2, \omega_3, \omega_6\} = K_B(V)$, at ω_1 Ann knows that Bob knows that $v \neq 5$: $\omega_1 \in K_A(K_B(V))$. Does this require any additional assumption about Ann’s knowledge of the way information is imparted to Bob? If

we think of the interpretation of Monotonicity suggested in Section 5.2, we see that this is not the case. In effect, since $P_A(\omega_1) \subseteq K_B(V)$, at ω_1 the fact that Bob knows that $v \neq 5$ makes itself manifest to Ann: Ann knows and cannot avoid knowing that Bob knows that $v \neq 5$, and again this happens without any additional assumption about Ann's knowledge of Bob's informational structure.

All this holds not only for event V , or the Ann-Bob pair: for any two agents i and j , if at ω i knows event E , i.e. $\omega \in K_i(E)$, and in all states j regards as possible at ω it happens that i knows E , i.e., $P_j(\omega) \subseteq K_i(E)$, then j knows that i knows E : $\omega \in K_j(K_i(E))$. This does not require any additional assumption about j 's knowledge of the way information is imparted to i , since by Monotonicity event $K_i(E)$ makes itself manifest to j . This clarifies the counterintuitive behavior of interactive knowledge in the state-space model.

7 COMMON KNOWLEDGE

As stated in the Introduction, an event is said to be common knowledge among a group of agents if all know it, all know that all know it, and so on, *ad infinitum*. Within the state-space model, an event E is said to be common knowledge between Ann and Bob in the state of the world ω – this is written as $\omega \in CK_{AB}(E)$ – if at ω Ann knows E in the sense of the operator K , Bob knows E in the sense of K , Ann knows that Bob knows E in the sense of K , and so on. Formally, $\omega \in CK_{AB}(E)$ if ω belongs to every set of the infinite sequence $K_A(E)$, $K_B(E)$, $K_A(K_B(E))$, $K_B(K_A(E))$, $K_A(K_B(K_A(E)))$, $K_B(K_A(K_B(E)))$, ...

If we look at this definition of common knowledge with the previous discussion in mind, it is natural to ask whether any additional assumption about common knowledge of their informational structures is required to state that event E is common knowledge between Ann and Bob. As in the case of interactive knowledge, the answer is in the negative. More specifically, sometimes it suffices to bring into play Substitutivity, whereas in general Monotonicity is needed.

To see that sometimes Substitutivity suffices, consider the event W “ $v = 6$ ”, which occurs only at ω_6 : $W = \{\omega_6\}$. At ω_6 , both Ann and Bob know W , since $P_A(\omega_6) = P_B(\omega_6) = \{\omega_6\} \subseteq W$. Note that ω_6 is also the only state where Ann and Bob know W , so that the events “ $v = 6$ ”, “Ann knows that $v = 6$ ”, and “Bob knows that $v = 6$ ” have the same extension: $K_A(W) = K_B(W) = W$. Hence Substitutivity applies, so that Ann and Bob reach level 1 of interactive knowledge: at ω_6 Ann (Bob) knows that Bob (Ann) knows that $v = 6$: $\omega_6 \in K_A(K_B(W))$ and $\omega_6 \in K_B(K_A(W))$. As explained in Section 6, this step does not involve

any additional assumption about the knowledge each player has about the informational structure of the other.

In effect, since $K_B(W) = W$, the set where Ann knows that Bob knows that $v = 6$, is again W , and the same holds for Bob: $K_A(K_B(W)) = K_B(K_A(W)) = W$. Hence, Substitutivity applies again, and level 2 of interactive knowledge is reached: at ω_6 Ann (Bob) knows that Bob (Ann) knows that she (he) knows that $v = 6$: $\omega_6 \in K_A(K_B(K_A(W)))$ and $\omega_6 \in K_B(K_A(K_B(W)))$. This step involves no additional assumptions about level 1 of interactive knowledge of the agents' informational structure.

In effect, it is easy to see that for each $i \in \{A, B\}$ and $j \neq i$, we have that $K_i(K_j(K_i \cdots K_i(W) \cdots)) = W$, so that by Substitutivity $\omega_6 \in K_i(K_j(K_i \cdots K_i(W) \cdots))$, which means that at ω_6 it is common knowledge among Ann and Bob that $v = 6$. Again, no step up this infinite staircase of interactive knowledge of W involves additional assumptions about lower levels of interactive knowledge of the way information is imparted to the agents. We can interpret this result in the sense that Substitutivity makes the entire hierarchy of "I know that you know that I know... that $v = 6$ " transparent for both Ann and Bob.

More generally, if at state ω event E is common knowledge between agent i and agent j , by the very definition of common knowledge $P_i(\omega)$ belongs to the infinite sequence $E, K_j(E), K_j(K_i(E)), K_j(K_i(K_j(E))), \dots$. Therefore, by Monotonicity, agent i knows and cannot avoid knowing all the events in the sequence. In a sense, $E, K_j(E), K_j(K_i(E)), K_j(K_i(K_j(E))), \dots$ make themselves manifest to i . Similarly, $E, K_i(E), K_i(K_j(E)), K_i(K_j(K_i(E))), \dots$ make themselves manifest to j . Hence, to state that E is common knowledge between i and j , no additional assumption about common knowledge of their informational structure is required.

8 CONCLUSION

This paper shows that, contrary to intuitive interpretations, in the state-space model interactive and common knowledge of an event do not entail additional assumptions about the knowledge agents have about the way information is imparted to others. This result is obtained by bringing into play Substitutivity, and sometimes the stronger Monotonicity, and holds for both canonical and non-canonical state spaces, as well as for partitional and non-partitional ones. When Substitutivity alone is involved, the result is robust even to modifications in the standard definition of K , since any set-theoretic knowledge operator satisfies Substitutivity.

The result is counterintuitive because neither commonsense nor philosophy regards Sub-

stitutivity and Monotonicity as plausible properties of actual knowledge. The original insight of the present contribution is in fact a methodological distinction between knowledge as understood by philosophy and commonsense on the one hand, and knowledge as modeled in the state-space formalism through the operator K on the other. When we keep in mind this methodological distinction, the counterintuitive behavior of interactive knowledge and common knowledge in the state-space model becomes intelligible.

This conclusion leaves us with the question about the relevance of the state-space model of knowledge: if the model necessarily endows the agents with implausible epistemic capabilities like those implied by Substitutivity and Monotonicity, to what extent is it useful for studying interactions among real agents? Or, to put the same question in more fashionable terms: if “an interpretation is a mapping which links a formal theory with everyday language” (Rubinstein, 1991, p. 909), what interpretations of the state-space model of knowledge are interesting and/or useful for economists and philosophers? Although the answer to this question is beyond the scope of the current paper, the analysis presented here shows the need for generalizations of the standard state-space model that could accommodate more realistic formal treatments of knowledge. In effect, a number of such generalizations have already been proposed, especially in order to overcome the circumstance that the standard state-space model precludes unawareness.²¹ The task of examining what happens to Substitutivity and Monotonicity in these generalizations of the standard state-space will be left for future research.

²¹ See, among others, Modica and Rustichini (1999); Halpern (2001); Heifetz, Meier and Schipper (2006); Li (2006); Galanis (2007).

REFERENCES

- Aumann, Robert J. 1976. Agreeing to disagree. *Annals of Statistics* 4:1236–9.
- Aumann, Robert J. 1987. Correlated equilibrium as an expression of Bayesian rationality. *Econometrica* 55: 1–18.
- Aumann, Robert J. 1989. Notes on interactive epistemology. Mimeo, Cowles Foundation for Research in Economics.
- Aumann, Robert J. 1999. Interactive epistemology I: Knowledge. *International Journal of Game Theory* 28: 263–300.
- Aumann, Robert J. 2005. Musings on information and knowledge. *Econ Journal Watch* 2: 88–96.
- Aumann, Robert J., and Aviad Heifetz. 2002. Incomplete information. In *Handbook of game theory*, Robert J. Aumann and Sergiu Hart (eds.), vol. 3, 1665–86. Elsevier.
- Bacharach, Michael. 1985. Some extensions to a claim of Aumann in an axiomatic model of knowledge. *Journal of Economic Theory* 37: 167–90.
- Bacharach, Michael. 1986. The problem of agent’s beliefs in economic theory. In *Foundations of economics*, Mauro Baranzini and Roberto Scazzieri (eds.), 175–203. Blackwell.
- Battigalli, Pierpaolo, and Giacomo Bonanno. 1999. Recent results on belief, knowledge and the epistemic foundations of game theory. *Research in Economics* 53: 149–225.
- Bealer, George. 1998. Intensional entities. In *Routledge Encyclopedia of Philosophy*, Edward Craig (ed.), vol. 4, 803–7. Routledge.
- Binmore, Ken, and Adam Brandenburger. 1989. Common knowledge and game theory. In Ken Binmore, *Essays on the foundations of game theory*, 105–50. Blackwell.
- Brandenburger, Adam. 1992. Knowledge and equilibrium in games. *Journal of Economic Perspectives* 6: 83–101.
- Brandenburger, Adam. 2007. The power of paradox: some recent developments in interactive epistemology. *International Journal of Game Theory* 35: 465–92.
- Brandenburger, Adam, and Eddie Dekel. 1993. Hierarchies of beliefs and common knowledge. *Journal of Economic Theory* 59: 189–98.
- Carnap, Rudolf. 1947. *Meaning and necessity*. University of Chicago Press.
- Christmas, Simon. 1998. Intensional logics. In *Routledge Encyclopedia of Philosophy*, Edward Craig (ed.), vol. 4, 807–11. Routledge.
- Church, Alonzo. 1951. A formulation of the logic of sense and denotation. In *Structure, method and meaning*, Paul Henle, Horace M. Kallen, and Susanne K. Langer (eds.), 3–24. The Liberal Arts Press.
- Cubitt, Robin P., and Robert Sugden. 2003. Common knowledge, salience and convention: a reconstruction of David Lewis’ game theory. *Economics and Philosophy* 19: 175–210.
- Dekel, Eddie, and Faruk Gul. 1997. Rationality and knowledge in game theory. In *Advances in economics and econometrics: theory and applications. Seven world congress*, David M. Kreps and Kenneth F. Wallis (eds.), vol. 1, 87–172. Cambridge University Press.
- Dekel, Eddie, Barton L. Lipman, and Aldo Rustichini. 1998. Standard state-space models preclude unawareness. *Econometrica* 66: 159–73.
- Fagin, Ronald, John Geanakoplos, Joseph Y. Halpern, and Moshe Y. Vardi. 1999. The hierarchical approach to modeling knowledge and common knowledge. *International Journal of Game Theory* 28: 331–65.
- Fagin, Ronald, Joseph Y. Halpern, Yoram Moses, and Moshe Y. Vardi. 1995. *Reasoning about knowledge*. MIT Press.
- Ferrante, Vittorioemanuele. 1996. A sound interpretation of minimality properties of common belief in minimal semantics. *Theory and Decision* 41: 179–85.
- Fitting, Melvin. 2007. Intensional logic. In *Stanford Encyclopedia of Philosophy (Spring 2007 edition)*, Edward N. Zalta (ed.). Available online at: <http://plato.stanford.edu/archives/spr2007/entries/logic-intensional/>
- Frege, Gottlob. 1892. Über Sinn und Bedeutung. *Zeitschrift für Philosophie and philosophische Kritik* 100: 25–50.
- Galanis, Spyros. 2007. Unawareness of theorems. Mimeo, University of Southampton.
- Gallin, Daniel. 1975. *Intensional and higher-order modal logic*. North-Holland.
- Geanakoplos, John. 1992. Common knowledge. *Journal of Economic Perspectives* 6: 53–82.
- Geanakoplos, John. 1994. Common knowledge. In *Handbook of game theory*, Robert J. Aumann and Sergiu Hart (eds.), vol. 2, 1437–96. Elsevier.
- Gilboa, Itzhak. 1988. Information and meta information. In *Theoretical aspects of reasoning about knowledge. Proceedings of the second conference (TARK 1988)*, Moshe Y. Vardi (ed.), 227–43.

- Kaufmann.
- Halpern, Joseph. 2001. Alternative semantics for unawareness. *Games and Economic Behavior* 37: 321–39
- Hart, Sergiu, Aviad Heifetz, and Dov Samet. 1996. “Knowing whether,” “Knowing that,” and the cardinality of state spaces. *Journal of Economic Theory* 70: 249–56.
- Heifetz, Aviad. 1999. How canonical is the canonical model? A comment on Aumann’s interactive epistemology. *International Journal of Game Theory* 28: 435–42.
- Heifetz, Aviad, Martin Meier, and Burkhard C. Schipper. 2006. Interactive unawareness. *Journal of Economic Theory* 127: 78–94.
- Heifetz, Aviad, and Dov Samet. 1998. Knowledge spaces with arbitrarily high rank. *Games and Economic Behavior* 22: 260–73.
- Heifetz, Aviad, and Dov Samet. 1999a. Hierarchies of knowledge: an unbounded stairway. *Mathematical Social Sciences* 38: 157–70.
- Heifetz, Aviad, and Dov Samet. 1999b. Coherent beliefs are not always types. *Journal of Mathematical Economics* 32: 475–88.
- Hintikka, Jaakko. 1962. *Knowledge and belief*. Cornell University Press.
- Li, Jing. 2006. Information structures with unawareness. Mimeo, University of Pennsylvania.
- Lismont, Luc, and Philippe Mongin. 1994. A non-minimal but very weak axiomatization of common belief. *Artificial Intelligence* 70: 363–74.
- Lismont, Luc, and Philippe Mongin. 2003. Strong completeness theorems for weak logics of common belief. *Journal of Philosophical Logic* 32: 115–37.
- Margalit, Avishai, and Menahem Yaari. 1996. Rationality and comprehension. In *The rational foundations of economic behaviour*, Kenneth J. Arrow, Enrico Colombatto, Mark Perlman, and Christian Schmidt (eds.), 89–100. Macmillan.
- Mariotti, Thomas, Martin Meier, and Michele Piccione. 2005. Hierarchies of beliefs for compact possibility models. *Journal of Mathematical Economics* 41: 303–24.
- Meier, Martin. 2005. On the nonexistence of universal information structures. *Journal of Economic Theory* 122: 132–9.
- Mertens, Jean-Francois, and Shmuel Zamir. 1985. Formulation of Bayesian analysis for games with incomplete information. *International Journal of Game Theory* 14: 1–29.
- Milgrom, Paul. 1981. An Axiomatic Characterization of Common Knowledge. *Econometrica* 49: 219–22.
- Milgrom, Paul, and Nancy Stokey. 1982. Information, trade and common knowledge. *Journal of Economic Theory* 26: 17–27.
- Mill, John Stuart. 1843. *A system of logic*. Parker.
- Modica, Salvatore, and Aldo Rustichini. 1999. Unawareness and partitional information structures. *Games and Economic Behavior*: 265–298.
- Monderer, Dov, and Dov Samet. 1989. Approximating common knowledge with common belief. *Games and Economic Behavior* 1: 170–90.
- Montague, Richard. 1960. On the nature of certain philosophical entities. *The Monist* 53: 159–94.
- Montague, Richard. 1970. Pragmatics and intensional logic. *Synthese* 22: 68–94.
- Osborne, Martin J., and Ariel Rubinstein. 1994. *A Course in Game Theory*. MIT Press.
- Rubinstein, Ariel. 1991. Comments on the interpretation of game theory. *Econometrica* 59: 909–24.
- Samuelson, Larry. 2004. Modeling knowledge in economic analysis. *Journal of Economic Literature* 42: 367–403.
- Sent, Esther-Mirjam. 2006. The tricks of the (no-)trade (theorem). *History of Political Economy* 38 (supplement): 305–21.
- Steup, Matthias. 2006. The analysis of knowledge. In *Stanford Encyclopedia of Philosophy (Spring 2006 edition)*, Edward N. Zalta (ed.). Available online at: <http://plato.stanford.edu/archives/spr2006/entries/knowledge-analysis/>
- Vanderschraaf, Peter, and Giacomo Sillari. 2005. Common knowledge. In *Stanford Encyclopedia of Philosophy (Winter 2005 edition)*, Edward N. Zalta (ed.). Available online at: <http://plato.stanford.edu/archives/win2005/entries/common-knowledge/>
- Vilks, Arnis. 1995. On mathematics and mathematical economics. *Greek Economic Review* 17: 177–204.
- Vilks, Arnis. 2007. Axiomatization, immunization, and convention in economics. Mimeo, Leipzig Graduate School of Management.
- Zalta, Edward N. 1988. *Intensional logic and the metaphysics of intensionality*. MIT Press.