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Diego Iribarren

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EERI
Economics and Econometrics Research Institute
Avenue de Beaulieu
1160 Brussels
Belgium

Tel: +322 299 3523
Fax: +322 299 3523
www.eeri.eu

FROM ECONOMIC ACTIVITY TO UNDERSTANDING SPACES[♣]

Diego Iribarren

ESRC Centre for Research
on Innovation and Competition

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Abstract: This paper constructs the probability space underlying the random variable of any time dependent econometric specification. The construction links concrete economic activity, both perceived and recorded, and econometric formulations. Furthermore, it is argued that the probability events belonging to this space are forms of understanding economic activity held by each agent. The model establishes two aspects of any econometric formulation. Mainly, that learning must be unique between any two ticks of the clock and that not all forms of understandings can indeed become events in the random variable's probability space. Finally, a model of the dependencies based on agent-based understandings, and evolution thereof, is presented as well.

Author's email: diribarren@estudios-tecnicos.com

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Introduction

The aim of this paper is two fold. First, it describes and develops a mathematical model of the formation, constitution and evolution of the probability space associated to the individual disturbance factor λ^t_i of any time dependent statistical system. Second, it describes the consequences that history bears upon it, i.e. its evolution. To do so, a detailed exploration into the nature and constitution of learning, as defined in here, is done. This requires some basic definitions on the nature of knowledge to be employed in here. The role of recorded information is highlighted as a conditioning factor of learning. Moreover, a difference is made between what is termed as biological and conventional time[1] in order to fully examine mutual influences amongst agents.

Knowledge and intuitions is characterised as abstract mathematical entities, just as other approaches have[2]. Our approach, however, is based on elementary set theory and topology. Individual understandings are linked to concrete records of economic activity through formal operators, i.e. dialectics, learning and probabilities. In order to technically construct the agent's probability measure, certain assumptions, to be spelled later on, are made in order to invoke the Riesz Representation Theorem (RRT). This approach, which can be found in all its details in the appendix, is only one of possibly many constructions. We purposely chose it in order to highlight the core aspects of the dynamics of understandings. That is, to bring history to the front of knowledge, and intuition generation within a concrete model. The assumptions brought into play will concern the events of understandings disposable to the agent at every moment, which will reflect, by construction, their "spot on" cognitive relation with the environment. Perhaps, the most striking outcome of the entire description is that not all forms of understandings can in fact become probable forms of understandings and learning cannot be arbitrary either. The entire description will be done for one random individual agent. The aggregate pool of agents follows directly as the mathematical "product" of each individual agent's behaviour. In this respect the evolution of the agents' collected understandings, as well as mutual dependencies, are represented using elementary homotopy theory.

The paper begins with a set of basic definitions, i.e. knowledge, intuitions and understandings. Then a result from logic on the limits to, what we have termed, dialectical understandings is stated. The next section describes the basic constituents of the dynamic process of understandings generation and their mathematical formulation. Furthermore, it establishes the basic abstract spaces and the corresponding operators that link them. We also present general results that provide additional characterisations of these spaces. The next section is entirely devoted to a detailed account of the nature of learning, as defined here, and the construction of the probability measure associated to the disturbance term of any time series statistical system. Next, specific use of homotopies is made to model and study the evolution of the probability space previously described. Finally, conclusions are drawn in the last section.

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1 Biological time is a notion of time that arises out of the periodicity of nature. Conventional time, as the name suggests, is a convention previously agreed upon for a specific purpose and use in measurements of events that "pass".

2 For example, the mathematical psychology and game theory literatures.

I Knowledge, intuitions and understandings

In any capitalist market with at least two interacting economic agents, individual beliefs are always based on forms of understandings at least partially dependent on the others' understandings, beliefs and expected behaviour. In the economics literature of knowledge this can be traced back to at least Von Hayek (1937). Game theory, particularly evolutionary game theory, relies heavily on this feature of individual behaviour, e.g. Weibull (1995), Mailath (1998). Furthermore, those expressions of understandings relevant to the agents are acquired through particular idiosyncratic processes, i.e. learning, in each case unique to the agent, e.g. Loasby (1999), Slembeck (1999). Learning, in this sense, allows individual economic agents to apprehend economic history. Whence, this begs the question of what knowledge is. Although many different definitions and approaches can be invoked, the following will be used for the purposes of the present paper:

Knowledge. We assume the perspective of the materialistic theory of knowledge. Knowledge is the understood (or apprehended) portion of the synthesis of a materially based dialectical contradiction.

Intuitions. These are forms of knowledge as well. They are non-dialectical and originate from reality itself. Formally, a direct relation between the mind and something abstract, therefore, not accessible through the senses (Oxford Companion of Philosophy).

Understandings. These can be either knowledge or intuitions.

Our definitions allow us to highlight the totally subjective nature of the economic agents' understandings. In this sense, the approach to be followed in the paper is based on the idea of events of understandings rather than problem-solving abilities. It is more or less in the spirit of Hintikka (1962)[3] but different in that the main emphasis is placed on the inner structure of the understanding spaces rather than in the construction of systems of understandings through logical operators.

As time develops, whatever its meaning and notion of, economic history unravels and agents (at least attempt to) understand it. Time, history and understandings are indeed unavoidably inseparable. If time were already eternal, history would cease to exist and there would be nothing further to discover and understand, Metcalfe (1998). Moreover, if there were no time, not only would economic agents not grasp the entirety of their surrounding reality (because human learning is not enough to discover all of the forces that drive economic activity) but also they would have no possibility to expand their horizon of understanding. Only Shacklian time would continue to exist as a reflection of individual "silent contemplation". In such a situation, understandings would reduce themselves to a set of rules and recipes upon which routines can follow into perpetuity. A time-independent attractor will have been reached in which economic activity will have been reduced to timeless, unchanging, economic patterns. Without change in time there cannot be change in the facts of history beyond the equilibrated activity of historical attractors; understandings would also remain in the same resilient state of affairs. Whence, change (beyond that of stable equilibrium) in any one of the three variables, i.e. time, history and

3 The full description of Hintikka's approach can be found in Rubinstein (1998)'s Modelling Bounded Rationality.

understandings, necessarily implies change in the other two. In this sense, only past dependent processes can induce new forms of understandings, Metcalfe (1998).

Human limitations, e.g. of storage, learning, searching and processing capacities, etc., allow the possibility of further knowledge to be pursued at all time, Loasby (1999). In particular, if self-consistent propositional systems are constructed to generate knowledge, it can be proven that such systems are always incomplete[4] and hence the knowledge derived thereof is incomplete as well. The consequences of this theorem, rightly called Gödel's theorem, are immensely profound and are certainly beyond the scope of this work. However, one corollary (of this theorem) does have direct bearings on this work. Science is all one gigantic set of (sometimes different) systems of self-consistent propositions. In particular, economics is, as a matter of practice, a very elaborate system of propositions based on the axioms of formal mathematics and additional assumptions about the real world. Now, if the systems of propositions constructed through binary logic are always incomplete then knowledge is always potentially expandable although in a limited boundedly rational fashion, e.g. Morgenstern (1935), Koppl and Rosser (2000)[5]. That is, there is always the possibility that additional yet-undiscovered knowledge and intuitions can influence recorded economic activity as a matter of empirical evidence. Thus, the very own non-quantifiable nature of knowledge and intuitions implies that, at any point in time, it is the disturbance term, i.e. λ^t_i, of any time dependent econometric formulation such as

$$h_i^t = f(h_i^{t-1}, z_i^{t-1}) + \lambda_i^t$$

the only measurable channel of influences and dependencies based on understandings. If the agent did know it all then he/she could specify each and every influence that affected him/her (through h^{t-1} or z^{t-1}). Whence, there would be no need for a disturbance term in the specification of his data generating mechanism. In this sense,

4 This most remarkable result is what is known as the Theorem On The Incompleteness of Propositional System. It is considered to be, perhaps, the most important theorem of all of mathematics. The main idea is this: given n axioms, i.e. absolute truths or assumptions, upon which a system of self-consistent propositions can be constructed exclusively through the systematic applications of logical operators, one can always construct a proposition that is unprovable given the n axioms and subsequent systems developed thereafter (in fact, the construction of such a proposition *is* the proof of the theorem). Hence, in order to complete the system the unprovable proposition must be added as an additional axiom thus raising the number of axioms to n + 1. However, the same argument repeats itself. Another unprovable proposition can be constructed given the new system of propositions and its n + 1 axioms. Therefore, the new unprovable proposition has to be assumed as a new axiom again in order to complete the system thus raising the total to n + 2. Again the same argument repeats in this new system of n + 2 axioms and so on for any number of axioms. It is an inductive argument. Hence, no propositional system can ever be complete. One corollary is immediate: any knowledge constructed thereof as a self-consistent system through the application of formal logic given *any* number of axioms (assumptions) is thus **proven** to be incomplete, i.e. limited. This great result is due to the Czech logician Kurt Gödel.

5 Some very interesting research has been done in this respect in related areas. For example, Wolpert (1996) presents a theorem on the impossibility of building a machine capable of calculating future specifications of a physical system. Hut, Ruelle and Traub (1998) study the limits to knowledge in physics and biology. Minkler (1993) argues on the limits that dispersed knowledge creates for modelling and managerial purposes; he proposes a core capability approach in stead. Langlois (2001) reviews the main concepts of rationality theory and suggests that institutions shape the emergence of rules and routines undertaken by agents. Although not stated in his paper, this is a clear statement on the limitations of human searching mechanisms through which institutions constrain as much as they stimulate further knowledge. Loasby (2000) argues the same point on the role of institutions in the question of preference formation in demand in an evolving market.

the consequences of the agents' cognitive limitations (and the modeller's as well) justify the existence of the random variable in the specification above. The coming sections will in fact examine, construct and model the evolution of the underlying probability space associated to the disturbance term, the key to modelling the dependencies.

There already exists a huge literature in rational choice theory and mathematical psychology on the construction of understandings spaces. The traditional mathematical psychology literature conceptualises understandings spaces in relation to problem solving schema. This general conception of understandings is not really concerned with knowledge's (nor intuitions') origin. This view faces further additional difficulties in that it does not take a deep concern as to the effects of individual search for knowledge. In all fairness, though, this approach, which is based on behaviourist psychology, is not aimed at studying knowledge in itself but rather how it is handled and dealt with. Hence, the difference in the type of agents, e.g. atomistic unintended neoclassical agents, Schumpeterian individual entrepreneurs or Penrosian conglomerates of peoples gathered in firms, is not in question. Coordination is not generally an issue either. However, in as far as understandings are concerned this is not a trivial matter for clearly, problem solving abilities differ from individuals to uncoordinated groups of people to coordinated groups of people. In any case, this literature defines the set of all understandings states as the agent's knowledge structure for the particular set of questions. If this understandings structure is closed under arbitrary unions then it is called the agent's knowledge space for the set of questions[6]. Further structures of order can be considered if it is also closed under intersections. A thorough exposition of this approach can be found in Albert, Schrepp and Held (1994) and Doignon and Falmagne (1998). In particular, Suck (1999) provides an interesting metric for discrete knowledge spaces which, in principle, allows the definition of distance of understandings among agents. This distance is constructed on a "lattice of understandings". That is, given certain tasks there will exist certain answers, however good or bad. Hence, these answers can be mapped onto an ordered set of points in a vector space, i.e. a lattice. Through this measure, the differences, i.e. the distances, in the possibilities of knowledge between two spaces based on recorded know-how can be estimated. Therefore, the consequences of the differences in problem solving abilities may be estimated.

As previously mentioned, once the nature and composition of understanding spaces has been established, the agents' understanding spaces' evolution, through time, is studied. This resembles, although unrelated in spirit, Schrepp (1997) where he studies "degrees of correctness" as a measure of the agent's evolving understanding of the problem domain. In our case, the core idea invoked to study changes in the agents' understandings structure is the notion of *deformations* of the agent's understanding space through time, rather than the expansion of the agent's understandings. By conceptualising the understanding space as being deformed and not just as expanding (or contracting) the core emphasis is placed on the notion of change in the inner composition of the space and the effects of history as channelled through learning. Hence, these deformations reflect the inherent mechanisms of adaptation to the evolving material conditions in the industry, i.e. the evolving *context* within which

6 This is our understanding space.

understandings arise, similar in spirit to Nonaka, Toyama and Konno (2000)[7]. The following construction is general and applies to any agent.

II The Dynamics of Understandings Formation

When the agent is considered at time t he/she, in reality, faces, as a constituting member of the industry, a given history of relevant facts in the industry, a sort of factography of the industry. This factography is objective in nature as is encapsulated information, e.g. letters, numbers, equations, combinations thereof, etc. Therefore, factographic phenomena, i.e. data, are in principle common to all agents although not necessarily true[8]. In fact, at t, the agent has more than just a given accumulated history of data; he also inherits accumulated, concentrated and clustered expressions of past reconstructable understandings related to information up to t. With the sole purpose of enhancing his/her position in the market he/she will necessarily strive for further understanding[9]. Possible future economic activity pulls (past) history to the front, very much, as Von Hayek (1937), Loasby (1999) imply it and Marx (1867) and Morgenstern (1935) clearly state it.

From an ontological stance, at least three features define an economic agent that participates in market activity and interacts with other agents; these are: history, facts thereof and understandings. Consider an interval of time during which economic activity develops. History's own changing constitution will induce changes upon the available recorded facts of history. Simultaneously, changes in economic activity, determined by the forces of history, will induce changes upon the systems of dialectical contradictions that so reflect economic activity as well. Hence, what is at the agents' disposal, as a reservoir of understandings, will change as a consequence of the existence of history. In other words, the domain of the entities to be learnt, i.e. syntheses, will change due to the forces of history. Furthermore, reality, whatever it entails for each agent, is a much deeper, extensive and complex universe of investigation. It is infinitely dense, i.e. it is a continuum, and carries with it all possible notions of human existence. In any case, the origin of knowledge will evolve (as history unscrambles forward in time) as a consequence of the fact that the space upon which dialectical materialism operates, i.e. economic activity, changes. Distinct instants of time have different systems of dialectical contradictions[10] associated to them and hence different dialectical realities emerge; ultimately, due to history never repeating itself. This idea of non-repetition can traced back to at least V century B.C, to Anaxagoras', the ancient Greek philosopher, concept of continuity in reality. Conceptually, in our context, dialectical materialism is a dynamic operator defined over material economic reality. This will be defined later on.

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7 In their paper, the context within which understandings arise is given a specific name: *Ba*, which is the name assigned to the idea of "surroundings" in traditional Japanese philosophy.

8 This is the issue of asymmetric information. Although not explicitly dealt with here it certainly constitutes an important part of the dynamics described here. Essentially, available information partly determines individual learning. For an introduction to problems of asymmetric information in economics see Hillier (1997) and of course Stigler (1961).

9 The aggregated outcomes of the individual intent in the search for understanding will be the unintended resulting economic activity, as observed by Adam Smith (1776). That is, the invisible coordinating hand.

10 A system of dialectical contradictions at any moment in time should be understood as being composed of two categories of contradictions: a principal contradiction and peripheral contradictions. The principal contradiction is the contradiction that allows for the existence of the phenomenon in question. Peripheral, or secondary, contradictions define the particular expression of the phenomenon.

Since agents attempt to understand from history, then it is only historical records that solely matter to the agent. Partly through them, history presents itself to the agents and hence (partly) determines how the agents can learn from the systems of dialectical contradictions. In this sense, as it was mentioned above, dialectical materialism is a mechanism, which operates over material reality, i.e. economic history. The role of data banks is that of helping and aiding learning. The more information held, the more aid data banks provide and more extensive individual learning becomes. In practice, the data banks are distributed asymmetrically and hence, just based on this point, learning is asymmetrical, i.e. heterogeneous. As mentioned in footnote 8, the case for asymmetric information and hence heterogeneous behaviour purely based on this, is not really an issue in this work, although it is certainly acknowledged. The reason for this is that heterogeneity is embedded in learning as the defining constituting components of learning deliver the heterogeneity from an understandings perspective. That is, heterogeneous life experiences, talents, intuitions, etc. all determine the composition of learning. Asymmetric information delivers further heterogeneity to the process. In any case information is assumed to be asymmetric although it is not relevant for the entire argument since any asymmetry in information is always reflected in learning.

As history develops itself, new potential forms of understandings emerge, which, with the aid of information, amongst other things, are apprehended through learning. Once learning has taken place, the agents hold forms of understandings, whether knowledge or intuitions, related to their surrounding economic activity. These present forms of understandings influence and partly determine existing beliefs and decisions about the future. The consequences of these decisions will be later recorded as information. The point to emphasise is that presently held forms of understandings are in fact probable future influences of understandings. Moreover, under certain conditions to be spelled in a moment, they constitute a probability space; the probability space associated to the disturbance term λ^t_i. As it will be seen later, technical requirements in the construction of the operators restrict the possible events of understanding to be considered. This generates a natural filter on understandings. There are also some requirements on formal learning, which will be presented later on as well. Certain inherent constraints in the agents' core capabilities do not allow them to use all possible forms of understandings to make them probable understandings. These requirements, which in this setting have sprouted entirely from the internal logic of the problem, generate differences across firms and constitute uneven core capabilities in the market as in Nelson (1991).

To summarise: at any moment in time, learning takes place formally from systems of dialectical contradictions and delivers knowledge. Informal learning operates over reality itself and delivers intuitions. Data banks partially influence both forms of learning in a positively correlated manner. At any moment in time, if learning is unique (between two "ticks" of the conventional clock) a probability distribution for the disturbance term of (any) the time dependent econometric formulation can be constructed (this will be proven later on). The next recorded facts of history will carry the influences of present understandings, channelled through learning and the probability distribution. These data will constitute (a very small) part of the recorded economic activity in the future. Future economic activity will have fed on itself and the spiral of history will have induced a self-feeding loop of cognition in the agents.

III The Role of Learning

Learning plays a fundamental role in any evolving and adapting process of interacting heterogeneous agents, Von Hayek (1937), Metcalfe (1998), Loasby (1999). It is not only a process of observation, experience, internalisation and ultimate adaptation. It is also a process of discovery (in the Austrian sense, see Kirzner (1994, 1997)) and reconnaissance, or as Shackle (1961) calls it, silent observation. This is so since, through time (whether conventional or otherwise) the agents, amongst other things, in fact recognise, classify and categorise their surroundings. They gather and associate relevant phenomena around them. As Loasby (1999, 2000, 2001) has repeatedly emphasised, agents relate perceived economic phenomena through different mental processes. Furthermore, through this process of reconnaissance they can indistinctly and inductively further discover the forces that drive economic activity. In the context of the previous section, it means that they can force ν to become smaller. In this respect, discovering new dimensions of reality, just like discovering new emergent markets, inevitably induces some sort of adaptation of the means by which learning takes place.

Two issues arise in this respect. The first one is that learning is an evolving adapting process itself. It responds to an evolving track record on the competence with which the agent learns and incorporates new experiences of cognition. In other words, changes in learning are a reflection on the past success of the agent's mechanisms and procedures used, in the past, to enhance his/her understandings about the industry. That is, a reflection on how far he/she has fulfilled his/her cognitive potential. In this sense agents learn to learn over time. This is what the management literature calls "double learning" or "double loop". The reasons for the changes in the manner in which agents learn, and this is the second issue, imply that learning need not be unique at all. In other words, an agent could possibly learn through more than one methodology at the same time[11] if so desired. Whence, in principle, learning <u>is</u> <u>not</u> <u>a</u> <u>sole</u> <u>unique</u> time-dependent process and each of its multifaceted forms evolves and adapts through time. This view is concomitant with the tradition, which conceptualises learning as a dynamic process, many times a repeated process, found in dynamic and evolutionary game theory; see Slembeck (1999) and Weibull (1995). In other words, the passage from history to understanding spaces is, in principle, as n-fold as the n procedures by which the agent formally learns.

To further blur the distinctive features of individual learning, intuitions only reinforce the idiosyncraticity of understandings. Indeed, in our context, intuitions have only one recognisable characteristic: they always represent forms of understanding. An experience of intuition is always an experience of cognition. They help, aim and ultimately influence belief formation and decisions concerning the individual and collective cognitive characteristics of economic phenomena. However, the origins of intuitions have been debated and assumed by philosophers of all ages (see the Oxford

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11 For example, the agent can repeat his/her previous procedure or he/she can innovate and incur in new knowledge acquiring techniques or imitate what is perceived as successful, etc. All three situations differ from the previous attempt to learn and represent at least three different methods to learn from the past. Other methods of learning have indeed been identified in the literature, for details see Slembeck (1999).

Companion to Philosophy (1995)). More importantly, the very nature of intuitions does not permit for a specification on the mechanism through which they emerge.

Dialectics defines syntheses from which understandings sprout by identifying opposing forces. If all of the forces were identified then the continuum of forces would be known. Hence, reality would, in this situation, be a continuum of knowledge. Nevertheless, as Penrose (1959), Lachmann (1976) and Polanyi (1946) hint and Von Mises (1957) and Popper (1950, 1956) clearly state, economic agents are limited in their search to know the world. They are constrained because of limited abilities to know the world (Penrose (1959)) and because the world is largely unknowable, Lachmann (1976), Polanyi (1946), Von Mises (1957)[12], Popper (1950, 1956). In this sense, as far as economic agents are concerned, knowledge is never complete since we cannot know it all. It is rather, as far as an experience of cognition is concerned, always bounded. Economic agents cannot learn beyond a certain (human) limit, i.e. they never reach the limit of the asymptote. Intuitions, however, are different. Basically, intuitions are intuitions as long as they are not dialectical in nature. Without any pretension whatsoever to enter a rather philosophical debate on the possible origin of intuitions, we will only take up issue as far as learning is concerned.

In reality, if the mechanisms by which intuitions are apprehended were known, then their emergence could be framed in terms of formal learning. This is the reason for our definition of intuitions at the beginning[13]. In order to retain consistency with our entire exposition and also our belief on the complete inadequacy of idealistic understandings, we are restricting ourselves to intuitions emanated only from material economic activity. That is, $\mathbf{I}^{t}_{i}[\text{History}] = \mathbf{\Omega}^{in}_{i}$. The operator $\mathbf{I}^{t}_{i}$ is some abstract operator, unique to each individual agent, which delivers intuitions originated in (economic) history, about economic phenomena. Its composition, by construction, is unknown.

Since intuitions, because of their very nature, cannot be constructed as self-consistent systems of propositions, Gödel's theorem does not induce any limitation to their scope. Indeed, in principle, there is no limit to intuitions although they are bounded. That is, in principle, they are infinite even though they are constrained by human senses. Furthermore, limitation on intuitions would require restrictions (of logic or ontology) that do not exist in the agent's mental intuitive processes. They are not a subject of matter simply because if the questions of logic or ontology ever became an issue then intuitions would not be intuitions any more. For logic or ontology to become issues, intuitions must be conceptualised as dialectical phenomena, that is, (in

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12 Von Mises (1957), pp. 8-9, develops a further argument on the limitations to human knowledge. In it, he expands on the idea that what we can know is limited by the universe in which our senses operate. That is, we can only know the universe of which our senses tell us that we are a part of. It is intrinsically Kant's argument on the limited use of pure-reason. In his own words: "Human knowledge is conditioned by the power of the human mind and the extent of the sphere in which objects evoke human sensations…There may also exist outside of the orbit we call the universe other systems of things about which we cannot learn anything because, for the time being, no traces of their existence penetrate into our sphere in a way that can modify our sensations". In other words, there may be universes that we have not yet discovered and hence do not know of.

13 That is, intuitions are the outcome of yet-to-be discovered protocols of learning; protocols that we have called informal learning in this paper.

our present scheme) as knowledge. The moment an intuition is framed into dialectics then all previously mentioned issues on the limits of dialectics repeat themselves. Whence, if intuitions are boundless, because of their non-dialectical nature, then they can, in principle, converge to the limit of possible understandings of economic activity, i.e. the continuum of cognition. In this sense it is that intuitions may be infinite (not in that there may be an infinite amount of them). In as far as economic agents, only intuitions can hence guarantee the possibility of understanding infinitely dense dimensions of economic reality. In this sense as well, if an economic agent ever desired to further know economic activity, intuition presents itself as a guiding mechanism, a compass of cognition of sorts, in the search for further knowledge. For our descriptive purposes, it represents one of the components of the two-fold process of individual learning.

IV The Model

In order to formalise mathematically some of the notions of the dynamics of understandings presented above, it will be necessary to assume certain aspects of the same; in particular, with respect to history and systems of dialectical contradictions. However, none of the assumptions concerning their mathematical representation undermine the description. Nor is the essence of the argument compromised. They are all, basically, requirements of formalisation. Each one of these will be carefully explained whenever necessary.

History and "Factography"

From now, unless otherwise stated, time is to be considered as a convention. Furthermore, assume that this convention is such that time is discrete (and hence countable) and that measurements of economic activity take place in this frame. Assume that history is a continuous space irrespective of the measurements of time associated to it. Call this space of history H_t where the subscript t merely signifies that it is history considered up to time t.

Associated to the passage of history, records of it arise. Indeed, assume that all facts of history can be stored and arrayed in some manner so that these recorded facts, i.e. factography, can be grouped as

$$F_t = \{f_j : f_j \text{ is a recorded fact of history at time } j \leq t\}$$

Furthermore, assume that F_t is countable and that the cardinality of F_t is less than or equal than the first infinite countable cardinal, i.e. $\aleph_0$, for any t, i.e. $\mathcal{C}(F_t) \leq \aleph_0$. This set varies with time, as new facts of history are included with the passage of time. It may or may not be connected (in the topological sense) depending on the manner in which data is stored and arrayed. Operations amongst the elements of F_t, i.e. essentially unions and intersections, are always well defined provided that there exists a well-defined form of storage of information, e.g. bits, symbols, numbers, functions, etc. Once coherence in the storage of records has been achieved handling the data is a matter of practical concern only. For the argument's sake, if need be, simply assume that there exists a well-defined self-consistent storage mechanism that allows data to be handled.

Systems Of Dialectical Contradictions

Dialectical contradictions are, in essence, abstractions, that reflect opposing forces present in concrete economic activity, i.e. unities of opposing forces. When these unities of opposites are concatenated they form a system. Thus, let

$\mathbf{K}_t$ = {k : k is a dialectical contradiction and k $\leftrightarrow$ (a,b) where a and b are the two [2.a]
opposing forces within concrete material economic activity that define k}

In other words, each element of $\mathbf{K}_t$ is an abstract point defined by two opposing forces within concrete economic activity. The definition above in no way should be understood to mean that these are vectors or 2-ples[14]. They are not vectors. The symbols used were purposely chosen to emphasise that the unit, i.e. k, the contradiction, depends on two opposing material forces, i.e. a and b. An alternative description could be constructed in which k is actually an interval of the real line. That is,

$\mathbf{K}_t$ = {k : k is a dialectical contradiction and k = [x_α y_β] where the specific interval [2.b]
itself depends on the forces a and b that define k}

where x_α and y_β depend on t. This alternative construction has the advantage (as it will be seen shortly) that the notion of connectivity in the system of dialectical contradictions can be made very precise and intuitive. It carries, however, the limitation that it forces a particular context onto the emergent system of dialectical contradictions, i.e. $\Re^2$ (and all other isomorphic spaces) that may not be appropriate in each case. The point to emphasise, though, is the concatenated nature of the syntheses in the system. Since $\mathbf{K}_t$ gathers all of the syntheses of the system of dialectical contradictions then $\mathbf{K}_t$ can be interpreted as the system of dialectical contradictions. As it will become clear later on, the actual definition is not relevant to the whole argument. In any case, unless stated, the definition to be used throughout the work is [2.a].

Note first that the set $\mathbf{K}_t$ is also time dependent for as history induces change on economic activity so does economic activity on the system that reflects it. Second, note that this set is connected for all t. The reason is simple: the syntheses of each of the dialectical contradictions concatenate, i.e. bind, the system as one single, inseparable abstract entity that reflects all of the relevant economic reality at once. If the system were not concatenated as one inseparable entity then there would exist two non-intercepting sets of syntheses of dialectical contradictions $\mathbf{K}^1{}_t$, $\mathbf{K}^2{}_t$, i.e. $\mathbf{K}^1{}_t \cap \mathbf{K}^2{}_t = \phi$. That is, dialectical materialism would be reflecting two parallel material realities. This is, by any consideration of individual cognition, wholly absurd for it can only be a pathological cognitive scenario (two parallel realities are, by definition, a schizophrenic world). Reality, is one and inseparable, at any point in time.

Possible And Probable Understandings

Expressions of individual understandings give meaning to the disturbance term in [1]. Indeed, if it is assumed that recorded facts of history are available up to t then the disturbance term $\lambda^t{}_i$ is in fact, and rigour, a random variable that associates possible events of understanding to real numbers. The construction of this variable highlights

14 2-ples are a mathematical expression completely identified with two dimension vector spaces, i.e. isomorphic, defined by (α, β) where α and β can be any mathematical entity. An n-tuple is identified with an n dimensional vector space, i.e. $\Re^n$.

one of the keys in modelling individual novelty. If an agent is to always possess the possibility of coming up with a revolutionary understanding, capable of perhaps changing the whole industry, then such an event of understanding must be present in the understanding space, even if it possesses only a very small probability. Otherwise, the possibility of novelty could not in general be guaranteed. In practice, psychological or cultural attributes are probably the most deterministic characteristics in the agent's ability to discern the surrounding economic activity. However, for modelling purposes, it is necessary to maintain the very same possibility as a probable event. In that manner it could always happen, as any external modeller should expect. Ultimately, culture and individual psychological development will condition individual understandings but only probabilities will guarantee novelty.

For expositional purposes we will first define the underlying understanding space and then proceed to construct it. In this particular context, it means that λ^t_i is a measurable function from a (probability) space, i.e. $\mathbf{\Omega}^t_i$, to the real numbers $\Re$, where the measure is defined on a sigma-field $D_t \subseteq \mathbf{\Omega}^t_i$. In other words, for every element (i.e. event of understanding) $\omega_i \in D_t \subseteq \mathbf{\Omega}^t_i$ there exists a probability (measure) π^t_i associated such that

$$1.\ \pi^t_i(\omega_i) = r_i \in \Re \quad \text{with} \quad \text{probability} \quad \pi_i \quad \forall i$$
$$2.\ \pi^t_i(\mathbf{\Omega}^t_i) = \bigcup_{1\le i\le q} r_i \in \Re \quad \text{with} \quad \text{probability} \quad \sum_{1\le i\le q} \pi_i \equiv 1$$

The upper limit of the index set, i.e. q, can of course be $\aleph_0$. Different probabilities may be constructed for different levels of generality and abstraction. The relevant issue in this case refers to the nature of the probability space ($\mathbf{\Omega}^t_i$, D_t, π^t_i), in particular to the elements of $\mathbf{\Omega}^t_i$. At time t, as a new system of dialectical contradictions emerges, the agent learns from it and whatever he/she learns is knowledge that is translated into an (possible) n-fold composite event of knowledge that belongs to $\mathbf{\Omega}^t_i$. As it was mentioned previously, these possible events of knowledge are themselves complemented by events of intuition. Hence, the n-fold composite event of knowledge and accompanying intuitions determine the sigma-algebra D_t that will (under certain restrictions) itself determine the composition of the probability space ($\mathbf{\Omega}^t_i$, D_t, π^t_i) at t +1. Formally, this space, i.e. $\mathbf{\Omega}^t_i$, is defined as

$$\begin{aligned}\mathbf{\Omega}^t_i &= \{u : u \text{ is an understanding}\}\\ &= \{u : u \in \mathbf{\Omega}^k_t \ \vee \ u \in \mathbf{\Omega}^{in}_t\}\end{aligned}$$

where

$$\mathbf{\Omega}^k_t = \{u : u \text{ is an event of knowledge}\}$$
$$\mathbf{\Omega}^{in}_t = \{u : u \text{ is an event of intuition}\}$$

and

$$m(\mathbf{\Omega}^{in}_t) < \infty$$

for any measure m defined on $\mathbf{\Omega}^{in}_t$. Whenever necessary, superscripts as well as subscripts will be used to identify the appropriate agent. All that this definition requires is that an event of understanding at time t must be either knowledge or intuition, and that the subset of intuitions must be finite under any measure. Under this scheme, history up to time t feeds the understanding space (and the underlying sigma-algebra) that determines the value of the random variable λ^t_i at t + 1. The requirements on the measure of $\boldsymbol{\Omega}^{in}_t$ will be explained later on in section VI.

The Links

Having defined the sets H_t, F_t, K_t and $\boldsymbol{\Omega}^t_i$ heterogeneity and diversity of knowledge follows directly through the following operators

$$\Delta_t : H_t \to K_t$$
$$\Xi^t_i : K_t \to \boldsymbol{\Omega}^t_i$$
$$\mathbf{I}^t_i : H_t \to \boldsymbol{\Omega}^t_i$$

termed the dialectical operator (Δ_t), formal learning (Ξ^t_i) and informal learning ($\mathbf{I}^t_i$) respectively, that link these sets amongst them. We can describe the composition of an agent's understanding space in terms of its subspaces. The passage from history to knowledge at t, for agent i, through formal learning, is thus the composite operator

$$\boldsymbol{\Theta}^t_i = \Xi^t_i \circ \boldsymbol{\Delta}_t : H_t \to \boldsymbol{\Omega}^k_i$$

The passage from history to intuitions is just

$$\mathbf{I}^t_i(H_t) = \boldsymbol{\Omega}^{in}_i$$

The passage from history to understandings spaces (through any form of learning) is different for every agent. The space of contradictions, i.e. $\mathbf{K}_t$, is not, however, different for each agent. This is a result, basically and unequivocally, of the fact that everyone faces the same history. In other words, every one faces the same history and at the same time every one understands it differently (because every one learns differently). If we assume for the time being that $\boldsymbol{\Omega}^t_i$ is indeed a proper probability space and define TS^i_{t+1} to be agent i's time series recorded up to t + 1 then we can represent the dynamics of understandings for agent i as

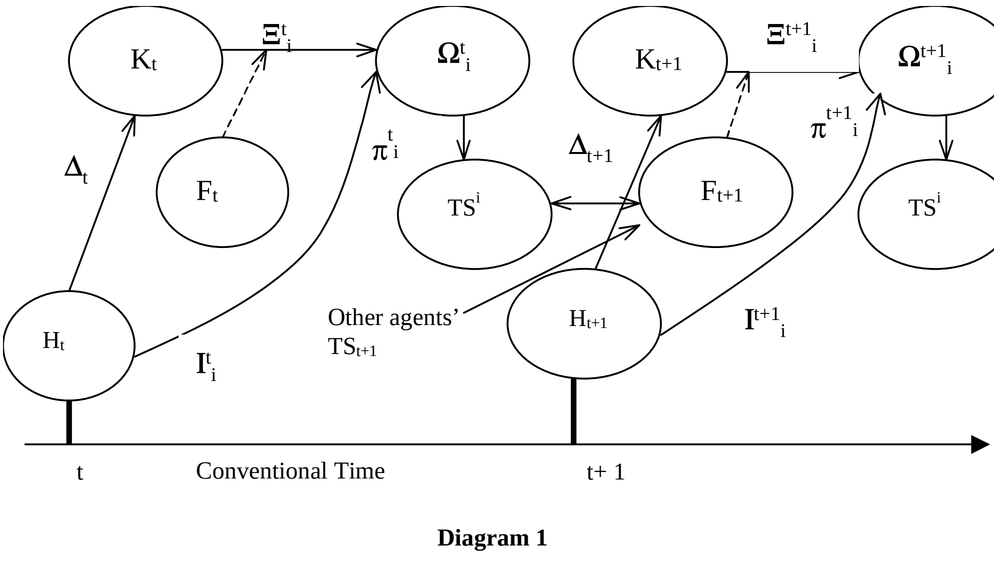

Diagram 1

What this diagram conveys is a graphical representation of the previous section. At t, agents face a history that allows them to learn (formally and informally). The understandings they gather are transformed into probable influences (of knowledge and intuitions) to be recorded through times series in the next period. The double arrow between TS^i_{t+1} and F_{t+1} is used only to emphasise that, as long as there are more than one agent, TS^i_{t+1} is a proper subset of F_{t+1}, i.e. $TS^i_{t+1} \subset F_{t+1}$.

The autopoetic nature of a process driven by the agents' understandings of the world is fully exposed in diagram 1. At time t more detailed, possibly new, (imperfect) expressions of recorded economic activity are brought forth from the last period. These new expressions enhance and possibly alter the manner in which the agents can learn from the forces that determine economic activity up to t. Hence, both history and its recorded facts determine what and how the agents learn. Furthermore, what the agents learn from these new systems may change the nature and composition of their understandings space, however extended or limited is the dialectical knowledge derived thereof. Innovative non-dialectical knowledge may emerge as well. In either case, the agents' understanding space will ultimately evolve from its previous state at t - 1. Hence, the agent's own record of their economic activity, i.e. the time series, will be influenced through these new expressions of understandings brought forward as events of understandings in $\mathbf{\Omega}^t_i$. The outcome of all the agents' actions, managed through between t and t + 1 will be recorded at t + 1. These records of history will possibly have evolved from t onto t + 1 and hence will have possibly influence the emergence of new dialectical understandings and intuitions about the world. In other words, understandings at t will have generated new understandings at t + 1 through knowledgeable economic praxis between t and t + 1. The description presented so far attains to individual agents and their respective personalised processes of understanding.

The scheme of understandings generation just describe is general enough to account for all possible sources of heterogeneity. In fact, it brings to the front the need for assumptions about the system's status quo, its mechanisms of change (including its rate of change) and possible direction of change, i.e. biasness, if there was one. In other words, it brings forth the need for assumption about what Metcalfe (1998) calls the fundamental determinants of evolutionary change. In terms of individuals this translates into a question of what they *can* understand, how they can understand and what may ultimately influence their beliefs. It means making assumptions about (individual) distinctions and (collective) similarities in between the agents. This is contrary to the traditional orthodox neoclassical approach to microeconomic modelling where just the opposite is assumed or pursued; that is, individual similarities and collective distinctions.

This should not be taken to mean that variety should be chased at all cost and to the limit. Indeed, some structure is always needed in order to be able to say something about heterogeneous interacting economic activity. The key is to seek and discover the sensible limit of variety in understandings and structure in the system. This should be a limit that can bind all economic agents under one analytical category, with common attributes and individual differences, flexible enough to allow for structural change yet, at the same time, rigid enough to retain it as a functional economic system.

V Further Characterisations

In this section we discuss some further characterisations concerning the spaces and operators used in the previous section. We will ascribe only to a general discussion and leave the mathematical details for the appendix. Whenever necessary we will make certain restrictive assumptions in order to make the discussions manageable and the proofs mathematically consistent.

The definitions provided in the previous section allow us to characterise a fundamental fact, stated before, and claimed by the Theory of Knowledge. That is, that (material) reality, in principle, is a <u>continuum</u> <u>of</u> <u>cognition</u>. Indeed, if we assume that the forces of history, at t, can be counted and that there exists a suitable form of aggregation of these forces (at least theoretically) then we can define a "conjugate" space of history. Let ν be the level of aggregation of forces of history (at least theoretically) so that the conjugate space can be defined as

$$H^{*}_{t,\nu} = \{(a, b): (a, b) \text{ are a pair of opposing forces of history considered at the level of aggregation } \nu, \text{ at time t}\}$$

It will soon become apparent that the issue of the level of aggregation is fundamental in the study of agents' capabilities in understanding economic activity. For the time being it can be established that the dialectical operator Δ_t can thus be defined indistinctively over history, i.e. H_t, or its conjugate, i.e. $H^{*}_{t,\nu}$. If defined over $H^{*}_{t,\nu}$, we can prove the three following complementary propositions:

<u>Proposition</u> <u>1</u> $\mathbf{K}_t$ is closed with respect to unions and intersections, that is

$$k_1 \cup k_2 \in \mathbf{K}_t \quad \wedge \quad k_1 \cap k_2 \in \mathbf{K}_t \text{ for any } k_1, k_2 \in \mathbf{K}_t.$$

<u>Proposition</u> <u>2</u> The dialectical operator Δ_t: $H^{*}_{t,\nu} \to \mathbf{K}_t$ is a one-to-one map for every fixed level of aggregation ν.

<u>Proposition</u> <u>3</u> The operator Δ_t: $H^{*}_{t,\nu} \to \mathbf{K}_t$ is continuous over $H^{*}_{t,\nu}$ (not *t*!).

What these three propositions, whose proofs can be found in sections A, B and C of the appendix respectively, entail is a characterisation (given our mathematical conceptualisation) on the availability of knowledge to the agents. Indeed, if $\mathbf{K}_t$ is closed under unions and interceptions it implies that $\mathbf{K}_t$ contains all possible "dialectical constructions". In other words, it contains all possible assemblages that may be structured upon more basic, primitive dialectical contradictions. Additionally $\mathbf{K}_t$ contains, by definition, all the syntheses associated to the (opposing) forces that define reality (dialectically). When combined with the fact that $\mathbf{K}_t$ contains all the constructs thereof (as stated in proposition 1), no matter how infinitesimally small or aggregated are the phenomena in question, $\mathbf{K}_t$ in fact becomes a direct reflection of reality. A sort of abstract mirror upon economic activity where each component in the mirror is a synthesis associated to a pair of (opposing) economic forces. This cognitive reflection, i.e. $\mathbf{K}_t$, indeed makes up a continuum, as claimed by the Theory of Knowledge. In the process no part of the economic activity has been lost through dialectics, i.e. the dialectical operator is continuous.

In as far as learning is concerned the key issue, in this case, lies in whether there exists or not a lower bound on the level of disaggregation of the forces, i.e. a lower bound on ν. A continuum of cognition implies necessarily that ν must, in principle, be capable of becoming infinitesimally small. It is up to the economic agents to pursue that lower bound. This brings to the front an old idea in a different disguise. That is that agents, in theory, have at their disposal the entirety of (dialectical) reality from which to learn. If they do not hold complete knowledge it is necessarily because they have not, or they cannot, learn it all (except neoclassical agents). The core competence literature recognises this fact and takes it as a corner stone of their approach to the study of the firm, e.g. Prahalad and Hamel (1990), Foss and Knudsen (1996). It has also been recognised by some authors in the economics literature as well. For example, Von Hayek (1937, 1945) spoke of agents incapable of knowing-it-all and hence of agents that hold at most focalised, subjective, bits of knowledge disseminated throughout the economy and coordinated through a pricing mechanism. Penrose (1959) and Richardson (1972) also spoke of limited expressions of knowledge within a firm that only holds what its constituent individuals can learn. The reader by Putterman and Kroszner (1996) contains several investigations on the nature of the firm as well, including Alchian and Demsetz (1972) study on industrial organisation.

Given a fixed level of aggregation ν then the syntheses in $\mathbf{K}_t$ are necessarily uniquely identified with its constituent forces at that level ν of aggregation. If we let $f_1 = (a_1, b_1)$, $f_2 = (a_2, b_2)$, $f^* = (a^*, b^*)$, $f^\wedge = (a^\wedge, b^\wedge)$ represent four pairs of opposing forces aggregated at level ν, then we can graph their relationship with their syntheses in the following manner:

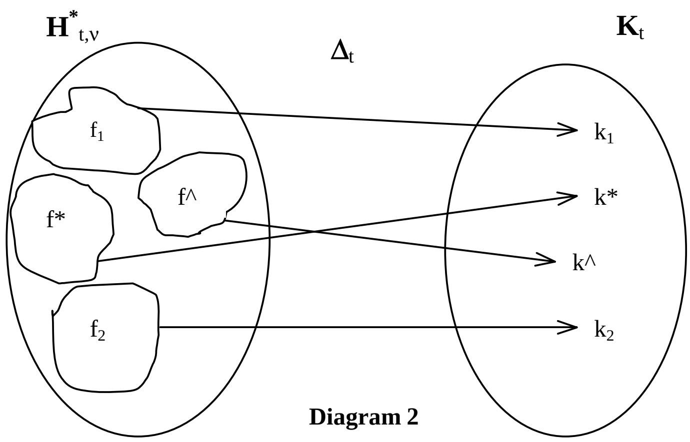

Diagram 2

We can characterise perfectly knowledgeable neoclassical agents based on the above discussions in the following lemma, which we state without proof.

<u>Lemma</u>. Perfect knowledge, a la neoclassical, implies that the agents can learn as if $\nu = 0$. That is, neoclassical agents hold an <u>infinitely</u> <u>dense</u>, continuum, cluster of knowledge at a single moment in time.

This is a direct consequence of propositions 2,3 and 4. If an agent learns as if total disaggregation had taken place, i.e. $\nu = 0$ at a moment in time, then the agents will have discovered and understood <u>all</u> of the components of economic activity. Hence, the agent will have (formally) learnt it all and thus will hold perfect knowledge of the material economic activity, as defined through dialectics. What in reality takes place, that is in a non-neoclassical world, is that agents learn from $\mathbf{K}_t$ with $\nu > 0$. In other words, they learn with constraints (for the reasons stated at the beginning) thus generating bounded understandings of the world with incomplete beliefs and limited inner logic. That is, they learn in a manner as to induce bounded rationality.

Finally, the present context presents a natural argument against any idealistic conception of understandings in economics. In terms of the description presented here this implies $\mathbf{\Omega}^t_i$ is an open set. Our argument is, again, by contradiction. If $\mathbf{\Omega}^t_i$ were closed then there could exist an event of understanding with a neighbourhood containing some element that might not be an event of understanding. This would imply that there exists a space $\mathbf{\Omega}^*$ such that $\mathbf{\Omega}^t_i \subseteq \mathbf{\Omega}^*$. But since $\mathbf{\Omega}^t_i$ incorporates all possible understandings arising from material economic activity (including intuitions) then $\mathbf{\Omega}^t_i$'s complement (with respect to $\mathbf{\Omega}^*$) can only be composed of that which has no origin in material reality[15]. This consideration allows for the possibility of understandings to exist and originate as a possibly idealistic cognisant phenomenon, i.e. economic understandings might originate from non-material foundations, even if dialectical in nature. This possibility, although philosophically valid, attempts against any common sense upon economic activity. Loasby's (1999, 2000, 2001) insistence on the mental patterns that help individual associate perceived economic phenomena only reaffirms our argument.

Moreover, idealistic knowledge eliminates the possibility of determining some of the core dimensions of the unobserved underlying driving structures of an economic system. The cycle (or loop) defined through the stages of economic activity-abstraction-economic activity is lost. Because of its very nature, idealistic understandings could never provide the necessary context for the emergence of this cycle. Without this cycle one could never guarantee the effects that new emergent understanding will bring onto the industry later on. In other words, history would be lost as a definitive source of understandings. Indeed, some of these effects, and this is the key, are responses to observed past economic activity. Therefore, it is only through this cycle that a causal chain of events can be guaranteed to exist.

The next section is entirely devoted to the construction of π^t_i, that is the probability distribution of agent i at time t. To do this, learning must be examined first so that the required structure, in $\mathbf{\Omega}^t_i$, can be identified. At the same time it will also require making some concrete assumptions about the possible events of knowledge.

VI The Probability Space of Individual Understandings

So far the space $\mathbf{\Omega}^t_i$ is just the union of the image of Ξ^t_i and $\mathbf{I}^t_i$. That is, the space that gathers all events of understanding. These are, forms of understandings concerning the state of affairs up to t – 1, achieved through formal and informal learning between periods t - 1 and t. Even without an explicit recognition of these influences, the definitions above do not, by themselves, guarantee that a probability space could be constructed from $\mathbf{\Omega}^t_i$ for the random variable λ^t_i at t. The evolution of the present

15 As it was mentioned before intuitions are an acceptable (very much real) form of understanding originated in material reality. In particular, economically relevant intuitions necessarily emanate from material economic activity itself for, otherwise, agents could understand something about economic processes without observing them. That is, out of coincidence which, for present ontological purposes, is meaningless.

state of dependencies of knowledge across the industry will depend on the industry's ability to perceive and measure i's knowledgeable-supported praxis. That is, it will depend on the rest of the agents' abilities to inform themselves of i's knowledgeable behaviour. In order for that to happen, understandings must somehow become available to the rest of the industry, e.g. econometric formulations, data banks. For that, understandings must be conformed in some probability space. Therefore the question concerning the existence of a Borel σ-algebra necessary in the construction of a probability is fundamental. For this, further conditions on learning are required. In fact,

<u>Proposition</u> <u>4</u> If the (formal) learning operator, Ξ^t_i, is <u>unique</u> (to the agent) at a given moment in time then $\mathbf{\Omega}^t_i$ is in fact a σ-algebra at that moment. Equivalently, for every (individual) learning process there exists a unique σ-algebra, i.e. $\mathbf{\Omega}^t_i$.

<u>Proof</u> Consider an arbitrary agent i at an arbitrary moment in time. In order to prove that $\mathbf{\Omega}^t_i$ is in fact a σ-algebra at t it must be proven that

1. $\mathbf{\Omega}^t_i \in \mathbf{\Omega}^t_i$
2. For any subset of $\mathbf{\Omega}^t_i$ its complement also lies in $\mathbf{\Omega}^t_i$. That is, for all $S \in \mathbf{\Omega}^t_i$, $S^C \in \mathbf{\Omega}^t_i$.
3. Any (infinitely) countable union of subsets of $\mathbf{\Omega}^t_i$ lies in $\mathbf{\Omega}^t_i$. That is, if $\{S_i\}$ is a countable collection of subsets of $\mathbf{\Omega}^t_i$ then
$$\bigcup_{1 \le i \le \infty} S_i \in \mathbf{\Omega}^t_i$$

Notice that we have used the belong-to symbol, i.e. $\in$, instead of the inclusion symbol, i.e. $\subseteq$, because $\mathbf{\Omega}^t_i$ is being considered as a collection of (sub)sets and not a set in itself.

We can trivially ascertain that $\mathbf{\Omega}^t_i \in \mathbf{\Omega}^t_i$ (as a sigma algebra) since $\mathbf{\Omega}^t_i \subseteq \mathbf{\Omega}^t_i$. Hence, the rest of the proof deals with 2) and 3).

Consider two arbitrary events in $\mathbf{\Omega}^t_i$ such that $\omega_j, \omega_k \neq \phi$. If one of these objects were the null set, say ω_k, then we would trivially have
$$\omega_j \cup \omega_k = \omega_j \in \mathbf{\Omega}^t_i$$
for all j, k. The same is true if countably infinite unions were considered. Now, consider the pre-image of $\omega_j \cup \omega_k$. There are three possible situations in this case: $\omega_j, \omega_k \in \mathbf{\Omega}^k_i$ or $\omega_j, \omega_k \in \mathbf{\Omega}^{in}_i$ or $\omega_j \in \mathbf{\Omega}^k_i$ and $\omega_k \in \mathbf{\Omega}^{in}_{i,}$. Let's see them case-by-case:

If $\omega_j, \omega_k \in \mathbf{\Omega}^k_i$ then the pre-image of
$$\omega_j \cup \omega_k = \Xi^t_i[k_j] \cup \Xi^t_i[k_k]$$
is
$$\left[\Xi^t_i[k_j] \cup \Xi^t_i[k_k]\right]^{-1} \qquad \text{[a]}$$
Since Ξ^t_i is unique, it defines a one-to-one relationship between syntheses and events of understanding. Henceforth, [a] must equal
$$k_j \cup k_k \in \mathbf{K}_t$$
by virtue of proposition 1. In other words, there exists a class of synthesis of dialectical contradictions, namely $k^* = k_j \cup k_k$ that must have an event of knowledge associated. Whence,

$$\Xi_i^t[k_j \cup k_k] = \Xi_i^t[k_j] \cup \Xi_i^t[k_k] = \omega_j \cup \omega_k = \Xi_i^t[k^*] \in \mathbf{\Omega}_i$$

This is valid for an infinitely countable union since

$$\bigcup_{1 \le i \le \infty} k_i \in \mathbf{K}_t$$

If $\omega_j, \omega_k \in \mathbf{\Omega}^{in}{}_i$ then

$$\omega_j \cup \omega_k = \mathbf{I}_i^t[\text{History}] \cup \mathbf{I}_i^t[\text{History}]$$

Now, since intuitions arise out of yet to be discovered protocols of learning we can identify, at least mathematically, each intuition with a different informal operator. That is,

$$\mathbf{I}_i^t[\text{History}] = \omega_i$$

$$\tilde{\mathbf{I}}_i{}^t[\text{History}] = \omega_k$$

Now, the union of both intuitions must necessarily be an intuition because, otherwise, it would become knowledge. That is, one of the operators $\mathbf{I}_i^t, \tilde{\mathbf{I}}_i{}^t$ would be captured (or transformed into) in $\Xi^t{}_i$. In other words, the union of intuitions could possibly alter the uniqueness of the formal operator at t. Furthermore, the same is true for any infinitely countable union of intuitions. If the (infinitely countable) union were not an intuition it would mean that one of the undiscovered operators would become part of formal learning and hence violate our initial assumption. Hence,

$$\bigcup_{1 \le i \le \infty} \omega_i \in \mathbf{\Omega}_i^{in} \subseteq \mathbf{\Omega}_i^t$$

Finally, if $\omega_j \in \mathbf{\Omega}^k{}_i$ and $\omega_k \in \mathbf{\Omega}^{in}{}_i$, that is, one event is knowledge and the other intuition, then

$$\omega_j \cup \omega_k \in (\mathbf{\Omega}_i^k \cup \mathbf{\Omega}_i^k) = \mathbf{\Omega}_i^t$$

for the two events ω_j and ω_k. In general, any infinitely countable unions of events must lie in either $\mathbf{\Omega}^k{}_i$ or $\mathbf{\Omega}^{in}{}_i$; that is, it must lie in $\mathbf{\Omega}^t{}_i$.

Finally, $\omega \in \mathbf{\Omega}^t{}_i$ then we can define its complement as $\omega^C = \omega_k \cup \omega_{in}$ where ω_k and ω_{in} are possibly composite (unions of) events of knowledge and intuitions respectively. By virtue of the previous arguments we have that $\omega_k \in \mathbf{\Omega}^k{}_i$ and $\omega_{in} \in \mathbf{\Omega}^{in}{}_i$. Hence,

$$\omega^C = \omega_k \cup \omega_{in}(\mathbf{\Omega}^k{}_i \cup \mathbf{\Omega}^{in}{}_i) = \mathbf{\Omega}^t{}_i$$

for an arbitrary event of understanding $\omega \in \mathbf{\Omega}^t{}_i$. Hence, $\mathbf{\Omega}^t{}_i$ is a Borel σ-algebra. Q.E.D.

What this proposition, whose proof can be found in section D of the appendix, states is that, as long as there is a uniquely identifiable path between systems of dialectical contradictions and dialectical understandings, then the understanding space itself is a σ - algebra. To use previous terminology it means that $D_t = \mathbf{\Omega}^t{}_i$. It is a requirement of identification. As long as (formal) learning is unique the events of knowledge that it delivers are identifiable, and hence they can be counted. It does not require the learning operator $\Xi^t{}_i$ to be the same throughout time, it only requires that it not change in between ticks of the (conventional) clock. There could possibly be different learning operator $\Xi^t{}_i$ for every t; however, in the present context the change must take place instantaneously at t.

Since by definition the subspace of intuitions is of finite measure for any measure constructed on $\mathbf{\Omega}^t_i$, then a proper probability measure is guaranteed to exist if the elements in $\mathbf{\Omega}^t_i$ possess certain further characteristics, i.e. essentially topological. A topology in this case constitutes a set of possible understandings. Moreover, different topologies give rise to different characterisations of the understandings space because, ultimately, they represent different possibilities. This is very much in the spirit of Belianin's (2000) approach to the question of topologies in economically related abstract sets. In fact, we can prove the following:

Proposition 5 If formal learning is unique between t – 1 and t then the space ($\mathbf{\Omega}^t_i$, D_t, π^t_i) is indeed a probability space for any t.

Proof. Proposition 4 proved that if learning is unique then $D_t = \mathbf{\Omega}^t_i$ is a indeed a sigma algebra. Whence, the proof will be concerned with the construction of the underlying probability distribution π^t_i. The coming arguments are based on Ash(1972), Edwards (1972) and Weir (1974), particularly Edwards (1972).

There are two possible approaches for the construction, the difference being whether the space of understandings are considered to be continuous of discrete. Both require assumptions that somehow restrict the generality of the representation. Nevertheless, they do not compromise in any way whatsoever the previous arguments. The requirements on the appropriate topology will be made precise whenever necessary.

First Approach. Assume that $\mathbf{\Omega}^t_i = [a^t_i, b^t_i] = I^t_i \subset \Re$ where $0 \in [a^t_i, b^t_i]$. Whence, the elements of $\mathbf{\Omega}^t_i$ are the subintervals of I^t_i. These subintervals are closed although they need not be in principle since the difference is constituted by a set of measure zero, i.e. discrete points. Now, consider then following:

1. Let the space of continuous functions over I^t_i be S and fix an arbitrary function $f \in S$. Now, consider a partition of I^t_i

$$a^t_i < x_1 < x_2 < \ldots < x_n < b^t_i$$

   where $p_i = x_i - x_{i-1}$. Call this partition $P_j = \{p_i\}$.
2. Let

$$\int_{p_i} f \, dx = \xi_{p_i}$$

The measure dx is a Lebesgue measure.

3. Let

$$\xi^{MAX}_{P_j} = \max_{P_j} \{\xi_{p_i}\}$$

This is the maximum value of the integral over the different subintervals defined by the partition P_j. In fact, the quantity defined above is valid for *any* partition P_j of I^t_i. Hence,

4. The quantity

$$\xi^*_f = \max_{P_j} \{\xi^{MAX}_{P_j}\}$$

   is well defined. This quantity ξ^*_f is the maximum value of the area under f for an interval, given *all possible* partitions of I^t_i.

Now, define $F_i : S \rightarrow \Re$ through which each $f \in S$ has a real number associated, i.e. ξ_f^*. F_i is continuous and linear since the process that defines Fi, i.e. steps 1-4, only involve integration and choosing. Additionally, all functions in S are continuous as well. Finally, choose a topology τ so that I^t_i is both compact and Hausdorff; then the RRT ensures that we can represent F_i as

$$F_i(f) = \int f \, d\mu$$

where μ is a regular Borel measure that depends on the representation. Take careful notice that the integral in the above representation of F_i _is_ _not_ the integral of step 2. This is so because step 2 is only the integral over a specific subinterval of the partition P_j whereas F_i involves maximising the value of the integrals over all possible subintervals, defined by P_j, throughout the family of possible partitions of I^t_i, i.e. $\{P_j\}$. F_i is, in fact, much more than an integral. It provides an approximation to a central tendency in the functions considered.

In particular, the identity function over this interval $Id_i : [a^t_i, b^t_i] \rightarrow [a^t_i, b^t_i]$ is continuous under any topology. Then we can ascertain that

$$F_i(Id_i) = \int_{[a_i^t, b_i^t]} Id_i d\mu = m_i$$

Notice that

$$\frac{1}{m_i} F_i(Id_i) \equiv 1$$

We can now consider the same argument on all the subsets of $\boldsymbol{\Omega}^t_i$ defined as stated at the beginning. Then the identity function over a subinterval $subI^t_i$, denoted $Id_i(\omega_j) : subI^t_i \rightarrow subI^t_i$ is again bounded and continuous and hence the RRT is applicable. In fact we can define a probability measure for (any) $\omega_j = [\alpha_j \; \beta_j]$ (where of course $a^t_i \leq \alpha_j \leq \beta_j \leq b^t_i$) by considering

$$\pi_i^t(\omega_j) = \frac{1}{m_i} F_i(Id_i(\omega_j)) = \frac{1}{m_i} \int_{[\alpha_j \beta_j]} Id_i(\omega_j) d\mu = \alpha_j < 1 \qquad \text{[b]}$$

If we define non-intercepting events of understanding to be $\omega_j \cap \omega_l = [\alpha_j \; \beta_j] \cap [\alpha_l \; \beta_l] = \phi$ then

$$\pi_i^t(\omega_j + \omega_l) = \frac{1}{m_i} F_i(Id_i(\omega_j + \omega_l)) = \frac{1}{m_i} \int_{[\alpha_j \beta_j]} Id_i(\omega_j) d\mu + \frac{1}{m_i} \int_{[\alpha_l \beta_l]} Id_i(\omega_l) d\mu = \alpha_j + \alpha_l$$

In fact if $\bigcap_{i \in \tilde{J}} subI_i^t = \phi$ where $\tilde{\mathbf{J}}$ is some indexing set for $[a^t_i, b^t_i]$ then

$$\pi_i^t(\bigcup_{i \in \tilde{J}} \omega_i) = \pi_i^t(\sum_{i \in \tilde{J}} subI_i^t) = \frac{1}{m_i} \sum_{i \in \tilde{J}} \int Id_i(\omega_i) d\mu \equiv 1 \qquad \text{[c]}$$

where $\tilde{J}$ is the i^{th} member of the indexing set $\tilde{J}$. Finally, if $\{subI^t_i\}$ are such that they do not reduce integration to a set of measure zero, i.e.

$$\pi_i^t(\sum_{i \in \tilde{J}} subI_i^t) = 0$$

then, π^t_i has finite variation given by

$$V(\pi_i^t) = \int_{[a_i^t, b_i^t]} Id_i d\mu = \|F_i\|$$

Equations [E.1] and [E.2] guarantee that π^t_i is indeed a well defined probability measure over $\mathbf{\Omega}^t_i = [a^t_i, b^t_i]$. This construction specifies the values in probability for each event of understanding through [b]. Conceptually, it represents an approximation of the central tendency in the occurrence of the event.

The probability distribution π^t_{system} for the product space, here termed $\mathbf{\Omega}^t_{system} = \mathbf{\Omega}^t_1 \times \mathbf{\Omega}^t_2 \times \cdots \times \mathbf{\Omega}^t_n$, is directly induced by each component. Indeed, if each agent's understanding space $\mathbf{\Omega}^t_i = [a^t_i\ b^t_i]$ has an associated topology τ_i that makes it compact and Hausdorff then the product topology will make $\mathbf{\Omega}^t_{system}$ compact and Hausdorff (by virtue of Tychonoff's theorem and the natural separation of points in $\mathbf{\Omega}^t_{system}$). The analogous linear functional F_{system} is now defined the product space $S_{system} = S_1 \times S_2 \times \cdots \times S_n$ so that

$$F_{system} : S_{system} \to \mathfrak{R}^{\mathbf{n}}$$

where

$$F_{system} = F_1 \times F_2 \times \cdots \times F_n$$

Each one of the components of F_P defines the marginal probability upon that particular agent. The probability distribution π^t_{system} is conditioned on the individual probability distributions and is thus defined as

$$\begin{aligned}\pi^t_{system}\left(\theta_j \middle| \mathbf{\Omega}^t_1, \mathbf{\Omega}^t_2, \ldots, \mathbf{\Omega}^t_n\right) &= \sqrt{\sum_i \left(\pi^t_i(\omega_{j,i})\right)^2} \\ &= \sqrt{\sum_i \left(\frac{1}{m_i} F_i(Id_i(\omega_{j,i}))\right)^2} \\ &= \sqrt{\sum_i \left(\frac{1}{m_i} \int Id_i(\omega_{j,i}) d\mu\right)^2}\end{aligned}$$

where the event of understanding in the entire system is the composite event

$$\theta_j = \omega_{j,1} \times \omega_{j,2} \times \cdots \times \omega_{j,n}$$

The value of each one of the components of $\pi^t_{system}\left(\theta_j \middle| \mathbf{\Omega}^t_1, \mathbf{\Omega}^t_2, \ldots, \mathbf{\Omega}^t_n\right)$ lies, by construction, between 0 and 1. Geometrically, the probability density may be graphed as an n-dimensional cube.

<u>Second Approach</u>. Assume that $\mathbf{\Omega}^t_i$ is countable. Hence, there exists an isomorphism between $\mathbf{\Omega}^t_i$ and the natural numbers **N** that identifies each event of understanding with an (ordered) natural number. Whence, we can write $\mathbf{\Omega}^t_i = \{\omega_j\}$. This identification also leads naturally to identification with the rational numbers **Q** as well (because of the isomorphism between **N** and **Q**). In any case, consider an arbitrary function $\varphi : \mathbf{\Omega}^t_i \to \mathfrak{R}$ such that

$$\varphi(\omega_j) = r_j$$

where r_j are discrete real numbers. Now, we can order the r_j's using the induced order from the real numbers, i.e. $\leq$ so that $T_i = \{r_j\}$ is an ordered countable set. Because of this order T_i thus has a minimum and a maximum value. Call them r^{MIN} and r^{MAX} respectively. Also, T_i is trivially σ - algebra because of the identificability with $\mathbf{\Omega}^t_i$. Only unions and complements of elements in T_i that can be "traced back" to $\mathbf{\Omega}^t_i$ have any meaning, i.e. are defined. Hence, T_i is necessarily a σ - algebra.

Consider now,

1. Let $f\in S^*$ ={set of continuous functions on T_i} be an arbitrary function. The existence of this set is equivalent to saying that there exists a topology δ that gives rise to a set (actually a Hilbert space of functions) of continuous functions. Hence, define

$$m(f(r_k)) = \left|f(r_k) - r^{MIN}\right|$$

so that

$$m(f(r_k) + f(r_l)) = \left|\max\{f(r_k), f(r_l)\} - r^{MIN}\right|$$

2. Let

$$m_f^* = \max_{T_i}\{m(f(r_k))\}$$

Hence, for every $f\in S^*$ there exists an associated real number, i.e. m^*_f. We can then define the following functional

$$F_i(f) = m^*_f$$

F_i is not just the difference between numbers but actually involves choosing the biggest difference amongst them as determined by T_i and f. It is a continuous linear functional since given a fixed set T_i a sequence $\{g_i\}$ can always be defined to approximate any $f\in S^*$. This is so because of the Cauchy property of the real numbers and any subset thereof. The Cauchy property states that given any fixed real number there exists a convergent sequence of real numbers whose difference amongst the members of the sequence shrinks the further the convergence. Finally, any $f\in S^*$ is defined over the real numbers and has a range of real numbers. Hence, choose a topology τ over T_i that: 1) maintains the continuity of any $f\in S^*$, 2) makes T_i compact and Hausdorff. Then, the RRT, allows us to represent F_i as

$$F_i(f) = \int f\, d\mu$$

In particular,

$$F_i(Id_{T_i}) = \int Id_{T_i}\, d\mu = \left|r^{MAX} - r^{MIN}\right| = \xi_i$$

so that

$$\frac{1}{\xi_i}F_i(Id_{T_i}) \equiv 1$$

Therefore, if $r_l < r^{MAX}$ then

$$F_i(Id_{T_i}(r_l)) = \int Id_{T_i}(r_l)d\mu = \left|r_l - r^{MIN}\right|$$

We can define a probability measure for the event ω_l as

$$\pi_i^t(\omega_l) = \frac{1}{\xi_i}F_i(Id_{T_i}(r_l)) = \frac{1}{\xi_i}\int Id_{T_i}(r_l)d\mu = \frac{\left|r_l - r^{MIN}\right|}{\xi_i} = \alpha_l < 1 \qquad [d]$$

If ω_l and ω_k are two non-intercepting events then $r_l \cap r_k = \phi$ then we can define the probability of the union as

$$\pi_i^t(\omega_l \cup \omega_k) = \frac{1}{\xi_i}F_i(Id_{T_i}(r_l \cup r_k)) = \frac{1}{\xi_i}\int Id_{T_i}(r_l \cup r_k)d\mu = \max\left\{\frac{1}{\xi_i}\int Id_{T_i}(r_l)d\mu, \frac{1}{\xi_i}\int Id_{T_i}(r_k)d\mu\right\}$$

From this we can ascertain, using the identificability of $\mathbf{\Omega}^t_i$ with T_i, that if $\bigcap \omega_j = \phi$ then

$$\pi_i^t(\bigcup \omega_j) = \frac{1}{\xi_i} F_i(Id_{T_i}(\bigcup r_l)) = \frac{1}{\xi_i} \int Id_{T_i}(\bigcup r_l) d\mu = \sum \alpha_l \le 1 \qquad [e]$$

In particular, if $\bigcup r_l = T_i$ then

$$\pi_i^t(\bigcup \omega_j) \equiv 1 \qquad [f]$$

Equations [c], [d], [e] and [f] ensure that π^t_i is a proper probability distribution for $\boldsymbol{\Omega}^t_i$ at time t.

Just as in the previous construction, the topological properties of the product space $\boldsymbol{\Omega}^t_{system} = T_1 \times T_2 \times \cdot\cdot\cdot \times T_n$ follow directly from the properties of each one its component. A (product) topology τ_{system} that makes $\boldsymbol{\Omega}^t_{system}$ Hausdorff and compact is one determined by the product of those topologies τ_i that make each $\boldsymbol{\Omega}^t_i$ Hausdorff and compact. Again, the analogous linear functional is defined as

$$F_{system} : S_P^* \to \Re^n$$

where

$$S_P^* = S_1^* \times S_2^* \times \cdots \times S_n^*$$

and

$$F_{system} = F_1 \times F_2 \times \cdots \times F_n$$

The probability distribution for an arbitrary composite event θ_j in $\boldsymbol{\Omega}^t_{system}$ is thus

$$\begin{aligned}
\pi^t_{system}\left(\theta_j \middle| \boldsymbol{\Omega}^t_1, \boldsymbol{\Omega}^t_2, \ldots, \boldsymbol{\Omega}^t_n\right) &= \sqrt{\sum_i \left(\pi^t_i(\omega_{j,i})\right)^2} \\
&= \sqrt{\sum_i \left(\frac{1}{\xi_i} F_i(Id_i(r_{j,i}))\right)^2} \\
&= \sqrt{\sum_i \left(\frac{1}{\xi_i} \int Id_i(r_{j,i}) d\mu\right)^2} \\
&= \sqrt{\sum_i \left(\frac{\left|r_{j,i} - r_i^{MIN}\right|}{\xi_i}\right)^2} \\
&= \sqrt{\sum_i \left(\alpha_{j,i}\right)^2}
\end{aligned}$$

where $\theta_j = \omega_{j,1} \times \omega_{j,2} \times \cdots \times \omega_{j,n}$ and $\alpha_{j,k} \le 1$ for k = 1, 2, …, n. Again, each component $\frac{\left|r_{j,k} - r_k^{MIN}\right|}{\xi_k}$ defines the marginal probabilities of π^t_{system} for each k = 1,2,…, n.
Q.E.D.

It can be seen that a number of different probabilities can be constructed. We have only shown two in order to emphasise the role of learning and understandings in general in the process of formation of the understanding space. Moreover, assumptions about the agents' understanding spaces and their relation to the real numbers are not so restrictive either; especially in light of the equivalence of all measures in $\Re$. The crystallizations of understandings are always, at any time, real numbers simply because they are part of measured economic activity. Hence, it is by no means arbitrary to conceptualise the agents' understandings spaces, in as far as modelling is concerned, as subsets of $\Re$. As stated previously, this particular construction of the probability measure was chosen for $\mathbf{\Omega}^t_i$, which is ultimately equivalent to a Lebesgue measure (because of the need to crystallise understandings as measured recordings), essentially to highlight the dynamics of understandings. Having stated this, the constructions are by no means unique. Notwithstanding, all constructions are equivalent in as much as knowledge and intuitions are concerned.

The probability distribution, constructed at time t, based on the dynamics of understandings generation is of course π^t_i, the probability just constructed, i.e.

$$\pi^t_i : \mathbf{\Omega}^t_i \rightarrow [0,1] \qquad [2]$$

This probability, i.e. [2], is the underlying probability measure of the random variable λ^t in in our general statistical specification of section I. Moreover, they may have different specific forms as far as empirical issues are concerned. However, for the present context, it is not a concern simply because the basis of this probability are not (possibly) repeated experiments nor combinatorial deducts. It is not a subjective measure of anything either. They represent the <u>effects</u> <u>of</u> <u>history</u>, as channelled through learning, borne by the different agents. Anything else concerning the nature of these probabilities is an assumption. In particular, for agent i, we have that

$$\lambda^t_i(\omega) = r_\omega \in \Re$$

where

$$\pi^t_i(\omega) = x \in [0,1], \forall \omega \in \mathbf{\Omega}^t_i$$

given, of course, that a suitable topology τ has been chosen to represent the set of possible events of understanding by the agent at t. In this case, then, the event ω takes place as $r_\omega \in \Re$ with probability $0 \leq x \leq 1$. In terms of our econometric specification we have

$$h^t_i = f(h^t_i, z^t_i) + r_{\omega*}$$

with probability $x_{\omega*}$ for some event of understanding $\omega*$ acquired at t – 1; that is, $\lambda^t_i(\omega*) = r_{\omega*}$ and $\pi^t_i(\omega*) = x_{\omega*}$.

The construction of the probability distributions associated to the random disturbance term λ^t_i emphasised a couple of points previously hinted at. First of all, it made explicit the requirement that learning be unique in between measurements of economic activity, i.e. between ticks of the conventional clock. Second, the requirement, at every t, on the topology of $\mathbf{\Omega}^t_i$ highlighted the fact, recognised in the core competence literature, that not all forms of understandings do indeed become relevant in individual decision making procedures. There are capabilities constraints inherent in the internal workings of individuals and firms that restrict the availability in use of understandings to agents and firms. The process was also seen to be entirely subjective in respect to learning. Individual (or firm) limitations on understanding processing capabilities were reflected in the spread of possible topologies to consider. Hence, we can always interpret limitations in individual understandings as arising from either learning or inabilities in handling understandings (thus defining coarser and coarser topologies). There will always exist at least one topology, i.e. the coarsest, that will guarantee the applicability of the RRT at each t. However, this case is very uninteresting and trivial. Agents normally operate anywhere in a position between the most trivial topology, i.e. the coarsest, and the finest, i.e. the one that contains all possible subsets of $\mathbf{\Omega}^t_i$[16]. In general, the topologies associated to each $\mathbf{\Omega}^t_i$ specify the possibilities of understandings. The RRT specifies their probable occurrence.

VII An Example

We will develop the example based on the second approach in the proof of proposition 5. Hence, consider 4 distinct agents each with a countable understanding space, i.e. $\mathbf{\Omega}^t_i$ is countable for i = 1,2,3,4. We can (arbitrarily) define the functions φ_i such that

$$\mathbf{\Omega}^t_1 : \varphi_1(\omega_j) = \frac{q_j}{2p_j} = r_j \quad \text{where} \begin{cases} r^{MIN} = 0 \\ r^{MAX} = 4 \end{cases}$$

$$\mathbf{\Omega}^t_2 : \varphi_2(\omega_j) = \frac{q_j}{3p_j} = r_j \quad \text{where} \begin{cases} r^{MIN} = 1 \\ r^{MAX} = 7 \end{cases}$$

$$\mathbf{\Omega}^t_3 : \varphi_3(\omega_j) = \frac{q_j}{5p_j} = r_j \quad \text{where} \begin{cases} r^{MIN} = 3 \\ r^{MAX} = 10 \end{cases}$$

$$\mathbf{\Omega}^t_4 : \varphi_4(\omega_j) = \frac{q_j}{9p_j} = r_j \quad \text{where} \begin{cases} r^{MIN} = \frac{1}{2} \\ r^{MAX} = \frac{3}{2} \end{cases}$$

where q_j and p_j are whole numbers, i.e. $q_j, p_j \in \mathbf{Z}$. Whence, at t, the agents' probability distributions are

$$\pi^t_1(\omega_j) = \frac{1}{4}\left[\frac{q_j}{2p_j}\right] = \frac{q_j}{8p_j} \quad \text{for all events } j$$

$$\pi^t_2(\omega_j) = \frac{1}{6}\left[\frac{q_j}{3p_j} - 1\right] = \frac{q_j - 3p_j}{18p_j} \quad \text{for all events } j$$

$$\pi^t_3(\omega_j) = \frac{1}{7}\left[\frac{q_j}{5p_j} - 3\right] = \frac{q_j - 15p_j}{35p_j} \quad \text{for all events } j$$

$$\pi^t_4(\omega_j) = \frac{1}{1}\left[\frac{q_j}{9p_j} - \frac{1}{2}\right] = \frac{2q_j - 9p_j}{18p_j} \quad \text{for all events } j$$

Each one of the sets T_i is in fact a strict subset of, i.e. it can never be equal to, the rational numbers that exist between the maximum and minimum values in T_i. This is simply because of the definitions of φ_i. We can depict this graphically for T_i as in the following diagram

16 This topology is called the Power Set of $\mathbf{\Omega}^t_i$ and it is denoted as $P(\mathbf{\Omega}^t_i)$.

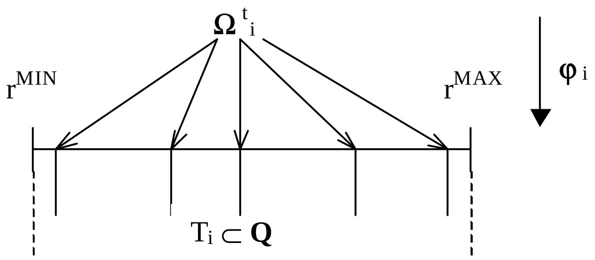

Diagram 3

The set **Q** represents the rational numbers, i.e. all numbers that can be written as $\frac{q_i}{p_i}$.

The probability distribution of the system, i.e. $\mathbf{\Omega}^t_i$, is defined as the conditional probability, which in this case is,

$$\pi_{system}\left(\theta \middle| \mathbf{\Omega}^t_1, \mathbf{\Omega}^t_2, \mathbf{\Omega}^t_3, \mathbf{\Omega}^t_4\right)$$

Whence, we can define the probability measure for the system as

$$\pi^t_{system}\left(\theta_j \middle| \mathbf{\Omega}^t_1, \mathbf{\Omega}^t_2, \mathbf{\Omega}^t_3, \mathbf{\Omega}^t_4\right) = \sqrt{\left(\pi^t_1(\omega_{j,1})\right)^2 + \left(\pi^t_1(\omega_{j,2})\right)^2 + \left(\pi^t_1(\omega_{j,3})\right)^2 + \left(\pi^t_1(\omega_{j,4})\right)^2}$$

$$= \sqrt{\left(\frac{q_{j,1}}{8p_{j,1}}\right)^2 + \left(\frac{q_{j,2} - 3p_{j,2}}{18p_{j,2}}\right)^2 + \left(\frac{q_{j,3} - 15p_{j,3}}{35p_{j,3}}\right)^2 + \left(\frac{2q_{j,4} - 9p_{j,4}}{18p_{j,4}}\right)^2}$$

where, $\theta_j = \omega_{j,1} \times \omega_{j,2} \times \omega_{j,3} \times \omega_{j,4}$ and $q_{j,k}$, $p_{j,k}$ are whole numbers for k = 1,2,…,∞ and $\omega_{j,i} \in \mathbf{\Omega}^t_i$. The cumulative probability density function associated to π^t_{system} may be depicted as the surface (up to a set of measure zero) of a four-dimensional cube defined by (x_1, x_2, x_3, x_4) where $0 \le x_k \le 1$ for k = 1, 2, 3, 4.

VIII The Evolution of an Agent's Understanding Space

An agent's understandings space is defined at any moment in time by the present economic activity, as reflected by $\mathbf{K}_t$, and his/her learning capacity. Furthermore, all forms of path dependency are incorporated in K_t. Hence, the understandings space's evolution will follow the path determined by history still-to-come. This still-to-come history will define what he/she can learn. The evolution of his/her learning process, on the other hand, will determine how and to what extent the agent learns. Given certain restrictions the agent's understanding space's evolution can be modelled using elementary homotopy theory.

Consider the following: let $\mathbf{\Omega}^t_i$ be agent i's understandings space, where

$$\mathbf{\Theta}^t_i[\text{History}] = \mathbf{\Omega}^k_i$$

$$\mathbf{I}^t_i[\text{History}] = \mathbf{\Omega}^{in}_i$$

then if there exists a homotopy that can deform the identity over $\mathbf{\Omega}^t_i$ into another function, i.e. some other space's identity function, then i's understandings space will have been deformed. That is, i's understandings space will have been deformed if there exists a homotopy H such that

$$\mathrm{Id}_{\boldsymbol{\Omega}_i^t} \approx \mathrm{Id}_{\tilde{\boldsymbol{\Omega}}_i^t} \qquad \mathrm{rel}(A)$$

where $A \subset \boldsymbol{\Omega}_i^t, \tilde{\boldsymbol{\Omega}}_i^t$. A is a subset of both understanding spaces and is of course allowed to be the null set. Note that the set $\Re \cup \{+\infty, -\infty\}$ is topologically the same, i.e. homeomorphic, to the unit interval I = [0 1]. Therefore if i's understandings space were to be constituted by $\Re \cup \{+\infty, -\infty\}$ then it would only be necessary to study the evolution of the interval [0,1].

Observe an important consideration with respect to time. First of all, the process just described is pertinent to both biological and conventional time. Through both frames understandings may emerge and hence the agents' understandings spaces be deformed. For a matter of concreteness, though, let time be considered in its continuous conventional frame, i.e. $t \in [0,1]$. Second, the deformation, or evolution, takes place in its entirety within the unit interval since $-\infty$ is identified with 0 and $+\infty$ with 1. The process is thus a continuous deformation through history reflecting the agent's evolving adapting expressions of understandings. At t = 0, Δ_0 and $\Xi^0{}_i$ are identified with the beginning of i's history and at t = 1, Δ_1 and $\Xi^1{}_i$ with his/her end. In this sense all forms of initial path dependencies of understandings are incorporated in $\Xi^0{}_i$, i.e. in the agent's status quo at t = 0. Finally, the issue of the continuity of the deformation will become crucial when specifying equations as well as the continuous nature of the space through which the deformation takes place.

Consider now the following example: let $\boldsymbol{\Omega}^t{}_i = [0, 1]$ so that

$$\mathrm{Id}_{\boldsymbol{\Omega}_i^t}(\omega) = \omega \qquad \forall \omega \in \boldsymbol{\Omega}_i^t = [0,1]$$

Now if the evolution of i's understandings space is such that it contracts, i.e. it collapses, to a point $\overline{\omega}$ in the space then the homotopy

$$\mathrm{H}(\omega, t) = t\overline{\omega} + \omega(1-t) \qquad [4]$$

continuously deform the identity over $\boldsymbol{\Omega}^t{}_i$ = [0 1] into the identity over $\tilde{\boldsymbol{\Omega}}_i^t = \{\overline{\omega}\}$since

$$t = 0 \Rightarrow \mathrm{H}(\omega,0) = \omega$$

$$t = 1 \Rightarrow \mathrm{H}(\omega,1) = \overline{\omega}$$

If $\overline{\omega} \notin I$, i.e. if it were the case that $\boldsymbol{\Omega}^t{}_i = I \cup \{\overline{\omega}\}$, the homotopy defined in [4] would still be valid. Moreover, geometrically, it would represent the net effect of a reduction of the space itself, i.e.

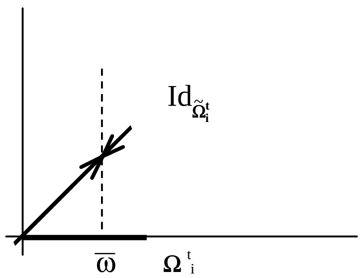

Diagram 4

The homotopy in [4] actually represents the collapse of a probability density function to a single point. Graphically

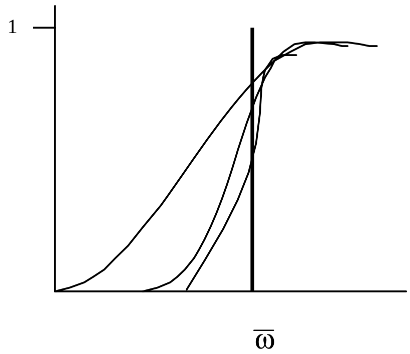

Diagram 5

The understandings space $\mathbf{\Omega}^t{}_i$ itself may be continuous or discrete, i.e. it may define a continuous or a discrete probability distribution. This depends on the composition of available forms of understandings. Furthermore, if the space is discrete, homotopies are still well defined since continuity of paths over the space depends on the existence of an appropriate topology on $\mathbf{\Omega}^t{}_i$ so as to make them continuous (which, tautologically, $\mathbf{\Omega}^t{}_i$ will always possess). We can express the deformation of the agent's understanding space, i.e. $\mathbf{\Omega}^t{}_{t,}$ in terms of the deformation of both of its constituent subspaces. In fact, for $\mathbf{\Omega}^t{}_t$ and $\mathbf{\Omega}^t{}_t$ we can define the following homotopies (over $H^*_{t,\nu}$) respectively

$$H(\varsigma, t) = t\overline{\Xi^1_i(\Delta_1(\varsigma))} + (1-t)\Xi^t_i(\Delta^t(\varsigma))$$
$$\tilde{H}(\varsigma, t) = t\overline{\mathbf{I}^1_i(\varsigma)} + (1-t)\mathbf{I}^t_i(\varsigma)$$

where $\varsigma \in H^*_{t,\nu}$. Note that by construction history has been identified with the unit interval. Hence, it can be seen that the homotopy that represents $\mathbf{\Omega}^t{}_i$'s evolution is determined through the dialectical operator Δ_t and the formal learning operator $\Xi^t{}_i$ and the informal operator $\mathbf{I}^t{}_i$.

This can be taken a step further if we make the further assumption that $\mathbf{\Omega}^t{}_i \subseteq \Re^n$ for some n. Then the homotopy in [4] can be thought of as a specific case of a more general type of homotopies. In fact, the function that acts as the identity of the space at t = 1 is unknown a-priori and is also continuous for every $t \in [0,1]$. Contrary to the identity of the initial understandings space $\mathbf{\Omega}^t{}_i$, whose structure and formation is known, through the learning and dialectical operators, the only thing known about the structure and formation of the final understanding space is that it is the result of continuous deformations induced through changes in the composition of understandings acquired by the agent. In this respect understandings create a backward linking systemic web through time. If understandings were complete, i.e. they did form a continuum, then each one of constituting elements at t = 1 can be traced back to its origin at t = 0. Since future understandings can at most be speculated about at present then every future emergent expression of cognition incorporated in the understandings space can at most be interpreted, from our present position in time and space, as future (statistical) noise to be adhered. Indeed, from our present perspective, future history has not happen yet. It is at most a possibility, one in the infinitely (perhaps, uncountably) many possibilities. Hence, from our perspective, future history is structureless. History that has not taken place cannot account for anything that might or might not induce, especially as time goes on into infinity. This is the essence of Popper (1950) argument: we cannot know the future because history has not yet happened as of yet. This was also insinuated in Polanyi (1946). Along these lines Popper (1956) proposes an interpretation of scientific knowledge as one made up of conjectures and hypotheses that are subject to verifiability. In this context we can interpret future history as one composed of pure statistical noise. This requires a hypothesis (or conjecture) concerning the underlying probability distribution of this noise. For a matter of expositional clarity we will assume that this noise has a Gaussian probability distribution with varying degrees of scatteredness, i.e. varying variances[17]. Hence, we can interpret our ignorance of the future in such a manner that, from our position in time, i.e. t = 0, the identity function of the future knowledge space throughout [0,1] is a Wiener process, i.e. it is Brownian Motion realisation. Hence the (convex) deformation of the original space $\mathbf{\Omega}^t_i$ is given by

$$H(\omega, t) = t\,BM_t + (1-t)Id_{\mathbf{\Omega}^1_i} \qquad [5]$$

as long as there exists a space $\mathbf{\Omega}_U$ such that $BM_t, \mathbf{\Omega}^t_i \subset \mathbf{\Omega}_U$. In this case the Brownian Motion BM_t represents a realisation through the space of deformation. That is, an agent's understandings space's deformation can always be modelled through a homotopy as long as there exists a larger understandings space that contains both the initial and the final understandings space, e.g. $BM_t \cup \mathbf{\Omega}^t_i = \mathbf{\Omega}_U$ upon which H(ω, t) is continuous under some topology. At time 1> t^* > 0 we can represent $\mathbf{\Omega}^t_i$'s deformation as

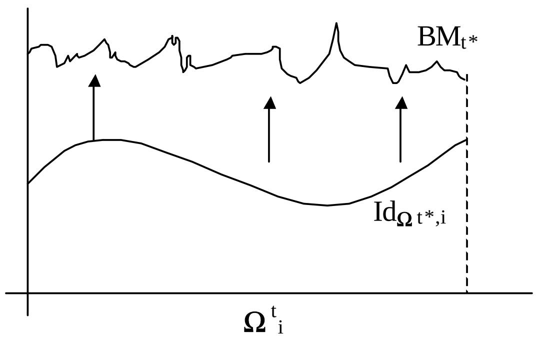

Diagram 6

Diagram 6 is meant to highlight the simple dynamics of the deformation. At our present point in time t*, future history has not taken place yet. The deformation of the space up to t* carries with it the path dependencies generated by past history. Since we cannot possibly know or intuit (at most speculate) about future history then, from our perspective, the future identity can be, at most and from our position, a Brownian Motion process evaluated at t*. The arrows indicate that as time goes on, i.e. t → 1,

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17 In this sense, our conjecture becomes that the identity of the agents' understanding spaces in the future is a Brownian Motion process from our present stance. This is the claim subjected to verifiability.

the agent is reaching his/her economic destiny and the dynamics of understandings are leading him/her to it.

Brownian Motion processes reflect further undertakings to the processes of evolving individual understanding. For any points in time $0 \leq t_1 < t_2 <\ldots< t_k \leq 1$ the changes in Brownian Motion $[BM_{t2} - BM_{t1}]$, $[BM_{t3} - BM_{t2}],\ldots,[BM_{tk} - BM_{tk-1}]$ are all independent multivariate Gaussian such that $[BM_s - BM_t] \sim N(0, s\text{-}t)$ for all s and t in [0,1]. That is, changes in the realisations of the Brownian Motion process, i.e. changes in the future understanding space, are not expected to occur and the dispersion of these changes increases with time. These characteristics of Brownian Motion, conceptually, reflect the precise nature of the changes to an agent's understandings space. Understandings become a possibility only through history, which, in the future, has not taken place yet. Hence, understandings should not be expected to change from their actual form and essence when looked at from the present. To put it differently, changes in understandings should be expected to take place incrementally; that is, as white noise processes.

In order to model an agent's understandings space in a somewhat realistic fashion it should be assumed that the agent in fact holds limited information that contributes to some individual articulation of his/her bounded rationality. In particular, the agent's learning mechanisms allow him/her to develop and adapt a proper understandings space that reflects what he/she is capable of learning from past observations on the environment. This mere postulate eliminates the possibility of using continuous mathematical tools, e.g. continuous real analysis, differential equations (stochastic or not), etc., in order to model the evolution of understanding spaces. Notwithstanding, homotopies can be used to describe the changes to the agents' understanding spaces by assuming (and constructing if so desired) that there exists a coarse enough topology as to make all function defined over $\mathbf{\Omega}_U$ continuous. If $\mathbf{\Omega}^t_i$ is a discrete then the BM_t term in [5] must be replaced[18]. This is so because all of the events in $\mathbf{\Omega}^t_i$, according to the logic from above, will follow themselves a BM_t process. In section G of the appendix further details of this issue can be found. We can represent this diagrammatically as

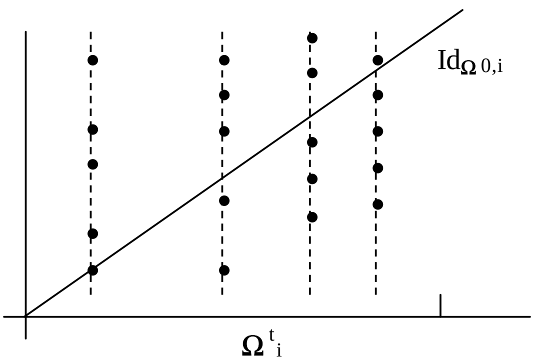

Diagram 7

The vertical lines are meant to represent the Brownian Motion process for each event and the black dots simply realisations of this events at different points in time after 0,

18 Basically, this implies that the entire space would be described as a family of Brownian motion.

i.e. $t > 0$. Initially, all events in the understanding space lie in the 45-degree line. As time goes on they move up and down in an entirely continuously random manner, i.e. following Brownian Motion processes.

The discrete-continuity question concerning the composition of the initial understandings space brings forth an additional issue, the one concerning the connectivity of understandings (within the bounded limits of our problem herein). For if the understanding space is connected (in the topological sense) then it would be possible to define connected paths of understandings. If this were the case then it must imply that there exist appropriate syntheses of dialectical contradictions that connect differentiated individual manifestations of understandings across the understandings space. Since the system of dialectical contradictions that reflects economic activity, i.e. $\mathbf{K}_t$, is concatenated, i.e. connected then $\mathbf{\Omega}^t_i$ is connected which implies that individual learning must be intricate and refined enough to give rise to links that can bind the entire space. These links are local in nature and may develop to create an interconnected network of understandings. Through the connectivity in this space, understandings disperse and ultimately bind themselves, in a sort of enveloping event. Furthermore, in the limit, connectivity (of knowledge and intuitions) creates *structure*. If this structure of understandings is indeed found then understandings-driven non-random developmental evolution within the industry can be, at least conceptually, guaranteed to give rise to some abstract model of it. Moreover, in the presence of structure, this type of understandings-related evolution will inevitably emerge. Section IX is devoted to this topic. First, though, we make further use of homotopies to study mutual influences amongst agents through understandings. Then, we deal with some bounded rationality issues as they emerge in this context.

IX Mutual Influences Through Understandings

If there are n agents at time t and each agent's adapting process of understanding economic activity around him/her is as described in diagram 1 then there are n simultaneous such processes taking place between $t - 1$ and t (assuming of course that there has been no entry or exit in between $t - 1$ and t). In order to guarantee that each agent's understanding space $\mathbf{\Omega}^t_i$ was a σ-algebra at t, the learning process was required to be unique. In other words, it could not change between $t - 1$ and t if $\mathbf{\Omega}^t_i$ is to define a σ-algebra upon which, through the RRT, a probability measure could be constructed. No matter what the rest of the agents' bearing is on an agent, the (formal) learning operator must be one and only one if it is to allow a proper probability space to be constructed. This implies that learning routines must not change in between ticks. The learning mechanisms that the agent had at $t - 1$ must carry him/her to t. This situation implies that no other agent can aid the agent in learning, essentially, because aid could possibly change the agent's learning routine and the required uniqueness of the process would be lost. There cannot be any mutual influence between agents in between ticks of (conventional) time in a way to alter the nature of the (formal) learning operator of either agent. In this section we relax this requirement upon learning and study the consequences. That is, we further study the learning operator by including the possibility of mutual influence and hence leave open the possibility that a probability space might not be constructible at all times. More importantly, if the agents can influence each other's learning routines, i.e. operators, then they can influence what each understands. In this way, they can influence each other's understandings. Also, the necessary assumptions to assure a probability space at all times will be revealed in light of mutual influence of understandings.

What do mutual influences amongst agents really mean? Consider two agents, say j and i, at time t. If one of them, say j, does something, whether purposely or not, upon whose actions, i changes his/her learning practices, between t - 1 and t, then j directly or not, intended or not, will have influenced i's learning practice. In the context of diagram 1 this means that the nature of the learning operator, i.e. Ξ^t_i and possibly its functional specification, has somehow changed. This "influence" on the learning operator takes place through a possibly distinct mode of time from that in which dependencies in understandings are represented. Indeed, influences between j and i need not occur through any convened expression of time. Nor do influences need to be measured in arbitrary sequentially spaced units of time. Hence, to incorporate features by which agents could influence each other implies a consideration of possibly two distinct frames of time: a conventional one and a biological one. Furthermore, as mentioned before, in principle, these two time frames need not intersect although conventional time can always be interpreted as an approximation of biological time. That is

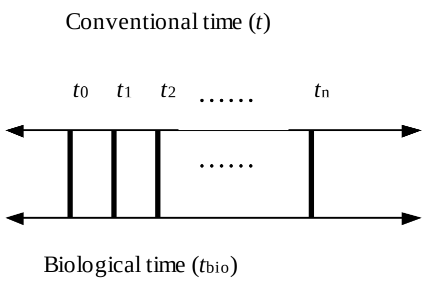

Diagram 8

Whatever the manner through which these influences play themselves out the outcome is always, at any point in (conventional) time, the ultimate possibility of changes to $\mathbf{\Omega}^t_i$. Additionally, if this influence, developed amongst the agents through biological time, <u>coincides</u> in the time frame with that of conventional time, i.e. they are both the same, then the required uniqueness of the learning operator might be lost.

Consider two agents, say i and j; if agent i begins influencing j's learning operator indefinitely, *ad perpetum* through t_{bio}, then there will come a point where j's learning modes and routines will have changed to a point of being equivalent to that of i's. This process, when taken to the limit, will have completely altered j's learning operator into i's operator. Hence, in the present context, mutual influences <u>always</u> deform the agents' formal learning operators into the other's formal learning operator, however small is the induced change. Note that the complete deformation of one formal learning operator into the other does not imply that the agents' understandings spaces must ever be the same, i.e. $\mathbf{\Omega}_{t,j} \neq \mathbf{\Omega}_{t,i}$. This is so, ultimately, because intuitions are always and at any point in time, through any frame of it, different and hence heterogeneous in the population. Nevertheless, mutual influences can be modelled mathematically with the aid of a (special type of) homotopy; one that can deform one operator, i.e. Ξ^t_i, into the other, i.e. Ξ^t_j. In fact, if we further require that

- The homotopy that deforms these operators $H_{i,j}(t_{bio}, k)$, where $k \in \mathbf{K}_t$, also be continuously differentiable at all points
- Biological time t_{bio} be also normalise to an interval (just as in the last section), basically by identifying $\pm\infty$ with the end points of an interval, call it I_{bio}

Noting that, although different, conventional time is contained in biological time then

$$H_{i,j}(t_{bio}, k): I_{bio} \times \tilde{\mathbf{K}} \to \mathbf{\Omega}_U^t \qquad [6]$$

where $\mathbf{\Omega}_U^t = \mathbf{\Omega}_i^t \cup \mathbf{\Omega}_j^t$ and $\tilde{\mathbf{K}} = \bigcup_{t \in I_{bio}} \mathbf{K}_t$ completely describes any mutual influence through biological time between i and j by deforming Ξ^t_i into Ξ^t_j and, if need be, vice-versa. Note that the subscripts depicted in the learning operators and the understandings spaces are measurements of conventional time whereas possible deformations of these operators into each other take place through a different frame of time, i.e. biological. Formal learning, in both i's and j's cases, is the composite outcome of the two-fold process taking place through convention and nature, i.e. factography and the biological concourse of nature respectively. This can be represented graphically as

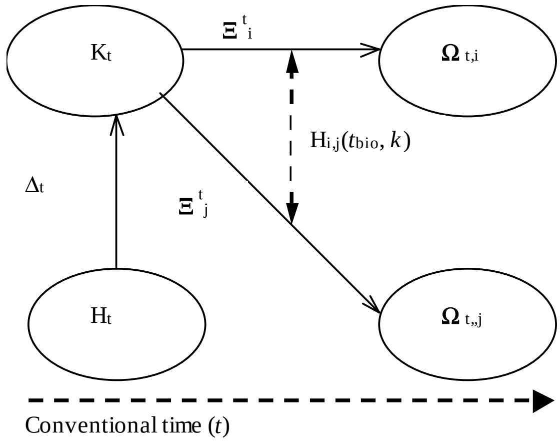

Diagram 9

When can $\mathbf{\Omega}^t_i$ become a σ-algebra, in light of mutual influences through biological and conventional time, then? If we assume that conventional time and biological time develop through the same frame irrespective of how discrete are the measurements that define conventional time, then the homotopy's derivative with respect to (biological) time

$$\frac{d}{dt_{bio}} H_{i,j} = \dot{H}_{i.j} \qquad [7]$$

defines the rate of change of Ξ^t_i into Ξ^t_j and vice versa. In other words, [7] determines how one learning operator is being transformed into the other through continuous biological time. To put it differently, [7] tells whether at one point in time (which is the same for both frames of time by assumption) there is actual deformation, i.e. any mutual influence between i and j, at all or not. Therefore, if at $\bar{t} = \bar{t}_{bio}$, i.e. both conventional and biological, we have that

$$\dot{H}_{i,j}\Big|_{\bar{t}} = 0$$

then each formal learning operator, at $\bar{t}$, bears no influence from the other agent. Hence, $\boldsymbol{\Omega}^t{}_i$ may become a proper σ-algebra at t. From this argument, it becomes clear that either the evolution of mutual influence (or its functional representation) must be known, i.e. the evolution of $\Xi^t{}_i$, or the influences must be specified through a particular (everywhere differentiable) homotopy $H_{i,j}$ (actually these two things are logically equivalent). Hence, having assumed that conventional time and biological are the same, then the learning operator for (say) agent i could be finally (and definitively) be specified as

$$\Xi_i^t = \Xi_i^t(t, H_{i,j}(t,k)) \qquad [8]$$

where t and $H_{i,j}$ develop through the same frame of time.

In general, note that in order to discover whether $\boldsymbol{\Omega}^t{}_i$ can indeed become a σ-algebra it is sufficient to evaluate $\dot{H}_{i,j}$ at every $\{t_n\}_{n\in N}$ where each t_n is a measurement in conventional time. If $\dot{H}_{i,j}\big|_{t_n} = 0$ then $\boldsymbol{\Omega}_{t,i}$ is a σ-algebra at t_n.

The above arguments can be easily generalised to the entire system of n agents. With n agents then (any) an arbitrary agent, say j, faces possible influences from n – 1 agents. Hence, both n - 1 homotopies and their rate of change must be known or n - 1 possibly distinct mutual influences, with j, must be known. Agent j's learning operator is thence

$$\Xi_j^t = \Xi_j^t(t, H_{j,1}, H_{j,2}, \ldots, H_{j,j-1}, H_{j,j+1}, \ldots, H_{j,n}) \qquad [9]$$

In this case, for j's understanding space $\boldsymbol{\Omega}^t{}_i$ to actually be a σ-algebra at $\bar{t}$ it must be true that

$$\dot{H}_{j,l}\big|_{\bar{t}} = 0$$

for l = 1, 2,…, j - 1, j + 1,…, n. This is a highly unlikely situation in actual, real and concrete, economic activity to say the least. If we assume that $I_{bio} = [t_1, t_2]$ then we can depict the possibility of deformations for agent i's learning operator in reference to the other agents, throughout biological time $t_1 < t < t_2$ as

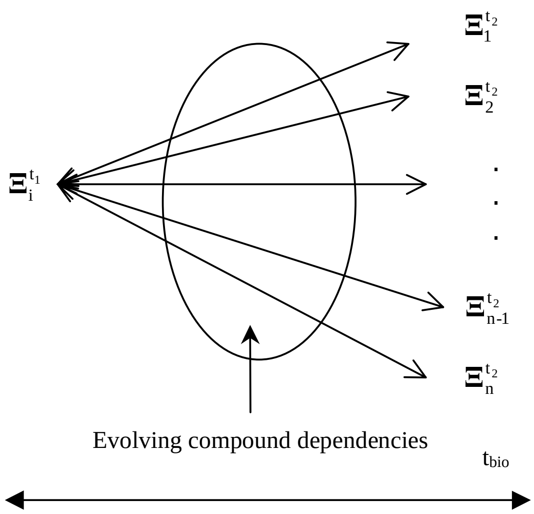

Diagram 10

Finally, the need for assumptions should be noted. It is necessary to make assumptions in order to make any sensible economic model of understandings interdependencies in a system composed of heterogeneous agents that learn through conventional and biological time, influence each other through biological (and possible conventional) periods and keep records of their activity through conventional stances. These assumptions, for the case of formal learning, could be summarised as

- Conventional time and biological time develop through the same frame irrespective of how they are both measured.
- Either the evolution of mutual influences (manifested through the formal learning operators) or the corresponding homotopies and their derivatives must be known. These requirements are logically equivalent.

X Conclusions

The passage from history to understandings spaces was seen to be essentially a two-fold sequential process. First, an elaboration of a system of materially-based dialectical contradictions takes place. Second, formal individual learning takes place over this system. The result is knowledge, which is complemented through informal learning that determines intuitions. Moreover, it is at this latter stage, the learning stage, in which heterogeneity is guaranteed. Knowledge becomes an individual human attribute once it is internalised. All forms of individual understandings are gathered in the agents' understandings space.

History, it was argued, exposes new facts, which are incorporated to the "stock" of information. Ergo, time presents itself as the required vertex through which possibilities of understanding new syntheses sprout. These new possible understandings, depending on each individual agent, may or may not become new probable events of understandings. The key issues in this respect were seen to be

- The required uniqueness of learning
- A suitably chosen topology in the space of understandings.

This implied that not all possible forms of understandings could become probable understandings. In other words, for every suitable topology there is a set of (possibly distinct) probable understandings. A suitable topology was seen to be one that made the agent's understandings space a Borel σ-algebra so that a proper probability space can be defined through the application of the RRT. In short, the first fundamental result was that for every different learning procedure and every different topology (and any combination thereof) there exists a possibly distinct probability distribution associated to λ^t_i. The details of this construction can be found in section E of the appendix.

It was seen that the effects that history bears upon the agents' understandings space, i.e. their evolution, can be modelled as a continuous deformation of the space through time. This required several assumptions as well. If it is assumed that

- Biological and conventional times take place in the same frame of time
- Either all learning operators are known or the homotopies that deforms them are known
- The homotopies are continuously differentiable

then mutual influences through understandings can be studied as deformation of one (formal) learning operator into another. The value of the (time) derivative homotopies at each moment determines the rate of deformation, i.e. mutual influence. Finally, and this is the second fundamental result of the paper, section VII establishes that if the three assumptions above hold and the value of all the derivatives at a moment in time are zero then the agent's understandings space is a Borel σ-algebra. This requirement is logically equivalent to the first fundamental result, established in section VI.

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APPENDIX

A. Proposition 1

Consider two arbitrary contradictions from $\mathbf{K}_t$, say k_i and k_j. Beyond the expression's digital symbolism, the union (of anything) captures the idea of two aggregated entities, in this case of two contradictions. When put together these two contradictions can represent either a more extensive system than the one defined by each of the contradictions separately[19] or a new more aggregated (and less specific) contradiction. Furthermore, two facts stand out: there exist at least four (opposing) forces that define the new contradiction (or system) and there exists a *path* that concatenates k_i and k_j together (because of the connectivity of $\mathbf{K}_t$). Let $(a, b) \leftrightarrow k_i$ and $(d, e) \leftrightarrow k_j$ be the four forces that underlie the new (aggregated) context. If the aggregation of forces is (arbitrarily) defined to be $(a, b) + (d, e) = (a \oplus d, b \oplus e)$ where $\oplus$ is some form of aggregation of forces (at least conceptually) then what the union of contradictions, i.e. $k^* = k_i \cup k_j$, represents is simply a new unity of opposites where the thesis and antithesis are $a \oplus d$ and $b \oplus e$ respectively. The particular definition of the aggregation of forces just presented only attempts to maintain consistency in the notation and conceptual coherence. However, it need not be the only form of aggregation. Moreover, the case for the interception of contradictions, i.e. $k_i \cap k_j$, is even more direct. Indeed, if the interception in non-empty, i.e. $\phi \neq k_i \cap k_j$, then the interception of either the theses, i.e. a and c, or the antitheses, i.e. b and d, or both is non-empty. That is, either $a \cap c \neq \phi$ or $b \cap d \neq \phi$ or both. Hence, the interception $k^\wedge = k_i \cap k_j$ of both contradictions is itself a contradiction defined by the (opposing) forces $(a \cap c \setminus a \oplus c, b \cap d \setminus b \oplus d)$. In both case the forces are defined to be what is mutual minus what is not shared. In this case, the commonality in the theses and antitheses is what defines the possibly "smaller" contradiction $k^\wedge$. Finally, notice a subtlety in the manner that the elements of $\mathbf{K}_t$ are handled: aggregation is always indirect and not necessarily unique whereas interception is not. The reason is that the interception of forces is always readily defined in terms of commonality whereas aggregation of forces requires a mechanism, or rule, of aggregation.

B. Proposition 2

Our argument is intrinsically by contradiction. If Δ_t is defined over $H^*_{t,\nu}$, i.e. $\Delta_t{:}H^*_{t,\nu} \rightarrow \mathbf{K}_t$, and it is not a one-to-one map then there exists at least one $k \in \mathbf{K}_t$ such that $[k]^{-1} = [\Delta_t(\varsigma_t)]^{-1}$ is not a singleton. That is, the preimage of k is not a singleton. To put it differently, there exist at least 2 distinct pairs of economic opposing forces, at t, i.e. $\varsigma_t \neq \varsigma^*_t$, such that $\Delta_t(\varsigma_t) = k = \Delta_t(\varsigma^*_t)$. But, since the level of aggregation ν of forces is fixed and both ς_t and ς^*_t give rise to the same synthesis, i.e. k, then necessarily they must both represent the same aggregate of forces. That is, $\varsigma_t = \varsigma^*_t$. This is a contradiction since Δ_t was assumed not to be one to one. Whence, it must be one to one for a fixed level of aggregation ν.

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19 In strict rigour, what the two aggregated contradictions, in fact, form is a *subsystem* of the entire system.

C. Proposition 3

In order to prove the continuity of Δ_t it is sufficient to consider an arbitrary convergent sequence $\{\varsigma_t\}$ of events in Δ_t's domain. If $\{\Delta_t(\varsigma_t)\}$ is convergent as well then Δ_t is continuous. As before Δ_t is defined over $H^*_{t,\nu}$ (this means that ς_t is a pair of opposing economic forces at t). The level ν of aggregation of economic forces induces a natural topology on $H^*_{t,\nu}$ that may be termed $\tau_{t,\nu}$. This topology is, by construction and nature, countable and is composed of all of the countable subsets of $H^*_{t,\nu}$ at a fixed level ν of aggregation. This topology is the finest topology induced by the level of aggregation ν. Also, it should be clear that this (finest) topology changes with ν. Hence, if $\{\varsigma_t\}$ is a convergent sequence then it means that the sequence has a cluster point under some topology, possibly coarser than $\tau_{t,\nu}$. Call this topology $\rho_{t,\nu}$, i.e. $\rho_{t,\nu} \subseteq \tau_{t,\nu}$, and the sequence's cluster point $\bar{\varsigma}_t$. That is, for all neighbourhoods $N(\bar{\varsigma}_t)$ of $\bar{\varsigma}_t$, under $\rho_{t,\nu}$, there exists at least one $\varsigma_l \in \{\varsigma_t\}$ different from $\bar{\varsigma}_t$ such that

$$\varsigma_l \in N(\bar{\varsigma}_t)$$

Now, since Δ_t is one to one (for the fixed level of aggregation ν) then $\Delta_t(\bar{\varsigma}_t)$, $\{\Delta_t(\varsigma_t)\}$ and $\{\Delta_t(p)\}$, where $p \subset \rho_{t,\nu}$ are the components of the topology, are all well defined quantities in $\mathbf{K}_t$. In fact, the collection $\{\Delta_t(p)\}$ forms a topology for $\mathbf{K}_t$ (because Δ_t is one to one). This topology is, again, possibly coarser than the topology defined by the image, under Δ_t, of $\tau_{t,\nu}$. Call the topology in $\mathbf{K}_t$, that arises from the image of $\rho_{t,\nu}$ under Δ_t, $\delta(\rho_{t,\nu})$. This topology $\delta(\rho_{t,\nu})$ is also countable and completely identified in $H^*_{t,\nu}$. For this topology every neighbourhood $N(\Delta_t(\bar{\varsigma}_t))$ of $\Delta_t(\bar{\varsigma}_t)$, under $\delta(\rho_{t,\nu})$, contains at least one member $\Delta_t(\varsigma_l) \in \{\Delta_t(\varsigma_t)\}$, different from $\bar{\varsigma}_t$. To see this, it is enough to consider the preimages of the objects in K_t. That is, pick an arbitrary neighbourhood of $N(\Delta_t(\bar{\varsigma}_t))$ and consider its preimage $[N(\Delta_t(\bar{\varsigma}_t))]^{-1}$. Since Δ_t is one to one and $\delta(\rho_{t,\nu})$ is discrete topology then $[N(\Delta_t(\bar{\varsigma}_t))]^{-1}$ must be a neighbourhood of $\bar{\varsigma}_t$. Hence, it must contain a member of the series $\{\varsigma_t\}$, i.e. there exists at least one $\varsigma_l \in \{\varsigma_t\}$ such that $\varsigma_l \in [N(\Delta_t(\bar{\varsigma}_t))]^{-1}$. This implies that there exist an open set in $[N(\Delta_t(\bar{\varsigma}_t))]^{-1}$, say O_{pre}, such that $\varsigma_l \in O_{pre}$. Since Δ_t is one to one and $[N(\Delta_t(\bar{\varsigma}_t))]^{-1}$ is a discrete set we therefore have

$$\Delta_t(\varsigma_l) \in \Delta_t(O_{pre}) \subset N[\Delta_t(\bar{\varsigma}_t)]$$

That is, given an arbitrary neighbourhood $N(\Delta_t(\bar{\varsigma}_t))$ of $\bar{\varsigma}_t$, under $\delta(\rho_{t,\nu})$, then there exists at least one $\Delta_t(\varsigma_l) \in \{\Delta_t(\varsigma_t)\}$, different from $\bar{\varsigma}_t$, such that $\Delta_t(\varsigma_l) \in N(\Delta_t(\bar{\varsigma}_t))$. Hence, $\Delta_t(\bar{\varsigma}_t)$ is a cluster point of $\{\Delta_t(\varsigma_t)\}$, under the "induced" topology $\delta(\rho_{t,\nu})$. That is, $\{\Delta_t(\varsigma_t)\}$converges and Δ_t is thus continuous for the level for aggregation ν.

D An Unrealistic Situation: An Agent That (Formally) Learns it All

The passage of time carries with it the passage of history. Through this pathway more facts and records of it are incorporated into the available stock of information. Whence, in terms of the learning operator and the dialectical operators, changes in time occur if and only if there occur changes in available information. Otherwise there would exist a static world: no time necessarily implies no new records of history and vice-versa, i.e. Δtime $\Leftrightarrow$ Δinformation. Therefore, to say that the operators reflect change through time is logically equivalent to saying that new information is arising and that history is transcribing. The following is an extension within the category of homotopies defined in [5], that is, of convex homotopies[20] used to study the continuous evolution of a subset of an agent's understanding space, i.e. the knowledge space.

Indeed, assume that

- That the formal learning, dialectical and initial identity functions are all differentiable at all points through time, i.e.

$$\varphi, \Delta^t, \Xi_i^t \in D^1 = \{ f \mid D(f) \; exists\}$$

  where $D:C^0 \to C^0$ is the differentiable operator over the set of continuous functions. Among other things this implies that formal learning takes place through a differentiable process (in the calculus sense).
- Formal learning is, throughout the evolution of the knowledge space, a unique single process.
- The domain of the initial identity function φ is [0,1].

As it was previously mentioned, future knowledge spaces can be conceived at present at most as a reflection of the accumulated knowledge through time, all of which can be "unrolled" backwards. This implies that at t = 0 the process by which a knowledge space evolves to its constituting stage is a noise process[21], i.e. the evolution of the knowledge space, look at t = 0, is Brownian Motion. Furthermore, the identity of the present knowledge space is, as it was mention defined by the learning and dialectical operator. Hence, its time evolution is

$$\overset{\bullet}{\left(\varphi(\Xi_i^t(\mathbf{\Delta}_t(\varsigma_t)))\right)} = \dot{\varphi}(\Xi_i^t(\mathbf{\Delta}_t(\varsigma_t)))\dot{\Xi}_i^t(\mathbf{\Delta}_t(\varsigma_t))\dot{\mathbf{\Delta}}_t(\varsigma_t) \qquad \text{[D.1]}$$

where φ represents the identity over the initial knowledge space $\mathbf{\Omega}^k_i$, $\varsigma_t \in H^*_{t,\nu}$ and Δ_t: $H^*_{t,\nu} \to K_t$. Now, the rate of change of the future identity function must be $BM_t(dBM_t)$ since this quantity represents the state of function at *t* times its differential change at *t*. Hence, the rate of change of the initial identity function through time, i.e. $\dot{\varphi}$, is the difference between the rate of change of the future identity function and the rate of change of the initial identity function determined by formal learning (and dialectics), i.e. [F.1]. Hence

$$\dot{\varphi} = BM_t \, dBM_t - \dot{\varphi}(\Xi_i^t(\Delta^t(\xi)))\dot{\Xi}_i^t(\Delta^t(\xi))\dot{\Delta}^t(\xi) \qquad \text{[D.2]}$$

Equation [F.2] is a special case of a more general category of stochastic differential equations called "noise" processes, i.e.

$$\frac{dX_t}{dt} = b(t, X_t) + \sigma(t, X_t)BM_t$$

where in this case[22],

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20 By convex homotopy we mean a homotopy such that the expression $H = t\psi + (1\text{-}t)\,\varphi$ is always well defined over the spaces where ψ and φ are defined, e.g. $\mathbf{\Omega}_U$.

21 In our scheme, the origin of noise is not-understood, untraced (backwardly), unrecorded expressions of, knowledge.

22 Also, note that if BM and BM* are two Brownian Motion processes then so are BM·BM* and BM/dt.

$$\begin{aligned} \varphi_t &\equiv X_t \\ b(s, X_t) &\equiv -\dot{\varphi}(\Xi_i^t(\Delta^t(\xi)))\dot{\Xi}_i^t(\Delta^t(\xi))\dot{\Delta}^t(\xi) \\ \sigma(s, X_t) &\equiv 1 \end{aligned} \qquad \text{[D.3]}$$

The term $BM_t dBM_t$ in [D.2] is distributed as a $t\cdot\chi^2(1)$ variable across realisations. The Ito interpretation of equation [D.2] is that there exists a function φ such that φ satisfies the stochastic integral

$$\varphi_t = \varphi_0 - \int_{[0,t]} \left(\dot{\varphi}(\Xi_i^t(\Delta^t(\xi)))\dot{\Xi}_i^t(\Delta^t(\xi))\dot{\Delta}^t(\xi)\right) ds + \int_{[0,t]} BM_t \, dBM_t \qquad \text{[D.4]}$$

The stochastic process henceforth defined in [D.4] is called an Ito diffusion. The key issue in finding a solution to [D.4] is the second integral for the rules of integration of stochastic calculus have nothing to do with those of deterministic calculus. In this sense, there arises the issue as to what type of integral should be used to solve the problem, Øksendal (1985). For a matter of clarity and exposition the Ito integral (and Ito's one dimensional formula) will be used in this instance.

Notice that [D.2] as well as [D.4] contain a deterministic part (defined by the learning and dialectics operators) and a stochastic part. Hence, using Ito's formula the solution to [D.4] is

$$\varphi_t = \varphi_0 + \int_{[0,1]} b(s,\varphi) ds + \tfrac{1}{2} BM_t^2 - \tfrac{1}{2} t \qquad \text{[D.5]}$$

where

$$b(s,\varphi) = \dot{\varphi}(\Xi_i^t(\Delta^t(\xi)))\dot{\Xi}_i^t(\Delta^t(\xi))\dot{\Delta}^t(\xi)$$

Is there always a solution to this (Ito) diffusion problem? If so, is it unique? In order to determine the existence and uniqueness of the solution certain conditions must be met by the $b(s, X_t)$ and $\sigma(s, X_t)$ functions in [F.3], Øksendal (1985). Concretely in this case, if $b(s, X_t) : [0,T] \times \Re \to \Re$ is such that

$$\left| b(t,x) \right| \le C(1 + |x|) - 1; \quad x \in \Re, t \in [0,T]$$

and

$$\left| b(t,x) - b(s,y) \right| \le D \left| x - y \right|; \quad x, y \in \Re, t \in [0,T]$$

for real constants C and D and there exists a random variable Z with finite second moment, i.e. $E[|Z|^2] < \infty$ then the stochastic differential equation

$$d\varphi_t = b(t,\varphi_t)\, dt + BM_t \, dBM_t; \quad 0 \le t \le T, X_0 = Z$$

has a unique t-continuous solution φ_t, Øksendal (1985). These conditions essentially require that the composite outcome of formal learning and dialectics be represented by a bounded process, (in the mathematical sense), throughout the entire deformation of the knowledge space. Hence, the existence and uniqueness of the solution of [D.2] depends entirely on the nature of both the learning and dialectical operators. Also, notice again, that for every formal learning process there is a different stochastic differential equation.

To summarise: if the nature of formal learning and dialectics are such that the previously stated conditions at the beginning are met and the composite synthesis of learning and dialectics is a bounded process then there exists a unique stochastic differential equation that represents the continuous deformation of the original knowledge space. Its family of solutions will depend on the composite representation of learning and dialectics and the type of stochastic integral used, i.e. whether it is an Ito integral or any other integral such as the Strationovich integral, Øksendal (1985).

The main difference between the two types of deformation, i.e. homotopies and stochastic differential equations, resides in that in the homotopic case one can always fine a coarse (or fine) enough topology υ on $\mathbf{\Omega}_U$ so as to make any path or function defined on it continuous. This is a mechanical mathematical procedure that holds no relationship to the nature of the economic problem in hand. This situation allows, of course, the existence of *discrete* knowledge spaces (in the sense that its elements can be counted) that define discrete probability distributions. In this situation the measure of the integral must be modified. Usually a Lebesgue measure will permit integration over highly discrete function, e.g. the characteristic function over the rationals that lie in [0,1]. However, if the degree of discreteness is such that no integration is possible and hence the cumulative probability functions must be replaced with fine sums of point-wise accumulated probabilities. On the other hand, stochastic differential equations can be used only in a continuous context, i.e. continuous flow of information and continuous knowledge spaces.

Finally, as a matter of practical concern, if the learning, dialectics and the initial identity function are defined then the solution to [D.2] will require some specification of a Brownian Motion process BM_t. It is impossible to develop such a specification a-priori because of the very nature of process itself (i.e. probabilistic realisations with $N(0,\Delta)$ increments between t and $t + \Delta$). However, a Brownian Motion process can always be approximated by discrete random walk processes by virtue of Kolmogorov's continuity theorem, Øksendal (1985), McCabe and Tremayne (1993), Hamilton (1994). Computer simulations can develop refined enough process to actually approximate Brownian Motion through random walks, as found in Grimmett and Stirzaker (1982), p. 492. In effect, equation [D.5] makes explicit the **impossibility** of knowing a-priori what will be known tomorrow (this is Popper's (1956) argument). It is rather through observation and experience, which tautologically must occur in the past, however near it might be, and later reflection and learning that an agent's understanding of the surroundings may be enhanced. In this sense, the Brownian Motion process in [D.5] sets a bound to human understanding of the economic environment and reaffirms the limited available possibilities for adaptation to the evolving material conditions of economic activity.