EERI
Economics and Econometrics Research Institute

Maximin-optimal sustainable growth with nonrenewable resource and externalities

Andrei V. Bazhanov

EERI Research Paper Series No 11/2008

EERI
Economics and Econometrics Research Institute
Avenue de Beaulieu
1160 Brussels
Belgium

Tel: +322 299 3523
Fax: +322 299 3523
www.eeri.eu

Maximin-optimal sustainable growth with nonrenewable resource and externalities*

Andrei V. Bazhanov[a,b†]

[a]Department of Economics, Queen's University, Kingston, ON, K7L 3N6, Canada

[b]Far Eastern National University, 8 Ulitsa Sukhanova, Vladivostok, 690600, Russia

August 08, 2008

Abstract

I offer an approach linking a welfare criterion to the "sustainable development opportunities" of the economy. This implies a dependence of a criterion on the information about the current state. I consider the problem for the Dasgupta-Heal-Solow-Stiglitz model with externalities. The economy-linked criterion is constructed on an example of the maximin principle applied to a hybrid level-growth measure. This measure includes as special cases the conventional measures of consumption level and percent change as a measure of growth. The hybrid measure or geometrically weighted percent can be used for measuring sustainable growth as an alternative to percent. The closed form solutions are obtained for the optimal paths including the paths, dynamically consistent with the updates in reserve estimates.

JEL Classification: O13; O47; Q32; Q38

Keywords: essential nonrenewable resource, modified Hotelling Rule, economy-linked criterion, geometrically weighted percent, normative resource peak

*The earlier version of the paper was presented at the AFSE Annual Thematic Meeting "Frontiers in Environmental Economics and Natural Resources Management", Université de Toulouse, June 9-11, 2008 and at the 16th Annual Conference of the EAERE, June 25 - 28, 2008, Gothenburg, Sweden.

†Tel.: 1-(613)-533-6000 ext. 75468; fax: 1-(613)-533-6668.
E-mail: bazhanov@econ.queensu.ca

1 Introduction

Koopmans (1964) claimed that "a decision maker should wish to retain some flexibility with regard to his future preference... to be able to make consistent responses to hypothetical choice situations." He argued that preferences should be adjusted to the economic opportunities, "viewing physical assets as opportunities," (Koopmans 1964, p. 253) and that the economic specificity of normative problems "imposes mathematical limits on the autonomy of ethical thought" (Koopmans 1965, p. 254). Koopmans illustrated in a simple model with discounted criterion that the optimal path could not exist depending on the choice of the discount factor (preferences). This implies that "Ignoring realities in adopting 'principles' may lead one to search for a nonexistent optimum, or to adopt an 'optimum' that is open to unanticipated objections" (Koopmans 1965, p. 229). It is known, that sustainability of consumption over time depends on the initial value of capital for the maximin programs (e.g. Leininger, 1985). A recent example of unacceptable consequences of using a criterion that is "not adjusted to opportunities" can be found by analyzing Stollery (1998). He examined the problem of a resource-extracting economy causing global warming and following the constant-utility optimal path. One can easily check that this criterion is not compatible with the Cobb-Douglas technology for plausible initial states by assuming constant extraction during some period. The plausible initial states imply in this framework unsustainable extraction, fast growth of temperature and collapse of the economy.

The approach offered in the current paper implies that the final expression for the criterion and therefore the optimal *sustainable* (in a weak sense) growth path for a specific economy with the given initial state is defined via the economy's technological parameters and the initial values for the resource reserve, the rate of the resource extraction, and capital. I assume here, following Koopmans, that "the *initial opportunity* is given by objective circumstances of technology and resources... independently of the ordering" (Koopmans 1964, p. 251). This approach to construction of a criterion is consistent with the Bellman's Principle of Optimality: "An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision" (Bellman 1957 p. 83). Koopmans (1964) concentrated his attention only on uncertainty in the chooser's own future preferences assuming complete certainty in physical assets of the economy. I consider here an example with a certain general form of preferences (criterion), parametrically linked to the initial state, while the resource reserve can be reevaluated over time. This implies flexibility of future preferences with respect to unpredictable changes in reserves.

The economy-linked criterion is constructed in this paper on an example of the maximin principle applied to a hybrid level-growth utility measure that I call "geometrically weighted percent." The use of the maximin in the problems of intergenerational justice implies that some social welfare measure should be maintained constant over time. Therefore, it is natural to use this convenient property of the maximin for formulating the long-run programs of sustainable development.[1] The hybrid measure, to which the maximin is applied in this paper, includes as particular cases the level of consumption and the rate of growth. In general case it includes all intermediate forms

[1] Solow (1974) applied the (Rawls 1971) maximin to the level of consumption as a simple social welfare measure that implied the constant-per-capita-consumption criterion. On the other hand, there is a conventional practice to formulate some long-run development goals in terms of constant percent change of GDP (e.g. World 1987, p. 169, p. 173).

for measuring the level and/or the rate of growth of consumption. This family of measures implies a corresponding family of patterns of optimal growth that can vary from stagnation and quasiarithmetic growth to linear and exponential growth. Using this approach, I answer the question: what is the “best” pattern of growth from this family that a specific economy with the given initial conditions can maintain forever? The approach differs from the conventional methodology in resource economics in that usually the optimal economy is being constructed under the given criterion.

I obtain the patterns of feasible and optimal sustainable growth for the extended Dasgupta-Heal-Solow-Stiglitz (DHSS) model (Dasgupta and Heal 1974; Solow 1974; Stiglitz 1974) with an essential nonrenewable resource under the standard Hartwick Investment Rule (Hartwick 1977). The extension is that the Hotelling Rule is modified by some phenomena whose total influence can be expressed in terms of an equivalent tax or subsidy.[2] I show that the feasible patterns of growth for this economy are between the constant consumption and the quasiarithmetic growth with parameters depending on the technological properties of production function.

The paper is structured as follows. Sections 2 introduces the methodology of specification of the generalized criterion for the given initial conditions; Section 3 describes the model; Section 4 provides the closed form solutions for the optimal paths in the DHSS model; Section 5 gives the condition

[2]There is extensive literature on a discrepancy between the standard Hotelling Rule and the observed data. The Rule implies that the path of the resource extraction must be decreasing and the resource price must grow at the rate of interest. However, this is not the case in the real economy (see e.g. (Gaudet 2007)). Gaudet (2007) considered different phenomena such as changes in the cost of extraction, durability, peculiarities of the market, and uncertainties. These phenomena can influence both the price dynamics and the paths of extraction, but they were not considered by Hotelling in his seminal paper (1931). Therefore, introduction of these effects into the model of Hotelling can reconcile his approach with empirical data for different kinds of resources including oil.

defining the feasible patterns of sustainable growth; Section 6 examines the unacceptable consequences of applying the criterion beyond its feasible limits; Section 7 gives the details of calibration on the world's oil extraction data; the optimal paths dynamically consistent with the updates in reserve estimates are constructed in Section 9. The conclusion is in Section 10.

2 The maximin variant of an economy-linked criterion

Criterion that is not adjusted to the economic opportunities can imply nonexistent or unsustainable optimal path for a specific economy. For example, the constant GDP percent change implies exponential growth that cannot be sustained infinitely under the assumptions of the essential nonrenewable resource and a plausible pattern of technical change (Dasgupta and Heal 1979). Stollery (1998) considered another example combining the constant-utility criterion $U(c,T) = c^{1-\gamma}T^{-1}/(1-\gamma) = const$ with the global temperature rising exponentially $T(t) = T_0 \exp\left[\phi \int_0^t r(\xi)d\xi\right]$ with the resource extraction. Assume that for some small initial period extraction r is constant that is close to the current pattern of the world's oil extraction. Then we obtain that T and (according to the criterion) c must grow exponentially over time that is not possible, say, for the Cobb-Douglas production function with constant extraction. This combination implies unsustainable behavior of the economy unless the rates of extraction decline very quickly in the initial period. One more example I studied in (Bazhanov 2008), where I have shown that an economy can enter an inferior path if it follows a criterion that is not linked to the "opportunities" of the economy expressed in the properties of the production function and the initial state.

In order to avoid these unacceptable consequences, I construct here the economy-linked criterion on an example of the maximin principle applied to a generalized level-growth utility measure.[3] The use of the maximin in the problems of intergenerational justice implies that some social welfare measure must be constant over time. Therefore, it is natural to use maximin for formulating the long-run programs of sustainable development.[4]

Solow (1974) showed that the maximin applied to the LEVEL of consumption implies constant consumption and no growth in output. I apply the same approach to a more general measure that takes into account not only the LEVEL of consumption but also the rate of its change.[5] I introduce a variant of generalized measure of consumption that includes as the specific cases conventional measures for the LEVEL or for the GROWTH of consumption depending on the values of parameters. Then I estimate the values of these parameters for the "initial opportunities" in the specific economy on an example of the DHSS economy with a nonrenewable resource, externality, and the tax, internalizing the externality in the optimal way. The closed form solutions for the optimal paths in this economy are provided in Lemma

[3] This approach was also considered in (Bazhanov 2007).

[4] One can claim that the overall wealth of an economy could be higher as a result of the alternate ups and downs, however, I will stick here to the evidence that "loss aversion favors social arrangements that provide a steady improvement of rewards or benefits over time, in preference to schedules in which the same total benefit is handed out in equal or diminishing quantities" (Kahneman and Varey, 1991, p. 152).

[5] There are findings supporting the idea that for estimating a consumer's perception of consumption and, consequently, the utility, it is not enough to calculate a vector of measurable static indicators. "We can ask, ... how well a person's life is going and whether that person is...better off than he or she was a year ago" (Scanlon 1991, p. 18). There is also evidence that has "documented the claim that people are relatively insensitive to steady states, but highly sensitive to changes" and that "the main carriers of value are gains and losses rather than overall wealth" (Kahneman and Varey 1991, p. 148). Here I take into account prehistory of consumption in the form of derivative $\dot{c}$.

1, Proposition 1, and Corollary 1 (Section 4).

The expression $\dot{c}^{\gamma} c^{\mu}$ is considered here as an example of a hybrid level-growth measure. The maximin applied to this expression implies that already this expression, not consumption per se, must be kept constant over time. Assume for simplicity that $\mu = 1 - \gamma$ and then we obtain the constant-utility criterion or the criterion of just growth[6] of consumption in a form[7] of

$$\dot{c}^{\gamma} c^{1-\gamma} = \overline{U} = const \tag{1}$$

that implies quasiarithmetic growth

$$c(t) = c_0 (1 + \varphi t)^{\gamma}, \tag{2}$$

where $\varphi = \left(\overline{U}/c_0\right)^{1/\gamma}/\gamma$.

Note that criterion (1) includes constant consumption as a specific case for $\gamma = 0$. More general expression $\dot{c}^{\gamma} c^{\mu}$ includes as specific cases

(a) conventional function for measuring the utility of the LEVEL of unlimitedly growing consumption $c^{1-\eta}/(1-\eta)$ for $\gamma = 0$, $\mu = 1 - \eta$, and $\overline{U} = \widehat{U}(1-\eta)$;

(b) percent change as a conventional measure of the GROWTH of consumption for $\gamma = 1$ and $\mu = -1$;

(c) a sample value function that relates value to an initial consumption c and to a change of consumption $\dot{c}$ (Kahneman and Varey, 1991, p. 157): $V(\dot{c}, c) = b\dot{c}^{a}/c$ for $\dot{c} > 0$, where $a < 1$ and $b > 0$; $V(0, c) = 0$; $V(\dot{c}, c) = -Kb(-\dot{c})^{a}/c$ for $\dot{c} < 0$, where $K > 1$.

The important property of criterion (1) is that it allows for the growth of the economy and that the parameters of the criterion must be specified

[6]For $\gamma > 0$ this version of criterion is applicable only to growth ($\dot{c} > 0$) because at the steady states ($\dot{c} = 0$) this expression is always zero (not sensitive to the LEVEL).

[7]This form can be written as follows $(\dot{c}/c)^{\gamma} c = \overline{U}$ that implies that the decline in the rate of growth in our hybrid utility is compensated by the growing level of consumption.

for the economy's initial conditions. This means that using this criterion, we can consider numerical examples that resemble the behavior of the real economy. The importance of the mechanism of matching of the criterion for just allocation of some scarce resource to the context was emphasized, for example, in (Konow 2003): "the most significant challenge to ... any theory ... is to incorporate the impact of context on justice evaluation, and much work remains in this regard."

3 The model

The analysis is provided for a decentralized economy with some externalities and government interventions, expressed in a general form, and based on the DHSS model with zero population growth, zero extraction cost, and with the Cobb-Douglas production function[8]

$$q(t) = f(k(t), r(t)) = k^{\alpha}(t)r^{\beta}(t), \tag{3}$$

where q - output, k - produced capital, r - current resource use, α, $\beta \in (0,1)$, $\alpha+\beta < 1$, are constants. The assumption about technical change $A(t)$ or TFP (Total Factor Productivity) exactly compensating for capital depreciation δk allows for considering the basic DHSS model with no capital depreciation and no technical progress. At the same time, this assumption makes it possible to examine correctly various patterns of growth in the economy. The pattern of this specific TFP is provided below in a separate section.

Without losing generality, assume that population equals to unity and then the lower-case variables are in per capita units. Then $r = -\dot{s}$, s - per

[8] There is mixed evidence about the elasticity of factor substitution between capital and resource including the results showing that this value is close to unity (Griffin and Gregory 1976; Pindyck 1979) that means that the use of the Cobb-Douglas technology is not implausible in this framework.

capita resource stock ($\dot{s} = ds/dt$). Prices of per capita capital and the resource are $f_k = \alpha q/k$ and $f_r = \beta q/r$, where $f_x = \partial f/\partial x$. Per capita consumption is $c = q - \dot{k}$.

As an example of the specific economy with "the *initial opportunity* ... given by objective circumstances" consider the economy with the growing rates of extraction at the initial moment that is consistent with the world's oil extraction. This extraction is the result of influence of various phenomena (including externalities and government policy), which can be expressed in terms of tax $T(t)$ and which result in modification of the Hotelling Rule. This implies that if $p(t)$ is the "equilibrium Hotelling price" without distorting phenomena and $f_r(t) \equiv f_r[p(t), T(t)] = p(t) + T(t)$ is the observable price with distortions, then the ratio $\dot{f}_r/f_r$ is not already equal to the rate of interest. I divide here the phenomena modifying the Hotelling Rule in two groups:

(a) "natural" processes; for example, technical progress and the worsening quality of resources that influence the cost of extraction;

(b) "externalities", which are the result of the specific market structure, insecure property rights, or common property.

I assume that

(1) the effects from the first group are "uncontrollable" essential parts of the process of the resource extraction and they must be included in the modification of the Hotelling Rule as a necessary condition of efficient (in terms of consumption) extraction.[9]

(2) The influence of the phenomena from the second group can be eliminated by institutional changes and environmental policies influencing the

[9] The necessity of the Hotelling Rule for efficient extraction is shown e.g. in (Dasgupta and Heal 1979).

resource demand (Caillaud et al. 1988; Pezzey 2002), including compensating tax in such a way that the resulting resource extraction will bring more social welfare to the economy. Hence, I am going to consider the effects of the second group separately from the effects of the first one and call them the "distortions" of the Hotelling Rule or "externalities."

(3) All the effects from the second group ("distortions") can be expressed in terms of equivalent amount of tax/subsidy.

For example, insecure property rights lead to shifting extraction from the future towards the present (Long 1975) or to "overexploitation" (in terms of consumption lost) that is happening also in a common property situation. I assume that the same effect can be obtained by subsidizing the oil-using production. Thus, I will consider all the phenomena modifying the Hotelling Rule in the same terms of tax/subsidy including the subsidies themselves.[10]

The assumptions imply that in general case the Hotelling Rule can be written as follows:

$$\frac{\dot{f}_r(t)}{f_r(t)} = F\left[f_k(t)\right] + \tau(t), \tag{4}$$

where F - "natural" modification of the Hotelling Rule, τ - distortion. In the simplest case that will be examined below, $F(f_k) \equiv f_k$.

Assume also that the "initial opportunity" of the specific economy includes also the pattern of saving, namely, that the economy follows the Hartwick Saving Rule that is consistent with the IMF data (world's saving (excluding the U.S.A.) fluctuates between 0.24 and 0.26 of GDP since 1980). Then the Hotelling Rule (4) for $\tau \equiv 0$ with the Hartwick Rule $\dot{k} = rf_r = \beta q$ implies constant consumption over time (Hartwick 1977). In general case,

[10] In fact, subsidies were being applied to stimulate oil use not only in the past but even today "the world fossil fuel industry is still being subsidized by taxpayers at more than $210 billion per year" (Brown 2006).

for $\tau \neq 0$, equation (4) follows

$$\frac{d\dot{k}}{dt} = \dot{r}f_r + r\dot{f}_r = \dot{r}f_r + r\left(f_k f_r + \tau f_r\right) \tag{5}$$

and $\dot{c} = f_k\dot{k} + f_r\dot{r} - \ddot{k}$. Substituting (5) for $\ddot{k}$ we have $\dot{c} = f_k\dot{k} + f_r\dot{r} - \dot{r}f_r - rf_kf_r - \tau rf_r = -\tau rf_r$ that goes to zero if $\tau/(rf_r) = \tau/(\beta q)$ goes to zero with $t \to \infty$. Realizing some declining "program" path for modifier τ we can approach the sustainable and optimal path of extraction in a desirable way.

Equation (5) and the saving rule also follow $\dot{f}_r/f_r = \beta\left[f_k + (\dot{r}/r)\left(1 - 1/\beta\right)\right] = f_k + \tau$ or $f_k(\beta - 1) + (\dot{r}/r)(\beta - 1) = \tau$ that gives us

$$\alpha\frac{q}{k} + \frac{\dot{r}}{r} = \frac{\tau}{\beta - 1}. \tag{6}$$

Then

$$\frac{\dot{q}}{q} = \alpha\frac{\dot{k}}{k} + \beta\frac{\dot{r}}{r} = \beta\left(\alpha\frac{q}{k} + \frac{\dot{r}}{r}\right) = \frac{\beta}{\beta - 1}\tau \tag{7}$$

that means that

1) growth is associated with negative $\tau(t)$ in the DHSS economy with the standard Hartwick Rule;

2) GDP percent change $\dot{q}/q \to 0$ with any $\tau(t) \to 0$.

According to assumption, modifier $\tau(t)$ can be expressed in terms of tax/subsidy. This implies that there exists a Pigovian tax $T(t)$ such that for $F(f_k) \equiv f_k$ equation (4) can take the form[11]

$$\frac{\dot{f}_r + \dot{T}}{f_r + T} = \frac{\dot{f}_r}{f_r} - \tau = f_k \tag{8}$$

This equation can be rewritten as follows:

$$\frac{\dot{f}_r + \dot{T}}{f_r + T} - \frac{\dot{f}_r - \tau f_r}{f_r} = 0$$

[11] This dynamic efficiency condition was used in (Hamilton, 1994) in the form $\dot{n}/n = f_k$ for the net rent per unit of resource $n = f_r - c - T$ with c - marginal cost of extraction.

or, for $f_r(f_r+T)\neq 0$, we have $\dot{f}f_r + \dot{T}f_r - \dot{f}_r f_r - T\dot{f}_r + \tau f_r(f_r+T) = 0$. This implies ($f_r \neq 0$ and $\dot{f}_r/f_r - \tau = f_k$) the dynamic condition for the tax

$$\dot{T} - Tf_k + \tau f_r = 0. \tag{9}$$

General solution of (9) is

$$T(t) = e^{\int f_k(t)dt}\left[\widehat{T} - \int \tau f_r e^{-\int f_k(t)dt}dt\right]. \tag{10}$$

The equation (9) and its solution (10) can be considered with the two types of initial conditions, associated with the two different interpretations of the equation (8).

Initial condition I. If we are looking for the path of tax $T(t)$ corresponding to the "program" decrease in distortion $\tau(t)$ then we will set $T(0) = T_0$. Since we introduce $T(t)$ as a new tax that compensates for the distorting phenomena and that

(a) is continuous,

(b) was not applied before ($T(t) = 0$ for $t \leq 0$),

then we will assume that $T_0 = 0$ that gives us $\dot{T}(0) = -\tau(0)f_r(0)$.

Initial condition II. If we want to estimate the effect of the distortions in terms of tax/subsidy at the current moment $t = 0$, in other words we want to find $T(0)$, then we assume that the distorting combination is continuous at $t = 0$ and we must use the given estimation of $\dot{T}(0) = \dot{T}_0$ in order to obtain $T(0) = \left[\tau(0)f_r(0) + \dot{T}_0\right]/f_k(0)$.

In problem I (equation (9) with the initial condition I), the observable resource price at $t = 0$ is $f_r(0)$, while in problem II (the initial condition II), the observable price is $f_r(0) + T(0)$ and $f_r(0)$ is the value of the price that it would be without distortions expressed in terms of the tax/subsidy $T(0)$.

4 Optimal paths in the DHSS economy

The social planner keeps the value of $\dot{c}^{\gamma}c^{1-\gamma}$ constant over time with the restriction on the extraction $\int_0^{\infty} r(t)dt = s_0$, production function in a form (3), the Hotelling Rule modified in a form (4), saving rule $\dot{k} = \beta q$, and nonnegative capital, output, and consumption. The Koopmans's claim "the *initial opportunity* is given" implies that the initial values of all variables in the problem are given. In this framework these values cannot be obtained as the optimal ones since then they could conflict with the values in some real economy, for which we would apply the results. I assume that even the initial value of tax is zero (the tax is new) in order to obtain smooth continuations for all the paths in the economy, rendering them consistent with the initial state. Otherwise, discontinuous shift can change the *initial opportunity* right at the initial point violating the Koopmans's prerequisite. The optimal paths in the DHSS economy with the specific initial conditions are provided in the following Lemma 1, Proposition 1, and Corollary 1.

Lemma 1. *For the economy $q = k^{\alpha}r^{\beta}$ with the saving rule $\dot{k} = \beta q$ and the Hotelling Rule modified in a way $\dot{f}_r/f_r = f_k + \tau$, the unique path of the Hotelling Rule modifier*

$$\tau(t) = \frac{\beta - 1}{\beta}\frac{1}{\lambda_1 t + \lambda_0}$$

is socially optimal with respect to (1) with $\gamma = 1/\lambda_1$ and $\overline{U} = c_0/\lambda_0^{1/\lambda_1}$.

Proof. Condition (1) implies that $\dot{c}^{\gamma}c^{1-\gamma} = (1-\beta)^{\gamma}\dot{q}^{\gamma}(1-\beta)^{1-\gamma}q^{1-\gamma}$ $= (1-\beta)\dot{q}^{\gamma}q^{1-\gamma} = \overline{U}$ or

$$\dot{q}^{\gamma}q^{1-\gamma} = \overline{U}/(1-\beta) \tag{11}$$

The equation (2) gives us $q = c/(1-\beta) = c_0(1+\varphi t)^{\gamma}/(1-\beta)$ and from the equation (7) we have $\dot{q} = \beta q\tau/(\beta-1)$. Substituting for $\dot{q}$ we obtain

$$\left(\frac{\beta}{\beta-1}q\tau\right)^{\gamma} q^{1-\gamma} = \left(\frac{\beta}{\beta-1}\tau\right)^{\gamma} q = \frac{\overline{U}}{1-\beta}.$$

Then substitution for q gives us $[\tau\beta/(\beta-1)\,(1+\varphi t)]^{\gamma} = \overline{U}/c_0$ or

$$\tau = \left(\frac{\overline{U}}{c_0}\right)^{\frac{1}{\gamma}} \frac{\beta-1}{\beta}\frac{1}{1+\varphi t} = \frac{\beta-1}{\beta}\frac{\varphi\gamma}{1+\varphi t} = \frac{\beta-1}{\beta}\frac{1}{1/(\varphi\gamma)+t/\gamma},$$

where $\lambda_0 = 1/(\varphi\gamma)$ and $\lambda_1 = 1/\gamma$. Substitution for $\varphi = \left(\overline{U}/c_0\right)^{1/\gamma}/\gamma$ into the expression for λ_0 gives us the expression for $\overline{U}$ via λ_0 and λ_1■

Proposition 1. *Let the economy* $q = k^{\alpha}r^{\beta}$ *follow the Hartwick Rule* $\dot{k} = \beta q$; *the Hotelling Rule is modified in a way* $\dot{f}_r/f_r = f_k + \tau$ *and the initial conditions are:* $\dot{q}_0/q_0$ *- the initial rate of growth;* $q_0 = q(0) = k_0^{\alpha}r_0^{\beta}$ *- the initial output, where* $k_0 = k(0), r_0 = r(0)$, *and* $s_0 = s(0)$ *are the initial values of capital, the resource extraction and the reserve estimate.*

Then the unique path of tax, introduced at $t = 0$ *with* $T(0) = 0$ *in the following way:*

$$T(t) = \beta\left\{[k(t)\,(1+\lambda_1)]^{\alpha}\, q_0^{\beta-1}\right\}^{1/\beta}\left[1-(t\lambda_1/\lambda_0+1)^{(\beta-1)/(\beta\lambda_1)}\right]$$

is socially optimal with respect to (1) with $\gamma = 1/\lambda_1$ *and* $\overline{U} = c_0/\lambda_0^{1/\lambda_1}$. *The optimal tax implies the following paths of capital and the resource use:*

$$k(t) = k_0 + \frac{\beta q_0}{\lambda_0^{1/\lambda_1}(1+\lambda_1)}\left[(\lambda_1 t+\lambda_0)^{(1+1/\lambda_1)} - \lambda_0^{(1+1/\lambda_1)}\right],$$

$$r(t) = \left(\frac{q_0}{\lambda_0^{1/\lambda_1}}\right)^{1/\beta}(\lambda_1 t+\lambda_0)^{1/(\beta\lambda_1)}\,k^{-\alpha/\beta},$$

where $\lambda_0 = q_0/\dot{q}_0,\ \lambda_1 = \lambda_1(s_0)$.

Proof: Appendix 1.

The optimal paths, obtained in Proposition 1, are smooth continuations of the initial conditions. Indeed, the tax is zero at the initial moment since it is a new tax, "additional" to the already existing taxes or subsidies that are expressed in the Hotelling Rule modifier τ and in the corresponding distortion in price f_r. Another interesting property of the economy-linked solution is the path of extraction r that includes growing multiplier $(\lambda_1 t + \lambda_0)^{1/(\beta\lambda_1)}$ allowing for the growing extraction in the neighborhood of the initial point.

Corollary 1. *In conditions of Proposition 1, the optimal path of consumption implied by (1) is*

$$c(t) = c_0 \left(1 + \frac{\dot{q}_0}{q_0} t\right)^{\gamma}$$

i.e. the optimal sustainable growth rate of consumption is defined by the initial GDP percent change $\dot{q}_0/q_0$ *and* $\gamma = 1/\lambda_1(s_0)$*;*

the expression for the Hotelling Rule is

$$\dot{f}_r(t)/f_r(t) = f_k(t) + \frac{\beta - 1}{\beta} \frac{1}{\lambda_1(s_0)t + q_0/\dot{q}_0},$$

where $\lambda_1(s_0)$ *is uniquely defined from the equation*

$$\begin{aligned} s_0 \;=\; & \frac{\lambda_1 + 1}{(\lambda_1 + 1)(\alpha - \beta) - 1 + \beta} \cdot \frac{k_0 r_0}{q_0} \\ & \times \left\{1 + (1 - \beta)\left(k_0 \beta \dot{q}_0 (\lambda_1 + 1) - \beta^2 q_0^2\right) \right. \\ & \left. \times\, {}_2F_0\left(\left[1, \frac{\beta(\lambda_1 + 2) - 1}{\beta(\lambda_1 + 1)}\right], [\,], \left(\beta q_0^2 - k_0 \dot{q}_0 (\lambda_1 + 1)\right)(\lambda_1 + 1)\beta^2\right)\right\}, \end{aligned} \tag{12}$$

where ${}_2F_0(\cdot)$ *is the hypergeometric function with 2 upper parameters and an empty list of lower parameters.*

Proof is the result of straightforward substitution of the expressions for $\overline{U}$, λ_0, and $\lambda_1(s_0)$ obtained in Lemma 1, Proposition 1, and Appendix 2 ■

Note that equation (12) defines a monotonically decreasing dependence between λ_1 and s_0 (Fig. 11). This implies an intuitive result that the larger is the initial reserve s_0, the greater is the optimal growth rate of consumption $\dot{c}/c = (1/\lambda_1)(\dot{q}_0/q_0)(1 + t\dot{q}_0/q_0)^{-1}$. Note also that the optimal tax results in the asymptotical satisfaction of the standard Hotelling Rule.

5 Compatibility of the criterion with the initial conditions

Before considering the numerical examples, I will examine possible limitations of criterion (1) that can prevent us from calibrating the model on some specific data from the real economy. It is known, that in the particular case of this criterion, for $\gamma = 0$ (constant consumption), we cannot use in numerical examples the data from a growing economy with the growing extraction $r(t)$. This is because $\dot{r}(0)$ must be negative in this case and it is strictly defined by the initial values of extraction $r(0)$, reserve $s(0)$, and parameters α and β. That is why the economy pursuing this specific type of intergenerational justice must adjust its extraction and saving during a transition period in order to switch to the optimal path in finite time (Bazhanov 2008).

In general case ($\gamma > 0$), the economy is already allowed to have different patterns of sustainable growth, and the specific type of growth corresponds to the specific set of initial data. This implies that the economy's initial conditions are already not strictly fixed by the criterion but they can belong to some range or satisfy some restricting relationship. In Appendix 1, I have shown that for the ratio $\dot{r}/r$ to be negative (declining extraction) in the long run, the value of λ_1 must be greater than $1/\alpha - 1$ that implies $\gamma < 1/(1/\alpha - 1) = \alpha/(1 - \alpha)$ (for $\alpha = 0.3$ we have $\gamma < 0.43$). Now we will examine how the value of λ_1 is restricted by the requirement of convergence of the integral $\int_0^\infty r(t)dt$. We can express $r(t)$ as follows:

$$\begin{aligned} r(t) &= q^{1/\beta} k^{-\alpha/\beta} \\ &= \widehat{q}^{1/\beta} \left[\widehat{k}(\lambda_1 t + \lambda_0)^{-1/(\alpha\lambda_1)} + \frac{\beta\widehat{q}}{1+\lambda_1}(\lambda_1 t + \lambda_0)^{((\lambda_1+1)/\lambda_1 - 1/(\alpha\lambda_1))} \right]^{-\alpha/\beta}. \end{aligned}$$

Convergence of the integral is defined by the behavior of the second term in bracket, since $\lim_{t\to\infty}(\lambda_1 t + \lambda_0)^{-1/(\alpha\lambda_1)} = 0$. This gives us the convergence condition $\left[\alpha^2(\lambda_1+1) - \alpha\right]/(\alpha\beta\lambda_1) > 1$ or

$$\lambda_1 > (1-\alpha)/(\alpha-\beta). \tag{13}$$

For example, it requires $\lambda_1 > 14$ ($\gamma < 0.0714$) for $\alpha = 0.3$ and $\beta = 0.25$, while the requirement of negative ratio $\dot{r}/r$ implies only $\lambda_1 > (1-\alpha)/\alpha = 2.333$. Note that the combination of condition (13) with the requirement of declining extraction ($\lambda_1 > (1-\alpha)/\alpha > 0$) implies $\alpha > \beta$ (Solow, 1974).

Groth et al (2006) argued that the notion of regular growth should be more general than that of exponential growth. Inequality (13) shows that in the DHSS model the value of γ must be less than $(\alpha-\beta)/(1-\alpha)$ regardless of the values of initial conditions. This restriction prevents the model from the patterns of growth that are close to linear if $\alpha < 0.5$, let alone for the exponential growth. The economy can realize only some variants of quasi-arithmetic growth including stagnation ($\gamma = 0$). The set of these feasible sustainable paths is located in Figure 1 between the constant ($\gamma = 0$) and the path for $\gamma = (\alpha-\beta)/(1-\alpha)$.

Condition (13) gives us only the lower bound for finding λ_1. The exact value of λ_1 must be defined from the equation $\int_0^\infty r(t,\lambda_1)dt = s_0$. Therefore, the question of existence of this solution is the main source of possible incompatibility of criterion (1) with some sets of the initial conditions. Hence, I will define the applicability of a criterion for formulating a long-run (sustainable) development program for the specific economy in the following way.

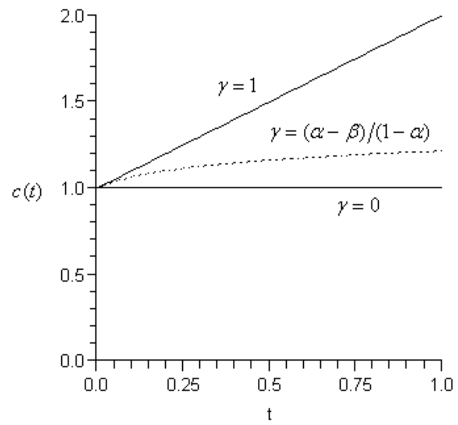

Figure 1: Patterns of feasible growth for the Cobb-Douglas economy with $\alpha = 0.3$ are between the constant ($\gamma = 0$) and the path with $\gamma = (\alpha - \beta)/(1 - \alpha)$

Definition 1 *We will say that a criterion is applicable for a long-run development program*[12] *in an economy* $q = f(k, r)$ *with the given initial state if there exists at least one optimal with respect to this criterion program* $\langle q^*, k^*, r^* \rangle$ *that satisfies the economy's initial conditions.*

The answer to the question about the applicability of criterion (1) for a long-run program in the DHSS economy is formulated in the following

Proposition 2. *Criterion* $\dot{c}^\gamma c^{1-\gamma} = const$ *is applicable for a long-run development program in the economy* $q = k^\alpha r^\beta$ *with* $\dot{k} = \beta q$ *if the initial reserve* s_0 *satisfies the condition*

$$s_0 \geq \frac{k_0 r_0}{q_0(\alpha - \beta)}, \tag{14}$$

[12] A criterion can be *applicable* for selecting the best path among the feasible paths in an economy, but it can be *not applicable* for a long-run development program because the optimal path that it implies can be *not realizable* in this economy in the long run.

where q_0, k_0, *and* r_0 *are the initial values of output, capital, and the rate of extraction.*

Proof. In Appendix 2, I have shown that the following formula can be used for defining λ_1 as a good approximation to the solution of the equation $\int_0^\infty r(t,\lambda_1)dt = s_0$ with respect to λ_1 :

$$\lambda_1 = \frac{(1-\alpha)s_0q_0 + k_0r_0}{(\alpha-\beta)s_0q_0 - k_0r_0}. \qquad (15)$$

This formula captures the main peculiarities of behavior of the exact solution. In particular, it shows that the denominator can be zero for some sets of parameters that follows the value of λ_1 going to infinity. This implies that denominator must be positive or $s_0 > k_0r_0/\left[q_0(\alpha-\beta)\right]$ that coincides with the condition (14). This means that the value of $\lambda_1(s_0)$ is a decreasing function from infinity at the minimal value for $s_0 = k_0r_0/\left[q_0(\alpha-\beta)\right]$ to the minimal value $\lambda_{1\min} = (1-\alpha)/(\alpha-\beta)$ for s_0 going to infinity (Fig. 2).

Indeed, considering the limiting case for the path of extraction with λ_1 going to infinity (corresponds to the smallest possible s_0), we obtain

$$\begin{aligned} r_\infty(t) &\triangleq \lim_{\lambda_1\to\infty} r(t,\lambda_1) = \lim_{\lambda_1\to\infty}\left(\frac{q_0}{\lambda_0^{1/\lambda_1}}\right)^{1/\beta}(\lambda_1 t+\lambda_0)^{1/(\beta\lambda_1)}\,k^{-\alpha/\beta} \\ &= q_0^{1/\beta}\left[k_0+\beta q_0 t\right]^{-\alpha/\beta} \end{aligned}$$

The total amount of reserve, extracted along this path is

$$\int_0^\infty r_\infty(t)dt = \frac{q_0^{1/\beta}}{\beta q_0\left(1-\frac{\alpha}{\beta}\right)}\left[k_0+\beta q_0 t\right]^{1-\alpha/\beta}|_0^\infty = -\frac{q_0^{1/\beta}k_0^{1-\alpha/\beta}}{q_0\left(\beta-\alpha\right)} = \frac{k_0r_0}{q_0\left(\alpha-\beta\right)}$$

that is the greatest lower bound for the feasible reserve s_0 ■

If the initial conditions in an economy do not satisfy (14), then the economy needs a transition period for adjusting its patterns of extraction and

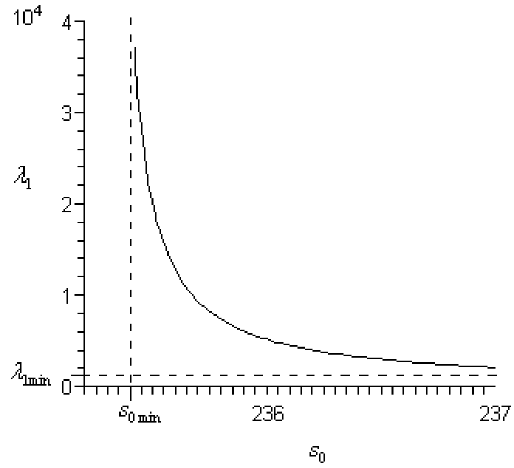

Figure 2: λ_1 as a function of the initial reserve s_0

saving in order to meet the minimum requirements expressed in (14) and then it can enter a sustainable path (Bazhanov 2008).

It would be interesting to analyze the practical applicability of the hybrid measure in general form $\dot{c}^{\gamma} c^{\mu}$ if we had had some opportunities for γ to be close to unity for the plausible values of α. However, our analysis for the simple case with $\mu = 1 - \gamma$ and with the conventional value of $\alpha = 0.3$ (see e.g. Nordhaus and Boyer 2000) shows that the DHSS economy in our framework can exhibit only the patterns of quasiarithmetic growth that are closer to constant than to linear function ($\gamma \ll 1$).

Moreover, these patterns of sustainable growth, including constant consumption, are affordable not for all initial conditions. If the economy overuses the resource having relatively small amount of the reserve, then it needs some transition period to adjust the extraction and saving in order to have an opportunity to enter a sustainable path in finite time. This result implies the impossibility of exponential growth for the DHSS model and therefore the inconvenience of the percent change as a measure for sustainable growth.

This follows an important practical application of the hybrid measure. This expression can be called *geometrically weighted percent,* and it can be used as a measure for sustainable growth of some economic indicators e.g. social welfare function or NNP (Hartwick 1990) instead of regular percent change. The rate of regular percent change declines for sustainable growth if this growth is not exponential. Indefiniteness of the rate of this decline makes regular percent an inconvenient and even a misleading measure for sustainable development. For example, this inherently unsustainable indicator was used as a necessary condition for sustainability even in such a seminal document for sustainable development as the Brundtland Report (World 1987): “The key elements of sustainability are: a minimum of 3 percent per capita income growth in developing countries” (p. 169) and “annual global per capita GDP growth rates of around 3 percent can be achieved. This growth is at least as great as that regarded in this report as a minimum for reasonable development” (p. 173). Besides contradictions with the environmental goals, which were noticed e.g. in (Hueting 1990), measuring growth in GDP percent change can conflict with theoretical possibility of realization of this program. In this sense, geometrically weighted percent in the form of (1) is more convenient for formulating the long-run economic goals because maintaining this expression constant implies feasible and sustainable growth.

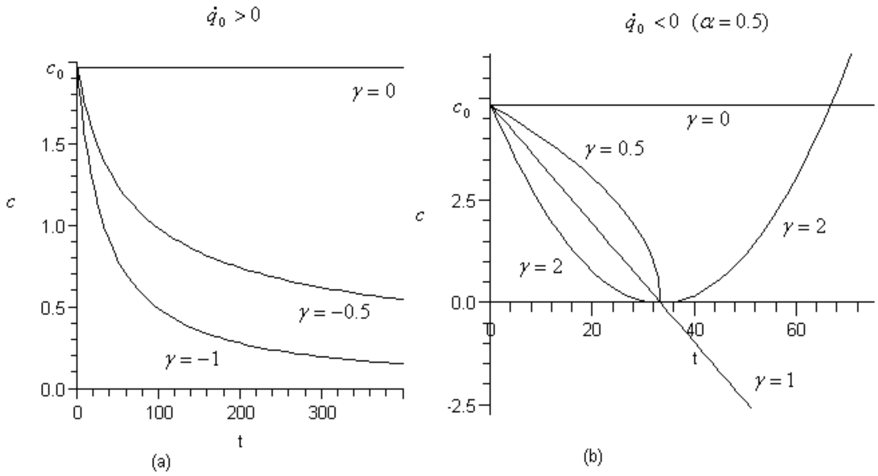

Figure 3: Unacceptable paths of consumption, optimal with respect to criterion (1): (a) growing economy and $\gamma < 0$; (b) declining economy and $\gamma > 0$

6 An economy with declining output and/or small reserve s_0

In order to complete the analysis of applicability of the economy-linked criterion in the form of (1) to formulating long-run development programs, I will show that this criterion leads to unacceptable implications for the cases when an economy has declining output ($\dot{q}_0/q_0 < 0$) at the initial moment and/or $\gamma < 0$.[13] The optimal paths of consumption for these cases can be obtained by plotting the formula for consumption in Corollary 1.

For a growing economy ($\dot{q}_0/q_0 > 0$) with $\gamma < 0$ criterion (1) implies optimal consumption asymptotically approaching zero (Fig. 3a). If the economy's output is declining at the initial moment and $\gamma > 0$, then we obtain

[13]Negative γ for the Cobb-Douglas technology is equivalent to the initial conditions not satisfying (14).

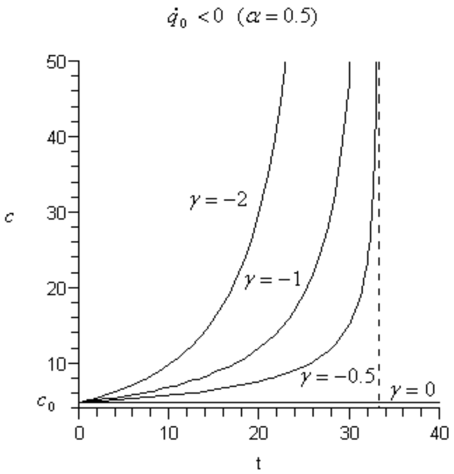

Figure 4: Paths of consumption, assumed by criterion (1) for declining economy and $\gamma < 0$

that the optimal paths of consumption must be decreasing to zero in finite time for all positive γ. However, for the even integer values of $\gamma > 1$, the optimal path after hitting zero must have unbounded polynomial growth (Fig. 3b). Note again that $\gamma > 1$ cannot be obtained in the DHSS model for the conventional values of α. In the last, presumably the most pessimistic case where the economy has declining output and can rely only on negative γ, we obtain that criterion (1) requires the consumption to be growing to infinity in a finite period (Fig. 4). This scenario can be realized only in the short run because growing consumption with decreasing output implies negative investment and subsequent collapse of the economy.

Hence, the only case when criterion (1) leads to ethically acceptable paths of consumption is growing output at the initial moment and $\gamma > 0$ (or satisfaction of condition (14)). The paths of consumption for this case are depicted in Figure 1.

7 Numerical example

I will start with problem II that estimates the effect of the distorting externalities in terms of tax/subsidy at the current moment $t = 0$. Assume that the distorting combination of externalities is continuous at $t = 0$, and that it is constant or $\dot{T}_0 = 0$ that implies $T(0) = \tau(0) f_r(0)/f_k(0)$.

The primary initial values are: $\alpha = 0.3$, $\beta = 0.25$ (this value gives us the reasonable interest rate $f_k(0)$ and at the same time it is close to the world's pattern of saving given $\dot{k} = \beta q$), GDP percent change $\dot{q}_0/q_0 = 0.03$, the initial rate of extraction $r_0 = 3.6243$, the initial reserve $s_0 = 2 \cdot 180.4722 = 360.9444$.[14] The rate of extraction is growing with $\dot{r}_0 = 0.1$. Note that there is a connection between the initial values (Bazhanov, 2006b) in the DHSS model, so we can express k_0 in terms of the readily available data

$$k_0 = \left\{ \left[\frac{\dot{q}_0}{q_0} \frac{1}{\beta} - \frac{\dot{r}_0}{r_0} \right] / \left(\alpha r_0^\beta \right) \right\}^{\frac{1}{\alpha - 1}} = 8.5174.$$

Then $\lambda_0 = q_0/\dot{q}_0 = 33.3333$. This follows $q_0 = k_0^\alpha r_0^\beta = 2.6236$, $c_0 = (1-\beta) q_0 = 1.9677$, $\dot{q}_0 = (\dot{q}_0/q_0)\, q_0 = 0.0787$, $\tau(0) = (\dot{q}_0/q_0)\, (\beta - 1)/\beta = -0.09$. For these values, condition (14) is satisfied (for our example $s_{0\,\min} = 235.3$)[15] and we have $q_0/k_0 = 0.308$, the rate of interest $f_k(0) = \alpha q_0/k_0 = 0.092$ and the resource price (that would be in problem II without distortions) $f_r(0) =$

[14]I use the world oil extraction on January 1, 2007 as r_0 and the world reserves as s_0 (Radler, 2006): $r_0 = 72{,}486.5$ [1,000 bbl/day] $\times 365 = 26{,}457{,}572$ [1,000 bbl/year] (or 3.6243 bln t/year); $s_0 = 1{,}317{,}447{,}415$ [1,000 bbl] (or 180.4722 bln t). I use coefficient 1 ton of crude oil = 7.3 barrel. According to the report of Cambridge Energy Research Associates (CERA, 2006), actual world reserves (3.74 trillion barrels) are about three times more than the conventional estimate being published in December issues of *Oil & Gas Journal*. I use in the example the "average" of the two estimates.

[15]If we take s_0 equal to 180.4722 bln t (*Oil & Gas Journal* estimate) then condition (14) will be not satisfied or our model of the world economy will be not compatible with the sustainable growth in the sense of criterion (1) and it will need a transition period in order to adjust the initial state.

$\beta q_0/r_0 = 0.18097$. Note that $\dot{f}_r(0) = f_r(0)\left(\dot{q}_0/q_0 - \dot{r}_0/r_0\right) = 0.0004$ (price is growing but very slowly). The assumption $\dot{T}_0 = 0$ implies $T(0) = -.1763$. This means that for our simplified economy

1) the current distortions are equivalent to subsidy rather than to tax;

2) the observable price $f_r(0) + T(0) = 0.0047$ is about 38.4 times less than it would be without this subsidy.

We turn to solving problem I, where we will estimate the optimal tax $T(t)$ and the paths of capital and extraction. This problem implies that there is no tax at the initial moment ($T_0 = 0$) that gives us $\dot{T}_0 = 0.016$ (growing optimal tax). Then we estimate $\lambda_1 = 60.11$[16] using the feasibility condition $\int_0^\infty r(t)dt = s_0$ (Appendix 2). This gives us the optimal path of capital that is very close to linear (solid line in Fig. 8), $k(t) = 8.16 + 0.0101 \cdot (60.11t + 33.33)^{1.0166}$, and the paths of the resource extraction (solid line in Fig. 9) and tax (solid line in Fig. 7). Quasiarithmetic growth of consumption is depicted in solid line in Fig. 10. We will proceed with comparative analysis of these paths in Section 9.

8 Technical progress compensating for capital depreciation

The assumption about no capital depreciation and no technical progress can be interpreted as an equivalent assumption about the specific TFP that exactly compensates for the capital decay. The path of this TFP can be constructed in order to estimate its plausibility. Our assumption implies that

[16] Numerical calculation (procedure _d01amc in Maple) of the integral gives $\lambda_1 = 60.11$; the expression via the hypergeometric function (Appendix 2) implies $\lambda_1 = 72.33$, and the approximate formula (15) gives $\lambda_1 = 42.1$.

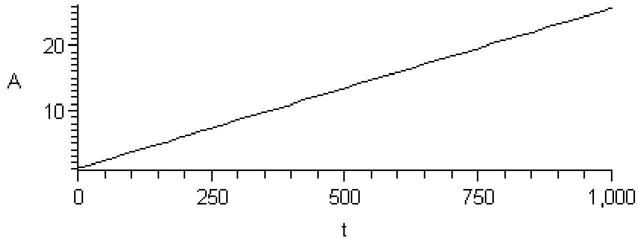

Figure 5: Technical progress compensating for capital depreciation

$$q(t) = A(t)k^{\alpha}r^{\beta} - \delta k$$

and technical progress $A(t)$ is such that $A(t)k^{\alpha}r^{\beta} - \delta k = k^{\alpha}r^{\beta}$. This follows

$$A(t) = 1 + \delta k^{1-\alpha}r^{-\beta}.$$

Substituting for $r = \widehat{r}\left(\lambda_1 t + \lambda_0\right)^{1/(\beta\lambda_1)} k^{-\alpha/\beta}$, where $\widehat{r} = \widehat{q}^{1/\beta}$ and $\widehat{q} = q_0/\lambda_0^{1/\lambda_1}$ we have

$$A(t) = 1 + \frac{\delta}{\widehat{q}}\left[\frac{\widehat{k}}{\left(\lambda_1 t + \lambda_0\right)^{1/\lambda_1}} + \frac{\beta\widehat{q}}{1+\lambda_1}\left(\lambda_1 t + \lambda_0\right)\right]$$

that is asymptotically linear with the slope $\delta\beta/\left(1+\lambda_1\right)$. For our example, given $\delta = 0.1$, the slope is $0.1 \cdot 0.25/(1 + 60.11) = 0.000409$ (Fig. 5).

9 Variable reserves and dynamic corrections

The amount of reserve s_0 was considered so far as a constant, though in practice the value of the proven recoverable reserve is being updated annually. This value decreases because of the extraction and it can increase due to the discovery of new oil fields and due to the changes in oil prices and in extracting technologies. Nevertheless, in many theoretical problems we can consider s_0 as all the amount of the reserve including proven, unproven, and as yet not discovered so we can assume correctly that s_0 is a constant in these problems. However, if we are going to estimate numerically the path of tax that depends on s_0 and that controls the economy in the optimal way, we should estimate s_0 as accurately as possible. Otherwise, the economy will follow an inferior path in the case of underestimation of s_0 or it will overconsume if s_0 is overestimated.

In this section, I will examine a procedure of dynamic policy correction that will depend on the information about the changes in the resource reserves over time. The economy-linked criterion is flexible with respect to this changes since the parameter $\gamma = 1/\lambda_1$ can be recalculated depending on changes in reserve. In our example with the DHSS model the paths are defined by the value of s_0 (via $\lambda_1(s_0)$) at the initial moment $t = 0$. With time, we obtain additional information about s_0 that was not available at the initial moment. Using this information at each moment $t > 0$ we will reestimate s_0 that will imply the dynamic correction of the tax and of all the paths in the economy according to the changes in the criterion.

Assume that with time our revaluation of s_0 is growing and asymptotically approaches a constant $\widehat{s}_0$, for example, in the following way (Fig. 6):

$$s_0(t) = \widehat{s}_0 - e^{-wt}(\widehat{s}_0 - \overline{s}_0) \qquad (16)$$

I will take for the numerical example $s_0(0) = \overline{s}_0 = 2 \cdot 180.4722 = 360.94$ [bln t] and $\widehat{s}_0 = \lim_{t\to\infty} s_0(t) = 3 \cdot 180.4722 = 541.41$ (CERA's reserve estimate). The parameter w here is $w = 0.001$. Then we can make use of the explicit expression (15) for $\lambda_1(s_0)$. Substituting (16) for s_0 in (15) and then substituting it into (1) we obtain the measure of the optimal sustainable growth dynamically responding to the new information about the reserves. Substitution of

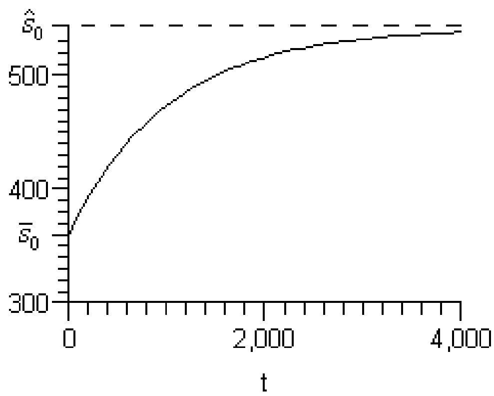

Figure 6: Information updates about the reserve estimate

the dynamically changing $\lambda_1(s_0(t))$ implies corresponding updates in paths of tax, capital, extraction, and consumption (Figs. 7 - 10, time in years). The paths corresponding to the precommitment policy with $s_0(t) \equiv \overline{s}_0$ are depicted as a solid line, precommitment paths with $s_0(t) \equiv \widehat{s}_0$ (assuming that we know everything about reserves at the initial moment) are in crosses, and the dynamically updated paths are in circles.

We can see that the reaction of the economy on the larger amount of the initial reserve ($s_0(t) \equiv \widehat{s}_0$, paths in crosses) is rather plausible. The level of tax is lower, the levels of capital and rates of extraction are higher and, as a result, the level of the optimal per capita consumption is also higher. Note that the economy-linked criterion combined with the modified Hotelling Rule can imply hump-shaped optimal paths of extraction. This result implies the notion of *the normative resource peak.* This peak can be compared with the one, being forecasted from the point of view of "physical possibility" of reaching the maximum level of extraction.[17]

[17]The theories of estimating the "physical" oil peak have been developing since the work of geologist M.K. Hubbert (1956). A methodology different from the Hubbert's oil-peak

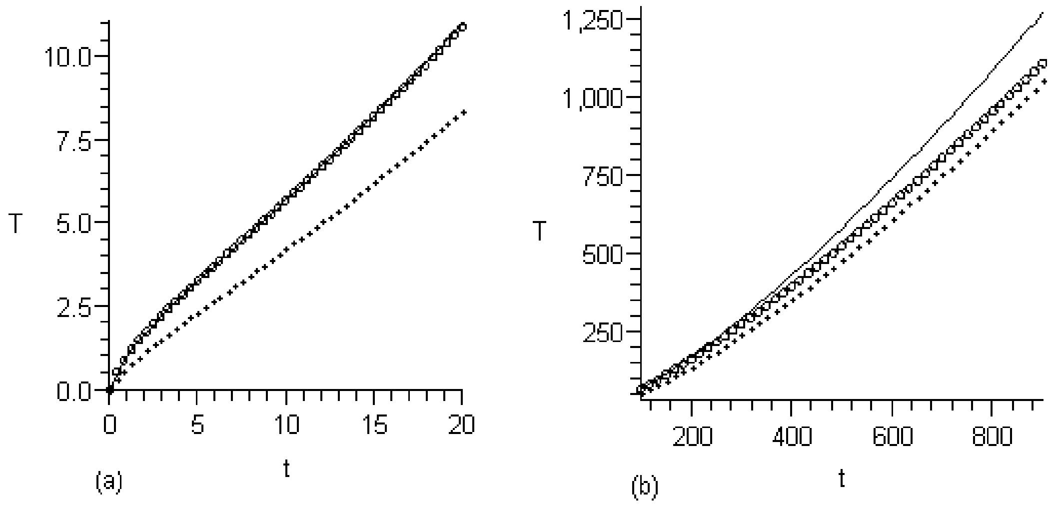

Figure 7: The optimal paths of tax: (a) in the short run; (b) in the long run. For fixed reserve $\overline{s}_0$ - as a solid line; for fixed reserve $\widehat{s}_0 = 1.5\overline{s}_0$ - in crosses; dynamically changing path - in circles

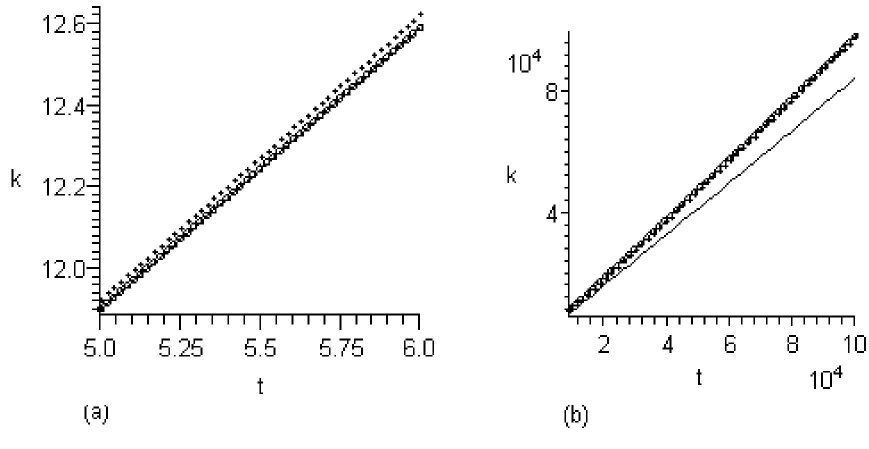

Figure 8: The optimal paths of capital: (a) in the short run; (b) in the long run. For fixed reserve $\overline{s}_0$ - as a solid line; for fixed reserve $\widehat{s}_0$ - in crosses; dynamically changing path - in circles

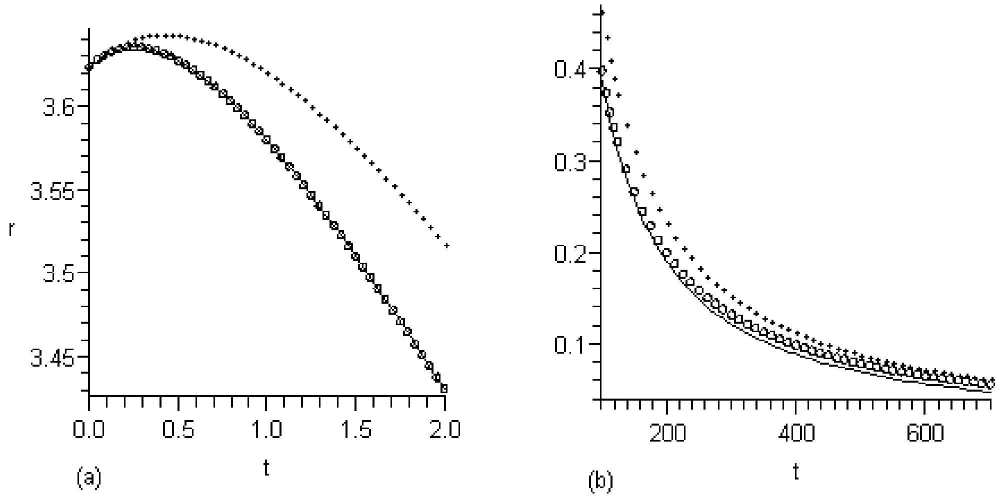

Figure 9: The optimal paths of extraction: (a) in the short run; (b) in the long run. For fixed reserve $\overline{s}_0$ - as a solid line; for fixed reserve $\widehat{s}_0$ - in crosses; dynamically changing path - in circles

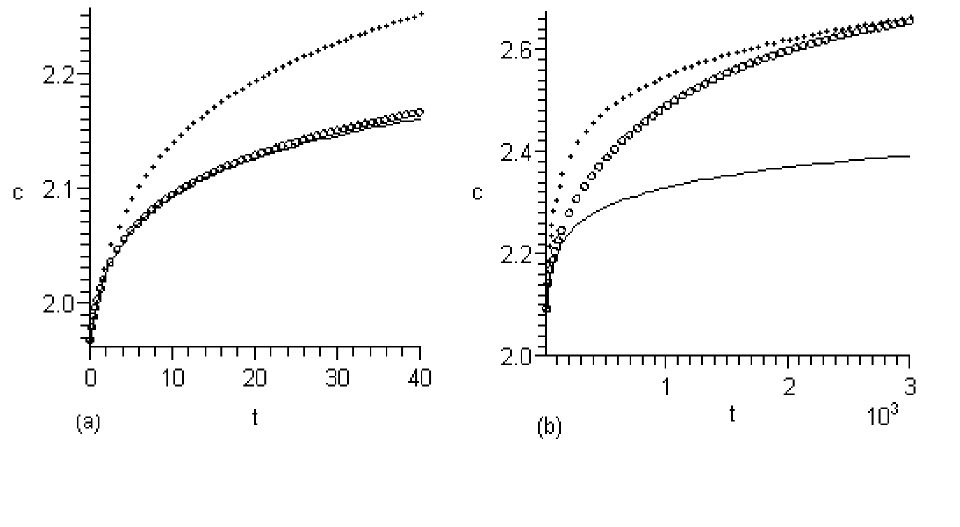

Figure 10: The optimal paths of consumption: (a) in the short run; (b) in the long run. For fixed reserve $\overline{s}_0$ - as a solid line; for fixed reserve $\widehat{s}_0$ - in crosses; dynamically changing path - in circles

One could expect that if an economy chooses an inferior path at the initial point due to the lack of knowledge about the reserve, then the difference in consumption with respect to the optimal "full-knowledge" path (line in crosses, Fig. 10) will only increase with time unless we correct the saving rule. However, the example shows that under the standard Hartwick Rule the consumption in the economy with the dynamically defined parameters (line in circles) is asymptotically "catching-up" to the optimal level of consumption in the process of updating the information about the reserve. The maximum difference in consumption during this process is less than 5%.

Another implication of the dynamically updated parameters is that the level of $\overline{U}$ in criterion (1) becomes variable ($\overline{U}(t) = c_0/\lambda_0^{1/\lambda_1(s_0(t))}$). This could undermine the argument about convenience of the geometrically weighted percent as a measure for sustainable growth. However, in our numerical example with substantially changing information about the reserve, the change in $\overline{U}$ is nothing more then 5% (from $\overline{U}(0) = 1.81$ to $\overline{U}(\infty) = 1.71$) that is negligible in comparison with the mismeasurements in the real economy.

10 Concluding remarks

Koopmans wrote that "The economist's traditional model of choice... is based on an analytical separation of preference and opportunity" (Koopmans 1964, p. 243). This paper offers an approach of linking a criterion (preference) with the opportunity of the specific economy. General form of the criterion is assumed to be fixed and parametrically connected with the

approach was used in the CERA's report (CERA 2006) according to which the world oil reserves are about three times larger than the conventional estimates and the "physical" oil peak is not expected before 2030. However, the optimal paths of extraction obtained in this paper imply that the normative oil peak must be much closer, namely, in 6 months even for the CERA's reserve estimate.

uncertain resource reserve and technological properties, while Koopmans assumed uncertain future preferences themselves with certain physical assets. Using this economy-linked criterion, it has been shown that from all patterns of growth offered in (Groth et al 2006) as regular growth, the extended Dasgupta-Heal-Solow-Stiglitz (DHSS) model can realize for the conventional value of $\alpha = 0.3$ only the (sustainable) paths of quasiarithmetic growth that are much closer to constant consumption than to linear function (Fig. 1). The DHSS model is extended here by the assumption that the Hotelling Rule is modified by the phenomena whose total influence can be expressed in terms of an equivalent tax or subsidy (Section 3). I interpreted the absence in the model of both technical change (TFP) and capital depreciation as presence of the specific TFP exactly compensating for the capital decay (Section 8).

The economy-linked criterion is constructed on an example of the maximin applied to a generalized level-growth measure (geometrically weighted percent). The parameter of this measure was calibrated on the economy's technological parameters and the initial conditions. The paper provides the closed form solutions for the optimal paths including the path of the Hotelling Rule modifier and the tax internalizing the externalities under the standard Hartwick Rule. I have derived the closed-form expression for the dependance of the parameter, specifying the criterion, on the reserve estimate. This formula was used to examine the optimal paths dynamically responding to the updates in the reserve estimates (Section 9).

The assumption about the generalized form of the Hotelling Rule modifier made it possible to calibrate the model on the world's oil extraction data (Sections 7 and 9). In particular, this modification allowed for nondecreasing extraction in the initial period. This property of the problem introduces the notion of the normative oil (resource) peak. It turned out that in the framework of this paper the optimal oil peak must be in 2-6 months depending on the amount of reserve. In other words, the socially-optimal oil peak is much closer to the initial moment than the various forecasts of the "physical" oil peak that show for how long the rates of extraction can grow.

It would be interesting to apply

(1) the economy-linked criterion for the problem with the specific externality like Stollery's (1998) and Hamilton's (1994) global warming, where the rising temperature affects not only the Hotelling Rule but also the utility and/or the production function;

(2) the methodology of linking a criterion to the specific economy for different hybrid measures and different criteria of justice;

(3) the methodology of linking a criterion to the specific economy with the specific patterns of endogenous technical change.

I think these problems deserve separate consideration.

11 Acknowledgments

I am grateful to Koichi Suga, Dan Usher, John M. Hartwick, Cees Withagen, and Maurice J. Roche for very useful comments and advice.

References

[1] Bazhanov, A.V. (2006a). Decreasing of oil extraction: Consumption behavior along transition paths. MPRA Paper No. 469, October 16, 2006.

[2] Bazhanov, A.V. (2006b). The peak of oil extraction and a modified maximin principle. MPRA Paper No. 2019, Proc. of the Intern. Conf. "Comparative Institution and Political-Economy: Theoretical, Experi-

mental and Empirical Analysis", Waseda University, Tokyo, Dec. 22-23, 2006, pp. 99-128.

[3] Bazhanov, A.V. (2007). The transition to an oil contraction economy. Ecological Economics, 64(1), 186-193.

[4] Bazhanov, A.V. (2008). Sustainable growth: compatibility between criterion and the initial state. MPRA Paper No. 9914, August 8, 2008.

[5] Bellman, R. (1957). Dynamic programming. Princeton, New Jersey: Princeton University Press.

[6] Brown, L. R. (2006). Plan B 2.0: Rescuing a Planet Under Stress and a Civilization in Trouble. New York: W.W. Norton & Company.

[7] Caillaud, B., Guesnerie, R., Rey, P., & Tirole, J. (1988). Government intervention in production and incentives theory: a review of recent contributions. RAND Journal of Economics. 19: 1-26.

[8] CERA, (2006). Why the Peak Oil Theory Falls Down: Myths, legends, and the future of oil resources. (November 10, 2006), http://www.cera.com/aspx/cda/client/report/reportpreview.aspx?CID=8437&KID= . Accessed 25 December 2007.

[9] Dasgupta, P. & Heal, G. (1974). The optimal depletion of exhaustible resources. Rev. Econ. Stud., 41, 3-28.

[10] Dasgupta, P. & Heal, G. (1979). Economic theory and exhaustible resources. Cambridge, Eng.: Cambridge University Press.

[11] Gaudet, G. (2007). Natural resource economics under the rule of Hotelling. Canadian Journal of Economics, 40(4), 1033-1059.

[12] Griffin, J.M. & Gregory, P.R. (1976). An intercountry translog model of energy substitution responses. Amer. Econ. Rev., 66, 845-857.

[13] Groth, C., Koch, K.J., Steger, T.M. (2006). Rethinking the concept of long-run economic growth. CESifo Working Paper N 1701.

[14] Hamilton, K. (1994). The Hartwick rule in a greenhouse world, Ch. 4 In Sustainable development and green national accounts, unpublished PhD thesis, University College London.

[15] Hartwick, J.M. (1977). Intergenerational equity and the investing of rents from exhaustible resources. Amer. Econ. Rev., 67, 972-974.

[16] Hartwick, J.M. (1990). Natural resources, national accounting and economic depreciation. Journal of Public Economics, 43, 291-304.

[17] Hotelling, H. (1931). The economics of exhaustible resources. The Journal of Political Economy, 39(2), 137 – 175.

[18] Hubbert, M.K. (1956). Nuclear energy and the fossil fuels. Amer. Petrol. Inst. Drilling & Production Practice. Proc. Spring Meeting, San Antonio, Texas., 7 - 25.

[19] Hueting, R. (1990). The Brundtland Report: A matter of conflicting goals. Ecological Economics, 2(2), 109-117.

[20] Kahneman, D., Varey, C. (1991). Notes on the psychology of utility. In J. Elster and J.E. Roemer (Eds), Interpersonal comparisons of well-being (pp. 127-163). New York: Cambridge University Press.

[21] Konow, J. (2003). Which is the fairest one of all? A positive analysis of justice theories. Journal of Economic Literature, 41(4), 1188 - 1239.

[22] Koopmans, T.C. (1964). On flexibility of future preference. In G. L. Bryan and M. W. Shelly, II, (Eds), Human judgments and optimality (pp. 243-254), New York, John Wiley.

[23] Koopmans, T.C. (1965). On the concept of optimal economic growth. In Pontificiae Academiae Scientiarum scripta varia, Vol.28, 225-300.

[24] Leininger, W. (1985). Rawls' maximin criterion and time-consistency: further results. Rev. Econ. Stud., 52, 505-513.

[25] Long, N. V. (1975). Resource extraction under the uncertainty about possible nationalization. Journal of Economic Theory, 10, 42-53.

[26] Nordhaus, W.D., Boyer, J. (2000). Warming the world: Economic models of global warming. Cambridge, Mass. : MIT Press.

[27] Pezzey, J.C.V. (2002). Sustainability policy and environmental policy. Australian National University. Economics and Environment Network Working Paper EEN0211.

[28] Pindyck, R.S. (1979). Interfuel Substitution and the Demand for Energy: An international comparison. Rev. Econ. Statist., 61, 169-179.

[29] Radler, M. (2006). Oil production, reserves increase slightly in 2006. Oil & Gas J., 104(47), 20-23.

[30] Rawls, J. (1971). A Theory of Justice. Cambridge, MA: Belknap Press of Harvard University Press.

[31] Scanlon, T.M. (1991). The moral basis of interpersonal comparisons. In J. Elster & J.E. Roemer (Eds), Interpersonal comparisons of well-being (pp. 17-44). New York: Cambridge University Press.

[32] Solow, R.M. (1974). Intergenerational equity and exhaustible resources. Rev. Econ. Stud., 41, 29-45.

[33] Stiglitz, J. (1974). Growth with exhaustible natural resources: Efficient and optimal growth paths. Rev. Econ. Stud., 41, 123-137.

[34] Stollery, K.R. (1998). Constant utility paths and irreversible global warming. The Canadian Journal of Economics, 31(3), 730-742.

[35] World, (1987). World Commission on environment and development. Our common future. Oxford/New York: Oxford University Press.

12 Appendix 1 (Proof of Proposition 1)

Lemma 1 gives the optimal pattern of the Hotelling Rule modifier $\tau(t) = [(\beta-1)/\beta]/(\lambda_1 t + \lambda_0)$. Indeed, equation (7) implies $\dot{q}/q = \tau\beta/(\beta-1) = 1/(\lambda_1 t + \lambda_0)$ that gives us $\lambda_0 = q_0/\dot{q}_0$ (for $\dot{q}_0 \neq 0$) and (solving it for $q(t)$) $q(t) = \widehat{q}\,(\lambda_1 t + \lambda_0)^{1/\lambda_1}$, where the constant of integration $\widehat{q}$ is defined from the initial condition $q(0) = q_0 : \widehat{q} = q_0/\lambda_0^{1/\lambda_1} = (\dot{q}_0/q_0)^{1/\lambda_1} q_0$. Then $\dot{q}(t) = \widehat{q}\,(\lambda_1 t + \lambda_0)^{1/\lambda_1 - 1}$ and expression $\dot{q}^\gamma q^{1-\gamma}$ with $\gamma = 1/\lambda_1$ gives us

$$\begin{aligned}\dot{q}^\gamma q^{1-\gamma} &= \widehat{q}^{1/\lambda_1}\,(\lambda_1 t + \lambda_0)^{(1/\lambda_1 - 1)/\lambda_1}\,\widehat{q}^{1-1/\lambda_1}\,(\lambda_1 t + \lambda_0)^{(1-1/\lambda_1)/\lambda_1} \\ &= \widehat{q} = const = \overline{U}/(1-\beta)\end{aligned}$$

We can rewrite $q(t)$ as follows $q(t) = q_0\,(1 + t\lambda_1/\lambda_0)^{1/\lambda_1}$.

Given expression for q and the saving rule $\dot{k} = \beta\widehat{q}\,(\lambda_1 t + \lambda_0)^{1/\lambda_1}$ we have the path for capital $k(t) = \widehat{k} + [\beta\widehat{q}/(1+\lambda_1)]\,(\lambda_1 t + \lambda_0)^{(1+1/\lambda_1)}$, where the initial condition $k(0) = k_0$ gives us the constant of integration $\widehat{k} = k_0 - \beta\widehat{q}\lambda_0^{(1+1/\lambda_1)}/(1+\lambda_1) = k_0 - \beta q_0\lambda_0/(1+\lambda_1)$. Then we have

$$k(t) = k_0 + \frac{\beta\widehat{q}}{(1+\lambda_1)}\left[(\lambda_1 t + \lambda_0)^{(1+1/\lambda_1)} - \lambda_0^{(1+1/\lambda_1)}\right].$$

The expressions for q and k give us the path of extraction $r(t) = \widehat{r}(\lambda_1 t + \lambda_0)^{1/\beta\lambda_1} k^{-\alpha/\beta}$ that implies the expression for the change of the rate of extraction[18]

$$\frac{\dot{r}}{r} = \frac{\widehat{k} + \beta\widehat{q}\left(\frac{1}{1+\lambda_1} - \alpha\right)(\lambda_1 t + \lambda_0)^{(1+1/\lambda_1)}}{\beta\widehat{k}\,(\lambda_1 t + \lambda_0) + \frac{\beta^2\widehat{q}}{1+\lambda_1}\,(\lambda_1 t + \lambda_0)^{(2+1/\lambda_1)}}. \tag{17}$$

The constant of integration $\widehat{r}$ can be defined via the initial value of extraction $r_0 : \widehat{r} = r_0\lambda_0^{-1/\beta\lambda_1}\left[\widehat{k} + \beta\widehat{q}\lambda_0^{(1+1/\lambda_1)}/(1+\lambda_1)\right]^{\alpha/\beta}$. The more simple expression

[18] The modified Hotelling Rule in form of (6) gives an equation for $\dot{r}/r$ that implies the same expression for r but in more cumbersome way.

for $\widehat{r}$ can be obtained using the production function $q = k^\alpha r^\beta$ that gives us $\widehat{r} = \widehat{q}^{1/\beta}$. Given the expression for $r(t)$ we can adjust parameter λ_1 using the feasibility and efficiency condition $s_0 = \int_0^\infty r(t)dt$ (Appendix 2).

Note that equation (17) implies that $\dot{r}/r \to 0$ with $t \to \infty$ and in order to obtain feasible behavior of $r(t)$ it is necessary that the ratio $\dot{r}/r$ is negative for t big enough. Assuming $\lambda_1 > 0$ we can see that for t big enough the denominator in (17) is positive and the nominator is negative if and only if $\alpha > 1/(1+\lambda_1)$ or $\lambda_1 > 1/\alpha - 1$ that justifies our assumption about the sign of λ_1 for $\alpha \in (0,1)$. This condition for $\lambda_1 = \lambda_1(s_0)$ can be interpreted as a possibility condition for realization of the economy-linked optimal (in a sense of criterion (1)) paths for the economy with technological parameter α and reserve s_0.

In order to express explicitly the path of tax from formula (10): $T(t) = \exp\left[\int f_k(t)dt\right]\left\{\widehat{T} - \int \tau f_r \exp\left[-\int f_k(\xi)d\xi\right]dt\right\}$ I will consider the following integral that for the case of the Hartwick Rule is $\int f_k(t)dt = \alpha\int(q/k)dt = (\alpha/\beta)\int(\dot{k}/k)dt = (\alpha/\beta)\ln k + C_1$. It implies $\exp\left[\int f_k(t)dt\right] = C_2 k(t)^{\alpha/\beta}$ and

$$\int \tau f_r \exp\left[-\int f_k(\xi)d\xi\right]dt = \frac{1}{C_2}\left[\frac{\beta\widehat{q}(1+\lambda_1)^{\alpha/\beta}}{\widehat{r}}(\lambda_1 t + \lambda_0)^{\frac{\beta-1}{\beta\lambda_1}}\right] + C_3$$

that gives us

$$T(t) = k(t)^{\alpha/\beta}\left[\widehat{T} - \frac{\beta\widehat{q}(1+\lambda_1)^{\alpha/\beta}}{\widehat{r}}(\lambda_1 t + \lambda_0)^{\frac{\beta-1}{\beta\lambda_1}}\right], \tag{18}$$

where $\widehat{T} = \widehat{T}(C_2, C_3)$. Since $\widehat{q} = q_0/\lambda_0^{1/\lambda_1}$ and $\widehat{r} = \widehat{q}^{1/\beta}$, and given $T_0 = T(0)$ we have $\widehat{T} = T_0 k_0^{-\alpha/\beta} + \beta\widehat{q}^{1-1/\beta}(1+\lambda_1)^{\alpha/\beta}\lambda_0^{(\beta-1)/(\beta\lambda_1)}$ or $\widehat{T} = T_0 k_0^{-\alpha/\beta} + \beta q_0^{(\beta-1)/\beta}(1+\lambda_1)^{\alpha/\beta}$. Substituting it into (18) we obtain

$$T(t) = k(t)^{\alpha/\beta}\left\{T_0 k_0^{-\alpha/\beta} + \beta(1+\lambda_1)^{\alpha/\beta} q_0^{(\beta-1)/\beta}\left[1 - \left(\frac{\lambda_1}{\lambda_0}t + 1\right)^{(\beta-1)/(\beta\lambda_1)}\right]\right\}$$

that for $T_0 = 0$ gives us the expression formulated in the proposition ■

13 Appendix 2 (Estimation of $\lambda_1(s_0)$)

The value of λ_1 can be expressed via reserve estimate s_0 using the feasibility-efficiency condition $\int_0^\infty r(t,\lambda_1)dt = s_0$. In order to find $\lambda_1(s_0)$ I will use sequential integration of $r(t,\lambda_1)$ by parts that will follow representation of s_0 as a series. For this I will express r in the following way $r = q^{1/\beta}k^{-\alpha/\beta} = (1/\beta)^{1/\beta}\,\dot{k}^{1/\beta-1}\dot{k}k^{-\alpha/\beta}$. Denote $u = \dot{k}^{1/\beta-1}$ and $dv = k^{-\alpha/\beta}\dot{k}dt$. Then

$$\begin{aligned}\int_0^\infty rdt &= (1/\beta)^{1/\beta}\int_0^\infty udv = (1/\beta)^{1/\beta}\left[uv - \int_0^\infty vdu\right]\\ &= (1/\beta)^{1/\beta}\left[-\frac{\dot{k}_0^{1/\beta-1}k_0^{1-\alpha/\beta}}{1-\alpha/\beta} - \frac{1-\beta}{\beta-\alpha}I_2\right],\end{aligned}$$

where $I_2 = \int_0^\infty k^{1-\alpha/\beta}\dot{k}^{1/\beta-2}\,\ddot{k}\,dt$. Substituting for $\ddot{k} = \beta\widehat{q}\,(\lambda_1 t+\lambda_0)^{1/\lambda_1-1} = (\beta\widehat{q})^{\lambda_1}\,\dot{k}^{1-\lambda_1}$ we have $I_2 = (\beta\widehat{q})^{\lambda_1}\,I_3$, where $I_3 = \int_0^\infty k^{1-\alpha/\beta}\dot{k}^{1/\beta-1-\lambda_1}dt$. Since $k/\dot{k}^{(1+\lambda_1)} = \widehat{k}\dot{k}^{-1-\lambda_1} + (\beta\widehat{q})^{-\lambda_1}/(1+\lambda_1)$ then $k^{1-\alpha/\beta}\dot{k}^{1/\beta-1-\lambda_1} = k^{-\alpha/\beta}\dot{k}^{1/\beta}k/\dot{k}^{(1+\lambda_1)} = k^{-\alpha/\beta}\dot{k}^{1/\beta}\left[\widehat{k}\dot{k}^{-1-\lambda_1} + (\beta\widehat{q})^{-\lambda_1}/(1+\lambda_1)\right]$. It implies $I_3 = \widehat{k}\int_0^\infty k^{-\alpha/\beta}\dot{k}^{1/\beta-1-\lambda_1}dt + (\beta\widehat{q})^{-\lambda_1}/(1+\lambda_1)\int_0^\infty k^{-\alpha/\beta}\dot{k}^{1/\beta}dt$. The second integral, expressed via the original one, equals to $\beta^{1/\beta}\int_0^\infty rdt$. Then the original integral is

$$\begin{aligned}\int_0^\infty rdt &= (1/\beta)^{1/\beta}\left\{-\frac{k_0^{1-\alpha/\beta}\dot{k}_0^{1/\beta-1}}{1-\alpha/\beta} - \frac{1-\beta}{\beta-\alpha}(\beta\widehat{q})^{\lambda_1}\right.\\ &\quad\left.\times\left[\frac{(\beta\widehat{q})^{-\lambda_1}}{(1+\lambda_1)}\beta^{1/\beta}\int_0^\infty rdt + \widehat{k}I_4\right]\right\},\end{aligned} \tag{19}$$

where $I_4 = \int_0^\infty k^{-\alpha/\beta}\dot{k}^{1/\beta-(\lambda_1+1)}dt$. Expressing $\int_0^\infty rdt$ from (19) we obtain

$$\begin{aligned}\int_0^\infty rdt &= \frac{\lambda_1+1}{(\lambda_1+1)(-\alpha/\beta+1)-1+1/\beta}(1/\beta)^{1/\beta}\\ &\quad\times\left\{-k_0^{1-\alpha/\beta}\dot{k}_0^{1/\beta-1} - (1/\beta-1)(\beta\widehat{q})^{\lambda_1}\widehat{k}I_4\right\}.\end{aligned} \tag{20}$$

Integrating I_4 by parts with $u = \dot{k}^{1/\beta-1-(\lambda_1+1)}$, $dv = k^{-\alpha/\beta}\dot{k}dt$ and applying the same substitutions we have

$$\begin{aligned} I_4 &= \frac{\lambda_1+1}{(\lambda_1+1)(-\alpha/\beta)-1+1/\beta} \\ &\times \left[-k_0^{1-\alpha/\beta}\dot{k}_0^{1/\beta-1-(\lambda_1+1)} - (1/\beta-1-(\lambda_1+1))\left(\beta\widehat{q}\right)^{\lambda_1}\widehat{k}I_8\right], \end{aligned}$$

where $I_8 = \int_0^\infty k^{-\alpha/\beta}\dot{k}^{1/\beta-2(\lambda_1+1)}dt$. Substituting for I_4 in (20) we obtain

$$\begin{aligned} \int_0^\infty r dt &= \frac{\lambda_1+1}{(\lambda_1+1)(-\alpha/\beta+1)-1+1/\beta}\left(1/\beta\right)^{1/\beta} \\ &\times \left\{-k_0^{1-\alpha/\beta}\dot{k}_0^{1/\beta-1} - \frac{(\lambda_1+1)(1/\beta-1)}{(\lambda_1+1)(-\alpha/\beta)-1+1/\beta}\left(\beta\widehat{q}\right)^{\lambda_1}\widehat{k}\right. \\ &\times \left.\left[-k_0^{1-\alpha/\beta}\dot{k}_0^{1/\beta-1-(\lambda_1+1)} - (1/\beta-1-(\lambda_1+1))\left(\beta\widehat{q}\right)^{\lambda_1}\widehat{k}I_8\right]\right\} \end{aligned}$$

Integrating I_8 by parts with $u = \dot{k}^{1/\beta-1-2(\lambda_1+1)}$, $dv = k^{-\alpha/\beta}\dot{k}dt$ we have

$$\begin{aligned} I_8 &= \frac{\lambda_1+1}{(\lambda_1+1)(-\alpha/\beta-1)-1+1/\beta} \\ &\times \left[-k_0^{1-\alpha/\beta}\dot{k}_0^{1/\beta-1-2(\lambda_1+1)} - (1/\beta-1-2(\lambda_1+1))\left(\beta\widehat{q}\right)^{\lambda_1}\widehat{k}I_{12}\right]. \end{aligned}$$

This makes visible the pattern of expressions for integrals $I_4, I_8, I_{12}, \ldots$ and so (multiplying fractions by $-\beta$) we can show that the original integral is

$$\int_0^\infty r dt = \frac{\lambda_1+1}{(\lambda_1+1)(\alpha-\beta)-1+\beta}\cdot\beta^{(1-1/\beta)}$$

$$\times \left\{k_0^{1-\alpha/\beta}\dot{k}_0^{1/\beta-1} + \frac{(\lambda_1+1)(1-\beta)}{(\lambda_1+1)\alpha-1+\beta}\left(\beta\widehat{q}\right)^{\lambda_1}\widehat{k}\right.$$

$$\times \left[k_0^{1-\alpha/\beta}\dot{k}_0^{1/\beta-1-(\lambda_1+1)} + \frac{(\lambda_1+1)(1-\beta[1+(\lambda_1+1)])}{(\lambda_1+1)(\alpha+\beta)-1+\beta}\left(\beta\widehat{q}\right)^{\lambda_1}\widehat{k}\right.$$

$$\left.\left.\times \left\{k_0^{1-\alpha/\beta}\dot{k}_0^{1/\beta-1-2(\lambda_1+1)} + \frac{(\lambda_1+1)(1-\beta[1+2(\lambda_1+1)])}{(\lambda_1+1)(\alpha+2\beta)-1+\beta}\left(\beta\widehat{q}\right)^{\lambda_1}\widehat{k} \times \left[k_0^{1-\alpha/\beta}\dot{k}_0^{1/\beta-1-3(\lambda_1+1)} + \ldots\right]\right\}\right]\right\}$$

Substituting for $\widehat{q}, \lambda_0$, and for $\dot{k}_0 = \beta k_0^{\alpha} r_0^{\beta}$ we obtain

$$\int_0^{\infty} r dt = \frac{\lambda_1 + 1}{(\lambda_1 + 1)(\alpha - \beta) - 1 + \beta} \cdot \frac{k_0 r_0}{q_0}$$

$$\times \left\{ 1 + \frac{(\lambda_1 + 1)(1 - \beta)}{(\lambda_1 + 1)\alpha - 1 + \beta} \cdot \widehat{k} \cdot [\beta \dot{q}_0 \right.$$
$$+ \frac{(\lambda_1 + 1)(1 - \beta[1 + (\lambda_1 + 1)])}{(\lambda_1 + 1)(\alpha + \beta) - 1 + \beta} \cdot \widehat{k} \cdot \{(\beta \dot{q}_0)^2$$
$$\left. + \frac{(\lambda_1 + 1)(1 - \beta[1 + 2(\lambda_1 + 1)])}{(\lambda_1 + 1)(\alpha + 2\beta) - 1 + \beta} \cdot \widehat{k} \cdot [(\beta \dot{q}_0)^3 + \ldots] \}] \right\}$$

This gives us a closed form solution for our integral as a series

$$\begin{aligned} \int_0^{\infty} r dt &= \frac{\lambda_1 + 1}{(\lambda_1 + 1)(\alpha - \beta) - 1 + \beta} \cdot \frac{k_0 r_0}{q_0} \\ &\times \left\{ 1 + \sum_{i=1}^{\infty} \left[\widehat{k}(\lambda_1) \beta \dot{q}_0 (\lambda_1 + 1) \right]^i \cdot \prod_{j=0}^{i-1} \frac{1 - \beta [1 + j(\lambda_1 + 1)]}{(\lambda_1 + 1)(\alpha + j\beta) + \beta - 1} \right\} \end{aligned}$$

The series can be expressed via special functions,[19] namely,

$$\prod_{j=0}^{i-1} \frac{1 - \beta [1 + j(\lambda_1 + 1)]}{(\lambda_1 + 1)(\alpha + j\beta) + \beta - 1} = [-\beta (\lambda_1 + 1)]^i \, \Gamma \left(i - \frac{1 - \beta}{\beta (\lambda_1 + 1)} \right) / \Gamma \left(- \frac{1 - \beta}{\beta (\lambda_1 + 1)} \right)$$

and then

$$\begin{aligned} \int_0^{\infty} r dt &= \frac{\lambda_1 + 1}{(\lambda_1 + 1)(\alpha - \beta) - 1 + \beta} \cdot \frac{k_0 r_0}{q_0} \\ &\times \left\{ 1 + (1 - \beta) \beta \widehat{k}(\lambda_1) \dot{q}_0 (\lambda_1 + 1) \right. \\ &\times {}_2F_0 \left(\left[1, \frac{\beta(\lambda_1 + 2) - 1}{\beta(\lambda_1 + 1)} \right], [\,], -\widehat{k}(\lambda_1) \dot{q}_0 \beta^2 (\lambda_1 + 1)^2 \right) \left. \right\}, \end{aligned} \tag{21}$$

where ${}_2F_0(\cdot)$ is the hypergeometric function with 2 upper parameters and empty list of lower parameters. Substituting for $\widehat{k} = k_0 - \beta q_0^2 / [\dot{q}_0 (1 + \lambda_1)]$

[19] The expression of the series via special functions can be obtained in Maple.

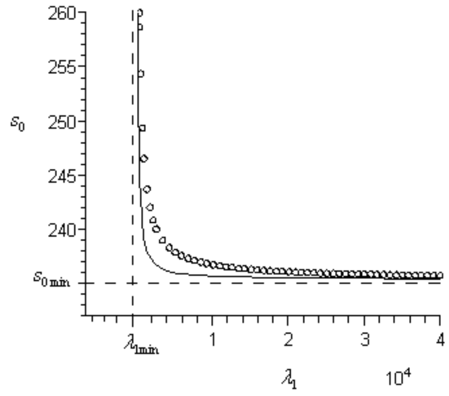

Figure 11: Dependence of reserve s_0 (the value of integral $\int_0^\infty r(t, \lambda_1)dt$) on λ_1: closed form solution (21) - in circles; approximate formula - solid line

(Appendix 1) we obtain equation (12) in Corollary 1. For our numerical example the second term in bracket $\{\cdot\}$ equals to 0.247 and so, taking into account the existing uncertainty in reserve estimate, we can consider as a good approximation for the value of reserve the following formula

$$s_0 = \int_0^\infty r dt = \frac{\lambda_1 + 1}{(\lambda_1 + 1)(\alpha - \beta) - 1 + \beta} \cdot \frac{k_0 r_0}{q_0}$$

that gives us an explicit expression for $\lambda_1(s_0)$:

$$\lambda_1 = \frac{(1 - \alpha)s_0 q_0 + k_0 r_0}{(\alpha - \beta)s_0 q_0 - k_0 r_0}.$$

This formula captures the main peculiarities of behavior of the exact solution. Particularly, it has the same horizontal and vertical asymptotes as the closed form solution (21) (Fig. 11).