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The Advance in Partial Distribution：A New Mathematical Tool for Economic Management

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Abstract: In this paper, the Partial Distribution (PD) and multivariate Partial Distribution (MPD) are presented in their concepts, properties and applications, and PD is compared with the lognormal and the levy distribution. Though the levy distribution is better to describe the exchange returns in security market on a moderately large volatility range, the lognormal is better in a region of low values of volatility. We shall try to elucidate that Partial Distribution is better than lognormal distribution and levy distribution in many respects, and PD and MPD have some interesting properties which some other probability distributions have not. From PD and MPD, lots of interesting results can be acquired and many interesting economic propositions could be interpreted in analytic way. These properties could describe analytically many of phenomena in economic management better, and the results based on PD and MPD could be applied to solve many problems in economic management.

Keywords: Partial Distribution; multivariate Partial Distribution; mathematical tool, economic management

1 Introduction

Mathematics is the important tool and basis to describe and analyze the management behaviors, strategies and activities, especially to economic management.

The economic problems, like the price behavior, pricing model, market risk and interpretation for economic propositions, are all the important and basic problems which researchers and scholars have been devoting to solve by the mathematical approaches. Gaussian, lognormal and Levy are commonly the probability distributions applied to do these works. Financial time series typically exhibit strong fluctuations that cannot be described by a Gaussian distribution. Recent empirical studies of stock market indices examined whether the distribution of returns can be described by a Levy-stable distribution with some index $0<\alpha\leq 2$. While the Levy distribution cannot be expressed in a closed form except the Gaussian and Cauchy. Ofer Biham et al. (2001) studied the distribution of returns in a generic model that describes the dynamics of stock market indices. For the distributions generated by this model, we observe that the scaling of the central peak is consistent with a Levy distribution while the tails exhibit a power-law distribution with an exponent $\alpha>2$, namely, beyond the range of Levy-stable distributions. The results are in agreement with both empirical studies and reconcile the apparent disagreement between their results.

Salvatore Micciche et al. (2002) investigated the historical volatility of the 100 most capitalized stocks traded in US equity markets. An empirical probability density function (pdf) of volatility is obtained and compared with the theoretical predictions of a lognormal model and of the Hull and White model. The lognormal model well describes the pdf in the region of low values of volatility_whereas the Hull and White model better approximates the empirical pdf for large values of volatility. Both models fail in describing the empirical pdf over a moderately large volatility range.

At the time when the volatility is large, we can use the Levy model. But, this does not mean that all of stock markets in the world will be in the violent fluctuation forever. Even if the stock market is generally in the violent fluctuation, the stock price is also in the low values of volatility at some periods of time. In fact, we need to use the most proper model to analyze the probability distribution of stocks price in case the stock price behavior is in the region of low values of volatility. In this case, people may accustom to the use the lognormal model. However, when a company collapses, the price of its stock will be zero. The lognormal model can't describe the possibility when price of a stock is zero, but the partial distribution (F. Dai, 2001) can. So the partial distribution should be applied to describe the price distribution of commodities and stocks at the lower values of volatility [4].

Also, it is very important to price objectively the product (capital asset, spot asset and options) in the modern economical society. The most outstanding studies and works have been done for the estimating and measuring of the price of capital asset, like CAPM (capital asset pricing model, W.F. Sharpe, 1964, J. Lintner, 1965) and APT (arbitrage pricing theory, S.A. Ross, 1976), etc. Coming a further considering, we see, CAPM needs a group of risk capitals. This is difficult to realize in reality because it is not easy to make a whole samples indexes in a larger financial market. In general, CAPM is regarded as an example of APT, because CAPM is a method for a single asset, and APT is a method for group assets. In fact, CAPM can also be extended to multiple assets, for example, the Consumer Service Model (R.C. Merton, 1973). This model also considered the risk premium of assets group, APT did not. Again, the consumption-based CAPM (T.Breeden, 1979) is more of imagination.

What we need to point out is, CAPM is based on the market equilibrium and is a result of investors behaving together. So CAPM must be under a series of assumptions, and some of assumptions are more rigorous in some time. APT is applicable to investment decision on group assets and emphasizes the rule of no-arbitrage. APT is based on the assets group, so it is not always right in pricing for single asset. In the other hand, CAPM and APT make the pricing on yield of asset or assets mainly. The prices of asset itself always change in a financial market. In many time, we need to know not only the yield of asset or assets, but also the current prices of asset itself, because both of them are influenced one another. A series of new models of pricing asset or assets could be given based on Partial Distribution and multivariate Partial Distribution in reference [11].

In another hand, although the model of pricing for European option is initiatively given (F. Black and M. Scholes, 1973), there is also no accurate analytic formula for pricing the American put option now. The American put value could be exercised at any time before its expiration, and European option must be exercised at end of expiration. Geske and Johnson (1984), however, obtained a solution as an n-fold compound option, using Geske's (1979) compound option formula and the equivalent martingale/risk neutrality assumption. D. S. Bunch and H. Johnson (1992) given a simple and numerically efficient valuation method for American put option. Y.Tian (1993, 1999) has given the binomial option pricing models. Up to now lots of numerical approximation procedures were proposed for pricing American put options (MacMillan 1986, Stapleton and Subrahmanyam 1997). Because of various difficulties in calculating the price of American put options, however, intensive efforts are still needed for developing new accurate formula to this problem. The problem of pricing for American options has been solved by the structure model (F. Dai, 2005) based on the Partial Distribution. Again, and also the Structure Models for Futures Options Pricing is given in reference [20].

Again, we can interpret some economic propositions in analytic way based on Partial Distribution and multivariate Partial Distribution, like “the more the risk is, the larger the possible profit is”, “the new asset will bring the higher sale margin”, and etc.

This paper will generalize the Partial Distribution and multivariate Partial Distribution in their concepts, properties, applications, and related models. The models given here can also be applied to price the virtual products, invisible asset and etc., and to analyze the price risk of them. Though the problems mentioned above may not be solved very well in a whole based on the Partial Distribution and multivariate Partial Distribution, it may be important for us to develop a series of new methods.

2 The Definitions of Partial Distribution AND Multivariate Partial Distribution

Definition 1 (Partial Distribution, PD for short) Let X be a non-negative stochastic variable, and it follows the distribution density

$$f(x)=\begin{cases} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \Big/ \int_0^\infty e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx & x\geq 0 \\ 0 & x<0 \end{cases} \tag{1}$$

where $\mu\geq 0$ and $\sigma>0$. Then, X is called to follow a Partial Distribution, and note as $X\in P(\mu, \sigma^2)$.

Definition 2 (Partial Process) Let $\{X(t), t\in[0,\infty)\}$ be a stochastic process. $\forall\ t\in [0,\infty)$, $X(t)$ follows the Partial Distribution $P(\mu(t), \sigma^2(\mathrm{t}))$, then the $\{X(t), t\in[0,\infty)\}$ is called a partial process.

Definition 3 (Multivariate Partial Distribution, MPD for short) if $X_1, \cdots, X_n$ ($n\geq 2$) are all the non-negative stochastic variables, and follow the multivariate distribution of density

$$f(x_1, \cdots, x_n)=\begin{cases} \dfrac{e^{-\frac{1}{2|M|}\left[\sum\limits_{i=1}^{n}|M_{ii}|(x_i-\mu_i)^2+\sum\limits_{i,j=1,i\neq j}^{n}|M_{ij}|\left(\sigma_i\left(x_j-\mu_j\right)\right)\right]}}{\int\limits_0^\infty\cdots\int\limits_0^\infty e^{-\frac{1}{2|M|}\left[\sum\limits_{i=1}^{n}|M_{ii}|(x_i-\mu_i)^2+\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n}\sigma_i|M_{ij}|\left(x_j-\mu_j\right)\right]}dx_1\cdots dx_n} & 0\leq x_1,\cdots,x_n<\infty \\ 0 & \text{other cases} \end{cases} \tag{2}$$

where, $M=\begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ \sigma_{n1} & \sigma_{n1} & \cdots & \sigma_{nn} \end{bmatrix}$, $\sigma_{ii}=\sigma_i^2$, $\sigma_{ij}=r_{ij}\sigma_i\sigma_i$ ($i\neq j$), $\sigma_i>0$, $|r_{ij}|\leq 1$, $i, j=1, \cdots, n$.

then $(X_1, \cdots, X_n)$ is called to follow n-dimensions Partial Distribution, and note as $\boldsymbol{X}\in P(\boldsymbol{\mu}, \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\sigma}, \boldsymbol{R})$. where, $\boldsymbol{X}=(X_1, \cdots, X_n)^{\mathrm{T}}$, $\boldsymbol{\mu}=(\mu_1, \cdots, \mu_n)^{\mathrm{T}}$, $\boldsymbol{\sigma}=(\sigma_1, \cdots, \sigma_n)^{\mathrm{T}}$, $\boldsymbol{R}=(r_{ij})_{n\times n}$, $\mu_1, \cdots, \mu_n\geq 0$, $\sigma_1, \cdots, \sigma_n>0$, r_{ij} is called the correlation coefficient between X_i and X_j, $r_{ii}=1$, $i, j=1, \cdots, n$.

As a special example of MPD, if the non-negative stochastic variables X and Y follow the multivariate distribution of density:

$$
f(x, y)=\begin{cases} \dfrac{e^{-\frac{1}{2(1-r^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2-2r\left(\left(\frac{x-\mu_1}{\sigma_1}\right)+\left(\frac{y-\mu_2}{\sigma_2}\right)\right)+\left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]}}{\int_0^\infty\int_0^\infty e^{-\frac{1}{2(1-r^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2-2r\left(\left(\frac{x-\mu_1}{\sigma_1}\right)+\left(\frac{y-\mu_2}{\sigma_2}\right)\right)+\left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]}dxdy}, & 0\le x, y<\infty \\ 0 & x<0 \text{ or } y<0 \end{cases} \tag{3}
$$

then, (X, Y) is called to follow 2-dimensions Partial Distribution, and note as $(X, Y) \in P(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,r)$, where, the constants μ_1, $\mu_2 \ge 0$, σ_1, $\sigma_2>0$, $-1<r<1$.

If $r=0$, thus we know, from expression (3) and (1), $f(x, y)=f_1(x)f_2(y)$, i.e. X is not correlating with Y. Where,

$$
f_1(x)=\begin{cases} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}}\Big/\int_0^\infty e^{-\frac{(u-\mu_1)^2}{2\sigma_1^2}}du & x\ge 0 \\ 0 & x<0 \end{cases},\quad f_2(y)=\begin{cases} e^{-\frac{(y-\mu_1)^2}{2\sigma_2^2}}\Big/\int_0^\infty e^{-\frac{(u-\mu_2)^2}{2\sigma_2^2}}du & y\ge 0 \\ 0 & y<0 \end{cases}
$$

When $r=1$, we know, according to reference [21], X is correlating with Y in linearity and on probability 1, i.e., the probability $P(Y=aX+b)=1$,where, $a>0$ if $r=1$, and $a<0$ if $r=-1$.

3 Some Basic results about PD and MPD

3.1 Some basic results about PD

According to references [3], we have two basic results about Partial Distribution as follow:

Theorem 1 For any $x\in[0,\infty)$, the following formulas are correct approximately:

1) $\int_0^x e^{-\frac{t^2}{2}}dt=\sqrt{\frac{\pi}{2}(1-e^{-\frac{2}{\pi}x^2})}$;

2) $\int_0^x e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx=\sqrt{\frac{\pi}{2}}\sigma\left((\sqrt{1-e^{-\frac{2}{\pi}(\frac{\mu}{\sigma})^2}}+\operatorname{sgn}(x-\mu)\sqrt{1-e^{-\frac{2}{\pi}(\frac{x-\mu}{\sigma})^2}}\right)$, where, $\operatorname{sgn}(x)=\begin{cases}1 & x>0\\ 0 & x=0\\ -1 & x<0\end{cases}$.

In the theorem 1, if $x\to\infty$, then we have

$$
\int_0^\infty e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx=\sqrt{\frac{\pi}{2}}\sigma\left((\sqrt{1-e^{-\frac{2}{\pi}(\frac{\mu}{\sigma})^2}}+1\right).
$$

Corollary 1 For any $x\in[a,\infty]$, a, μ and σ are constant, $0\le a<\mu$, $\sigma>0$, then the following equations are correct approximately:

$$
\int_a^x e^{-\frac{(u-\mu)^2}{2\sigma^2}}du=\sqrt{\frac{\pi}{2}}\sigma\times\left(\sqrt{1-e^{-\frac{2}{\pi}(\frac{\mu-a}{\sigma})^2}}+\operatorname{sgn}(x-\mu+a)\sqrt{1-e^{-\frac{2}{\pi}(\frac{x-\mu+a}{\sigma})^2}}\right), \text{ where, } \operatorname{sgn}(x)=\begin{cases}1 & x>0\\ 0 & x=0\\ -1 & x<0\end{cases}.
$$

Corollary 2 For any $x\in[a,\infty]$, a, μ and σ are constant, $0\le\mu<a$, $\sigma>0$, then the following equations are correct approximately:

$$\int_a^x e^{-\frac{(u-\mu)^2}{2\sigma^2}} du = \sqrt{\frac{\pi}{2}}\sigma \times \left(\sqrt{1-e^{-\frac{2}{\pi}(\frac{x-\mu}{\sigma})^2}} - \sqrt{1-e^{-\frac{2}{\pi}(\frac{a-\mu}{\sigma})^2}} \right).$$

From theorem 1, we have

Theorem 2 Let X follow the PD, $X \in P(\mu, \sigma^2)$, thus

1) The expected value of X, $E(X)$, is as follows

$$E(X) = \int_0^\infty x e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \Big/ \int_0^\infty e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx,$$

i.e., $E(X) = \mu + \sqrt{\frac{2}{\pi}} \frac{\sigma e^{-\frac{\mu^2}{2\sigma^2}}}{\sqrt{1-e^{-\frac{2}{\pi}(\frac{\mu}{\sigma})^2}} + 1}$ (4)

2) The variance of X, $D(X)$, is as follows

$$D(X) = \int_0^\infty [x - E(X)]^2 f(x) dx,$$

i.e., $D(X) = \sigma^2 + E(X)[\mu - E(X)]$ (5)

3.2 Some basic results about MPD

Similarly to theorem 1 and theorem 2, we obtain separately the theorem 3 and theorem 4 as follow:

Theorem 3 If both X_1 and X_2 are stochastic variables and follow 2-dimensions Partial Distribution, i.e., $(X_1, X_2) \in P(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, r)$, thus

1) $$\int_0^\infty \int_0^\infty e^{-\frac{1}{2(1-r^2)}\left[\left(\frac{u-\mu_1}{\sigma_1}\right)^2 - 2r\left(\left(\frac{u-\mu_1}{\sigma_1}\right)+\left(\frac{v-\mu_2}{\sigma_2}\right)\right)+\left(\frac{v-\mu_2}{\sigma_2}\right)^2\right]} dudv$$

$$= \frac{\pi}{2}\sigma_1\sigma_2(1-r^2)e^{\frac{r^2}{1-r^2}}\left(1+\sqrt{1-e^{-\frac{2}{\pi}\left(\frac{\mu_1+r\sigma_1}{\sigma_1\sqrt{1-r^2}}\right)^2}}\right)\left(1+\sqrt{1-e^{-\frac{2}{\pi}\left(\frac{\mu_2+r\sigma_2}{\sigma_2\sqrt{1-r^2}}\right)^2}}\right).$$

2) $$\int_0^{x_1}\int_0^{x_2} f(u,v) dudv = \frac{A_1(x_1)A_2(x_2)}{\left(1+\sqrt{1-e^{-\frac{2}{\pi}\left(\frac{\mu_1+r\sigma_1}{\sigma_1\sqrt{1-r^2}}\right)^2}}\right)\left(1+\sqrt{1-e^{-\frac{2}{\pi}\left(\frac{\mu_2+r\sigma_2}{\sigma_2\sqrt{1-r^2}}\right)^2}}\right)}, \ 0 \le x_1, x_2 < \infty$$

where, $A_i(x) = \sqrt{1-e^{-\frac{2}{\pi}\left(\frac{\mu_i+r\sigma_i}{\sigma_i\sqrt{1-r^2}}\right)^2}} + \mathrm{sgn}(x-\mu_i)\sqrt{1-e^{-\frac{2}{\pi}\left(\frac{x-(\mu_i+r\sigma_i)}{\sigma_i\sqrt{(1-r^2)}}\right)^2}}$, the meaning of sgn($t$) is similar to theorem 1, i=1, 2.

If denoting:

$$f_{1r}(x)=\int_0^\infty f(x,y)dy=\begin{cases} e^{-\frac{[x-(\mu_1+r\sigma_1)]^2}{2\sigma_1^2(1-r^2)}} \Bigg/ \int_0^\infty e^{-\frac{[u-(\mu_1+r\sigma_1)]^2}{2\sigma_1^2(1-r^2)}} du & x\ge 0 \\ 0 & x<0 \end{cases}$$

and $$f_{2r}(y)=\int_0^\infty f(x,y)dx=\begin{cases} e^{-\frac{[y-(\mu_2+r\sigma_2)]^2}{2\sigma_2^2(1-r^2)}} \Bigg/ \int_0^\infty e^{-\frac{[u-(\mu_2+r\sigma_2)]^2}{2\sigma_2^2(1-r^2)}} du & y\ge 0 \\ 0 & y<0 \end{cases}$$

Theorem 4 If both X_1 and X_2 are stochastic variables and follow 2-dimensions Partial Distribution, i.e., $(X_1, X_2) \in P(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,r)$, thus

1) The expected values of each stochastic variable are

$$E_r(X_1)=\int_0^\infty xf_{1r}(x)dx=\int_0^\infty\int_0^\infty xf(x,y)dxdy=\mu_1+r\sigma_1+\sqrt{\frac{2}{\pi}}\frac{\sigma_1\sqrt{1-r^2}\,e^{-\frac{1}{2}\left(\frac{\mu_1+r\sigma_1}{\sigma_1\sqrt{1-r^2}}\right)^2}}{1+\sqrt{1-e^{-\frac{2}{\pi}\left(\frac{\mu_1+r\sigma_1}{\sigma_1\sqrt{1-r^2}}\right)^2}}} \tag{6}$$

$$E_r(X_2)=\int_0^\infty yf_{2r}(y)dy=\int_0^\infty\int_0^\infty yf(x,y)dxdy=\mu_2+r\sigma_2+\sqrt{\frac{2}{\pi}}\frac{\sigma_2\sqrt{1-r^2}\,e^{-\frac{1}{2}\left(\frac{\mu_2+r\sigma_2}{\sigma_2\sqrt{1-r^2}}\right)^2}}{1+\sqrt{1-e^{-\frac{2}{\pi}\left(\frac{\mu_2+r\sigma_2}{\sigma_2\sqrt{1-r^2}}\right)^2}}} \tag{7}$$

2) The variances of each stochastic variable are

$$\begin{aligned} D_r(X_1)&=\int_0^\infty [x-E_r(X_1)]^2 f_{1r}(x)dx \\ &=\sigma_1^2(1-r^2)+E_r(X_1)\left[\mu_1+r\sigma_1-E_r(X_1)\right] \end{aligned} \tag{8}$$

$$\begin{aligned} D_r(X_2)&=\int_0^\infty [y-E_r(X_2)]^2 f_{2r}(y)dy \\ &=\sigma_2^2(1-r^2)+E_r(X_2)[\mu_2+r\sigma_2-E_r(X_2)] \end{aligned} \tag{9}$$

It can be validated that $D_r(X_1)=D(X_1)$ and $D_r(X_2)=D(X_2)$ if r=0.

We take the 2-dimension PD as an example, and the samples series of stochastic variable 1 and variable 2 are separately $x_{1,1}, x_{1,2}, \cdots, x_{1n}$ and $x_{21}, x_{22}, \cdots, x_{2n}$ (x_{1i}, x_{2i} >0, i=1,⋯, n).

According to the modified maximum likelihood estimation [10], we can obtain $\hat{\mu}_k$ (the estimated value of μ_k) and $\hat{\sigma}_k$ (the estimated value of σ_k), k=1, 2. Thus, the correlation coefficient can be estimated as follows:

$$\hat{r}_{1,2}=\frac{\sum_{i=1}^n (x_{1i}-\hat{\mu}_1)(x_{2i}-\hat{\mu}_2)}{\sqrt{\sum_{i=1}^n (x_{1i}-\hat{\mu}_1)^2\cdot\sum_{j=1}^n (x_{2j}-\hat{\mu}_2)^2}} \tag{10}$$

4 Comparing Analysis for Properties of Some Probability Distribution

4.1 The basic properties

Here we will make the comparing analysis between the Partial Distribution, lognormal and levy in basic properties.

4.1.1 The properties of lognormal distribution $Ln(\mu, \sigma^2)$.

The lognormal distribution is not symmetric, and

1) The stochastic variable of lognormal is non-negative, and the probability density is zero at x=0, namely, $f(0)=0$.

2) The shape of distribution curve is relevant mainly to parameter σ.

3) The expectation is $E(X) = e^{\mu+\frac{\sigma^2}{2}}$, and the variance is $D(X) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$.

4.1.2 The properties of Levy distribution $L_\alpha(a,x)$.

The Levy distribution is symmetric, and

1) Levy distribution is the probability distribution function $L_\alpha(a,x)$ with a characteristic function:

$$\hat{l}_\alpha(a,k) = e^{-a|k|^\alpha} \quad (0<\alpha\leq 2)$$

2) Gaussian (α=2, $a=\sigma^2/2$) and Cauchy (α=1) distribution are the only two levy distributions that can be inverted and expressed as elementary functions.

3) When $0<\alpha<2$, both expectation and variance of levy does not exist.

4) The center of the Levy distribution gets sharper and higher and the tails get fatter as $\alpha\to 0$ and $x\to 0$.

5) Most of applications of levy distribution are in the numeric way, so there are not very convenient in its use.

6) The predominance of levy distribution is which is able to fit better the returns of many of financial trading in shorter time field.

4.1.3 The properties of Partial Distribution $P(\mu, \sigma^2)$.

Partial Distribution is not symmetric, and

1) Stochastic variable is non-negative, and the probability is non-zero at x=0, namely,

$$f(0)= \sqrt{\frac{2}{\pi}} \frac{\sigma e^{-\frac{\mu^2}{2\sigma^2}}}{\sqrt{1-e^{-\frac{2}{\pi}(\frac{\mu}{\sigma})^2}}+1}$$

2) The shape of distribution curve is relevant to parameter σ and μ.

3) When $x\to\mu$, Partial distribution is sharper than Gaussian and lognormal distribution as μ is less.

4) If μ is big enough, the Partial distribution is approximately near to Gaussian distribution.

5) The expectation and variance are analytically expressed in theorem 2.

We see from above, the Partial Distribution has many characteristics which Gaussian distribution has not though it is a kind of truncated Gaussian distribution. What we need to point out emphatically is Partial Distribution has two basic characteristics. One characteristic is that the probability is equal to zero when the variable is less than zero, this is even corresponding to that the prices of any asset, like capitals, stocks, futures and commodity, are all non-negative. Another characteristic is that the probability is not equal to zero when the variable is equal to zero. This is even corresponding that the prices of some asset may become zero in market, like the price of stock of a company closed down, the price of overdue food or medicine, etc. Both Gaussion and lognormal distribution do not have above two characteristics at the same time. Levy distribution is a better one for fitting the value behaviors of trading returns in financial market, but it can not be expressed as the elementary function except Gaussion and Cauchy distribution. Cauchy distribution has an infinite variance, so it is of inconvenience in application.

4.2 The properties of expectation and variance

4.2.1 The properties of expectation and variance of PD.

According to the definition of PD, theorem 1and theorem 2, Partial Distribution has also the following properties about its expectation and variance, and these two important properties can not be got by other probability distribution at the same time.

1) The expectation is $E(x)=\mu+R(x)$, where, $R(X)=\sqrt{\frac{2}{\pi}}\frac{\sigma e^{-\frac{\mu^2}{2\sigma^2}}}{\sqrt{1-e^{-\frac{2}{\pi}(\frac{\mu}{\sigma})^2}}+1}$.

If we take μ as the cost price of a product, σ as the fluctuation of cost price, X as the market price, the $E(X)$ means the average market price of the product, and $R(X)$ means average selling profit of the product in market. $E(X)>\mu$ means that the average selling price of a product should be higher than its cost price. In the expression in $R(X)$, if we take the σ as the risk of cost price, the economic proposition "The more the risk is, the larger the possible profit is" can be interpreted analytically.

2) The variance is $D(X)=\sigma^2+E(X)(\mu-E(X))$.

From the expression of $D(X)$, we obtain $D(X)<\sigma^2$. This means the risk of cost price of a product is higher than the risk of its market price.

Suppose that X_i is the market price of a product in ith trading and $X_i\in P[E(X_{i-1}), D(X_{i-1})]$, $E(X_0)=\mu$, $D(X_0)=\sigma^2$, $i=1, \cdots$. We have $E(X_i)>E(X_{i-1})> \cdots >E(X_0)$, $R(X_i)<R(X_{i-1})< \cdots<R(X_0)$ and $D(X_i)<D(X_{i-1})< \cdots<D(X_0)$. These make clear that: if the economic environment and product quality do not change, the average trading price of a product would be higher and higher, the average sale profit will be lower and lower and the risk of market price will go down and down, but, the ranges in which the average trading price, the average sale profit and the price risk fluctuate will be smaller and smaller. So, we can interpret the economic proposition "the new asset must be developed continuously in order to acquire the higher sale profits" in analytic way.

4.2.2 The properties of expectation and variance of MPD.

If both X_1 and X_2 are stochastic variables and follow 2-dimensions Partial Distribution, i.e., $(X_1, X_2) \in P(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,r)$, there are some important differences between the $E(X)$ of PD and $E_r(X_1)$ in expression (6) or $E_r(X_2)$ in expression (7), and the $D(X)$ of PD and $D_r(X_1)$ in expression (8) or $D_r(X_2)$ in expression (9).

In summing up, there are two differences:

Difference 1 The μ is replaced by the $\mu_1+r\sigma_1$ or $\mu_2+r\sigma_2$.

Difference 2 The σ is replaced by $\sigma_1\sqrt{1-r^2}$ or $\sigma_2\sqrt{1-r^2}$.

In the MPD, the correlation coefficient $r>0$ means the positive correlation, and $r<0$ means the negative correlation. We take μ_1 and μ_2 separately as the cost prices of product 1 and product 2, and X_1 and X_2 separately as the market prices of product 1 and product 2, then $r>0$ means two products need the same cost resource, they will compete the same cost resource, so that the cost prices of two products become higher according to difference 1. In contrary, $r<0$ means two products need the reverse cost resource, they could use the different cost resources which are of complementarity, so that the cost prices of two products become lower according to the difference 1.

On the other hand, whether $r>0$ or $r<0$, the risk of cost prices of two products will become lower according to difference 2. This means that both the competition and cooperation will reduce the price risk of products.

6 The Applications of PD and MPD

Here, we will use the following basic notations:

μ——the cost price of commodity, or the average of holding price of all traders in the market to a stock.

σ——the standard variance of cost price of commodity.

X——the market price variable of a commodity or a stock.

By the PD or MPD, we could do some works like asset pricing, risk analysis and option pricing.

6.1 The model for pricing asset

6.1.1 The model for pricing single asset

If the market price of an asset follows PD, i.e., $X\in P(\mu, \sigma^2)$, according to Theorem 2, we have

1) The average market price of the asset can be evaluated as

$$E(X) = \mu + \sqrt{\frac{2}{\pi}}\frac{\sigma e^{-\frac{\mu^2}{2\sigma^2}}}{1+\sqrt{1-e^{-\frac{2}{\pi}\left(\frac{\mu}{\sigma}\right)^2}}}$$

where, $R(X) = \sqrt{\frac{2}{\pi}}\frac{\sigma e^{-\frac{\mu^2}{2\sigma^2}}}{1+\sqrt{1-e^{-\frac{2}{\pi}\left(\frac{\mu}{\sigma}\right)^2}}}$ is the average sale profit of the asset.

Because of $E(X)>\mu$, this means the average market price should be higher than the cost price of asset.

We also have the optimal pricing model for single asset [19].

6.1.1 The model for pricing group asset

If X_1 and X_2 are separately the asset 1 and asset 2, and follow 2-dimensions Partial Distribution, i.e., $(X_1, X_2)\in P(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,r)$, from reference [11] or the expression (6)and (7), we obtain the models for pricing the group assets as follow:

The average market price of asset 1 based on the correlation coefficient r is

$$E_r(X_1)=\mu_1+r\sigma_1+\sqrt{\frac{2}{\pi}}\frac{\sigma_1\sqrt{1-r^2}\,e^{-\frac{1}{2}\left(\frac{\mu_1+r\sigma_1}{\sigma_1\sqrt{1-r^2}}\right)^2}}{1+\sqrt{1-e^{-\frac{2}{\pi}\left(\frac{\mu_1+r\sigma_1}{\sigma_1\sqrt{1-r^2}}\right)^2}}}$$

Where, $R_r(X_1)=\sqrt{\frac{2}{\pi}}\frac{\sigma_1\sqrt{1-r^2}\,e^{-\frac{1}{2}\left(\frac{\mu_1+r\sigma_1}{\sigma_1\sqrt{1-r^2}}\right)^2}}{1+\sqrt{1-e^{-\frac{2}{\pi}\left(\frac{\mu_1+r\sigma_1}{\sigma_1\sqrt{1-r^2}}\right)^2}}}$ could evaluate the average selling profit of asset 1.

The average market price of asset 2 based on the correlation coefficient r is

$$E_r(X_2) = \mu_2 + r\sigma_2 + \sqrt{\frac{2}{\pi}} \frac{\sigma_2\sqrt{1-r^2}\, e^{-\frac{1}{2}\left(\frac{\mu_2+r\sigma_2}{\sigma_2\sqrt{1-r^2}}\right)^2}}{1+\sqrt{1-e^{-\frac{2}{\pi}\left(\frac{\mu_2+r\sigma_2}{\sigma_2\sqrt{1-r^2}}\right)^2}}}$$

Where, $R_r(X_2) = \sqrt{\frac{2}{\pi}} \frac{\sigma_2\sqrt{1-r^2}\, e^{-\frac{1}{2}\left(\frac{\mu_2+r\sigma_2}{\sigma_2\sqrt{1-r^2}}\right)^2}}{1+\sqrt{1-e^{-\frac{2}{\pi}\left(\frac{\mu_2+r\sigma_2}{\sigma_2\sqrt{1-r^2}}\right)^2}}}$ could evaluate the average selling profit of asset 2.

6.2 The risk analysis for asset price

The risk of market price of single asset can be evaluated as

$$D(X) = \sigma^2 + E(X)[\mu - E(X)]$$

Because of $D(X) < \sigma^2$, this means the trading risk is less than the cost risk of asset.

If we have two assets X_1 and X_2, and $(X_1, X_2) \in P(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, r)$, the expression (8) and (9) are separately the computing formulas to evaluate the market price risk of each asset in the meaning of correlation.

The computing formulas to evaluate the market price risk of each asset on its independence are separately

$$\overline{D}(X_1) = \int_0^\infty [x - E(X_1)]^2 f_{1r}(x)dx = D_r(X_1) + [E_r(X_1) - E(X_1)]^2 \tag{11}$$

$$\overline{D}(X_2) = \int_0^\infty [y - E(X_2)]^2 f_{2r}(y)dy = D_r(X_2) + [E_r(X_2) - E(X_2)]^2 \tag{12}$$

where, $E(X_i)$ comes from expression (4), $E_r(X_i)$ comes from expression (6) or (7), and $D_r(X_i)$ comes from expression (8) or (9), i=1,2.

We can validate that $\overline{D}(X_1) = D(X_1)$ and $\overline{D}(X_2) = D(X_2)$ if r=0.

6.3 The pricing model for American options

Here, we use the following notations:

t—the current time.

$S(t)$—market price of the stock at t.

X—strike price of option on $S(t)$.

T—time of expiration of option.

r—risk-free rate of interest to maturity T.

$C_S(t)$—value of call option to buy one share.

$P_S(t)$—value of put option to sell one share.

From the reference [18], we have the pricing models for American options as follow:

6.3.1 The price of call option. The price of call option at time *t* is

$$C_S(t) = (S(t) - Xe^{-r(T-t)}) \times$$

$$\times \left[\frac{\sqrt{1-e^{-\frac{2}{\pi}\frac{(Xe^{-r(T-t)})^2}{D[S(t)](T-t)}}} + \mathrm{sgn}(S(t)e^{r(T-t)} - X)\sqrt{1-e^{-\frac{2}{\pi}\frac{(S(t)-Xe^{-r(T-t)})^2}{D[S(t)](T-t)}}}}{1+\sqrt{1-e^{-\frac{2}{\pi}\frac{(Xe^{-r(T-t)})^2}{D[S(t)](T-t)}}}} \right] + \sqrt{\frac{2D[S(t)](T-t)}{\pi}} \left[\frac{e^{-\frac{(S(t)-Xe^{-r(T-t)})^2}{2D[S(t)](T-t)}} - e^{-\frac{(Xe^{-r(T-t)})^2}{2D[S(t)](T-t)}}}{1+\sqrt{1-e^{-\frac{2}{\pi}\frac{(Xe^{-r(T-t)})^2}{D[S(t)](T-t)}}}} \right]$$

6.3.2 The price of put option. The price of put option at time *t* is

$$P_S(t) = (Xe^{-r(T-t)} - S(t)) \times$$

$$\times \left[\frac{1 - \mathrm{sgn}[S(t)e^{r(T-t)} - X]\sqrt{1-e^{-\frac{2}{\pi}\frac{(S(t)-Xe^{-r(T-t)})^2}{D[S(t)](T-t)}}}}{1+\sqrt{1-e^{-\frac{2}{\pi}\frac{(Xe^{-r(T-t)})^2}{D[S(t)](T-t)}}}} \right] + \sqrt{\frac{2D[S(t)](T-t)}{\pi}} \left[\frac{e^{-\frac{(S(t)-Xe^{-r(T-t)})^2}{2D[S(t)](T-t)}}}{1+\sqrt{1-e^{-\frac{2}{\pi}\frac{(Xe^{-r(T-t)})^2}{D[S(t)](T-t)}}}} \right]$$

And from reference [20], we have also the structure models for futures options pricing.

5 Conclusions and Remarks

In this paper, we have generalized a new probability density—the Partial Distribution. It includes the Partial Distribution on single variable (PD for short) and the Partial Distribution on multivariable (MPD for short).

PD and MPD have some interesting properties, they are as follow

- Expectation value for a single PD variable. It could be applied to valuate the average price of product (asset, commodity, virtual products, invisible asset, etc.) in market, and the average profit of selling a product.
- Expectations for the MPD variables. They could be applied to valuate the average prices and selling profits of products which are correlated, and to show the effect of correlation coefficient to the cost price and selling profit of the products.
- Variance of a single PD variable. It could be applied to valuate the price risk of a product in market, and to interpret that the cost risk of a product is larger than the risk of market price of it.
- Variances of the MPD variables. They could be applied to valuate the market price risks of the product which are correlated, and to show that the effect of correlation coefficient to the risks of the products prices.
- Interpreting the economic propositions. Many economic propositions could be interpreted in analytic way by the expressions of above expectations and variances, such as “the more the risk is, the larger the possible profit is”, “the new asset will bring the higher sale margin”, and etc.

Further, we could obtain many of valuable results based on PD and MPD, and these can be applied to economic management, like product pricing, analysis for price risk, options pricing, also see references [4], [11], [18]-[20].

Of course, there should be many interesting mathematical properties about PD, MPD, Partial Process and

Multivariate Partial Process, like the statistical properties, process properties, etc. Here, we do not make the corresponding discussion. We shall do these works in the future if needed.

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