Reversibility in Dynamic Coordination Problems*

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Abstract

Agents at the beginning of a dynamic coordination process (1) are uncertain about actions of their fellow players and (2) anticipate receiving strategically relevant information later on in the process. In such environments, the (ir)reversibility of early actions plays an important role in the choice among them. We characterize the strategic effects of the reversibility option on the coordination outcome. Such an option can either enhance or hamper efficient coordination, and we determine the direction of the effect based only on simple features of the coordination problem. The analysis is based on a generalization of the Laplacian property known from static global games: players at the beginning of a dynamic game act as if they were entirely uninformed about aggregate play of fellow players in each stage of the coordination process.

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1 Introduction

An agent at the outset of an economic crisis is uncertain about the future evolution of the economy because she does not know how fellow agents perceive the odds of the crisis and, hence, how they will act. Additionally, such an agent anticipates to receive strategically valuable information in later stages of economic development and therefore the decisions at the outset of the crisis, as well as indirectly the final outcome of the crisis, may crucially depend on the reversibility of the early actions.

Reversibility of early decision is always beneficial in single person decision problems because later on, in a light of subsequent information, the decision may be viewed as detrimental. The effects of reversibility become more complex in strategic problems. While each agent would benefit from a unilateral provision of the reversibility option as it alleviates the adverse effects of uncertainty, the provision of the option to all agents can be harmful because the very source of the uncertainty — the actions of fellow players — becomes less predictable. In the case of coordination problems studied here, the reversibility option helps agents avoid participation in a coordination failure, but at the same time it may also increase the incidence of coordination failures. Thus, while the provision of the option to reverse an action unambiguously increases the incentive to take the action in non-strategic problems, the sign of the effect cannot be determined without careful analysis in strategic problems.

The starting point of our analysis — the observation that agents at the outset of crisis are uncertain about others' actions — is well formalized in the global games literature. A global game is an incomplete information coordination game that captures crises as coordination failures. Players receive private signals about an underlying economic fundamental and in the unique equilibrium, they invest at signals above a certain threshold signal and do not invest below that threshold. The critical agent at the outset of the crisis corresponds to the player receiving the threshold signal constituting the boundary between the sets of investing and non-investing types and who is uncertain about the realized proportions of the fellow players on each side of the boundary and so is uncertain about the aggregate investment. A key to the solution of static global games is an observation that the threshold type has *uniform* belief about the aggregate investment. To emphasize the connection to Laplace's principle of insufficient reason, Morris and Shin (2003) dub such belief as *Laplacian*, and we will refer to this observation as the *Laplacian property*. The main methodological contribution of this paper is a generalization of the Laplacian property to dynamic environments. The property will, in the generalized form, play a key role in our analysis of dynamic coordination problem with a reversible action.

The Laplacian property not only greatly enhances tractability of the static global games, but at the same time it captures the intuition of the strategic uncertainty during crises. Because of this attractiveness, the static global game framework is often applied even at a cost of abstracting from dynamic features of the analyzed problem. For example, Morris and Shin (2004) study debt crises as coordination failures arising among a group of creditors. Their model focuses on the interim stage of an investment project, after the creditors have invested in the project but before its completion. Each creditor has the option to exit the project in the interim stage, and the project fails if a critical mass of the creditors do exit. To fit the static global game framework, Morris and Shin (2004) keep entry decisions at the beginning of the project exogenous. The unique equilibrium of such a static game exhibits inefficient exit behavior, and thus it is natural to ask how the provision of the exit option influences players' ability to coordinate on efficient investment. The purpose of our paper is to provide a framework that addresses this very question. Such a question requires a dynamic model because variation in the provision of the exit option will affect the entry decisions. The total effect of the provision of the exit option is, without a formal analysis, ambiguous because the option will be beneficial if the project turns out to evolve towards a failure, but occurrence of the coordination failures may increase with the provision of the option.

In the previous paragraph the risky investment was reversible while the safe action was kept irreversible — the investment could not be delayed. A related question can be asked about coordination problems in which investing, or risky action in general, is irreversible but can be delayed in order to acquire additional information. As in the previous case, the effect of such delay option on the final coordination outcome can be determined only by a formal analysis. On one hand, the delay option is helpful because promising projects will attract large participation in at least in a late stage of the coordination process but, on the other hand, if too many players delay investment, projects that would have succeeded in the absence of the delay option may become unpromising.

We capture such dynamic problems by a coordination game in which players decide whether to participate in a project consisting of an early and a late stage. Players first decide on their participation at the beginning of the first stage based on their initial private information. During the first stage of the project players learn additional private information and can reverse the initial decision in between the two stages. More precisely, one of the two available actions, participation or non-participation, is irreversible and the other is reversible, which induces an option value to the reversible

action.

When evaluating the reversible action at the beginning of the game, each player has to form an expectation about the profits from the early and the late stage of the project. The latter expectation is more complex and so we focus on it in the introduction. When forming the expectation at the beginning of the game about the late stage profit, the player holding the reversibility option has to condition on her participating in the late stage. A direct characterization of this expectation is cumbersome because it involves computation of equilibrium belief about the fellow players' actions, and the belief has to be conditioned on the use of the option. Our main technical insight is that the characterization of this complicated belief can be circumvented by the use of *the Laplacian property generalized to dynamic games*. We find that the threshold type at the beginning of the game forms her expectation about the profit from the second stage of the project in a particularly simple way. Taking into account the reversibility option, she forms the expectation as if she had *uniform* belief about the participation level in the second stage of the project and as if she *did not have the option*.

Unlike in the static global game in which the threshold type truly has uniform belief, the Laplacian property in the dynamic game is a virtual "as if" property; the actual belief is not uniform, and players do have the option. This virtual characterization reflects our analytical approach. To avoid the direct characterization of the option value in the dynamic environment, we map a part of the dynamic game to a virtual static game with a mapping that does not distort the payoff expectation of the threshold type. We can then solve the virtual static game using existing static global game tools.

Our benchmark, to which we will compare the coordination outcome in the dynamic game, is a static game without the reversibility option in which the decisions at the beginning of the project are irreversible. Thanks to the generalized Laplacian property, the characterization of the reversibility effects becomes simple. As the Laplacian property holds in both games, we do not need to worry about the differences in the equilibrium beliefs across the two games and we can evaluate the differences in the expected payoffs of the threshold types and, hence, in equilibrium actions, based solely on certain simple mechanistic properties of the investment project.

We find that the provision of either the exit or the delay option can enhance or hamper efficient coordination and that the sign of the effect depends on an intertemporal payoff structure. We say that payoffs exhibit forward spillovers if production has inertia, so that profit in the late stage depends not only on the late but also on the

payoff spillovers		option to	
backward	forward	exit	delay
✓	×	more failures	less failures
×	✓	less failures	more failures
×	×	irrelevance result	

Table 1: Effect of the reversibility option on the occurrence of coordination failures.

early investment level. We say that payoffs exhibit backward spillovers if the profit from participation in the early stage of the project depends not only on the early but also on the late investment level.[1] Using this terminology, the effects are the following: the exit option enhances efficient coordination in projects with forward spillovers and hampers efficient coordination in projects with backward spillovers. The delay option has the opposite effects. As a corollary, neither the exit nor the delay option has any effect in projects without both backward and forward spillovers.

We keep the structure of the paper and the exposition of the generalized Laplacian property subordinated to the economic problem of reversible investment. However, the Laplacian property holds beyond our baseline setup. In Section 8, we sketch the extensions of the Laplacian property to dynamic environments with a more general option structure. We let players interact in a dynamic game with multiple rounds in which player's action choice in each round imposes constraints on the future play.

We share the focus on the effects of reversibility options on investment decisions with McDonald and Siegel (1986) or Dixit and Pindyck (1994), but we differ in the source of uncertainty and in the benchmark. Their literature on single-person investment decisions with delay option considers uncertainty coming from exogenous shocks, and their benchmark is the neoclassical setup with all actions reversible. In our framework, the main source of uncertainty is endogenous and strategic as the players are uncertain about others' actions and our benchmark is the static global game. The difference in the source of the uncertainty dictates differences in research questions and methods. Our main result characterizes the direction of the reversibility effect on the incentive to invest in the strategic environment. In the non-strategic environment, reversibility unambiguously increases the incentive to choose the reversible action, and hence such models can focus on the size of the effects. Regarding the method, the core of our analysis consists of the characterization of beliefs about the uncertain behavior of the fellow player, whereas the beliefs about the source of uncertainty are exogenous in the other literature.

[1] Backward spillovers can arise if players cannot exit the project to the full extent or under schemes which redistribute profits among the investors.

Our paper belongs to a booming literature on dynamic global games. One of the strands of this literature emphasizes intertemporal tradeoffs of players facing frictions in the adjustment of actions to an evolving environment (Burdzy, Frankel and Pauzner 2001, or Levin 2001). The second stream of this literature emphasizes equilibrium multiplicity induced by public learning stemming from observation of endogenously chosen public policy (Angeletos, Hellwig, and Pavan 2006), observation of prices (Angeletos and Werning 2006), or observation of earlier coordination outcomes (Angeletos, Hellwig, and Pavan 2007).

Our paper belongs to yet another stream of the dynamic global games literature in which one of the available actions is irreversible while another can be reverted frictionlessly which, together with learning, induces positive option value to the reversible action. Heidhues and Melissas (2006), Dasgupta (2007) and Dasgupta, Steiner, and Stewart (2007) allow players to delay their investment decisions in order to engage in learning. Learning is private, and hence, unlike in the second stream of the literature, equilibrium uniqueness may be preserved, which facilitates the characterization of the reversibility effects. The generalized Laplacian property described here unifies the characterization of the reversibility effects across a large class of setups without resorting to specific payoff functions.

One of the dynamic effects studied in the literature but not here is that investment by one player can trigger investment by her fellow players either through signalling or even absent of signalling via complementarities; see Corsetti, et al. (2004) or Hörner (2004) within the global games, and Chamley and Gale (1994), Gale (1995), or Gul and Lundholm (1995) outside of the global games literature. Our model abstracts from informational externalities because the amount of information revealed during coordination is assumed to be independent of players' actions. Moreover, our players are small and therefore cannot individually trigger investments by others.

The organization of the paper is as follows: Section 2 introduces the model; Section 3 provides an informal overview of the analysis; Section 4 contains the main technical contribution of the paper — it describes the generalized Laplacian property in dynamic games. The Laplacian property holds in monotone strategy profiles, and hence in Section 5 we constrain our attention to global games in which the monotone strategy profiles are relevant for the equilibrium analysis. Section 6 identifies the strategic effects of the reversibility option by comparing equilibria across the dynamic game and the static benchmark, and Section 7 continues in this comparison in the limit of small noise. In Section 8 we further explore generality of the Laplacian property in a large set of dynamic coordination games.

2 Model

We study a dynamic, binary action game, Γ_{dyn}, with one of the two actions being reversible and the other irreversible. A continuum of players indexed by $i \in [0,1]$ simultaneously choose action $a_1^i \in \{0,1\}$ in round 1. Players who played action 0 reach their final node and receive a payoff normalized to 0. Players who played action 1 choose simultaneously $a_2^i \in \{0,1\}$ in round 2. The payoff for private action history [2] $a_1^i a_2^i = 10$ is $u_1(\theta, l_1, l_2)$, and the payoff for private action history 11 is $u_1(\theta, l_1, l_2) + u_2(\theta, l_1, l_2)$. The letter θ denotes a payoff parameter which we refer to as the fundamental, l_1 denotes the measure of players playing $a_1^i = 1$ in round 1, and l_2 is the measure of players choosing 1 in both rounds. Functions u_1 and u_2 are real-valued, defined on the domain $\{(\theta, l_1, l_2) \in \mathbb{R} \times [0,1] \times [0,1] : l_2 \leq l_1\}$. We assume that u_t are continuous in all arguments.[3] The additive payoff structure is without loss of generality and facilitates the formulation of assumptions that we impose on the model below.

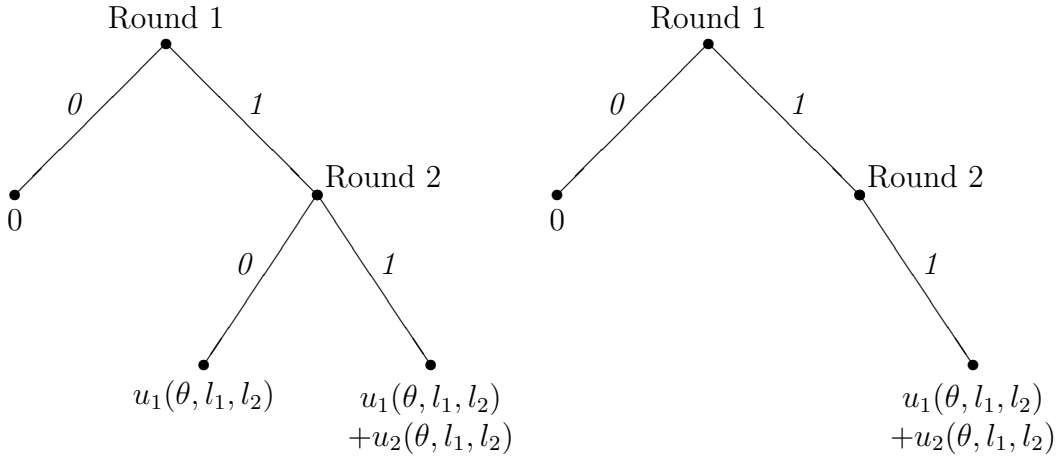


Figure 1: Decision tree in the dynamic game Γ_{dyn} (left) and in the benchmark static game Γ_{st} (right). Moves of Nature and of fellow players are not depicted.

This game can be interpreted as a process of investment in a project with two production stages. Round 1 takes place at the beginning of stage 1, and interpreting action 1 as investing, players decide whether to invest or take an outside option. Round 2 takes place in between production stages 1 and 2. In round 2, we interpret action 1 as staying in and 0 as exiting the project. Payoff u_t is interpreted as a profit from participating in the stage $t = 1, 2$ of the project, and l_t are the investment (participation) levels in stage t.

[2] For simplicity of notation we abbreviate the ordered pair (a_1^i, a_2^i) to $a_1^i a_2^i$.

[3] Results can be extended to allow for isolated payoffs discontinuities such as those used in the games of regime change.

Following the global games literature, we assume heterogeneity in players' private information. Nature draws the (common) fundamental θ from improper[4] uniform distribution on $\mathbb{R}$. At the beginning of round $t = 1, 2$ player i moving in round t observes a private signal $x_t^i = \theta + \sigma\eta_t^i$. The vector of errors (η_1^i, η_2^i) is distributed according to a continuous joint distribution with a compact convex support H, joint density f, and joint c.d.f. F. We assume that (η_1^i, η_2^i) are i.i.d. across players and independent from θ (but are not required to be independent across rounds). The supports of the marginal distributions of η_t^i are assumed to be symmetric intervals $[-h_t, h_t]$ where h_1 and h_2 are strictly positive constants. The symmetry is without loss of generality because if the supports of the marginal distributions were not symmetric around 0, players would simply subtract the bias of errors from their signals when forming posterior beliefs. Marginal c.d.f. of η_1^i and η_2^i are denoted by F_1 and F_2. In addition, we denote $\eta_\Delta^i = \eta_2^i - \eta_1^i$ the difference of the errors. The support of η_Δ^i is $[\underline{\eta}_\Delta, \overline{\eta}_\Delta]$ where $\underline{\eta}_\Delta = \min_{(\eta_1,\eta_2)\in H}(\eta_2 - \eta_1)$ and $\overline{\eta}_\Delta = \max_{(\eta_1,\eta_2)\in H}(\eta_2 - \eta_1)$. We denote the marginal c.d.f. of η_Δ^i by F_Δ. We assume no aggregate uncertainty about the realization of the errors — the realized population of errors is identical to the joint density f.

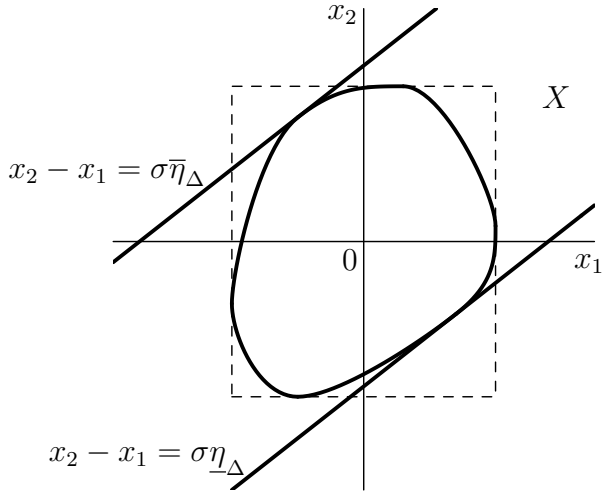


Figure 2: Type space X and related notation.

Bold letter $\mathbf{x}^i = (x_1^i, x_2^i)$ denotes the type (signal pair) of player i. The type set is $X = \left\{(x_1, x_2) : x_2 - x_1 \in [\sigma\underline{\eta}_\Delta, \sigma\overline{\eta}_\Delta]\right\}$; see Figure 2. We will use the usual incomplete product order $\leq$ to compare the types. A pure strategy is a pair of functions $\mathbf{s} = (s_1, s_2)$ with $s_t : X \to \{0, 1\}$ and with $s_1(x_1^i, x_2^i)$ depending only on the

[4]The use of the improper distribution does not cause any ambiguities, because we work only with probabilities conditional on the signals, and these are well defined. See Morris and Shin (2003) for the discussion of the use of uninformative prior in global games.

first signal x_1^i. Notice that the values of $s_2(\mathbf{x})$ for types $\mathbf{x}$ at which $s_1(\mathbf{x}) = 0$ are payoff irrelevant because such types do not reach round 2. Abusing terminology and notation, we will also call signal x_1^i in round 1 a type, and action rule $s_1(x_1^i)$ in round 1 a strategy.

Our main applied result characterizes the effect of provision of the reversibility option on the coordination outcome. To that end we compare the above *dynamic* game Γ_{dyn} with a benchmark *static* game Γ_{st} which differs from Γ_{dyn} only in the lack of the reversibility option: each player can move only in round 1; once a player invests in round 1, she must automatically stay in the project in round 2; see Figure 1 for the comparison of the games. To facilitate comparison with the dynamic game, we keep the lower index 1 when describing the signal x_1^i or strategy $s_1(x_1^i)$ in the static game despite it having only one non-trivial round.

2.1 Discussion of the setup

Let us briefly discuss the assumptions imposed on the model up to now. First, the uninformative prior together with the independence of errors with respect to θ imply that conditional distributions are invariant to diagonal translations on the type space, i.e., $(\theta, \mathbf{x}^j) | (\mathbf{x}^i + t \cdot (1,1)) = t \cdot (1,1,1) + (\theta, \mathbf{x}^j) | \mathbf{x}^i$. This translation invariance, which is necessary for the generalized Laplacian property, would be distorted by an informative prior. However, in the limit of small noise, as $\sigma \to 0^+$, any prior becomes approximately uninformative, and hence our results remain to be approximately valid under any prior, as long as the signals are sufficiently precise. This fact is also important for the interpretation of our comparative results that specify whether the provision of the reversibility option enlarges the set of investing types. Formally we cannot draw implication on the ex ante welfare because of the improper prior, but our results on the changes in equilibrium thresholds have unambiguous welfare consequences under any proper prior.

Second, we assume that the value of the fundamental θ is fixed throughout the game. The generalized Laplacian property would remain valid in a randomly evolving environment. We abstract from the fluctuations in θ because learning alone suffices to induce positive value to the reversibility option, and the arguments behind the generalized Laplacian property are orthogonal to the fluctuations.

Third, we assumed that investment — the risky action — is reversible and the safe action is irreversible. This choice is arbitrary, and we will also consider a simple variant of the above dynamic game in which we switch the reversibility of the actions. In this variant, we will keep the investment irreversible, whereas not investing will

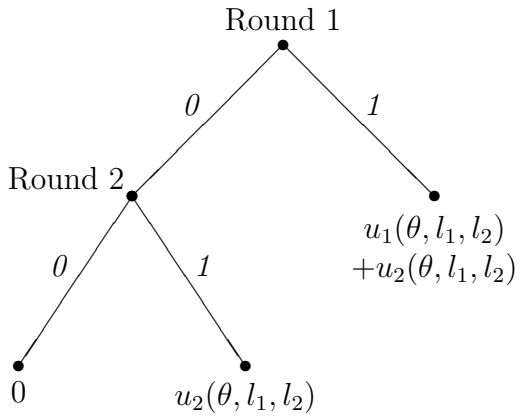


Figure 3: Variant of the dynamic game in which players can delay investment.

be reversible — players may delay investment; see Figure 3. The two variants of the dynamic game can be mapped to each other by a careful relabeling of the actions so we will formulate the whole analysis only in terms of the first variant. However, the studied effects turn out to have opposite signs across the two variants of the dynamic game, and hence we will sketch the results also for the second variant.

Fourth, let us look at the assumed information structure in round 2 and its connection to social learning. We specified above that players in round 2 receive additional information about θ, whereas the early investment level l_1 is unobserved. Obviously, the signal x_2^i provides in equilibrium indirect information about l_1 as well. For instance, if all players use a monotone strategy with threshold x_1^* in round 1, then θ and l_1 are related by the mapping $l_1 = 1 - F_1\left(\frac{x_1^* - \theta}{\sigma}\right)$. In fact, we can reverse the perspective and formulate an alternative model in which the primary source of information in round 2 is a noisy observation of l_1 and players learn about θ only indirectly. Assume in this alternative model that players in round 1 observe fundamental-based signal $x_1^i = \theta + \sigma\eta_1^i$ as above, but instead of the round 2 signal $x_2^i = \theta + \sigma\eta_2^i$, players observe a noisy aggregate statistic of the round 1 actions. The following specification is used for the tractability reasons in the literature:[5]

$$y^i = 1 - F_1^{-1}(1 - l_1) + \eta_2^i. \tag{1}$$

The advantage of this particular specification is that, in a symmetric monotone equilibrium, the observation of y^i turns out to be equivalent to the observation of $x_2^i = \theta + \sigma\eta_2^i$, as a player observing y^i can compute x_2^i in the equilibrium. Hence, the set of symmetric monotone equilibria must coincide across our model with fundamental-

[5]This specification has been first used in Dasgupta (2007), and later in Angeletos, Hellwig and Pavan (2007), Angeletos and Werning (2006) or in Goldstein, Ozdenoren and Yuan (2008).

based learning and the alternative model with social-based learning. Our model with fundamental-based learning turns out to have unique equilibrium (under assumptions from Section 5) which is monotone and symmetric, and so it remains to be unique equilibrium within the class of monotone symmetric equilibria in the model with social learning; though non-monotone equilibria cannot be precluded in the latter model.

Last, the learning in our model is assumed to be private which preserves the equilibrium uniqueness. Private as opposed to public learning is a reasonable assumption whenever information sources or evem perceptions of a common information source are heterogenous across the players.

3 Overview of the Argument

Our central goal is to compare investment behavior across the static and the dynamic game. Both games turn out to have unique rationalizable strategies under global game assumptions presented in Section 5, and hence we can focus on the comparison of conditions for the rationalizability of actions across the two games.

Let us first review the arguments in Morris and Shin (2003). They construct the rationalizability conditions in the static global game in three steps: In the first step they show that under any symmetric monotone strategy profile with threshold x_1^*, the threshold type x_1^* has Laplacian belief about the investment level; $l_1|x_1^*$ is uniformly distributed on $[0,1]$. The second step characterizes the rationalizability of each of the actions. For each $x_1 \in \mathbb{R}$, let $m_{st}(x_1)$ be the expected incentive to invest of the threshold type x_1 under the symmetric monotone profile in which all players use strategy with the threshold $x_1^* = x_1$. Then action 1/0 is the unique rationalizable action at signal x_1 in the game Γ_{st} if and only if $m_{st}(x_1)$ is positive/negative. Thanks to the Laplacian property, the payoff expectation $m_{st}(x_1)$ is a simple object — type x_1 has uniform belief about l_1 under the profile in which she has the position of the threshold type. The third step examines the limit of small noise in which players are almost certain about θ, but the threshold type remains to be entirely uncertain about l_1. In such a limit, all analytical complications coming from the underlying uncertainty about θ disappear, and the analysis can conveniently focus on the strategic uncertainty about l_1.

Our analysis of the dynamic game follows the above structure with the three steps as well. The value added lies primarily in our first step in which we show that the Laplacian property generalizes to the dynamic game. We examine the expectation of a threshold type x_1^* in round 1 under a monotone strategy $\mathbf{s} = (s_1, s_2)$ where

s_1 has the threshold x_1^*, and s_2 is a symmetric monotone equilibrium strategy in the continuation game of round 2 induced by s_1. That is, we are forcing players to use the threshold x_1^* in round 1, but assume equilibrium behavior afterwards. We introduce function $m_{dyn}(x_1^*) = D_1(x_1^*) + D_2(x_1^*)$ that again denotes the incentive to invest in round 1 as expected at the threshold signal x_1^* in round 1. It is a sum of the expected profits

$$D_1 = E[u_1|x_1^*], \qquad D_2 = E\left[s_2\left(\mathbf{x}^i\right) \cdot E\left[u_2|\mathbf{x}^i\right] \mid x_1^*\right]$$

for each of the two stages of the project, where in the case of D_2 the threshold type x_1^* anticipates her own action $s_2\left(\mathbf{x}^i\right) \in \{0,1\}$ optimally chosen in round 2 based on information $\mathbf{x}^i = (x_1^*, x_2^i)$.

We analyze the expectations D_1 and D_2 in Section 4. Expressing D_1 is simple because, exactly as in the static game, the threshold type x_1^* has uniform belief about the first stage investment level l_1. The analysis of the expected second-stage payoff D_2 formed in round 1 is more complex. The threshold type x_1^* in round 1 has to anticipate whether she stays in the project in round 2, and that is contingent on her signal x_2^i that she has yet to receive. Our central finding is that the threshold type x_1^* in round 1, taking into account her reversibility option in round 2, forms expectation D_2 *as if she had not had the reversibility option and believed that* l_2 *was uniform on* $[0,1]$:

$$D_2 = \int_0^1 u_2(\theta, l_1, l_2) dl_2,$$

where θ and l_1 are treated as functions of l_2 uniquely induced by the strategy profile $\mathbf{s}$ and by the error distributions. The intuition behind this result is more complex than the intuition behind the Laplacian property in the static setup. We first show that we can replace the reversibility option advantage that players enjoy by an informational advantage. That is, we deprive the players of the exit option, but we compensate them by manipulating their information at the beginning of the game in a way that preserves incentives of the threshold type x_1^*. This transforms the originally dynamic problem to a static one, in which the static Laplacian property applies.

In the second step, in Section 5, we examine the rationalizability of actions in round 1 of the dynamic game. Again, action 1/0 is the unique rationalizable action at signal x_1 in round 1 of Γ_{dyn} if and only if $m_{dyn}(x_1)$ is positive/negative. Note that neither $m_{dyn}(x_1)$ nor $m_{st}(x_1)$ are the equilibrium payoff expectations of type x_1 in the dynamic or the static game. Rather they are expectations in the imaginary situation in which all the players are forced to use the monotone strategy with the threshold x_1 in round 1.

As in the static case, the Laplacian property and the rationalizability condition fruitfully enrich each other in the dynamic game because both $D_1(x_1)$ and $D_2(x_1)$ are formed based on the uniform belief about l_1 and l_2 respectively and without the intricacies of the reversibility option. This is applied in Section 6 where we compare the investment behavior across the two games. It specifies simple and economically intuitive conditions under which the provision of the option enhances or hampers investment at the beginning of the project. The comparison is possible because, thanks to the Laplacian property, the functions $m_{st}(x_1)$ and $m_{dyn}(x_1)$ are, roughly speaking, based on identical beliefs about l_t across the two games. Under the identical beliefs, the threshold expectations can be compared across the two games based solely on qualitative characteristics of the project, without undergoing the equilibrium analysis of the continuation game in round 2. This is not only convenient, but it also implies that the comparison does not depend on details of the payoff functions.

In the third step, in Section 7, we continue with the analysis in the limit of precise signals. As in the static case, the analysis is simplified because players are almost certain about θ and so the analysis can focus on the strategic uncertainty about l_1 and l_2. This strategic uncertainty is preserved in the limit and so the reversibility effects do not vanish even if the noise becomes negligible. Additionally, in the limit of precise signals, it is possible to delineate rationalizable behavior in round 2 of the dynamic game. Under a simple condition, the investments from round 1 are not reverted in round 2. In such cases the provision of the reversibility option affects the final coordination outcome for a large set of realized fundamental θ, but the option is *not* exercised apart from in cases when Nature draws θ from a small neighborhood of the equilibrium threshold in round 1, and this neighborhood vanishes in the limit of precise signals.

4 The Laplacian Property

In this section, we analyze payoff expectations of a threshold type in round 1 under a symmetric monotone strategy profile. First, in Subsection 4.1, we review the Laplacian property in the static games as described in Morris and Shin (2003). Then, in Subsection 4.2, we describe how the Laplacian property generalizes to the dynamic game with reversible investment. The class of setups in which the Laplacian property holds is larger than the particular economic environment discussed here. In an extension introduced in Section 8 we further generalize the Laplacian property to dynamic environments in which players undergo a series of binary investment decisions,

with each decision influencing the degree of player's commitment to the investment project.

The analysis will pay close attention to monotone strategies $\mathbf{s}(\mathbf{x})$ weakly increasing in $\mathbf{x}$. To avoid ambiguity of the exposition, we assume throughout the paper that the types on the boundary of set $\{\mathbf{x} \in X : s_t(\mathbf{x}) = 1\}$, $t = 1, 2$, always invest. This only facilitates discussion, as manipulation of actions of the boundary types does not change the best response correspondence of any type.

4.1 Laplacian Belief in the Static Game

Let $s_1(x_1^i)$ be a symmetric monotone strategy profile with threshold x_1^*. The profile induces a non-decreasing function

$$\ell_1(\theta) = \Pr\left(x_1^i \geq x_1^* \mid \theta\right) \tag{2}$$

that specifies the investment level after round 1 as a function of realized θ.

The following theorem describes the Laplacian property in the static game Γ_{st}:

Theorem 1. *(Morris and Shin, 2003) The conditional belief $\ell_1(\theta) \mid x_1^*$ is uniform on $[0, 1]$.*

The Laplacian property is driven by the following intuition. The threshold type x_1^* constitutes a boundary between the sets of investing and non-investing types, and the type x_1^* is uncertain about the realized proportions of players on the each side of the boundary. These proportions are determined by the rank of the threshold type's signal within the realized population of player signals. The only information the threshold type receives is her own private signal, which is entirely uninformative about her rank and consequently about l_1. For future exposition, we emphasize that the Laplacian property holds for any noise distribution, as long as the the prior is uninformative and the errors are independent across players and of θ.

4.2 Laplacian Expectations in the Dynamic Game

We now examine the expected payoff of the threshold type in round 1 of the dynamic game Γ_{dyn}.

Let us first introduce necessary notation. We fix a symmetric monotone strategy profile $\mathbf{s}$, and denote the threshold signal in round 1 again by x_1^*. We let $I_t = \{\mathbf{x} \in X : s_t(\mathbf{x}) = 1\}$ denote the set of types that choose action 1 in round t. Sets $L_1 = I_1$ and $L_2 = I_1 \cap I_2$ denote the sets of types that participate in the first and in both stages,

respectively. The strategy $\mathbf{s}$ induces a pair of investment profiles $\ell_t(\theta) = \Pr(L_t|\theta)$ that specify investment levels in round $t = 1, 2$ for a realized fundamental θ. Note that the definition of $\ell_1(\cdot)$ is identical to the definition in (2) in the static game because L_1 is the set of types with the first signal of at least x_1^*. Both ℓ_1 and ℓ_2 are non-decreasing in θ because strategy $\mathbf{s}$ is monotone, errors are independent of θ, and the prior is uninformative. We define $\vartheta_t(l_t)$ on domain $(0, 1)$ as inverse functions to $\ell_t(\theta)$. We will also need to express l_1 as a function of l_2 and vice versa, for which we introduce $\lambda_1(l_2) = \ell_1(\vartheta_2(l_2))$, and similarly $\lambda_2(l_1) = \ell_2(\vartheta_1(l_1))$. To summarize, out of the triple of variables θ, l_1, l_2 we can choose any one as the independent one and express the remaining two variables as its non-decreasing functions. We introduce $\tilde{u}_t(l_t)$ that denotes the profit for stage t of the project when all the arguments of u_t are expressed as functions of l_t induced by the fixed strategy profile $\mathbf{s}$; $\tilde{u}_1(l_1) = u_1\left(\vartheta_1(l_1), l_1, \lambda_2(l_1)\right)$ and $\tilde{u}_2(l_2) = u_2\left(\vartheta_2(l_2), \lambda_1(l_2), l_2\right)$. Finally, let $U_2(\mathbf{x})$ be the conditional expectation of type $\mathbf{x}$ in round 2 about the second stage profit under the strategy $\mathbf{s}$:

$$U_2(\mathbf{x}) = E\left[u_2(\theta, l_1, l_2)|\mathbf{x}\right].$$

We now examine monotone symmetric strategy profiles under which (I) players behave optimally in round 2 but not necessarily in round 1, and (II) sufficiently high types invest in both rounds:

(I) **optimality in round 2**: For all $\mathbf{x} \in X$ such that $s_1(\mathbf{x}) = 1$: $s_2(\mathbf{x}) = 1$ if $U_2(\mathbf{x}) > 0$ and $s_2(\mathbf{x}) = 0$ if $U_2(\mathbf{x}) < 0$.

(II) **non-emptiness in round 2**: There exists $\overline{\mathbf{x}} \in X$ such that $s_1(\overline{\mathbf{x}}) = s_2(\overline{\mathbf{x}}) = 1$.

At this point we impose those assumptions directly on the strategy profile, and below, in Section 5, we specify assumptions on the primitives of the model that assure that the assumptions will be satisfied in the profiles relevant for the equilibrium analysis.

We let D_t denote the expected profit for stage $t = 1, 2$ of the project as expected in round 1 by the threshold type x_1^*. The boundary[6] ∂L_1 of the set L_1 is the set of types (x_1^*, x_2^i) with the first signal equal to the threshold in round 1. Using this, we write D_1 and D_2 as:

$$D_1 = E\left[u_1(\theta, l_1, l_2)|\partial L_1\right], \tag{3}$$

$$D_2 = E\left[s_2(\mathbf{x}^i) \cdot U_2(\mathbf{x}^i)|\partial L_1\right], \tag{4}$$

[6]When we refer to boundary ∂L of a set $L \subseteq X$ we mean the boundary with respect to the topological space X. That is, ∂L does not include parts of ∂X with respect to the topological space $\mathbb{R}^2$.

where in the case of D_2, a player in round 1 anticipates her own behavior $s_2(\mathbf{x}^i) = s_2(x_1^*, x_2^i) \in \{0,1\}$ which is contingent on the yet unreceived signal x_2^i. The expectations are computed under the fixed profile $\mathbf{s}$ which we omit from the notation.

The following theorem is the central technical insight of the paper:

Theorem 2 (**Generalized Laplacian Property**). *If a monotone strategy* $\mathbf{s}$ *satisfies (I)–(II) then the payoff* D_t *for the stage* $t = 1, 2$ *expected by the threshold type* x_1^* *in round 1 satisfies*

$$D_t = \int_0^1 \tilde{u}_t(l_t) dl_t. \tag{5}$$

Proof. Follows from auxiliary Lemmas 1 and 2 below. □

Equation (5) for the first stage payoff D_1 is an immediate consequence of the static Laplacian property from Theorem 1 because the threshold type x_1^* in round 1 of Γ_{dyn} has uniform belief about l_1, exactly as she had in the static case. However, the result for D_2 is not immediate because the relevant belief about l_2 is *not* uniform. Before proceeding to the proof of Theorem 2, it is instructive to attempt to solve for D_2 directly. We can write $D_2 = \int_0^1 \tilde{u}_2(l_2) dP(l_2)$ where for any $z \in [0,1]$

$$P(z) = \Pr(\ell_2(\theta) < z \mid L_2 \cap \partial L_1) \cdot \Pr(L_2 \mid \partial L_1).$$

In words, the player with the threshold signal x_1^* in round 1 first computes the probability that she stays in the project upon receiving x_2^i and then she forms belief about l_2 conditioning on staying. Generically, $P(\cdot)$ is *not* the c.d.f. of the uniform distribution. The direct characterization of D_2 via the function $P(\cdot)$ is cumbersome because $P(\cdot)$ is a complicated object reflecting both the distributional assumptions on the errors and the relative positions of the sets L_t. The advantage of Theorem 2 is that it circumvents the computation of the function $P(\cdot)$. The simple integral in (5) based on the *uniform* distribution of l_2 instead of on $P(\cdot)$ gives the correct value of D_2. The error distributions and the relative positions of L_1, L_2 still influence D_2 but they are summarized by the function $\lambda_1(l_2)$ that relates investment levels across rounds 1 and 2. This separation of the error and profile properties from the beliefs is convenient because below we will be able to make predictions independent of details of the functions $\lambda_t(\cdot)$.

We deal with the complications stemming from the provision of the reversibility option in two auxiliary lemmas. In Lemma 1 we transfer the players' advantage arising from the option into an advantage arising from a superior information. The transformed problem is static, and, broadly speaking, this transformation is useful

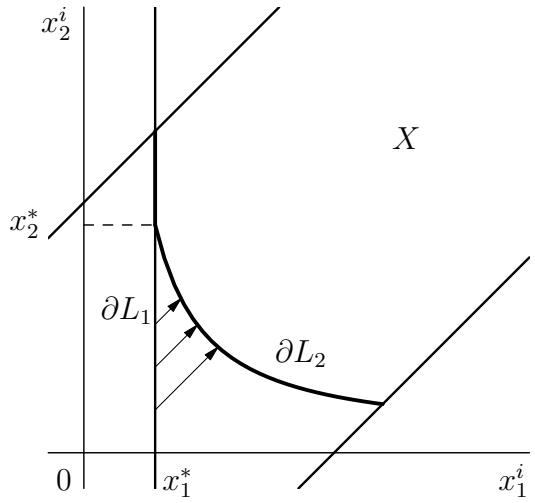


Figure 4: Illustration of the argument supporting Lemma 1.

because the variations in information structure do not distort the static Laplacian property. Indeed, in Lemma 2 we recognize that the transformed problem is essentially a static one in which the known static Laplacian property holds.

The first auxiliary lemma states that D_2 defined by the left-hand side of (6) satisfies a formula analogous to the definition of D_1:

Lemma 1. *If a monotone strategy* $\mathbf{s}$ *satisfies conditions (I)–(II), then*

$$E\left[s_2(\mathbf{x}^i) \cdot U_2\left(\mathbf{x}^i\right) | \partial L_1\right] = E\left[u_2\left(\theta, l_1, l_2\right) | \partial L_2\right]. \tag{6}$$

The player described by the left-hand side of (6) enjoys the advantage of the exit option. The right-hand side describes a player who enjoys, compared to the left-hand side, an advantage of superior information because the boundary ∂L_2 lies above ∂L_1. Lemma 1 claims that the information advantage precisely compensates for the loss of the option advantage.

The idea behind Lemma 1 is illustrated in Figure 4. Types that observed the threshold signal x_1^* in round 1 do or do not participate in the second stage of the project depending on whether their signal x_2^i in round 2 exceeds a critical signal x_2^*. The participating types (if any) — those on the part of ∂L_1 above x_2^* — belong also to the boundary ∂L_2. The types who exit — those on the part of ∂L_1 below x_2^* — receive payoff 0 for the second stage. Types $\mathbf{x}$ on ∂L_2 to the right of x_2^* who participate in stage 2 also receive expected payoff $U_2(\mathbf{x}) = 0$ because they must satisfy the indifference condition in round 2. We show that, when computing the expectation on the left-hand side of (6), we can replace the exiting types (if any) at ∂L_1 below x_2^* with the participating types on ∂L_2 to the right of x_2^*. Thus, we have arrived at the

expectation conditional on ∂L_2 of a player who does not exit in round 2 for any type $\mathbf{x} \in L_2$ — the right-hand side of (6).

Notice that Lemma 1 and the Laplacian property hold, trivially, even if the player who observed x_1^* always exits in round 2. Then, the continuation payoff is $D_2 = 0$ which is equal to the right-hand side of (6) because then the whole ∂L_2 satisfies the indifference condition.

Proof of Lemma 1. For convenience, we let $\sigma = 1$ in the proof.

Let $x_2^* = \inf\{x_2 \in [x_1^* + \underline{\eta}_\Delta, x_1^* + \overline{\eta}_\Delta] : s_2(x_1^*, x_2) = 1\}$ with a convention that $x_2^* = x_1^* + \overline{\eta}_\Delta$ if no type in ∂L_1 invests in round 2. We denote $\eta_\Delta^* = x_2^* - x_1^*$.

We prove (6) by showing that both its left- and right-hand side are equal to

$$\int_{\underline{\eta}_\Delta}^{\eta_\Delta^*} 0 \, dF_\Delta(\eta_\Delta) + \int_{\eta_\Delta^*}^{\overline{\eta}_\Delta} U_2(x_1^*, x_1^* + \eta_\Delta) \, dF_\Delta(\eta_\Delta). \tag{7}$$

In the proof we make use of the fact that η_Δ^i is independent of events ∂L_1 and ∂L_2, and therefore the conditional distribution of $\eta_\Delta^i | \partial L_t$ is equal to the unconditional distribution F_Δ. This independence will be demonstrated at the end of the proof.

The left-hand side of (6) equals (7) because $s_2(x_1^*, x_1^* + \eta_\Delta) = 0$ for $\eta_\Delta < \eta_\Delta^*$, $s_2(x_1^*, x_1^* + \eta_\Delta) = 1$ for $\eta_\Delta > \eta_\Delta^*$, and distribution of $\eta_\Delta | \partial L_1$ equals the unconditional distribution F_Δ.

Let us now turn to the right-hand side of (6). By the law of iterated expectations, we can write it as $E\left[U_2\left(\mathbf{x}\right) | \partial L_2\right]$. Next, for each value of $\eta_\Delta \in [\underline{\eta}_\Delta, \overline{\eta}_\Delta]$, define $\mathbf{x}(\eta_\Delta)$ as the intersection of the boundary ∂L_2 and line $x_2 = x_1 + \eta_\Delta$. (We do not introduce new symbol for the function $\mathbf{x}(\cdot)$ which is a slight abuse of notation.) The intersection exists and is unique. The existence is assured by the condition (II): for sufficiently high x_1, type $(x_1, x_1 + \eta_\Delta)$ exceeds $\overline{\mathbf{x}}$ and then $(x_1, x_1 + \eta_\Delta) \in L_2$. For sufficiently low x_1, $x_1 < x_1^*$ and then $(x_1, x_1 + \eta_\Delta) \notin L_1 \supseteq L_2$. The uniqueness follows from the fact that the strategy $\mathbf{s}$ is monotone and hence ∂L_2 cannot contain $\mathbf{x}$ and $\mathbf{x}'$ such that $\mathbf{x} > \mathbf{x}'$.

Using this notation, and the independence of η_Δ from the event ∂L_2, we can divide the expectation $E\left[U_2\left(\mathbf{x}\right) | \partial L_2\right]$ into

$$\int_{\underline{\eta}_\Delta}^{\eta_\Delta^*} U_2(\mathbf{x}(\eta_\Delta)) dF_\Delta(\eta_\Delta) + \int_{\eta_\Delta^*}^{\overline{\eta}_\Delta} U_2(\mathbf{x}(\eta_\Delta)) dF(\eta_\Delta). \tag{8}$$

The first integral is identical to the first integral in (7) because if $\eta_\Delta < \eta_\Delta^*$ then $\mathbf{x}(\eta_\Delta)$ satisfies the indifference condition in round 2, $U_2(\mathbf{x}(\eta_\Delta)) = 0$. To see this, note

that type $(x_1^*, x_1^* + \eta_\Delta) \notin L_2$ because by the definition of η_Δ^*, $s_2(x_1^*, x_1^* + \eta_\Delta) = 0$ for $\eta_\Delta < \eta_\Delta^*$. Then, by the monotonicity of s_2, $\mathbf{x}(\eta_\Delta)$ is in the interior of L_1 for $\eta_\Delta < \eta_\Delta^*$. Thus in any neighborhood of $\mathbf{x}(\eta_\Delta)$ there exist $\mathbf{x}'$ and $\mathbf{x}''$ such that $s_2(\mathbf{x}') = 0$ and $s_2(\mathbf{x}'') = 1$. Strategy $s_2(\mathbf{x})$ is assumed to be optimal in round 2 by the condition (I), and hence $U_2(\mathbf{x}') \leq 0$, $U_2(\mathbf{x}'') \geq 0$. Then $U_2(\mathbf{x}(\eta_\Delta)) = 0$ from the continuity of expectations with respect to the signals.

The second integral in (8) is identical to the second integral in (7) because if $\eta_\Delta > \eta^*$ then $\mathbf{x}(\eta_\Delta) = (x_1^*, x_1^* + \eta_\Delta)$ as the type $(x_1^*, x_1^* + \eta_\Delta)$ lies on the boundary of L_2. To see this, notice that $s_2(x_1^*, x_1^* + \eta_\Delta) = 1$ by the definition of η_Δ^*; therefore $(x_1^*, x_1^* + \eta_\Delta) \in L_2$. On the other hand, $(x_1^* - \delta, x_1^* + \eta_\Delta) \notin L_1 \supseteq L_2$ for any $\delta > 0$.

We now complete the proof by showing that η_1^i, η_2^i, and therefore $\eta_\Delta^i = \eta_2^i - \eta_1^i$, are independent of events ∂L_1 and ∂L_2. For $t = 1, 2$, we let $d_t(\mathbf{x}) = x_1^* + d$, where d is equal to the distance of $\mathbf{x}$ from the boundary ∂L_t along the diagonal, i.e., $\mathbf{x} - (d, d) \in \partial L_t$. Notice that $d_1(\mathbf{x})$ is simply the first coordinate, $d_1(x_1, x_2) = x_1$. For $t = 2$, mapping d_2 defines for each $\tilde{x}$ a set $\{\mathbf{x} \in X : d_2(\mathbf{x}) = \tilde{x}\}$ which we call an isosignal. We use the mapping d_2 to index the parallel isosignals, as seen in Figure 5. The conditional joint distribution of errors is invariant to diagonal translations: $(\eta_1^i, \eta_2^i)|\mathbf{x} = (\eta_1^i, \eta_2^i)|\big(\mathbf{x} + (d, d)\big)$ for any $d \in \mathbb{R}$. Hence, by the construction of the isosignals, distribution of $(\eta_1^i, \eta_2^i)|d_t(\mathbf{x}^i) = \tilde{x}$ is identical for each $\tilde{x}$ and thus also equal to the unconditional distribution of (η_1^i, η_2^i). □

The second auxiliary lemma is a direct extension of the static Laplacian property. Indeed, for $t = 1$ Lemma 2 coincides with the static case in Theorem 1. Thus, though we write Lemma 2 generally for $t = 1, 2$, the reader may focus on the case $t = 2$.

Lemma 2. *$\ell_t(\theta)|\partial L_t$ is uniformly distributed on $[0, 1]$.*

As the threshold type x_1^* in the static game, the set ∂L_2 constitutes a boundary between the sets of types who do and do not participate in stage 2, respectively. The set L_2 is an upper contour set,[7] and hence the types participating in the stage 2 are those above the boundary ∂L_2. As in the static case, the information that player's type is on the boundary turns out to be entirely uninformative about the realized proportion of players above the boundary. The following proof uses the invariance of the type space to diagonal translations to reduce the two-dimensional problem of Lemma 2 to a static problem, in which the one-dimensional Laplacian property holds.

Proof of Lemma 2. Let $\tilde{x}_t^i = d_t(\mathbf{x}^i)$, $\tilde{\eta}_t^i = \tilde{x}_t^i - \theta$, where d_t was defined in the proof of Lemma 1. We can interpret $\tilde{x}_t^i = d_t(\mathbf{x}^i)$ as a virtual private signal, and $\tilde{\eta}_t^i = \tilde{x}_t^i - \theta$

[7] We call $S \subseteq X$ an *upper contour set*, if for all $\mathbf{x}, \mathbf{x}' \in X$ such that $\mathbf{x}' \geq \mathbf{x}$: if $\mathbf{x} \in S$, then $\mathbf{x}' \in S$.

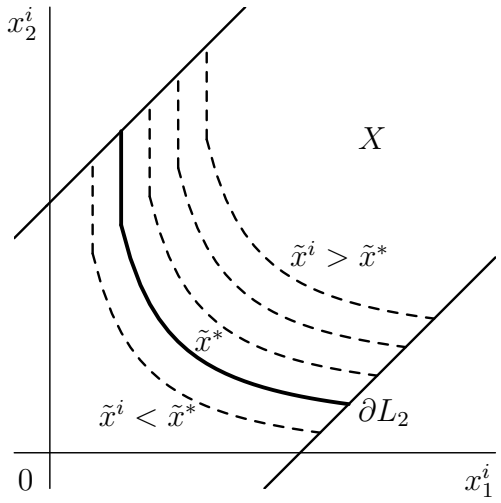


Figure 5: Illustration of the argument supporting Lemma 2.

as a virtual error. The conditional distribution of $\theta|(x_1^i, x_2^i)$ is invariant to diagonal translations, and therefore by the construction, the virtual error $\tilde{\eta}_t^i$ is independent of $\tilde{x}_t^i$ and θ. From the definition of the virtual signal $\tilde{x}_t^i = d_t(\mathbf{x}^i)$, event $\tilde{x}_t^i = x_1^*$ is identical to the event $\mathbf{x}^i \in \partial L_t$ and type $\mathbf{x}^i$ participates in the stage t of the project, $\mathbf{x}^i \in L_t$, if and only if $\tilde{x}_t^i \geq x_1^*$. See Figure 5 for an illustration. Therefore

$$\ell_t(\theta) \mid \partial L_t = \Pr\left(\mathbf{x}^j \in L_t | \theta\right) \mid \partial L_t = \Pr\left(\tilde{x}_t^j > x_1^* | \theta\right) \mid \tilde{x}_t^i = x_1^*,$$

and the last conditional random variable is uniformly distributed on $[0, 1]$ by the static Laplacian property in Theorem 1. □

The Laplacian property in the variant of the game from Figure 3 with the irreversible investment and the delay option has an identical formulation. Threshold type x_1^*'s incentive to invest in round 1, taking into account the delay option, is again $D_1 + D_2$ where D_t satisfies (5).

5 Equilibrium Uniqueness

In the previous section we constructed a convenient characterization of payoff expectations at the threshold in round 1. This characterization did not require any direct assumptions on the payoff functions. Rather, the Laplacian property was driven only by the assumptions imposed on the information structure and on the examined strategy profile. In this section, we introduce assumptions on the payoffs under which the generalized Laplacian property plays a central role for the equilibrium charac-

terization. We impose payoff monotonicities common in the global games literature under which the game becomes dominance solvable and the unique rationalizable actions are characterized in terms of payoff expectations of threshold types under monotone strategy profiles. Then we review the results of Morris and Shin (2003) on rationalizability in the static game (Proposition 1). The main result in this section characterizes rationalizable actions in the dynamic game (Proposition 2).

First, we introduce global game assumptions sufficient for dominance solvability of both the static and the dynamic game.

A1 Strict State Monotonicity: $u_1(\theta, l_1, l_2)$ and $u_2(\theta, l_1, l_2)$ are strictly increasing in θ.

A2 Weak Action Monotonicity: Both $u_1(\theta, l_1, l_2)$ and $u_2(\theta, l_1, l_2)$ are non-decreasing in l_1 and l_2.

A3 Dominance Regions:

A3a (lower and upper dominance regions in the static game): There exist $\underline{\theta}$, $\overline{\theta}$ such that $u_1(\theta, l_1, l_2) + u_2(\theta, l_1, l_2) < 0$ for all $\theta < \underline{\theta}$, and all $l_1, l_2 \in [0,1]$, $l_2 \leq l_1$; and $u_1(\theta, l_1, l_2) + u_2(\theta, l_1, l_2) > 0$ for all $\theta > \overline{\theta}$ and all $l_1, l_2 \in [0,1]$, $l_2 \leq l_1$.

A3b (lower dominance region in round 1): There exists $\underline{\theta}_1$ such that $u_1(\theta, l_1, l_2) < 0$ for all $\theta < \underline{\theta}_1$ and all $l_1, l_2 \in [0,1]$, $l_2 \leq l_1$.

A3c (upper dominance region in round 2): There exist $\overline{\theta}_2$ such that $u_2(\theta, l_1, l_2) > 0$ for all $\theta > \overline{\theta}_2$ and all $l_1, l_2 \in [0,1]$, $l_2 \leq l_1$.

Assumption A1 states that projects with higher parameter θ are, *ceteris paribus*, more profitable. Assumption A2 imposes rich strategic complementarities not only within each round but also across the rounds. It assures that investing by any player in any round increases the incentive to invest for all other players in both rounds. Finally, in Assumption A3a–A3c we assume existence of dominance regions. These assumption together assure that in both stages of the dynamic game and in the static game, players with very high signals participate in the project and those with very low signals do not participate. Assumption A3a assumes both dominance regions for the static game directly. In the case of the dynamic game, players with very high signals invest in round 1 by A3a, and so we only need to assure by A3b that those with very low signals will not invest. Similarly, in A3c we need to assume only the upper dominance region in round 2, because players with very low second signals will

not participate in the second stage as they have not invested already in round 1 by A3b.

We now review the results of Morris and Shin (2003) on rationalizability in the static game. To examine rationalizable actions of type x_1 in Γ_{st}, we return to the symmetric monotone strategy profile s_1 with threshold equal to x_1 and define $m_{st}(x_1)$ as the expected payoff for action 1 of the threshold type x_1. Using the Laplacian property in the static game, we get

$$m_{st}(x_1) = \int_0^1 \left(u_1\left(\vartheta_1(l_1), l_1, l_1\right) + u_2\left(\vartheta_1(l_1), l_1, l_1\right)\right) dl_1,$$

where the right-hand side depends on x_1 through $\vartheta_1\left(l_1; x_1\right)$. For the sake of brevity, we omit the threshold value from the arguments of ϑ_t. Function $m_{st}(x_1)$ is continuous, strictly monotone by A1, and hence it attains 0 at a unique point. The following proposition states that the static game Γ_{st} is dominance solvable, and it characterizes the unique rationalizable action at each signal x_1 (apart from the single point where $m_{st}(x_1) = 0$).

Proposition 1. *(Morris and Shin, 2003)*
Action 1 (0) is the unique rationalizable action for type x_1 in the static game Γ_{st} if and only if $m_{st}(x_1) > 0$ ($m_{st}(x_1) < 0$).

Proof. See proof of Proposition 2.1 in Morris and Shin (2003). □

Next we move to the dynamic setup. We use the concept of rationalizability in the extensive form games introduced in Pearce (1984). However, due to the specific features of our dynamic game, we avoid complications that generally arise in dynamic games. In particular, players in our game do not observe actions of the fellow players, and hence no conjectures about the opponents' strategies are ever refuted in the progress of the play. For convenience we state here a simplified definition of rationalizability particularly tailored to our game. Strategy profile is a mapping $\Sigma(i)$ specifying strategy for each player i. The best response set $BR_2(\mathbf{x}, \Sigma) \subseteq \{0, 1\}$ of type $\mathbf{x}$ in round 2 against profile Σ includes action 1 if $U_2(\mathbf{x}) \geq 0$ and action 0 if $U_2(\mathbf{x}) \leq 0$ under the profile Σ. In round 1, $BR_1(x_1, \Sigma) \subseteq \{0, 1\}$ includes action 1 if $E[\max\{0, U_2(x_1, x_2)\}|x_1] \geq 0$ and action 0 if $E[\max\{0, U_2(x_1, x_2)\}|x_1] \leq 0$ under the profile Σ. We write S^k for the set of pure strategies $\mathbf{s}$ that are not eliminated after k iterations. That is, we let S^0 denote the set of all strategies, and define S^k recursively for $k > 0$: Strategy $(s_1, s_2) \in S^k$ if and only if $(s_1, s_2) \in S^{k-1}$ and for each

type $\mathbf{x} = (x_1, x_2) \in X$ there exists strategy profile Σ such that $\Sigma(i) \in S^{k-1}$ for all players $i \in [0,1]$ and

1. $s_2(\mathbf{x}) \in BR_2(\mathbf{x}, \Sigma)$, or $s_1(x_1) = 0$,

2. $s_1(x_1) \in BR_1(x_1, \Sigma)$.

The *set of rationalizable strategies* is $S^* = \bigcap_k S^k$. The *set of rationalizable actions at signal x_1 at round 1* is the set of actions $a_1 \in \{0,1\}$ for which there exists $(s_1, s_2) \in S^*$ so that $s_1(x_1) = a_1$. The *set of rationalizable actions at type $\mathbf{x}$ at round 2* is the set of actions $a_2 \in \{0,1\}$ for which there exists $(s_1, s_2) \in S^*$ so that $s_2(\mathbf{x}) = a_2$.

As in the static case, we examine a monotone strategy profile with threshold x_1^* in round 1 and the central object of the analysis will be the payoff expectation $m_{dyn}(x_1^*)$ in round 1 at a threshold signal x_1^*. However, unlike in the static case, players have the reversibility option and hence, to fully specify profile (s_1, s_2), we first need to analyze the continuation game in round 2. Let $\Gamma_2(x_1^*)$ denote the continuation game induced from Γ_{dyn} by constraining players in round 1 to the monotone strategy $s_1(x_1^i)$ with the threshold x_1^*. The game $\Gamma_2(x_1^*)$ is a static Bayesian game, as players have no control about actions in round 1, and they only have a choice about $s_2(\mathbf{x}^i)$. We are interested in choices in round 2 only for types $\mathbf{x} \in L_1(x_1^*)$ as $s_2(\mathbf{x})$ of the types $\mathbf{x} \notin L_1(x_1^*)$ not investing in round 1 are not payoff relevant. For the sake of brevity, let us constrain types $\mathbf{x} \in X \setminus L_1(x_1^*)$ who have not invested in round 1 to $s_2(\mathbf{x}) = 0$. This constraint does not affect best response correspondence of any type in $L_1(x_1^*)$. The following lemma states that the continuation game $\Gamma_2(x_1^*)$ is dominance solvable, and therefore, by specifying the threshold x_1^* in round 1, we uniquely determine the strategy profile in the whole game. Moreover, the strategy profile is monotone and satisfies conditions (I)–(II) from Section 4 and so the Laplacian property holds.

Lemma 3. *For each x_1^*, the continuation game $\Gamma_2(x_1^*)$ has a unique rationalizable strategy s_2, which is monotone and $s_2(\mathbf{x}) = 1$ for sufficiently high types $\mathbf{x}$.*

Proof. In Appendix. □

The proof of Lemma 3 consists of constructing the largest and smallest rationalizable strategies by the contagion argument standardly used in the global game literature and then showing that they coincide by an adaptation of the translation argument from Frankel, Morris and Pauzner (2003).

We define the function $m_{dyn}(x_1^*)$ as the expected equilibrium payoff $D_1(x_1^*) + D_2(x_1^*)$ in the game $\Gamma_2(x_1^*)$ formed conditional on the threshold signal x_1^* in round 1. By Lemma 3, $m_{dyn}(x_1^*)$ is uniquely defined. Additionally:

Lemma 4. *Function $m_{dyn}(x_1)$ is strictly increasing and attains both positive and negative values.*

Proof. In Appendix. □

By Lemma 4, there exists a unique threshold x_1^{**} satisfying $m_{dyn}(x_1) > 0$ for $x_1 > x_1^{**}$ and $m_{dyn}(x_1) < 0$ for $x_1 < x_1^{**}$.

The following proposition states that the dynamic game Γ_{dyn} is dominance solvable. The dynamic game Γ_{dyn} has essentially a unique equilibrium; the equilibria can differ only at the boundaries ∂L_t and off the equilibrium path, that is, in $s_2(\mathbf{x})$ of those types $\mathbf{x}$ who did not invest in round 1.

Proposition 2. (i) *Action 1 (0) is the unique rationalizable action in round 1 at signal x_1 in the dynamic game Γ_{dyn} if and only if $m_{dyn}(x_1) > 0$ ($m_{dyn}(x_1) < 0$).*

(ii) *The unique rationalizable action of type (x_1, x_2) with $x_1 > x_1^{**}$ in round 2 is $s_2(x_1, x_2)$ where s_2 is the unique rationalizable strategy in the continuation game $\Gamma_2(x_1^{**})$.*

Proof. In Appendix. □

The proof is based on the usual contagion argument. Action 1 (symmetrically for 0) is dominant in round 1 of Γ_{dyn} at extreme signals in the upper dominance region. Moreover, if action 1 is dominant in round 1 at all signals above x_1 then $m_{dyn}(x_1)$ is a lower bound for payoff expectation of the type x_1 in round 1. Hence, if $m_{dyn}(x_1) > 0$ then action 1 is serially dominant at some x_1' below but close to x_1 by continuity of expectations with respect to signals. The set of the signals at which action 1 (0) is established to be serially dominant in round 1 can be iteratively expanded as long as $m_{dyn}(x_1) > 0$ ($m_{dyn}(x_1) < 0$) on the boundary of the set. Hence, the contagion of action 1 from above and of action 0 from below meet at the root of m_{dyn}.

6 Strategic Effects of Reversibility

In this section we compare the equilibrium investment behavior across the dynamic and the static game. The comparison will be simple thanks to the generalized Laplacian property assuring that the payoff expectations of threshold types are based on Laplacian belief in the both games.

From the previous section we know that the key objects for the analysis of investment behavior are the functions m_{st} and m_{dyn}. To facilitate the comparison of m_{st} and $m_{dyn} = D_1 + D_2$, we decompose m_{st} into $m_{st} = S_1 + S_2$, where

$S_t(x_1^*) = \int_0^1 u_t(\vartheta_1(l_1), l, l)dl$ is the expected payoff of the threshold type x_1^* in the static game for stage t of the project. The following Lemma compares $m_{st}(x_1)$ and $m_{dyn}(x_1)$ by parts.

Lemma 5. *For each $x_1 \in \mathbb{R}$:*

(i) $S_1(x_1) \geq D_1(x_1)$,

(ii) $S_2(x_1) \leq D_2(x_1)$.

Proof. (i) The monotonicity of u_1 implies

$$S_1 = \int_0^1 u_1(\vartheta_1(l_1), l_1, l_1)dl_1 \geq \int_0^1 u_1(\vartheta_1(l_1), l_1, \lambda_2(l_1))dl_1 = D_1, \tag{9}$$

as the integrals differ only in the third argument of u_1, and $l_1 \geq l_2 = \lambda_2(l_1)$.

(ii) Similarly, the monotonicity of u_2 implies

$$S_2 = \int_0^1 u_2(\vartheta_1(l_2), l_2, l_2)dl_2 \leq \int_0^1 u_2(\vartheta_2(l_2), \lambda_1(l_2), l_2)dl_2 = D_2. \tag{10}$$

In this case, we used inequalities $\vartheta_1(l_2) \leq \vartheta_2(l_2)$ and $l_2 \leq l_1 = \lambda_1(l_2)$. To obtain the first inequality recall that $\vartheta_1(l)$ is the inverse function to $\ell_1(\theta)$ and $\vartheta_2(l)$ the inverse function to $\ell_2(\theta)$. Both ℓ_1 and ℓ_2 are increasing and $\ell_1(\theta) \geq \ell_2(\theta)$ for all θ, so the opposite inequality holds for the inverse functions. □

Let us discuss in brief how the inequalities in Lemma 5 change in the variant of the game from Figure 3 with irreversible investment and the delay option. In the baseline game $l_2 \leq l_1$ because investment is reversible, but the opposite inequality $l_2 \geq l_1$ holds in the variant with the delay option as some of the players who have not invested in round 1 can join the project in round 2. For this reason the inequalities in Lemma 5 attain opposite signs in the variant with the delay option. Below we continue with the exposition only for the baseline game with the reversible investment, and the results for the variant with the delay option can be obtained simply by reverting the signs of the effects; see Table 1 for a summary.

We call the inequalities (i) and (ii) the first and the second stage effects. The two effects have opposite signs and therefore the comparison of m_{st} and m_{dyn} is possible only if we add further structure to the model. To discuss some structures that lead to unambiguous comparisons, we introduce the following terminology: the payoffs do *not* exhibit

1. *backward spillovers* if $u_1(\theta, l_1, l_2)$ does not depend on l_2.

2. *forward spillovers* if $u_2(\theta, l_1, l_2)$ does not depend on l_1.

Such restrictions on payoffs can naturally arise in many economic situations. If the players who exited the project do not hold any liability for the continuation of the project, then, arguably, payoffs do not exhibit backward spillovers because profits from the early stage are not casually influenced by subsequent investment behavior. Similarly, if production does not exhibit inertia, and profits are not redistributed among the early and the late investors, then, arguably, payoffs do not exhibit forward spillovers.

The first stage effect vanishes ($S_1 = D_1$) in the absence of backward spillovers. In that case, players invest in the first round of the dynamic game more often than in the static game:

Proposition 3. *If the payoffs do not exhibit backward spillovers then:*

(i) *If action 1 is the unique rationalizable action at x_1 in Γ_{st} then 1 is the unique rationalizable action at x_1 in round 1 of Γ_{dyn}.*

(ii) *If action 0 is the unique rationalizable action in round 1 at x_1 in Γ_{dyn} then 0 is the unique rationalizable action at x_1 in Γ_{st}.*

Proof. The proposition follows from the necessary and sufficient conditions for rationalizability stated in Propositions 1 and 2 and from the inequality $m_{dyn}(x_1) = D_1(x_1) + D_2(x_1) \geq S_1(x_1) + S_2(x_1) = m_{st}(x_1)$ which holds in the absence of backward spillovers. □

Proposition 3 is summarized by the second row in Table 1 in the introduction. Under the specified conditions, players invest on a larger set of signals in round 1 of the dynamic game than in the static game, and so in this case the provision of the exit option enhances investment in round 1. Provision of the delay option has the opposite effect.

When the signals are very precise, the second stage effect vanishes ($S_2 = D_2$) in the absence of forward spillovers, and we get a result analogous but opposite to the one in Proposition 3.[8] This will be formally formulated in the following section

[8]Unlike in the case of absence of backward spillovers in which the first stage effect vanishes even for positive σ, the second stage effect vanishes in the absence of forward spillovers only in the limit $\sigma \to 0$. This is because in the latter case S_2 and D_2 differ also in the first argument of the payoff function; $u_2(\vartheta_1(l_2), \cdot, \cdot)$ vs. $u_2(\vartheta_2(l_2), \cdot, \cdot)$. Proposition 8 below specifies a condition under which this difference disappears in the limit of precise signals.

where we continue in the comparison of the behavior across the dynamic and the static game in the limit of precise signals. Importantly, in this limit we can also characterize rationalizable behavior in round 2.

7 Limit Results

In this section we characterize the effects of reversibility in the limit of precise signals. This limit is a natural domain of our model because we assume the uninformative prior, which is a good approximation of a general prior if the private signals are precise. Formally, we will examine sequences of the static and the dynamic games with varying scale of noise, σ. Games with the scaling parameter equal to σ are denoted by $\Gamma_{st}(\sigma)$ or $\Gamma_{dyn}(\sigma)$, and Γ_{st}, Γ_{dyn} without the argument denote in this section whole classes of games $(\Gamma(\sigma))_{\sigma}$ rather than particular games. We will examine coordination outcome in games $\Gamma_{st}(\sigma)$ and $\Gamma_{dyn}(\sigma)$ when Nature draws fundamental with a value θ^*, when $\sigma \to 0^+$.

We say that *action (history)* $h \in \{0, 1, 10, 11\}$ *is selected at* θ^* *in* Γ_{st}, respectively in Γ_{dyn}, if there exists $\overline{\sigma} > 0$ such that for all $\sigma \in (0, \overline{\sigma}]$ all players in $\Gamma_{st}(\sigma)$, respectively in $\Gamma_{dyn}(\sigma)$, reach action history h whenever Nature draws the fundamental θ^* and all players play according to rationalizable strategies. Naturally, only actions $h \in \{0, 1\}$ can be selected in Γ_{st}. By saying that action 1 is selected at θ^* in Γ_{dyn} we specify that all players play action 1 in round 1, and we leave the continuation play in round 2 unspecified.

We start by describing action selection in the static game. There

$$\lim_{\sigma \to 0+} m_{st}(x_1, \sigma) = m^*_{st}(x_1) = \int_0^1 \big(u_1(x_1, l, l) + u_2(x_1, l, l)\big) dl,$$

and $m^*_{st}(x_1)$ is strictly increasing and continuous. The following proposition follows directly from Proposition 2.1 in Morris and Shin (2003) and from the assumption of bounded errors.

Proposition 4. *Action 1 is selected at* θ^* *in* Γ_{st} *if and only if* $m^*_{st}(\theta^*) > 0$. *Action 0 is selected at* θ^* *in* Γ_{st} *if and only if* $m^*_{st}(\theta^*) < 0$.

We now turn to the dynamic game. To simplify the formulation of limit results in the dynamic game, we will focus on setups in which the limit $m^*_{dyn}(x_1) = \lim_{\sigma \to 0+} m_{dyn}(x_1, \sigma)$ exists is continuous and strictly increasing in x_1. To assure the existence of the limit, in this section we constrain our attention to signal structures

in which η_2^i and $\eta_\Delta^i = \eta_2^i - \eta_1^i$ are independent.[9] In statistical terms, this means that x_2^i is sufficient statistics of (x_1^i, x_2^i) for θ. In such case, the first signal $x_1^i = x_2^i - \sigma\eta_\Delta^i$ is an uninformative coarsening of the second signal $x_2^i = \theta + \sigma\eta_2^i$. This signal structure is a special case of the signal structure considered up to now. The advantage of x_2^i being a sufficient statistic is that it keeps the analysis in round 2 one-dimensional as players in round 2 condition only on x_2^i and not on x_1^i.[10]

Lemma 6. *If η_2^i and $\eta_\Delta^i = \eta_2^i - \eta_1^i$ are independent and, in addition to the previous assumptions, $u_2(\theta, l_1, l_2)$ is strictly increasing l_2, then the limit $\lim_{\sigma\to 0^+} m_{dyn}(x_1, \sigma)$ exists is continuous and strictly increasing in x_1.*

Proof. In Appendix. □

All the remaining results in this section are proven under the assumption that the limit exists, is continuous and is strictly increasing, but without a direct use of the independency assumption (or the existence of the sufficient statistic). Therefore the results remain valid under any other sufficient condition for the statement in Lemma 6.

The characterization of the selected action in round 1 of the dynamic game is analogous to the one in the static game.

Proposition 5. *Action 1 is selected at θ^* in Γ_{dyn} if and only if $m^*_{dyn}(\theta^*) > 0$. Action 0 is selected at θ^* in Γ_{dyn} if and only if $m^*_{dyn}(\theta^*) < 0$.*

Proof. In Appendix. □

Proposition 5 specifies only the action played in round 1 of the dynamic game. In order to characterize the strategic effects of the reversibility option on the *final* coordination outcome, we also need to characterize the continuation play in round 2:

Proposition 6. *Suppose action 1 is selected at θ^* in Γ_{dyn}. Then:*

(i) *If $\int_0^1 u_2(\theta^*, 1, l_2)dl_2 < 0$ then action history 10 is selected at θ^* in Γ_{dyn}.*

(ii) *If $\int_0^1 u_2(\theta^*, 1, l_2)dl_2 > 0$ then action history 11 is selected at θ^* in Γ_{dyn}.*

Proof. In Appendix. □

[9] A special case of this signal structure is also used in Heidhues and Melissas (2006).

[10] In the general case, when players in round 2 need to condition on both signals, the set of indifference points in round 2 is characterized by a functional equation, and analysis of the limit becomes cumbersome.

Let us discuss Proposition 6. First, the selected continuation play in round 2 depends solely on the payoff function u_2 and is independent of u_1; there is no interplay in between the two stages of the project in this result. Second, the condition $\int_0^1 u_2(\theta^*, 1, l_2)dl_2 \lesseqgtr 0$ has an intuitive interpretation. Imagine first a game in which all players must participate in stage 1 of the project and can only choose whether to continue into stage 2. Then $\int_0^1 u_2(\theta^*, 1, l_2)dl_2 < 0$ is the condition under which the coordination failure is selected at θ^* in this imaginary static global game. The coordination problem of players in round 2 of the dynamic game with endogenous entry is more severe because they are unsure of participation $l_1 \leq 1$ in stage 1, and hence the condition $\int_0^1 u_2(\theta^*, 1, l_2)dl_2 < 0$ implies coordination on exit also in Γ_{dyn} with the voluntary participation in stage 1. On the other hand, if $\int_0^1 u_2(\theta^*, 1, l_2)dl_2 > 0$ then coordination on staying in the project is selected in the imaginary static global game. The same inequality happens to imply that coordination on staying in the project is selected at θ^* even if participation in stage 1 is voluntary.

We now examine the selected coordination outcome at θ^* under the condition $\int_0^1 u_2(\theta^*, 1, l_2)dl_2 > 0$.[11] The following two propositions extend the result in Proposition 3 by comparing coordination outcomes across the static and the dynamic game when payoffs do not exhibit either backward or forward spillovers. Suppose first that payoffs do not exhibit backward spillovers. Then, the first stage effect discussed in the previous section vanishes, and hence $m^*_{st}(\theta^*) \leq m^*_{dyn}(\theta^*)$ by the second stage effect. In that case, efficient coordination is selected in Γ_{dyn} on a larger set of fundamentals than in Γ_{st}:

Proposition 7. *If payoffs do not exhibit backward spillovers, and* $\int_0^1 u_2(\theta^*, 1, l_2)dl_2 > 0$ *then:*

(i) *If action 1 is selected at* θ^* *in* Γ_{st} *then action history 11 is selected at* θ^* *in* Γ_{dyn}.

(ii) *If action 0 is selected at* θ^* *in* Γ_{dyn} *then action 0 is selected at* θ^* *in* Γ_{st}.

Proof. We showed that $m_{dyn}(x_1, \sigma) \geq m_{st}(x_1, \sigma)$ for any x_1 and σ in the absence of backward spillovers, and so $m^*_{dyn}(x_1) \geq m^*_{st}(x_1)$ for all x_1.

(i) If action 1 is selected at θ^* in Γ_{st} then $0 < m^*_{st}(\theta^*) \leq m^*_{dyn}(\theta^*)$ and so action 1 is selected at θ^* in round 1 of Γ_{dyn}. Then, by Proposition 6, action history 11 is selected at θ^* in Γ_{dyn}.

[11] The analysis of the exit option becomes trivial in the other case, in which $\int_0^1 u_2(\theta^*, 1, l_2)dl_2 < 0$. Then, whenever Nature draws θ^* and σ is small, all players already know in round 1 that they will not participate in stage 2 of the project, and the second stage becomes vacuous.

(ii) If action 0 is selected at θ^* in Γ_{dyn} then $0 > m^*_{dyn}(\theta^*) \geq m^*_{st}(\theta^*)$ and hence action 0 is selected at θ^* in Γ_{st}. □

If payoffs do not exhibit forward spillovers then the second stage effect vanishes in the limit of precise signals, and hence $m^*_{st}(\theta^*) \geq m^*_{dyn}(\theta^*)$ by the first stage effect. Then efficient coordination is selected in Γ_{dyn} at a smaller set of fundamentals than in Γ_{st}. However, this mirror image of the previous proposition holds only in the limit of precise signals and under the condition in the following proposition:

Proposition 8. *If payoffs do not exhibit forward spillovers and* $\int_0^1 u_2(\theta^*, 1, l_2)dl_2 > 0$ *then:*

(i) *If action 1 is selected at* θ^* *in* Γ_{dyn} *then action 1 is selected at* θ^* *in* Γ_{st}.

(ii) *If action 0 is selected at* θ^* *in* Γ_{st} *then action 0 is selected at* θ^* *in* Γ_{dyn}.

Proof. In the supplement to this proof in Appendix we show that if $\int_0^1 u_2(x_1, 1, l_2)dl_2 > 0$ and payoffs do not exhibit forward spillovers then $m^*_{dyn}(x_1) \leq m^*_{st}(x_1)$.

(i) If action 1 is selected at θ^* in round 1 of Γ_{dyn} then $0 < m^*_{dyn}(\theta^*) \leq m^*_{st}(\theta^*)$ and so action 1 is selected at θ^* in Γ_{st}.

(ii) If action 0 is selected at θ^* in Γ_{st} then $0 > m^*_{st}(\theta^*) \geq m^*_{dyn}(\theta^*)$ and hence action 0 is selected at θ^* in Γ_{dyn}. □

A simple corollary of the last two propositions is that if payoffs exhibit neither backward nor forward spillovers and $\int_0^1 u_2(\theta^*, 1, l_2)dl_2 > 0$, then the selected coordination outcome is identical at θ^* across the static and the dynamic game. Another case in which the coordination outcome is identical at θ^* across the two games arises when $u_2(\theta^*, 0, 0) > 0$. Then, when Nature selects θ^*, each player knows already in round 1 that all players who participate in stage 1 will continue into stage 2 of the project, and so the reversibility option becomes vacuous.

Apart from those two cases, the option affects the coordination outcome on a large set of fundamentals, and the effect does not disappear as $\sigma \to 0^+$. Although the reversibility option is in the unique equilibrium of $\Gamma_{dyn}(\sigma)$ exercised by a vanishing set of types in the limit $\sigma \to 0^+$, the player observing the equilibrium threshold signal x_1^{**} will use the option with a positive probability which does not vanish as $\sigma \to 0^+$. Such a player has a pivotal role in the equilibrium analysis, and hence the option remains to have non-trivial consequences even as $\sigma \to 0^+$. We illustrate this point in the next subsection where we explicitly compute the equilibrium of the dynamic game in a particular but tractable limit.

7.1 Example

In this example we analyze the dynamic game in an ordered limit that sends precisions of both signals to infinity, but the precision of the second signal increases more quickly. We use the error structure $x_1^i = x_2^i - \sigma_1\eta_\Delta^i$, $x_2^i = \theta + \sigma_2\eta_2^i$, with η_Δ^i independent of η_2^i, and examine the ordered limit $\lim_{\sigma_1\to 0^+}\lim_{\sigma_2\to 0^+}$.

In the ordered limit, the relationship between l_1 and l_2 becomes simple. Let $x_2^* = x_1^* + \sigma_1\eta^*$ for some fixed x_1^*, σ_1, η^*, and let us introduce $l_1^* = \ell_1(x_2^*) = 1 - F_1(-\eta^*)$. Then, as $\sigma_2 \to 0^+$

$$l_2 = \lambda_2(l_1) = \begin{cases} 0 & \text{if } l_1 < l_1^*, \\ l_1 & \text{if } l_1 > l_1^*, \end{cases} \qquad l_1 = \lambda_1(l_2) = \begin{cases} l_1^* & \text{if } l_2 < l_1^*, \\ l_2 & \text{if } l_2 > l_1^*. \end{cases} \tag{11}$$

The intuition is the following: Under the described strategy profile, players in round 2 extract from their very precise signals x_2^i very precise information about l_1. In the limit $\sigma_2 \to 0^+$, all players exit in round 2 whenever $\ell_1(\theta) < l_1^*$ at the realized θ, and all players stay in round 2 whenever $\ell_1(\theta) > l_1^*$.

The expression for the expected incentive to invest of the threshold type x_2^* in round 2 is simple in the ordered limit as well. Using the above definition of l_1^*, the indifference condition in round 2 can be expressed in the ordered limit as

$$\frac{1}{l_1^*}\int_0^{l_1^*} u_2(x_1^*, l_1^*, l_2)dl_2 = 0, \tag{12}$$

when $u_2(x_1^*, 0, 0) < 0$ and $\int_0^1 u_2(x_1^*, 1, l_2)dl_2 > 0$. See (17) in Proof of Lemma 6, for the out of the limit expression.

Then, in the non-trivial cases,[12]

$$\lim_{\sigma_1\to 0^+}\lim_{\sigma_2\to 0^+} m_{dyn}(x_1, \sigma_1, \sigma_2) = \int_0^1 u_1(x_1, l_1, \lambda_2(l_1))dl_1 + \int_0^1 u_2(x_1, \lambda_1(l_2), l_2)dl_2,$$

where $\lambda_t(\cdot)$ are given by (11) and l_1^* is the unique solution of (12). Let us mention that in the ordered limit the function $m_{dyn}^*(x_1)$ and, consequently the selected coordination outcome, is noise independent — it does not depend on the assumed error distribution. The solution that we sketched here for the ordered limit can be extended also to the analysis of an unordered limit in which both $\sigma_1 \to 0$, $\sigma_2 \to 0$, with their ratio kept constant. However, the solution will depend on the error distribution in that case.

[12] See cases (a) or (b) in Proof of Lemma 6 for analysis of the trivial cases when $u_2(x_1^*, 0, 0) > 0$ or $\int_0^1 u_2(x_1^*, 1, l_2)dl_2 < 0$.

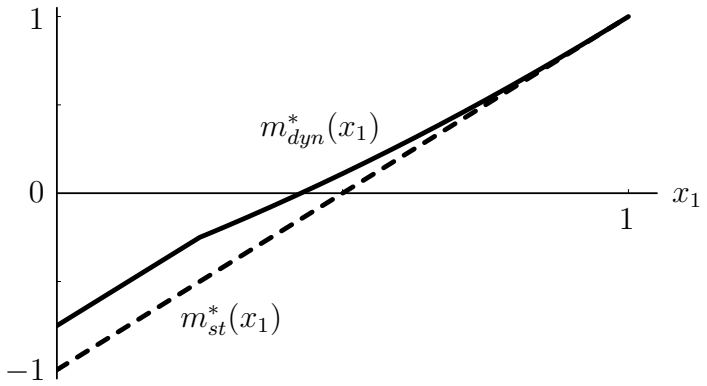


Figure 6: Comparison of functions $m^*_{st}(x_1)$ and $m^*_{dyn}(x_1)$ in the ordered limit.

As an illustration of the solution in the ordered limit, we depict in Figure 6 the limit functions $m^*_{st}(x_1)$ and $m^*_{dyn}(x_1)$ for the payoff functions

$$u_1(\theta, l_1, l_2) = \theta - 1 + l_1, \qquad u_2(\theta, l_1, l_2) = \theta - 1 + \frac{l_1 + l_2}{2}.$$

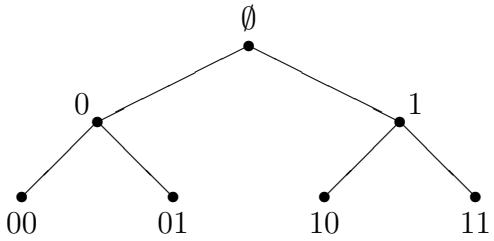
The illustrative payoffs do not exhibit backward spillovers and so, in accordance with Proposition 7, coordination on successful investment is selected in the dynamic game on a larger set of fundamentals than in the static one. This effect does not disappear in the limit despite the fact that, by Proposition 6, the investors do not exercise their exit option apart from the types in a vanishing neighborhood of the equilibrium threshold x_1^*. The effect remains significant in the limit because the types in the neighborhood of x_1^{**} use the exit option with a positive probability and hence the investment profiles $\lambda_1(l_2) \neq l_2$ and $\lambda_2(l_1) \neq l_1$ even in the limit, which translates into the significant difference between the functions $m^*_{st}(x_1)$ and $m^*_{dyn}(x_1)$.

8 Further Generalization of the Laplacian Property

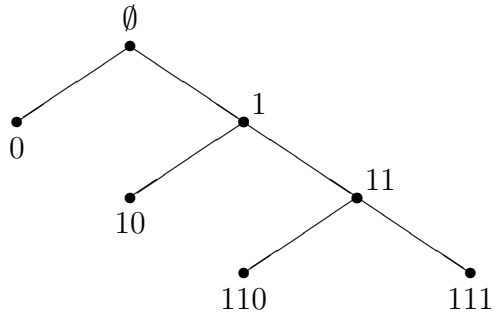
In this Section, we sketch out Laplacian property in environments extending the baseline dynamic game. We abstract here from the particular economic problem of reversible investment analyzed elsewhere in the paper and explore the generality of property.

A continuum of players indexed by $i \in [0, 1]$ make a finite sequence of binary decisions. Let h^i denote a private action history of player i: We write $\emptyset$ for the empty action history at the beginning of the game and write $h^i a^i$ for the private action

history of player i who has chosen action $a^i \in \{0,1\}$ at h^i; letter H denotes the set of attainable action histories. History $z \in H$ is called *terminal*, if $z0, z1 \notin H$. We denote Z the set of all terminal histories and $T = H \setminus Z$ the set of *transient* histories. We endow the set of terminal histories Z by the lexicographical order $\prec$. If two terminal histories z and z' differ in the length then their order is determined based on the first d actions, where d is length of the shorter history. Note that the set Z is completely and strictly ordered by the above order because otherwise there would exist z and z' such that z' would be a continuation of z which would contradict z being terminal. We denote the minimal and the maximal terminal history by $\underline{z}$ and $\overline{z}$. See Figure 7 for illustrations of this notation.



(a) Transient histories $T = \{\emptyset, 0, 1\}$, terminal hist. $Z = \{00, 01, 10, 11\}$, $00 \prec 01 \prec 10 \prec 11$.



(b) Transient histories $T = \{\emptyset, 1, 11\}$, terminal hist. $Z = \{0, 10, 110, 111\}$, $0 \prec 10 \prec 110 \prec 111$.

Figure 7: Examples of general games.

We again interpret this game as an investment game and $z \in Z$ as feasible levels of participation in an investment project with $\underline{z}$ being an outside option. We think of a player reaching $z' \succ z$ as participating more actively in the project than a player reaching z. For $z \succ \underline{z}$ let l_z denote measure of players whose participation level is *at least* z and $l = (l_z)_{z \succ \underline{z}}$ is the tuple of l_z.

The payoff at a terminal action history $z \in Z$ is $\sum_{z' \preceq z} u_{z'}(\theta, l)$, and we normalize payoff function $u_{\underline{z}}$ for the outside option $\underline{z}$ to 0. The function u_z represents the payoff for increasing participation from the predecessor of z to z. The additive payoff structure is without loss of generality.

The information structure is the extension of the one in the baseline model. Nature draws θ from improper uniform distribution on $\mathbb{R}$, and at each transient action history $h \in T$ that player reaches she observes a private signal $x_h^i = \theta + \sigma\eta_h^i$, with the errors η_h^i i.i.d. across players and independent from θ. The tuple of errors $(\eta_h^i)_{h \in T}$ is drawn from an atomless joint distribution with a compact convex support. Bold

letter $\mathbf{x}^i = (x_h^i)_{h\in T}$ denotes type of player i and $\mathbf{x}_h^i$ denotes history of private signals observed up to h, including the signal at h if h is transient. Strategy $\mathbf{s} = (s_h)_{h\in T}$ is a tuple of functions s_h each mapping signal history $\mathbf{x}_h^i$ to action in $\{0, 1\}$.

For a fixed symmetric strategy profile $\mathbf{s}$, let $v_h(\mathbf{x}_h)$ be the expected payoff of a player at history h with a signal history $\mathbf{x}_h$ who follows $\mathbf{s}$ from h onward:

$$v_h(\mathbf{x}_h) = E[v_{hs_h(\mathbf{x}_h)}(\mathbf{x}_{hs_h(\mathbf{x}_h)}) \mid \mathbf{x}_h],$$

for any transient history $h \in T$ and let $v_z(\mathbf{x}_z) = E[\sum_{z'\preceq z} u_{z'}(\theta, l) \mid \mathbf{x}_z]$ for any terminal history $z \in Z$.

As in the description of the Laplacian property in Section 4, we restrict our attention to monotone strategies $\mathbf{s}$. We denote the threshold signal at the beginning of the game by $x_\emptyset^*$. We again require $\mathbf{s}$ to be optimal at all histories *except for the starting history* $\emptyset$. For each $h \in T \setminus \{\emptyset\}$ and for all types $\mathbf{x}$ that reach h,

$$s_h(\mathbf{x}_h) \in \arg\max_{a\in\{0,1\}} E\left[v_{ha}(\mathbf{x}_{ha}) \mid \mathbf{x}_h\right].$$

Furthermore, we restrict attention to $\mathbf{s}$ under which sufficiently high types reach the maximal participation level $\overline{z}$.

For a fixed symmetric strategy profile $\mathbf{s}$, we let L_z be the set of types $\mathbf{x}$ that reach the participation of at least z. Notice that the lexicographical order on Z and the cumulative definition of l_z and of L_z are purely notational structures, and they do not impose any restrictions on the game. With this notation, and with the assumption that the considered stratefy $\mathbf{s}$ is monotone, L_z is a decreasing sequence of *upper contour sets*, $L_z \subseteq L_{z'}$ for $z \succeq z'$, which will play an important role below.

For any transient history h we introduce $\underline{z}_h$ to be the minimal participation attainable from h. That is, a player who has reached h has committed herself to participation of at least $\underline{z}_h$. For each h we abbreviate the boundary $\partial L_{\underline{z}_{h1}}$ to B_h. The set B_h is the boundary separating types who reach $z \succeq \underline{z}_{h1}$ from types who reach a participation below $\underline{z}_{h1}$. Note that $B_\emptyset$ is simply the set of types who receive the threshold signal $x_\emptyset^*$ at the beginning of the game.

Our aim is to compute the incentive to invest as expected by the threshold type $x_\emptyset^*$ at the beginning of the game. For that purpose, but more generally, we introduce the following objects. For each transient history h we define:

$$\Delta_h = E[v_{h1}(\mathbf{x}_{h1}) - v_{h0}(\mathbf{x}_{h0}) \mid B_h],$$

and let $\Delta_z = 0$ for all terminal histories z. Our main object of interest is $\Delta_\emptyset$ which is the incentive to play action 1 instead of 0 as expected by the threshold type $x^*_\emptyset$ at the beginning of the game. Our tool for finding $\Delta_\emptyset$ is the following recursive formula:

$$\Delta_h = \Delta_{h0} + \Delta_{h1} + E[v_{h10}(\mathbf{x}_{h10}) - v_{h01}(\mathbf{x}_{h01}) \mid B_h], \tag{13}$$

where we make a convention that if history ha is terminal then $v_{haa'}(\mathbf{x}_{haa'})$ is replaced by $v_{ha}(\mathbf{x}_{ha})$.

To establish (13) we explore the expected payoff for playing 1 or 0 at transient history h conditional on B_h:

$$E[v_{h1}(\mathbf{x}_{h1}) \mid B_h] = E\left[\big(1 - s_{h1}(\mathbf{x}_{h1})\big)v_{h10}(\mathbf{x}_{h10}) + s_{h1}(\mathbf{x}_{h1})v_{h11}(\mathbf{x}_{h11}) \mid B_h\right],$$

because a player reaches $h10$ if she plays $s_{h1}(\mathbf{x}_{h1}) = 0$ at $h1$ and reaches $h11$ if $s_{h1}(\mathbf{x}_{h1}) = 1$. The next, central step is to recognize that

$$E\left[s_{h1}(\mathbf{x}_{h1})\left(v_{h11}(\mathbf{x}_{h11}) - v_{h10}(\mathbf{x}_{h10})\right) \mid B_h\right] = E\left[v_{h11}(\mathbf{x}_{h11}) - v_{h10}(\mathbf{x}_{h10}) \mid B_{h1}\right] = \Delta_{h1}, \tag{14}$$

which is a generalization of Lemma 1.

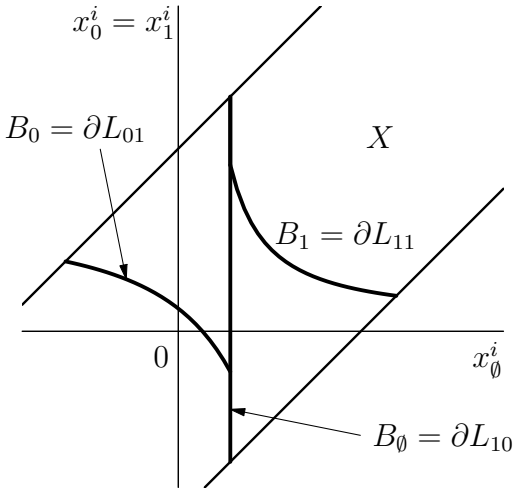


Figure 8: Type space and the boundaries B_h in the game from Figure 7a.

Let us discuss the identity (14) on a particular example of the game from Figure 7a. In this game, the type $(x^i_\emptyset, x^i_0, x^i_1)$ is a three-dimensional vector, but, to allow for a two-dimensional representation, let us consider a signal distribution under which x^i_0 and x^i_1 are identical. Then we can depict the type space X and boundaries $B_\emptyset = \partial L_{10}$, $B_0 = \partial L_{01}$, $B_1 = \partial L_{11}$ as in Figure 8. Let $h = \emptyset$. The left-hand side of (14)

is the expected value of the option to choose participation level $z = 11$ instead of 10 in history 1 with the expectation formed conditional on the boundary $B_\emptyset$ at the beginning of the game. As in the case of Lemma 1, the expression (14) claims that this option advantage can be transformed to an advantage of superior information B_1 instead of $B_\emptyset$. As in the proof of Lemma 1, this argument is based on replacing the types $\mathbf{x}$ on $B_\emptyset$ who do not play 1 at history 1 (so that $s_1(\mathbf{x}_1) = 0$) with the types $\mathbf{x}'$ on the boundary B_1 that play 1 but are indifferent between investing and not investing (so that $E[v_{11}(\mathbf{x}'_{11}) - v_{10}(\mathbf{x}'_{10}) \mid \mathbf{x}'_1] = 0$). In doing such a replacement, we use that the examined strategy profile is assumed to be optimal in the continuation game and that the sets L_z are upper contour sets. We omit the formal proof of (14).

Using (14) we get

$$E[v_{h1}(\mathbf{x}_{h1}) \mid B_h] = \Delta_{h1} + E\left[v_{h10}(\mathbf{x}_{h10}) \mid B_h\right],$$

and by a careful relabeling of actions we get

$$E[v_{h0}(\mathbf{x}_{h0}) \mid B_h] = -\Delta_{h0} + E\left[v_{h01}(\mathbf{x}_{h01}) \mid B_h\right].$$

The recursive formula (13) is the difference of the last two expressions.

Let us now apply the recursive formula (13) on the two exemplary games in Figure 7. We start with the game in Figure 7a. Applying (13) we get

$$\Delta_\emptyset = \Delta_0 + \Delta_1 + E[v_{10}(\mathbf{x}_{10}) - v_{01}(\mathbf{x}_{01}) \mid B_\emptyset] = E[u_{01} \mid B_0] + E[u_{11} \mid B_1] + E[u_{10} \mid B_\emptyset] = \\ = \sum_{z \in \{01,10,11\}} E[u_z(\theta, l) \mid \partial L_z].$$

Each of the sets L_{01}, L_{10}, and L_{11} is an upper contour set which allows for a simple generalization of Lemma 2. As in that case, information that a type is on the boundary ∂L_z turns out not to contain any information about proportion of the fellow players above the boundary, and hence $l_z \mid \partial L_z$ is uniformly distributed on $[0, 1]$. Therefore

$$\Delta_\emptyset = \sum_{z \in \{01,10,11\}} \int_0^1 \tilde{u}_z(l_z) dl_z,$$

where $\tilde{u}_z(l_z)$ denotes $u_z(\theta, \ldots, l_{z'}, \ldots,)$ with θ and $l_{z'}$, $z' \neq z$, treated as functions of l_z induced by the examined strategy profile $\mathbf{s}$.

Generally, we need to apply the recursive formula (13) repeatedly. As an illustration, consider the game in Figure 7b. Applying (13) the first time we get

$\Delta_\emptyset = E[u_{10}(\theta, l) \mid B_\emptyset] + \Delta_1$. When examining Δ_1 we can ignore the payoff term u_{10} because it is a part of the payoff at all terminal histories that can be reached from history 1. Applying (13) again, $\Delta_1 = E[u_{110}(\theta, l) \mid B_1] + \Delta_{11}$, and applying (13) the last time, we get

$$\Delta_\emptyset = E[u_{10}(\theta, l) \mid B_\emptyset] + E[u_{110}(\theta, l) \mid B_1] + E[u_{111}(\theta, l) \mid B_{11}] = \sum_{z \in \{10,110,111\}} E[u_z(\theta, l) \mid \partial L_z].$$

Using that $l_z \mid \partial L_z$ is uniformly distributed on $[0, 1]$ we get

$$\Delta_\emptyset = \sum_{z \in \{10,110,111\}} \int_0^1 \tilde{u}_z(l_z) dl_z.$$

This method is general. For any dynamic game described in this section, the iterative application of the recursive formula (13) will express the incentive $\Delta_\emptyset$ of the threshold type $x^*_\emptyset$ to invest at the beginning of the game as a linear combination of the integrals $\int_0^1 \tilde{u}_z(l_z) dl_z$ with positive integer weights.

If we, as in Section 5, impose the global game assumptions on the payoff functions u_z (state monotonicity, monotonicity in l_z, existence of the dominance regions) then the game becomes dominance solvable and the unique rationalizable action at history $\emptyset$ for type $x_\emptyset$ can be characterized by the sign of $\Delta_\emptyset(x_\emptyset)$ in the same manner as the sign of $m_{dyn}(x_1)$ determined the unique rationalizable action in Proposition 2. The comparison of investment across the static and the dynamic game in Section 6 was based on a further restriction of the payoff structure $u_z(\theta, \ldots, l_z, \ldots)$ motivated by our economic application. Comparative results of a similar nature can be based on alternative economically meaningful restrictions of the payoff structure stemming from different economic applications which we plan to explore in future research.

9 Conclusion

Economically relevant coordination problems are rarely static. Typically, they are dynamic processes in which economic agents can postpone irreversible decisions in order to acquire additional information. We developed a modeling framework that incorporates learning and (ir)reversibility without compromising analytical tractability. The framework allows for a qualitative assessment of the reversibility effects based only on two features observable by an outside modeler. The first relevant feature is the (ir)reversibility of actions available to the economic agents, and the second feature is the structure of the intertemporal payoff spillovers in between different stages of the

coordination process. Based on these two features, the modeler or a policy maker can assess the effects of the reversibility option as summarized in Table 1 on page 4.

The applicability of this dynamic framework can be demonstrated on the economic problems discussed in the introduction. The problem studied in Morris and Shin (2004) consisted of an investment project with reversible investment and irreversible safe action, which conforms to the left column in Table 1. It is conceivable that the investment project exhibits forward payoff spillovers because a higher level of investment in the early stage may *ceteris paribus* increase profits in the late stage of the project due to inertia in the production process. On the contrary, the backward payoff spillovers are unlikely because the instantaneous profit from the first stage should not be causally influenced by the investment level in the later stage. This structure of the payoff spillovers is corresponds to the second row of Table 1 and thus the players coordinate more efficiently in the dynamic game than in the benchmark static one without the option. Although, as found by Morris and Shin, the exit option could lead in the interim stage of the project to inefficient runs, this is in this case more than offset by the valuable flexibility provided by the option. The opposite effect arises if the structure of the payoff spillovers is preserved but, instead of the option to revert the investment, players have the option to delay (Heidhues and Melissas 2006, Section 3.2 falls into this category). The provision of the delay option hampers efficient coordination in this case.

In other cases, the structure of the payoff spillovers may differ. In some applications of the regime changes games,[13] the success of the attack and the payoffs for the participation in the early and the late wave of the attack will depend only on the final size of the attack l_2 (as assumed in Dasgupta 2007). In such cases the payoffs exhibit only backward but no forward spillovers, the delay option enhances and the exit option hampers the efficient coordination.

It is useful to summarize here the logical structure that leads from the generalized Laplacian property to the characterization of the reversibility effects. First, the Laplacian property is a property of the strategy profile and of the information structure; no direct assumptions on the payoffs are needed. We imposed the global game assumptions on the payoffs only in the second part of the argument. That assured that the monotone strategy profiles in which the Laplacian property holds are relevant for the equilibrium analysis of the examined game. The third part of the argument was based on the simple structure of the examined investment game

[13]Static games of regime change are used to model currency attacks (Morris and Shin 1998), bank runs (Goldstein and Pauzner 2005) or revolutions (Edmond 2008). The attack in these models succeeds if its size exceeds a critical level.

by which we could unambiguously compare investment levels l_1 and l_2. This last step allowed us to compare rationalizable actions across the dynamic and static game. Functions m_{st} and m_{dyn} can be compared also under other restrictions than those we imposed on the intertemporal payoff spillovers. For instance it is easy to show that if payoff satisfy $u_1 = \alpha u_2$ and they depend (apart on θ) only on the sum $l_1 + \alpha l_2$, where α is a positive constant, then $m_{st}(x_1) \leq m_{dyn}(x_1)$.

In Section 8 we sketched the Laplacian property in a large set of dynamic coordination problems. The second step, in which we combined Laplacian property with global games payoffs and have got equilibrium uniqueness, can be also generalized in a direct manner. A modification of the third step provides a promising opportunity for future research. That is, the new research should identify economically relevant restrictions of the game structure from Section 8 that would allow for comparison of incentives of threshold types, and thus also equilibrium behavior, across different games.

A Appendix

Proof of Lemma 3. For convenience, we let $\sigma = 1$ in this proof.

In the first step we analyze the maximal and the minimal rationalizable strategy in the continuation game $\Gamma_2(x_1^*)$. This problem is analyzed in van Zandt and Vives (2007) for general Bayesian games with strategic complementarities, but our problem differs in certain details of the setup, such as continuous vs. discrete set of players, and so we give a direct argument.

The existence of the upper dominance region (by A3c) assures that there exists $\overline{x}_1 \geq x_1^*$ such that for all $(x_1, x_2) \in X$ satisfying $x_1 \geq \overline{x}_1$ action 1 is dominant at (x_1, x_2) in $\Gamma_2(x_1^*)$.

Using notation $L_1(x_1) = \{(x_1', x_2') \in X : x_1' \geq x_1\}$, let us define: $\underline{L}_2^{(1)} = L_1(\overline{x}_1)$ and $\overline{L}_2^{(1)} = L_1(x_1^*)$. The sets $\underline{L}_2^{(1)}$, $\overline{L}_2^{(1)}$ are upper contour sets and $\underline{L}_2^{(1)} \subseteq \overline{L}_2^{(1)}$. Action 1 is dominant in $\Gamma_2(x_1^*)$ on $\underline{L}_2^{(1)}$ and action 0 is dominant everywhere on $L_1(x_1^*) \setminus \overline{L}_2^{(1)}$ (because this is an empty set).

For $L_2 \subseteq L_1 \subseteq X$, $\mathbf{x} \in X$ let $\tilde{U}_2(\mathbf{x}, L_1, L_2)$ be the expected payoff $E[u_2(\theta, l_1, l_2)|\mathbf{x}^i]$ if the set opponents' types who invest in round 1 is L_1 and the set opponents's types who invest in both rounds is L_2.

Next, let us define for $k = 1, 2 \ldots$

$$\underline{L}_2^{(k+1)} = \left\{ \mathbf{x} \in X : \tilde{U}_2 \left(\mathbf{x}, L_1(x_1^*), \underline{L}_2^{(k)} \right) > 0 \right\},$$
$$\overline{L}_2^{(k+1)} = \left\{ \mathbf{x} \in X : \tilde{U}_2 \left(\mathbf{x}, L_1(x_1^*), \overline{L}_2^{(k)} \right) \geq 0 \right\}.$$

By induction, action 1 is unique rationalizable action in $\Gamma_2(x_1^*)$ on types in $\underline{L}_2^{(k)}$ and 0 is unique rationalizable action in $\Gamma_2(x_1^*)$ on types in $L_1(x_1^*) \backslash \overline{L}_2^{(k)}$ after k iterations. The sets $\underline{L}_2^{(k)}$, $\overline{L}_2^{(k)}$ are upper contour sets and satisfy the following properties: $\underline{L}_2^{(k)} \subseteq \overline{L}_2^{(k)}$ and $\overline{L}_2^{(k+1)} \subseteq \overline{L}_2^{(k)}$, $\underline{L}_2^{(k+1)} \supseteq \underline{L}_2^{(k)}$ for all k.

Next we define sets $\underline{L}_2 = \bigcup_k \underline{L}_2^{(k)}$ and $\overline{L}_2 = \bigcap_k \overline{L}_2^{(k)}$. Action 1 is the unique rationalizable action on $\underline{L}_2$ and action 0 is the unique rationalizable action on $L_1(x_1^*) \setminus \overline{L}_2$. From the properties of sets $\underline{L}_2^{(k)}$, $\overline{L}_2^{(k)}$, the sets $\underline{L}_2$, $\overline{L}_2$ are upper contour sets and $\overline{L}_2 \supseteq \underline{L}_2$.

In the second step, based on the translation argument in Frankel, Morris, and Pauzner (2003), we prove that interiors of the sets $\underline{L}_2$ and $\overline{L}_2$ are identical. Because the sets are upper contour sets, it suffices to prove that their boundaries are equal; $\partial \overline{L}_2 = \partial \underline{L}_2$. Let us suppose by contradiction that $\partial \overline{L}_2 \neq \partial \underline{L}_2$.

Let us recall notation from Section 2: $\eta_\Delta^i \in [\underline{\eta}_\Delta, \overline{\eta}_\Delta]$ denotes errors' difference $\eta_2^i - \eta_1^i$, and with $\sigma = 1$ we also have $\eta_\Delta^i = x_2^i - x_1^i$. In addition, let $\underline{\mathbf{x}} : [\underline{\eta}_\Delta, \overline{\eta}_\Delta] \to \partial \underline{L}_2$ and $\overline{\mathbf{x}} : [\underline{\eta}_\Delta, \overline{\eta}_\Delta] \to \partial \overline{L}_2$ denote the intersections of the line $x_2 - x_1 = \eta_\Delta$ with $\partial \underline{L}_2$ and $\partial \overline{L}_2$, respectively.

By Milgrom and Roberts (1990), the largest and the smallest rationalizable strategy each constitutes a symmetric equilibrium in $\Gamma_2(x_1^*)$. Hence, for those η_Δ for which $\underline{\mathbf{x}}(\eta_\Delta)$ lies in the interior of $L_1(x_1^*)$, the type $\underline{\mathbf{x}}(\eta_\Delta)$ must satisfy the indifference condition, $\tilde{U}_2 \left(\underline{\mathbf{x}}(\eta_\Delta), L_1(x_1^*), \underline{L}_2 \right) = 0$.[14] Similarly, $\tilde{U}_2 \left(\overline{\mathbf{x}}(\eta_\Delta), L_1(x_1^*), \overline{L}_2 \right) = 0$ if $\overline{\mathbf{x}}(\eta_\Delta)$ lies in the interior of $L_1(x_1^*)$.

Another property of the functions $\underline{\mathbf{x}}(\eta_\Delta)$, $\overline{\mathbf{x}}(\eta_\Delta)$ is that $\underline{\mathbf{x}}(\eta_\Delta) \geq \overline{\mathbf{x}}(\eta_\Delta)$ for all η_Δ because the sets $\underline{L}_2$, $\overline{L}_2$ are upper contour sets and $\underline{L}_2 \subseteq \overline{L}_2$. As the last property we note that functions $\underline{\mathbf{x}}(\eta_\Delta)$, $\overline{\mathbf{x}}(\eta_\Delta)$ are continuous: take η_0 and $\mathbf{x}_0 = \underline{\mathbf{x}}(\eta_0)$ (symmetrically for $\overline{\mathbf{x}}$) and consider a ball in X with radius r around $\mathbf{x}_0$. Then $\underline{\mathbf{x}}(\eta)$ lies in this ball whenever $|\eta - \eta_0| < r/2$. This is because $\mathbf{x}_0 + (0, r') \in \underline{L}_2$ and $\mathbf{x}_0 + (-\frac{3}{2}r', -\frac{1}{2}r') \notin \underline{L}_2$ when $r' < r/2$, and both these points lie in the ball.

Now we define function $\zeta : [\underline{\eta}_\Delta, \overline{\eta}_\Delta] \to [0, +\infty)$ as $\zeta(\eta_\Delta) = \underline{x}_1(\eta_\Delta) - \overline{x}_1(\eta_\Delta)$ where $\underline{x}_1$, $\overline{x}_1$ are the first coordinates of $\underline{\mathbf{x}}$, $\overline{\mathbf{x}}$. Value $\zeta(\eta_\Delta) \geq 0$ for all η_Δ because $\underline{x}_1(\eta_\Delta) \geq \overline{x}_1(\eta_\Delta)$. The function ζ is continuous and hence it attains a maximum

[14] Note that $\tilde{U}_2 \left(\underline{\mathbf{x}}(\eta_\Delta), L_1(x_1^*), \underline{L}_2 \right) \geq 0$ may be positive if $\underline{\mathbf{x}}(\eta_\Delta) \in \partial L_1(x_1^*)$.

on the compact set $[\underline{\eta}_\Delta, \overline{\eta}_\Delta]$ at some value η^*_Δ. The maximal value $\zeta(\eta^*_\Delta)$ is strictly positive if the boundaries $\partial\underline{L}_2$ and $\partial\overline{L}_2$ differ.

Let $T_d(S)$ be a translation operator that translates a set $S \in X$ by distance d in the direction of diagonal: $T_d(S) = \{\mathbf{x} \in X : \mathbf{x} - d \cdot (1,1) \in S\}$. Let us introduce

$$\begin{aligned}
\underline{U}(\mathbf{x}) &= \tilde{U}_2(\mathbf{x}, L_1(x_1^*), \underline{L}_2), \\
\overline{U}(\mathbf{x}) &= \tilde{U}_2(\mathbf{x}, L_1(x_1^*), \overline{L}_2), \\
\overline{U}'(\mathbf{x}) &= \tilde{U}_2\left(\mathbf{x}, L_1(x_1^*), T_{\zeta(\eta^*_\Delta)}(\overline{L}_2)\right).
\end{aligned}$$

By construction $T_{\zeta(\eta^*_\Delta)}(\overline{L}_2)$ is a subset of the closure of $\underline{L}_2$ and therefore by action monotonicity $\overline{U}'(\mathbf{x}) \leq \underline{U}(\mathbf{x})$ for all $\mathbf{x}$. Also, $\overline{U}'(\mathbf{x}) > \overline{U}\left(\mathbf{x} - \zeta(\eta^*_\Delta) \cdot (1,1)\right)$ because type $\mathbf{x}$ under $T_{\zeta(\eta^*_\Delta)}(\overline{L}_2)$ has identical belief about the aggregate action in round 2 as type $\mathbf{x} - \zeta(\eta^*_\Delta) \cdot (1,1)$ under $\overline{L}_2$, but the belief of the latter about (θ, l_1) is stochastically dominated (in the sense of the first order stochastic dominance) by the belief about (θ, l_1) of the former type; strict inequality holds because u_2 strictly increases in θ.

Finally, let us consider the type $\underline{\mathbf{x}}(\eta^*_\Delta) = \overline{\mathbf{x}}(\eta^*_\Delta) + \zeta(\eta^*_\Delta)(1,1)$. If the boundaries $\partial\underline{L}_2$ and $\partial\overline{L}_2$ differ then the type $\underline{\mathbf{x}}(\eta^*_\Delta)$ lies in the interior of $L_1(x_1^*)$ because $x_1^* \leq \overline{x}_1(\eta^*_\Delta) < \underline{x}_1(\eta^*_\Delta)$. Hence, the type $\underline{\mathbf{x}}(\eta^*_\Delta)$ satisfies the indifference condition under $\underline{L}_2$ and hence $\underline{U}(\underline{\mathbf{x}}(\eta^*_\Delta)) = 0$. On one hand, $\underline{U}'(\overline{\mathbf{x}}(\eta^*_\Delta)) \leq \underline{U}(\underline{\mathbf{x}}(\eta^*_\Delta)) = 0$, but on the other hand $\underline{U}'(\overline{\mathbf{x}}(\eta^*_\Delta)) > \overline{U}(\overline{\mathbf{x}}(\eta^*_\Delta)) \geq 0$ which establishes the contradiction. □

Proof of Lemma 4. We again let $\sigma = 1$ in the proof.

We obtain $m_{dyn}(x_1^*) > 0$ for sufficiently high x_1^* by the existence of the upper dominance region (by A3a), and $m_{dyn}(x_1^*) < 0$ for sufficiently low x_1^* by the existence of the lower dominance region (by A3a in combination with A3b).

Let us consider sets $\underline{L}_2^{(k)}(x_1)$ as defined for the continuation game $\Gamma_2(x_1)$ in the proof of Lemma 3 when all players use strategy with threshold x_1 in round 1. From the definition of $\underline{L}_2^{(1)}$, we have $\underline{L}_2^{(1)}(x_1') = T_{x_1'-x_1}\left(\underline{L}_2^{(1)}(x_1)\right)$ where $T_d(\cdot)$ is the translation operator defined in Proof of Lemma 3. By the strict state monotonicity A1,

$$\tilde{U}_2(\mathbf{x}, L_1, L_2) < \tilde{U}_2\left(\mathbf{x} + (d,d), T_d(L_1), T_d(L_2)\right) \tag{15}$$

for any $d > 0$. Using this monotonicity property iteratively, we get $\underline{L}_2^{(k)}(x_1') \supseteq T_{x_1'-x_1}\left(\underline{L}_2^{(k)}(x_1)\right)$ for any $x_1' > x_1$, and therefore $\underline{L}_2(x_1') \supseteq T_{x_1'-x_1}\left(\underline{L}_2(x_1)\right)$ for any $x_1' > x_1$.

Then, if $x_1' > x_1$, $D_t(x_1') > D_t(x_1)$ for $t = 1, 2$ because round 1 belief at signal x_1' about (θ, l_1, l_2) under strategy profile induced by the sets $L_1(x_1')$, $L_2(x_1')$ dominate

round 1 belief at signal x_1 about (θ, l_1, l_2) under $L_1(x_1)$, $L_2(x_1)$. The strict inequality follows from the strict state monotonicity A1. Therefore $m_{dyn}(x_1) = D_1(x_1) + D_2(x_1)$ is strictly increasing in x_1. □

Proof of Proposition 2. From the existence of the dominance regions in round 1 of the dynamic game[15], there exists $\overline{x}_1$ ($\underline{x}_1$) such that action 1 (0) is strictly dominant in round 1 for signals $x_1 > \overline{x}_1$ ($x_1 < \underline{x}_1$). Function $m_{dyn}(x_1)$ is strictly positive (negative) for $x_1 \geq \overline{x}_1$ ($x_1 \leq \underline{x}_1$).

Suppose action 1 is the strict best response in round t at type $\mathbf{x}$ under the symmetric profile consisting of the minimal rationalizable strategy $\underline{\mathbf{s}}$. Then 1 is the unique rationalizable action in round t at type $\mathbf{x}$. For this to hold, there must exist k (dependent on $\mathbf{x}$) such that action 1 is at $\mathbf{x}$ in round t the strict best response against $\underline{\mathbf{s}}^k$ where $\underline{\mathbf{s}}^k$ is the minimal strategy in S^k. This holds because $\underline{\mathbf{s}}^k$ converges pointwise to $\underline{\mathbf{s}}$, and therefore $\underline{\mathbf{s}}^k$ differs from $\underline{\mathbf{s}}$ on a arbitrarily small set of types for a sufficiently high k.

Suppose next that there exists $\overline{x}'_1$ such that $m_{dyn}(\overline{x}'_1) > 0$ and action 1 is the unique rationalizable action in round 1 for all $x_1 \geq \overline{x}'_1$. Then the expected payoff for playing 1 in round 1 on signal $\overline{x}'_1$ against the minimal rationalizable strategy $\underline{\mathbf{s}}$ is at least $m_{dyn}(\overline{x}'_1) > 0$. The payoff expectation is continuous in the signal, and hence there exists $\overline{x}''_1 < \overline{x}'_1$ such that action 1 is the strict best response in round 1 against $\underline{\mathbf{s}}$ at all signals $x_1 \geq \overline{x}''_1$. Hence action 1 is the unique rationalizable action in round 1 for all $x_1 \geq \overline{x}''_1$. Iterating this argument, action 1 is the unique rationalizable action in round 1 at all x_1 such that $m_{dyn}(x_1) > 0$. Symmetric argument establishes that action 0 is the unique rationalizable action in round 1 at all x_1 such that $m_{dyn}(x_1) < 0$.

We have established that each serially undominated strategy $\mathbf{s} = (s_1, s_2)$ prescribes to play according to the threshold strategy s_1 with the threshold x_1^{**} in round 1. Then s_2 must be the unique serially undominated in the continuation game $\Gamma_2(x_1^{**})$ by Lemma 3. □

Proof of Lemma 6. Before proceeding with the proof, let us introduce some additional notation. Let $\tilde{F}(z_1, z_2) = \Pr(\eta_1^i \geq z_1 \wedge \eta_2^i \geq z_2)$ denote the complementary cumulative distribution function of (η_1^i, η_2^i). If η_2^i and η_Δ^i are independent, the set H is a parallelogram and $\overline{\eta}_\Delta = -\underline{\eta}_\Delta = h_1 - h_2$. Further note that, as x_2^i is sufficient statistic, player i's decision in round 2 depends only on x_2^i and not on x_1^i. Thus, $s_2(x_2^i)$ is a monotone function $\mathbb{R} \to \{0, 1\}$ with a threshold denoted by x_2^*. Let us also denote $\eta^* = \frac{x_2^* - x_1}{\sigma}$.

[15]Upper dominance region is implied by A3a and lower dominance region by A3a and A3b.

Let us divide the proof into three cases depending on the value of x_1.

Case (a). If $u_2(x_1, 0, 0) > 0$, then $u_2(x_1 - \sigma h_1, 0, 0) > 0$ for σ that is small enough. Then, all types with $x_1^i \geq x_1$ strictly prefer to stay in the project in the continuation game $\Gamma_2(x_1, \sigma)$. Then, $\ell_1(\theta) = \ell_2(\theta)$ and so $m_{dyn}(x_1, \sigma) = m_{st}(x_1, \sigma)$ and $\lim_{\sigma \to 0^+} m_{dyn}(x_1, \sigma) = m_{st}^*(x_1) = \int_0^1 \Big(u_1(x_1, l, l) + u_2(x_1, l, l)\Big) dl$.

Case (b). Consider the case when $\int_0^1 u_2(x_1, 1, l_2) dl_2 < 0$. We show that then $\lim_{\sigma \to 0^+} m_{dyn}(x_1, \sigma) = \int_0^1 u_1(x_1, l_1, 0) dl_1$. First, the inequality $\int_0^1 u_2(x_1, 1, l_2) dl_2 < 0$ implies that $x_2^*(\sigma) \geq x_1 + \sigma(h_1 + h_2)$ when σ is sufficiently small. Suppose the opposite inequality $x_2^*(\sigma) < x_1 + \sigma(h_1 + h_2)$ holds. Then, for type $\mathbf{x}$ who observes signal x_2^* in round 2 we obtain

$$U_2(\mathbf{x}) < \int_0^1 u_2(x_1 + \sigma(h_1 + 2h_2), 1, l_2) dl_2, \tag{16}$$

The above inequality is based on three observations. First, a player with signal x_2^* in round 2 knows with certainty that the fundamental does not exceed the value $x_2^* + \sigma h_2 < x_1 + \sigma(h_1 + h_2) + \sigma h_2 = x_1 + \sigma(h_1 + 2h_2)$. Second, the investment level l_1 can be at most 1. Third, the second round belief of the player who observes x_2^* about l_2 is stochastically dominated by the uniform distribution on $[0, 1]$, as only fellow players who observe second round signal $x_2^i \geq x_2^*$ can possibly participate in the second stage.

The right-hand side of (16) is negative for sufficiently small σ, which establishes a contradiction because the type $\mathbf{x}$ (with signal x_2^* in round 2) weakly prefers to stay in the project. Thus, indeed $x_2^*(\sigma) \geq x_1 + \sigma(h_1 + h_2)$. This inequality implies that the types with $x_1^i = x_1$ do not reach action history 11 in $\Gamma_2(x_1, \sigma)$, and hence $D_2(x_1, \sigma) = 0$. Moreover, $\vartheta_1(l_1; x_1, \sigma) \leq x_1 + \sigma h_1$ for all $l_1 \in (0, 1)$. Therefore, $\lambda_2(l_1) \leq \ell_2(x_1 + \sigma h_1)$. But $\ell_2(x_1 + \sigma h_1) = 0$ because we established that players with the second signal at most $x_1 + \sigma(h_1 + h_2)$ do not invest in the continuation game $\Gamma_2(x_1, \sigma)$. Thus, $\lambda_2(l_1) = 0$ for all $l_1 \in (0, 1)$ and for sufficiently small σ.

Case (c). Consider the case when $u_2(x_1, 0, 0) \leq 0 \leq \int_0^1 u_2(x_1, 1, l_2) dl_2$. This case requires some additional notation. Let $a_1 = \frac{x_1 - \theta}{\sigma}$, $a_2 = \frac{x_2^* - \theta}{\sigma} = a_1 + \eta^*$. Note that $\ell_1(x_1 - \sigma a_1)$ depends only on a_1 and is independent of x_1, x_2^* and σ. Similarly $\ell_2(x_1 - \sigma a_1)$ depends only on a_1 and η^*. To see this, recall that $\ell_1(\theta) = \Pr(x_1^i \geq x_1 \mid \theta) = \Pr\left(\eta_1^i \geq \frac{x_1 - \theta}{\sigma} \mid \theta\right)$, and thus, $\ell_1(x_1 - \sigma a_1) = \Pr(\eta_1^i \geq a_1) = \tilde{F}(a_1, -h_2)$. Similarly, $\ell_2(x_1 - \sigma a_1) = \Pr(\eta_1^i \geq a_1 \wedge \eta_2^i \geq a_1 + \eta^*) = \tilde{F}(a_1, a_1 + \eta^*)$.

Player receiving the threshold signal x_2^* in round 2 must be indifferent between actions 0 and 1 in the continuation game $\Gamma_2(x_1, \sigma)$. In the above notation, and after

transformation $\theta = x_1 - \sigma(a_2 - \eta^*)$, the indifference condition can be written as

$$J(\eta, \sigma) = \int_{-h_2}^{h_2} u_2\big(x_1 - \sigma(a_2 - \eta),\, \tilde{F}(a_2 - \eta, -h_2),\, \tilde{F}(a_2 - \eta, a_2)\big) g_2(a_2)\, da_2 = 0. \quad (17)$$

Observe that for $\sigma > 0$: $J(\eta, \sigma)$ is strictly increasing and continuous in η and due to the existence of dominance regions, it attains both positive and negative values. Thus, for every $\sigma > 0$ there exists unique $\eta = \eta^*(\sigma)$ such that $J(\eta, \sigma) = 0$. For $\sigma = 0$: $J(\eta, 0)$ is strictly increasing in η as well, by strict monotonicity of u_2 in l_2. Therefore, the equation $J(\eta, 0) = 0$ has at most one solution.

Now, for $\eta = -(h_1 + h_2)$ and for all $a_2 \geq -h_2$, we have $\tilde{F}(a_2 - \eta, -h_2) = \tilde{F}(a_2 - \eta, a_2) = 0$. Moreover, for $\eta = h_1 + h_2$ and $a_2 \leq h_2$, we have $\tilde{F}(a_2 - \eta, -h_2) = 1$ and $\tilde{F}(a_2 - \eta, a_2) = \Pr(\eta_2^i \geq a_2) = 1 - F_2(a_2)$, where F_2 is the cumulative distribution function of η_2^i. Summing up,

$$J\big(-(h_1 + h_2), 0\big) = u_2(x_1, 0, 0) \leq 0 \leq \int_0^1 u_2(x_1, 1, l_2) dl_2 = J(h_1 + h_2, 0).$$

Therefore, the equation $J(\eta, 0) = 0$ indeed has a unique solution and that solution lies in the interval $[-(h_1 + h_2),\, h_1 + h_2]$; denote it η^{**}. It follows that $\eta^*(\sigma) \to \eta^{**}$ as $\sigma \to 0^+$ and η^{**} is continuous and decreasing in x_1, for x_1 satisfying $u_2(x_1, 0, 0) \leq 0 \leq \int_0^1 u_2(x_1, 1, l_2) dl_2$.

Let us now study $\lim_{\sigma \to 0^+} D_t(x_1, \sigma)$, $t = 1, 2$, for the range of x_1 considered in (c). In order to study the limit $\lim_{\sigma \to 0^+} D_1(x_1, \sigma)$, let us first denote $\tilde{F}_1^{-1}(l)$ the inverse function to $\tilde{F}(z, -h_2)$ with respect to z. Then, $\vartheta_1(l_1) = x_1 - \sigma \tilde{F}_1^{-1}(l_1)$ and $\lambda_2(l_1) = \tilde{F}\big(\tilde{F}_1^{-1}(l_1),\, \tilde{F}_1^{-1}(l_1) + \eta^*(\sigma)\big)$. Both are continuous in x_1 and σ, and in the limit $\sigma \to 0^+$, we have $\vartheta_1(l_1) \to x_1$ and $\lambda_2(l_1) \to \lambda_2^*(l_1; x_1) = \tilde{F}\big(\tilde{F}_1^{-1}(l_1),\, \tilde{F}_1^{-1}(l_1) + \eta^{**}(x_1)\big)$. Note that the latter is non-increasing in η^{**}. Thus, the limit

$$\lim_{\sigma \to 0^+} D_1(x_1, \sigma) = \lim_{\sigma \to 0^+} \int_0^1 u_1(x_1 - \sigma \tilde{F}_1^{-1}(l_1), l_1, \lambda_2(l_1)) dl_1 = \int_0^1 u_1(x_1, l_1, \lambda_2^*(l_1; x_1)) dl_1$$

exists, is continuous in x_1, and strictly increasing in x_1. The monotonicity is strict by the assumption of the strict state monotonicity A1.

Similarly, if we denote $\tilde{F}_2^{-1}(l, \eta)$ the inverse function to $\tilde{F}(z, z + \eta)$ with respect to z, we obtain $\vartheta_2(l_2) = x_1 - \sigma \tilde{F}_2^{-1}(l_2, \eta)$ and $\lambda_1(l_2) = \tilde{F}\big(\tilde{F}_2^{-1}(l_2, \eta^*(\sigma)),\, -h_2\big)$. Again, both are continuous in x_1 and σ and in the limit $\sigma \to 0^+$ we obtain $\vartheta_2(l_2) \to x_1$ and $\lambda_1(l_2) = \tilde{F}\big(\tilde{F}_2^{-1}(l_2, \eta^{**}),\, -h_2\big)$ for x_1 considered in the case (c). Therefore also the limit $D_2^*(x_1) = \lim_{\sigma \to 0^+} D_2(x_1, \sigma)$ exists, and it is continuous in x_1. Moreover $D_2^*(x_1)$

is non-decreasing because we established in the Proof of Lemma 4 that $D_2(x_1, \sigma)$ increases in x_1 for each σ. Therefore the sum $m^*_{dyn}(x_1) = D^*_1(x_1) + D^*_2(x_1)$ exists, is continuous in x_1, and is strictly increasing in x_1 for x_1 in the range considered in case (c).

We found that the limit $m^*_{dyn}(x_1)$ is continuous in x_1 for ranges of x_1 considered in all the three cases (a), (b), and (c). Moreover, in case (c), $\eta^{**} = -(h_1 + h_2)$ if $u_2(x_1, 0, 0) = 0$ and $\eta^{**} = h_1 + h_2$ if $\int_0^1 u_2(x_1, 1, l_2)dl_2 = 0$, which implies that $m^*_{dyn}(x_1)$ continuously connects at the boundaries in between the cases (a) and (c), and in between cases (c) and (b). □

Proof of Proposition 5. If $m^*_{dyn}(\theta^*) > 0$ then $m^*_{dyn}(\theta)$ is positive on some δ-neighborhood of θ^*. Together with the monotonicity of $m_{dyn}(\theta, \sigma)$ with respect to θ it implies that $m_{dyn}(\theta, \sigma)$ is positive in the δ-neighborhood of θ^* for $\sigma < \overline{\sigma}$, for some $\overline{\sigma} > 0$. For sufficiently small $\overline{\sigma}$ all players receive signals x^i_1 above $\theta^* - \delta$ in round 1 of $\Gamma_{dyn}(\sigma)$ whenever Nature draws fundamental θ^* and $\sigma < \overline{\sigma}$. Then, by Proposition 2, action 1 is the unique rationalizable action for all players in round 1 of $\Gamma_{dyn}(\sigma)$. The symmetric argument implies that if $m^*_{dyn}(\theta^*) < 0$ then action 0 is selected.

Function $m^*_{dyn}(\theta)$ has a unique root at which none of the actions is selected in round 1 of Γ_{dyn}. Hence the reverse implications hold as well. □

Let us introduce and remind notation used in the proofs that follow. Let $\overline{x}_2(x_1; \sigma) = x_1 + \sigma(h_1 + h_2)$; If player receives $\overline{x}_2(x_1; \sigma)$ in round 2, then she knows that all the fellow players have received signals at least x_1 in round 1. Let $L_2(x_1; \sigma)$ be the set of types who reach action history 11 if they play rationalizable strategy in the continuation game $\Gamma_2(x_1; \sigma)$; let $\mathbf{x}(\eta; \sigma)$ be the intersection of $\partial L_2(x_1; \sigma)$ with the line $x_2 - x_1 = \eta$, where $\eta \in [\underline{\eta}_\Delta, \overline{\eta}_\Delta]$. Let $x_2(\eta; \sigma)$ be the second coordinate of $\mathbf{x}(\eta; \sigma)$. We will pay attention to the round 2 signal $x_2(\overline{\eta}_\Delta; \sigma)$ which has the following property implied by monotonicity of s_2: all types $(x'_1, x_2) \in X$ such that $x'_1 \geq x_1$ and $x_2 > x_2(\overline{\eta}_\Delta; \sigma)$ reach action history 11 in $\Gamma_2(x_1; \sigma)$.

Lemma 7. *If $\int_0^1 u_2(x_1, 1, l_2)dl_2 > 0$, then there exists $\overline{\sigma} > 0$ such that $x_2(\overline{\eta}_\Delta; \sigma) < \overline{x}_2(x_1; \sigma)$ for all $\sigma < \overline{\sigma}$ (the notation is introduced in the paragraph above).*

Proof of Lemma 7. Suppose the contrary. Then there exists a sequence $\sigma_k \to 0^+$ such that $x_2(\overline{\eta}_\Delta; \sigma_k) \geq \overline{x}_2(x_1; \sigma_k)$ for all k.

Let us explore beliefs of the type $\mathbf{x}(\overline{\eta}_\Delta; \sigma_k)$. First, she knows that $\theta \geq x_1 - \sigma_k h_1$. Second, she knows that $l_1 = 1$ because only fellow players' signals at least x_1 are compatible with her second signal which is $\overline{x}_2(x_1; \sigma_k)$ or larger. Third, her belief about

l_2 stochastically dominates uniform distribution on $[0,1]$. This is because all players with $x_2^i > x_2(\overline{\eta}_\Delta;\sigma_k)$ invest in both rounds and a player receiving $x_2^i = x_2(\overline{\eta}_\Delta;\sigma_k)$ has uniform belief about the proportion of players with the second signal above $x_2(\overline{\eta}_\Delta;\sigma_k)$. Using all the three observations we get

$$U_2\left(\mathbf{x}(\overline{\eta}_\Delta;\sigma_k)\right) \geq \int_0^1 u_2(x_1 - \sigma_k h_1, 1, l_2) dl_2. \tag{18}$$

The right-hand side of (18) is positive for sufficiently small σ_k because $\int_0^1 u_2(x_1, 1, l_2) dl_2$ is continuous in x_1.

This contradicts the indifference condition $U_2\left(\mathbf{x}(\overline{\eta}_\Delta;\sigma_k)\right) = 0$. The indifference condition must hold because the type $\mathbf{x}(\overline{\eta}_\Delta;\sigma_k)$ is in the interior of $L_1(x_1)$ by the assumption that $x_2(\overline{\eta}_\Delta;\sigma_k) \geq \overline{x}_2(x_1;\sigma_k)$. □

Proof of Proposition 6. Claim (i) Let $\underline{x}_2^*(\sigma)$ be the minimal second round signal at which a player can reach action history 11 under the (essentially) unique rationalizable strategy (s_1, s_2) in $\Gamma_{dyn}(\sigma)$:

$$\underline{x}_2^*(\sigma) = \min\{x_2 : \exists x_1 \text{ s.t. } (x_1, x_2) \in X, \text{ and } s_1(x_1, x_2) = s_2(x_1, x_2) = 1\}.$$

(The minimum is attained by our convention that L_1 and L_2 are closed sets.)

Suppose claim (i) does not hold. Then there exists a sequence $\sigma_k \to 0^+$, such that $\underline{x}_2^*(\sigma_k) \leq \theta^* + \sigma_k h_2$ so that some players choose action 1 in the both rounds when Nature draws θ^*. Let us examine payoff expectation of the type receiving the signal $\underline{x}_2^*(\sigma_k)$ in round 2. In particular, let $(x_1, \underline{x}_2^*(\sigma_k)) \in X$ be a type that reaches action history 11. The following inequality holds:

$$U_2(x_1, \underline{x}_2^*(\sigma_k); \sigma_k) \leq \int_0^1 u_2(\theta^* + 2\sigma_k h_2, 1, l_2) dl_2. \tag{19}$$

This holds by the following three arguments: First, a player with round 2 signal $\underline{x}_2^*(\sigma_k)$ knows that the fundamental lies below $\underline{x}_2^*(\sigma_k) + \sigma_k h_2 \leq \theta^* + 2\sigma_k h_2$. Second, investment level l_1 cannot exceed one. Third, only types with $x_2^i \geq \underline{x}_2^*(\sigma_k)$ can possibly invest in both rounds, and the type with the second signal equal to $\underline{x}_2^*(\sigma_k)$ has uniform belief about proportion of players with the second signal at least $\underline{x}_2^*(\sigma_k)$.

The right-hand side of (19) is negative for sufficiently small σ because $\int_0^1 u_2(\theta^*, 1, l_2) dl_2 < 0$ and $\int_0^1 u_2(\theta, 1, l_2) dl_2$ is continuous in θ. This contradicts the assumption that the type $(x_1, \underline{x}_2^*(\sigma_k))$ prefers to invest in round 2.

Claim (ii) (See page 44 for notation.) If action 1 is selected at θ^* in Γ_{dyn} then

$m^*_{dyn}(\theta^*) > 0$ by Proposition 5 and, because m^*_{dyn} is continuous, $m^*_{dyn}(\theta^* - \delta) > 0$ for some $\delta > 0$. Again by Proposition 5, action 1 is selected at $\theta^* - \delta$ in round 1 of Γ_{dyn}. Therefore, for $\sigma < \overline{\sigma}$ players invest in round 1 of $\Gamma_{dyn}(\sigma)$ for all signals $x_1^i \geq \theta^* - \delta$, and we can apply Lemma 7: choosing $\delta > 0$ small enough so that $\int_0^1 u_2(\theta^* - \delta, 1, l_2) > 0$, we have $s_2\left(\theta^* - \delta, \overline{x}_2(\theta^* - \delta, \sigma)\right) = 1$ in the unique equilibrium of $\Gamma_{dyn}(\sigma)$.

If Nature draws fundamental θ^* then types of all players exceed $(\theta^* - \sigma h_1, \theta^* - \sigma h_2)$. We can choose $\overline{\sigma}'$ small enough so that for $\sigma < \overline{\sigma}'$, $(\theta^* - \sigma h_1, \theta^* - \sigma h_2) > (\theta^* - \delta, \overline{x}_2(\theta^* - \delta, \sigma))$ and so that all players invest in the both rounds whenever Nature draws fundamental θ^*. □

Supplement to Proof of Proposition 8. If $\int_0^1 u_2(x_1, 1, l_2) dl_2 > 0$ and payoffs do not exhibit forward spillovers then

$$m^*_{dyn}(x_1) \leq m^*_{st}(x_1) \tag{20}$$

by the following argument: By Lemma 7 players reach action history 11 in $\Gamma_2(x_1, \sigma)$ whenever their round 2 signal exceeds $\overline{x}_2(x_1, \sigma)$. Hence if Nature draws $\theta > \overline{x}_2(x_1, \sigma) + \sigma h_2 = x_1 + \sigma(h_1 + 2h_2)$ all players reach action history 11 in the continuation game $\Gamma_2(x_1, \sigma)$. Then $\vartheta_2(l_2, x_1, \sigma) \leq x_1 + \sigma(h_1 + 2h_2)$ for all $l_2 \in (0, 1)$ where $\vartheta_2(l_2, x_1, \sigma)$ is the inverse function to the stage 2 investment profile $\ell_2(\theta)$ induced by the rationalizable strategy in the continuation game of $\Gamma_2(x_1, \sigma)$. Therefore we have

$$\begin{aligned}\int_0^1 u_2(\vartheta_2(l_2, x_1, \sigma), l_2) dl_2 &\leq \int_0^1 u_2(x_1 + \sigma(h_1 + 2h_2), l_2) dl_2 \leq \\ &\leq \int_0^1 u_2(\vartheta_1(l_2, x_1 + 2\sigma(h_1 + h_2), \sigma), l_2) dl_2.\end{aligned}$$

In the first inequality, we used $x_1 - \sigma h_1 \leq \vartheta_1(l, x_1, \sigma)$ for all $l \in (0, 1)$. We also have (trivially)

$$\begin{aligned}\int_0^1 u_1(\vartheta_1(l_1, x_1, \sigma), l_1, \lambda_2(l_1)) dl_1 &\leq \int_0^1 u_1(\vartheta_1(l_1, x_1, \sigma), l_1, l_1) dl_1 \leq \\ &\leq \int_0^1 u_1(\vartheta_1(l_1, x_1 + 2\sigma(h_1 + h_2), \sigma), l_1, l_1) dl_1.\end{aligned}$$

Summing the two inequalities we get $m_{dyn}(x_1, \sigma) \leq m_{st}(x_1 + 2\sigma(h_1 + h_2), \sigma)$ and, as $\sigma \to 0$, the inequality (20).

□

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