Randomly Available Outside Options in Bargaining*

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Abstract

We consider an extension of the standard Rubinstein model where both players are randomly allowed to leave the negotiation after a rejection, in which case they obtain a payoff of known value. We show that, when the value of the outside opportunities is of intermediate size, there exist a continuum of subgame-perfect equilibrium outcomes, including some with delayed agreements. Considering outside opportunities of significant value, we prove that efficient delays arise caused by the bargainers' aspirations in waiting for their outside option rather than by threats. Moreover, if taking the outside option decreases the probability that the opponent receives an outside option in the future, then it is possible that exactly two equilibrium payoffs coexist. In this latter case, inefficiencies may be created by agreeing too early.

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1. Introduction

We study bilateral negotiations where both parties have outside options. We give a complete characterization of the set of subgame-perfect equilibria (SPE) under different scenarios. First, we review results under the assumption that the options are always available and then we extend the analysis to the case where options arise at random. With fixed known options, when their value is small (even zero) there exist a continuum of subgame-perfect equilibrium outcomes, including some with significant delay. We show that qualitatively similar results hold for random options of small value. On the other hand, for options of significant value delay may arise caused by the bargainers' aspirations in waiting for their outside option to become available rather than by threats. Moreover, if taking the outside option decreases the probability that the opponent receives an outside option in the future, then it is possible that two stationary (!) equilibria coexist.

In a strategic analysis, it is most important whether players can obtain their "disagreement payoff" voluntarily or only in case of an exogenous breakdown of negotiations. That is why outside options cannot be regarded as a disagreement point is used in the axiomatic approach, and their effects have to be studied explicitly in the non-cooperative framework. All the initial work on incorporating outside options to non-cooperative bargaining theory assumed that if a player had the opportunity to take up an outside option she could do so only after she rejected the proposal of her opponent. In this case, if this responder's option is larger than her equilibrium share in the game without the possibility to opt out, there is a unique subgame-perfect equilibrium, in which the player who has the option obtains a payoff equal to the value of her option. Otherwise, the option has no effect on the outcome (the Outside Option Principle, see Shaked and Sutton [1984]). It was Shaked [1994] who recognized that the assumption that only the responder can opt out is not without loss of generality. In fact, if it is the proposer who can threaten to take his outside option, the strategic consequences are markedly different. This is so, because as long as his threat is credible (that is, his outside option exceeds his continuation value in case he does not leave the game) he can appropriate the entire surplus, by making a take-it-or-leave-it offer. As Shaked shows (see also Osborne and Rubinstein [1990]), when one of the players may take up his option in the periods where he is the proposer, there exist a range of outside options (strictly between zero and one) for which there exist multiple equilibria.

In a bargaining situation which is symmetric -- in the sense that both parties can make proposals -- we find it more realistic that both players be offered to voluntarily break off negotiations. Moreover, considering uncertainty about the outside options is a natural extension to the literature that deserves attention. We have shown elsewhere (Ponsatí and Sákovics [1998]) that moving from one-sided to two-sided outside options has important effects. For example, if players are allowed to walk away from negotiations (and commit to not coming back) then, even if they have no valuable outside option, a continuum of equilibria exist, and as the players become patient any division is possible in equilibrium. In this paper we combine that result with uncertainty about the availability of options, yielding a more complete analysis.

When options come and go, the question of their availability becomes an issue to be further clarified. For example, when a player without an option currently available is left without a bargaining partner because her opponent opts out, it is not necessarily the case that the player herself is left with a zero expected payoff, since she can still wait for her option to materialize in the future. On the other hand, we assume away the possibility that a player leaves the negotiations in exchange for a future outside option. We do this for expositional clarity, and based on the fact waiting for the option to appear without breaking up the negotiation is weakly dominant.

In our model, there is a stationary stochastic process which controls which combination of the players has an option available, in each period of an alternating offer bargaining game. Thus, bargaining takes place in four different states,[1] (x, x), (ø, x), (x, ø) and (ø, ø). In each of these states the equilibrium behavior of the players is different, reflecting the fact that they have different actions available to them. Consequently, unlike in the usual bargaining models where the issue is simply whether there will be agreement and if yes, under what terms, in the present model there is a wealth of qualitatively different equilibrium configurations. The 'types' of equilibria are distinguished according to whether the bargainers agree (A), take their option(s) (O) or wait for better future possibilities (D) in the four states. Of course, not all the 81 possible combinations can arise in equilibrium. However, as shown in Figure 1, spanning the parameter-space we can find seven different types of equilibria. The complexities of the equilibrium set are not circumscribed to their differing types, though. Within each type we can further distinguish qualitatively different equilibria (involving agreement), depending on the possible credibility configurations of the threats used to pin down an agreement.

	4.1	4.3a	4.3b	4.3b	4.3b	4.2	4.2
ø ø	A	A	D	D	D	A	D
ø x	A	A	D	D/ A	A	O	O
x ø	A	A	A	A	A	O	O
x x	A	O	O	O	O	O	O

Figure 1
The different actions in the states and the subsections that describe them

1 x denotes an available option, ø the absence of it. The first sign of the tuple refers to the proposer of the period.

It happens to be the case that the most useful way to span the parameter-space is centering our attention on the size of the options, x. When the sum of the two options is less than the potential gains from trade (normalized to one), we are in the usual type of equilibrium, since the options will never be taken (and therefore waiting will not be optimal either). This type of equilibria are characterized in Proposition 4.1. Observe, however, that the case of large outside options, i.e. $2x > 1$, is no longer trivial. It is true that in state (x, x) no agreement can give the players as much as the sum of their options and thus they must necessarily opt out.[2] Nevertheless, in the rest of the states the equilibrium behavior is not pinned down by this condition. Whether the options are taken in the states where only one of the players has them available depends on a different inequality. In this case we need to compare the sum of the payoffs in case one of them takes the option with the available gains from trade. Let y denote the expected gains of a player without an option when the opponent leaves to take the option. Then there are gains from trade provided that $x + y < 1$. When this inequality is not satisfied, in all states but (ø, ø) the options will be taken and thus the only remaining question is whether in that state they get to an agreement or prefer to wait for the options to become available (see Proposition 4.2). In the in between case, when $x + y < 1 < 2x$, the options will only be taken in state (x, x). If x is relatively low, there will be agreement in the rest of the states (see Proposition 4.3a) if not there will be delay in some of them (see Proposition 4.3b). These delays, caused by the players aspiration in waiting for very attractive outside options, are efficient. Equilibria do not always yield the delays that are appropriate to ensure efficiency, though. In fact, if the probability of future outside opportunities decreases after players see their opponent take the option, then it is possible that exactly two equilibrium payoffs coexist: one involves delay in both states (ø, ø) and (ø, x), the other involves delay only in state (ø, ø). In this latter case, inefficiencies may be created by agreeing too early.

Models of strategic bargaining with symmetric but imperfect information are very scarce. Wolinsky [1987] presents a model of bargaining where players enjoy symmetric uncertain outside opportunities: After a proposal/response round ends in rejection, both players may search for outside opportunities. A player that finds an outside option may take it, or leave it to return to another proposal/response round. Wolinsky's bargaining model is richer than ours in its consideration of the search for outside opportunities. Whereas in our model outside options simply fall from heaven, in Wolinsky's model players actively search for them, they decide the intensity of their search, and pay its cost. On the other hand, his bargaining procedure is simpler than ours in the strategic role played by outside options: since players do not have their outside options available during the proposal/response rounds these opportunities never interfere with the offer/response decisions of the players, and thus they are not used as a threat in bargaining. Vislie [1988] adds uncertainty about the presence of a second potential worker in the model of Shaked and Sutton [1984] and derives the corresponding unique equilibrium. Avery and

[2] Note that, whenever it is preferred to take the option to immediate agreement, it is also preferred to waiting.

Zemsky [1994a] consider a model in which exogenous shocks affect the gains from trade. The outcome of each shock is revealed after the proposer made her offer but before the responder's move. Consequently, the responder can use this "private" information to reduce the proposer's "first mover" advantage. In equilibrium, the first proposal is not accepted for all possible realizations of the shock and therefore there will be delay with positive probability before agreement occurs. This inefficiency in turn, may cause multiple equilibria, and hence deterministic delay, according to the Money Burning Principle (see, Avery and Zemsky [1994b]). Finally, Merlo and Wilson [1995,1998] also analyze a stochastic model of sequential bargaining, although they only consider the variability of the available surplus and do not allow for (random) outside options. They also obtain equilibria where the players disagree in some states of the world because they both expect to improve their shares in the future.

The paper is organized as follows. Section 2 presents the model. Sections 3 and 4 present the results for fixed and random outside options, respectively. Some illustrative examples are reviewed in Section 5, and Section 6 concludes.

2. The model

Player 1 and Player 2 are bargaining over the division of a fixed surplus, normalized to one, in an environment where outside opportunities arise at random during the process of bargaining. We assume that the existence of outside opportunities in the future is uncertain while the value of the options, whenever they are available, is common knowledge and equal to $x_i \geq 0$ $i = 1,2$. Time is running in discrete, equidistant periods, numbered by the natural numbers. At each t, the game is in one of four possible states: neither player has an option, state (ø, ø); only the responder has an option, state (ø, x); only the proposer has an option, state (x, ø); and both players have an option, state (x, x). The random variable that governs outside opportunities is the same for each period, while its realizations are independent across periods. Thus, Prob{both players have the outside option at t} = α, Prob{only Player i has the option at t} = β_i, and Prob{neither player has the option at t} = μ, $\alpha + \beta_1 + \beta_2 + \mu = 1$, for all t as long as no player takes the outside option. The situation where both players have permanent options arises when $\alpha = 1$. The games without options (Rubinstein [1982]) and with a permanent option for one of the players (Shaked [1994]) arise when $\mu = 1$ and $\beta_1 = 1$, respectively.

The players discount future payoffs via the (common) discount factor δ. In even (odd) periods Player 1 (Player 2) makes an offer which the other party may accept, thus terminating the game with agreement at the proposed shares, or upon rejection, either of the two parties may take their outside option, provided that it is available. In this case, the opponent’s payoff is the outside option if it is available or the discounted expected value of obtaining it later, otherwise.

We denote this second parameter by $y_i = \gamma_i x_i$, with $\gamma_i = p_i \sum_{n=0}^{\infty} (1-p_i)^n \delta^{n+1} = \frac{p_i \delta}{1-(1-p_i)\delta}$, where p_i denotes the probability that the option is available for i in any given period after j has taken the option. In principle, the outside opportunities available to i after j takes her option may remain constant, but they may also increase or decrease. Thus, p_i may be greater, smaller or equal to $\alpha + \beta_i$. If the offer is rejected but neither player opts out then bargaining goes on to the following round. Observe that we make no assumptions on whether the availability of outside options is correlated or uncorrelated between players. [3] On the other hand, for simplicity, we ignore the possibility of serially correlated options and we impose that the duration of options and offers to coincide.

3. Permanent, two-sided outside options

A detailed analysis of the game with fixed and known options, i.e. $\alpha = 1$, is in Ponsatí and Sákovics [1998]. Here we only present a brief discussion of the results. Observe that, if the sum of the two options is greater than the surplus to be divided, the unique Nash equilibrium outcome is that players take their outside options in period zero. Basically, in such a case the negotiation does not even begin, since in fact there are no "gains from trade." For the rest of this section we assume that $x_1 + x_2 \leq 1$.

The key observation that drives the results in the game with $\alpha = 1$ is that, when both players can opt out after a rejection, the proposer can always keep the responder's share down to the value of her outside option, *independent* of the size of the options. We show this via the following lemma:

Lemma 1 (Ponsatí and Sákovics [1998]) For any $0 \leq x_1 \leq 1\text{-}x_2 \leq 1$, immediate agreement at $(1\text{-}x_2, x_2)$ is an outcome that can be supported by a subgame-perfect equilibrium.

Proof: Consider the following strategies: if Player i is the proposer he always asks for $1\text{-}x_j$; the responder accepts any proposal that is not worse than the (candidate) equilibrium proposal; if the proposer asks for more, then the responder rejects and takes her outside option; if the responder does not accept a proposal the proposer opts out. It is straightforward to verify that these strategies constitute a subgame-perfect equilibrium. Q.E.D.

The intuition behind this result is easy to understand. Given that tomorrow's proposer is assumed to have a large bargaining power, today's proposer is to expect little tomorrow and thus his threat to take his option today is credible. When only one of the players has an option, such a threat is not credible for low option values, since today's proposer can obtain a decent payoff

[3] With perfectly correlated options, we have $\beta = 0$ and $\mu = 1\text{-}\alpha$. If the options are independent and they appear with probability p_i for player i, then we have $\beta_i = p_i (1\text{-} p_j)$, $\alpha = p_1 p_2$ and $\mu = (1\text{-}p_1)(1\text{-}p_2)$.

even when he is responder. This is so, because next period's proposer does not have an outside option and therefore the responder has some bargaining power, since she controls whether payoffs are to be discounted. When both players can opt out when they are proposers, the proposers can commit to opting out, thus depriving the responders of all their bargaining power -- except for the one given to them by the outside option principle -- and in turn making each other's commitment to opt out credible. Thus, the result in Lemma 1 depends exclusively on the possibility for both players to opt out when they are proposers, the responders' opting out only serves as "damage control," guaranteeing a minimum payoff to the second mover.

If the options of the players are of sufficiently high value then they must take them (upon a hypothetical rejection) in equilibrium and thus the equilibrium of Lemma 1 is the unique SPE. Otherwise, we have multiple equilibria, which are sustained by a threat to switch to an equilibrium which makes the proposer indifferent between opting out and continuing.[4] Proposition 1 gives a complete characterization of the equilibria.

Proposition 3.1 (Ponsatí and Sákovics [1998] , with $\delta_2=\delta_1$)

i) If $x_1 \leq \delta^2(1- x_2/\delta)$ and $x_2 \leq \delta^2(1- x_1/\delta)$, all immediate efficient agreements that give Player 1 a payoff in $[1-\delta(1-x_1/\delta), 1-x_2]$ can be supported as subgame-perfect equilibrium outcomes. Moreover , for any period t > 0, if the interval $[(1-\delta (1-x_1/\delta))\delta^{-t}, 1 - (1-\delta(1-x_2/\delta))\delta^{(1-t)}]$ is non-empty, delayed agreements that give Player 1 a share in this interval and the remainder to Player 2, can also be supported as subgame-perfect equilibrium outcomes.

ii) Otherwise the equilibrium in Lemma 1 is the unique SPE.

Proof: See Ponsatí and Sákovics[1998].

4. Randomly available outside options

In this section we characterize ex-ante expected SPE payoffs when options appear randomly. For simplicity, we focus on the symmetric case where $x_1 = x_2 = x$ and $\beta_1 = \beta_2 = \beta$. Thus, we will characterize the intervals of ex-ante expected SPE payoffs of the first proposer and responder, that is $[\underline{v}_P, \bar{v}_P]$ and $[\underline{v}_R, \bar{v}_R]$, respectively. Let $\underline{a}$ = inf{a, such that a is the SPE payoff of the proposer in a subgame starting in state (x, x)} and $\bar{a}$ = sup{a, such that a is the SPE payoff of the proposer in a subgame starting in state (x, x)}. Defining $(\underline{b}, \bar{b})$, $(\underline{c}, \bar{c})$ and $(\underline{d}, \bar{d})$ analogously for states (x, ø), (ø, x) and (ø, ø), respectively we have that

$$\underline{v}_P \equiv \alpha\, \underline{a} + \beta(\underline{b} + \underline{c}) + \mu\, \underline{d} ,$$

[4] Note that we cannot use a more extreme threat (like holding the proposer down to his outside option next period), since then he would simply opt out now. We are grateful to Margarida Corominas for this observation.

and

$$\bar{v}_P \equiv \alpha\bar{a} + \beta(\bar{b} + \bar{c}) + \mu\bar{d}.$$

The interval of payoffs to the responder $[\underline{v}_R, \bar{v}_R]$, can be defined analogously.

4.1 When options are only used as threats

Let us first consider the case of relatively small, $x < {}^1/_2$, outside options. Now the sum of the payoffs of the players cannot exceed one in any state and in any equilibrium. Consequently, all efficient[5] equilibrium payoffs can be implemented by immediate agreement in all states, implying that the responder always earns one minus the proposer's share (therefore, in this subsection we drop the sub indices in v_P and v_R, and we write v to denote a generic SPE payoff to the proposers in the first period, and we let $\underline{v}$ and $\bar{v}$ to denote the bounds on the interval such payoffs). The exact values of $\underline{v}$ and $\bar{v}$, depend on the strategic capabilities of each player in each state, that in turn, depend on whether threats to take the option in absence of an agreement are credible or not. This credibility is determined by the relative size of the option with respect to the continuation values, that is, $\delta(1\text{-}v)$ for the proposer and δv for the responder. If x is larger than these values then opting out is a credible threat, otherwise not. The full characterization of the set of SPE, requires that we completely explore the implications of each of the nine possible credibility configurations:[6]

Case I: $x > \delta(1\text{-}\underline{v})$ and $x > \delta\bar{v}$.

Case IIa: $\delta(1\text{-}\bar{v}) \leq x \leq \delta(1\text{-}\underline{v})$ and $\delta\underline{v} \leq x \leq \delta\bar{v}$.
Case IIb: $x < \delta(1\text{-}\bar{v})$ and $\delta\underline{v} \leq x \leq \delta\bar{v}$.
Case IIc: $\delta(1\text{-}\bar{v}) \leq x \leq \delta(1\text{-}\underline{v})$ and $x < \delta\underline{v}$.
Case IId: $x < \delta(1\text{-}\bar{v})$ and $x < \delta\underline{v}$.

Case IIIa: $x < \delta(1\text{-}\bar{v})$ and $x > \delta\bar{v}$.
Case IIIb: $\delta(1\text{-}\bar{v}) \leq x \leq \delta(1\text{-}\underline{v})$ and $x > \delta\bar{v}$.

Case IVa: $x > \delta(1\text{-}\underline{v})$ and $x < \delta\underline{v}$.
Case IVb: $x > \delta(1\text{-}\underline{v})$ and $\delta\underline{v} \leq x \leq \delta\bar{v}$.

Since this exploration is tedious and involved, we present it in the Appendix. Its result is summed up in the following Proposition:

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[5] We refer to inefficient equilibria at the end of Section 4.
[6] In Case I both players strictly prefer to take their options, whenever they can. In Case II neither of them has such strict preference. In Case III only the responder, while in Case IV only the proposer wants to take her option.

Proposition 4.1 *Let* $x < {}^1/_2$, *then for each parameter configuration, the set of SPE involving immediate agreement is fully characterized as follows:*[7]

i) if $x \quad [0, \min\{\underline{x}_{IIa} = \dfrac{\delta(1-\delta)}{1-\delta^2\left(1-\beta(1-\gamma)\right)}, \underline{x}_{IIc} = \dfrac{\delta^2(1-\delta)(\beta+\mu)}{1-\delta^2(\beta+\mu)-\delta(\alpha+\beta\gamma)}\}),$

then $\underline{v} = \bar{v} = \dfrac{1}{1+\delta}$,

ii) if $x \quad [\underline{x}_{IIc}, \min\{\underline{x}_{IIa}, \bar{x}_{IIc} = \dfrac{\delta^2(1-\delta(\beta+\mu))}{1-\delta^2(\beta(1-\gamma)+\mu-\alpha)}\}],$

then $[\underline{v}(x), \bar{v}(x)] = \left[\dfrac{1-\delta\left(1-(\alpha+\beta\gamma)x\right)}{1-\delta^2(\beta+\mu)}, \dfrac{1-(\alpha+\beta\gamma)x-\delta(\beta+\mu)}{1-\delta^2(\beta+\mu)}\right]$,

iii) if $x \quad (\bar{x}_{IIc}, \underline{x}_{IIa}),$ *then* $\underline{v}(x) = \bar{v}(x) = \dfrac{1-(\alpha+\beta\gamma)x}{1+\delta(\beta+\mu)}$,

iv) if $x \quad [\underline{x}_{IIa}, \min\{\bar{x}_{IIa} = \dfrac{\delta^2(1-\delta\mu)}{1+\delta^2\left(\alpha+\beta(1+\gamma)-\mu\right)}, {}^1/_2\}],$

then $[\underline{v}(x), \bar{v}(x)] = \left[\dfrac{1-\delta\left(1-(\alpha+\beta(1+\gamma))x\right)}{1-\delta^2\mu}, \dfrac{1-(\alpha+\beta(1+\gamma))x-\delta\mu}{1-\delta^2\mu}\right]$,

v) if $x \quad (\bar{x}_{IIa}, \min\{\bar{x}_{IVa} = \dfrac{\delta}{1+\delta\left(1-\beta(1-\gamma)\right)}, {}^1/_2\}),$ *then* $\underline{v}(x) = \bar{v}(x) = \dfrac{1-(\alpha+\beta\gamma)x}{1+\delta(\beta+\mu)}$,

vi) if $x \quad [\max\{\bar{x}_{IVa}, \bar{x}_{IIa}\}, {}^1/_2),$ *then* $\underline{v}(x) = \bar{v}(x) = \dfrac{1-(\alpha+\beta(1+\gamma))x}{1+\delta\mu}$.

Proof: See Appendix.

Since the result is very sensitive to changes in the parameter values, we will later illustrate it for some salient configurations in Section 5, together with the set of SPE outcomes for the case of large outside options. However, there are a few things that we can remark, without the scrutiny of the specific values. First, notice that some of the cases - namely, IIb, IIIa and IIIb - never arise in equilibrium.[8] This is not surprising, since in these cases it should be true that the responder expects more than the proposer despite of the commitment value of being the first mover. Second, again according with intuition, for sufficiently small values of the outside options they have no effect on the outcome and we obtain the Rubinstein result. For cases IIa and IIc, we have multiple equilibria. These multiplicities are a consequence of Proposition 3.1 type results for IIa (where both players may or may not prefer to opt out), and Shaked's [1994]

[7] With the convention that $[a, b) = \quad$ if $a \geq b$.

[8] IVb does arise for a single value of x.

result (there, options can be taken by the proposer but only one of the players enjoys them, and thus when the player with outside options acts as the proposer, a multiplicity of proposals can be sustained in SPE). If the options are relatively high, we can find ourselves in Case I where both players always want to opt out after a rejection, again yielding a unique outcome. For the remaining values of x we find ourselves in Case IVa, where only the proposer prefers to take his option. Since again, both players have strict preferences about the opting out decision, this equilibrium is also unique.

4.2 When options are always taken

Let us now turn to the case of large outside options (x+y >1). First, note that, whenever x > 1/2, in state (x, x) the unique equilibrium outcome is taking up the options, since otherwise in the continuation either there is agreement, in which case the sum of payoffs is one and therefore at least one player would prefer to deviate and take up her option; or there is disagreement, in which case (by discounting) it is strictly preferable to take the option now. In order to determine the equilibrium behavior in the rest of the states a relevant question is whether x+y exceeds one — or, equivalently, whether $x \geq q \equiv \frac{1}{1+\gamma}$.

If $x \geq q \equiv \frac{1}{1+\gamma}$, then in both states where one of the players has the option available, she will take it. To see this, note that agreement is not possible in these states, since at least one of the parties would prefer to refuse the deal, just as in state (x, x). By the same argument, any future agreement is dominated. The only remaining possibility is to take the option later, but then the player who has it today will strictly prefer to take it now. Thus, the only open question is what happens in state (ø, ø). In this case, they will agree at $1-\delta v_P$, unless it is Pareto superior to wait until next period. Waiting one period is not Pareto Superior if the discounted expected gains of proposer plus the discounted expected gains of the responder do not exceed 1, the current sum of proposer plus responder payoffs, that is, $v_P + v_R = 2\alpha x + 2\beta(x+y) + \mu\delta(v_P + v_R) \leq 1/\delta$ — or, equivalently $x \leq z \equiv \frac{1-\delta\mu}{2\delta(\alpha + \beta(1+\gamma))}$.

Proposition 4.2 *Let* $x \geq q \equiv \frac{1}{1+\gamma}$. *Then for each parameter configuration, the set of SPE is fully characterized as follows:*

i) if $x < z \equiv \frac{1-\delta\mu}{2\delta(\alpha+\beta(1+\gamma))}$, *then the unique equilibrium yields*

$$v_P(x) = \frac{(\alpha+\beta(1+\gamma))x+\mu}{1+\delta\mu} \quad \textit{and} \quad v_R(x) = \frac{\delta\mu^2+(\alpha+\beta(1+\gamma))(1+2\delta\mu)x}{1+\delta\mu};$$

ii) if x ≥ z, then the unique equilibrium yields

$$v_P(x) = v_R(x) = \frac{(\alpha + \beta(1+\gamma))x}{1-\delta\mu}.$$

Proof: Omitted.

The results of the proposition are quite intuitive. Note that z is decreasing in α, β, γ, and δ. That is, waiting is more likely if either the players are more patient or the options are more likely to appear. When they do wait in equilibrium, the expected payoffs of proposer and responder are the same, since they never get to agreement and thus the first-mover advantage disappears.

4.3 When the options are only taken if they are available simultaneously

Let us now consider the case $x < q \equiv \frac{1}{1+\gamma}$, where x+y is less than one. Now, in the state where only the proposer can opt out, agreement must be immediate in equilibrium. To prove this, we show that the proposer's option always exceeds her continuation value (and hence she obtains the full surplus net of y). First, observe that the sum of the expected payoffs of the bargainers cannot exceed 2x in any state and, therefore, it can never exceed this value. Now, if the proposer's threat to take her option were not credible it should be the case that she expects at least x/δ next period (as a responder). But this would imply that her opponent -- as a proposer -- expects no more than $2x - x/\delta = \frac{2\delta - 1}{\delta}x < x < x/\delta$, contradicting the fact that, ex-ante, the proposer never expects less than the responder in equilibrium. Moreover, observe that if agreement is delayed in state (ø, x), then the option cannot be credible and therefore in state (ø, ø) agreement is delayed too. Therefore, delay can be avoided if and only if $v_P + v_R = 2\alpha x + 2\beta + \mu\delta(v_P + v_R) \leq 1/\delta$, or, equivalently, $x \leq u \equiv \frac{(1-\delta) + \delta\alpha}{2\delta\alpha}$. Therefore if $x < \min\{q, u\}$, agreement is always reached in states (ø, ø), (ø, x) and (x, ø). In state (ø, ø) they agree at $1-\delta v_P$ and in state (ø, x) at $1-\max\{\delta v_P, x\}$. Finally, in state (x, ø) the proposer obtains 1-y.

Proposition 4.3a *Let x and γ be such that* $\frac{1}{2} < x < \min\{q \equiv \frac{1}{1+\gamma}, u \equiv \frac{(1-\delta)+\delta\alpha}{2\delta\alpha}\}$. *Then the set of SPE with immediate agreement (except in state (x, x)) is fully characterized as follows:*

i) If $x < \underline{x} \equiv \frac{\delta(1-\alpha)}{1 + \delta(\mu - \alpha + \beta(1+\gamma))}$, *then the unique equilibrium yields*

$$v_P(x) = \frac{1-\alpha + (\alpha - \beta\gamma)x}{1+\delta(\beta+\mu)} \quad \textit{and} \quad v_R(x) = \frac{\delta(\beta+\mu)(1-\alpha) + (\alpha + \beta\gamma + 2\delta\alpha(\beta+\mu))x}{1+\delta(\beta+\mu)};$$

ii) if $\bar{x} \equiv \dfrac{\delta(1-\alpha)}{1+\delta\big(\mu+\alpha-\beta(1+\gamma)\big)} > \underline{x}$*, then for* x $[\underline{x}, \bar{x}]$*, there is an interval of equilibria that yield payoffs to the proposer in* $[\underline{v}_P(x), \bar{v}_P(x)]$ *with*

$$\underline{v}_P(x) = \frac{\big(x(\alpha-\beta\gamma)+1-\alpha\big)\big(1-\delta(\beta+\mu)\big)+\delta\beta(\beta+\mu)x}{1-\delta^2\mu(\beta+\mu)},$$

$$\bar{v}_P(x) = \frac{\big(x(\alpha-\beta\gamma)+1-\alpha\big)(1-\delta\mu)-\beta x}{1-\delta^2\mu(\beta+\mu)};$$

iii) if $x > \max(\underline{x}, \bar{x})$*, then the unique equilibrium yields*

$$v_P(x) = \frac{1-\alpha+(\alpha-\beta(1+\gamma))x}{1+\delta\mu} \quad \textit{and} \quad v_R(x) = \frac{\delta\mu(1-\alpha)+\big(\alpha+\beta(1+\gamma)+2\delta\alpha\mu\big)x}{1+\delta\mu}.$$

Proof: See the Appendix.

This result is very much in the spirit of Shaked [1994]. If the option is relatively small the equilibrium is unique, since in the only state where the outcome is not yet determined it does not affect the outcome.[9] If the option is relatively large we have again a unique equilibrium , since now the option will become a credible threat. For the in between values, the credibility of the threat to opt out is indeterminate, so we have multiple equilibria. Let us next analyze what happens if delay is possible:

Proposition 4.3b *Assume that* $u \equiv \dfrac{(1-\delta)+\delta\alpha}{2\delta\alpha} \le x < q \equiv \dfrac{1}{1+\gamma}$*, and let* $w \equiv \dfrac{2\beta\delta}{1-\delta^2\mu^2-\delta\big[\alpha-\beta(1+\gamma)+\delta\mu\big(\alpha+\beta(1+\gamma)\big)\big]}$ *and* $w' \equiv \dfrac{\beta\delta}{1-\delta^2(\beta+\mu)\big(1-\beta(1-\gamma)\big)-\delta(\alpha-\beta\gamma)}$.

Then the following holds:

i) If $u < x < \min\{w,w'\}$ *there is a unique SPE. For all t, delay occurs in (ø, ø) and (ø, x), agreement where the proposer obtains 1-y occurs in state (x, ø) and the players disagree and take the options in (x, x).*

ii) If w < x < w', the above SPE coexists with a SPE where for all t, delay occurs in (ø, ø), agreement occurs in (x, ø) and (ø, x) where the proposer obtains 1-y and 1-x, respectively, and the players disagree and take the options in (x, x).

iii) If $x > \max\{w,w'\}$*, the second equilibrium in ii) is the unique SPE.*

[9] The equilibrium payoffs still depend on x since in other states it matters.

Proof: Following the previous discussion, the only open question is whether agreement is delayed or not in state (ø, x). This decision corresponds to the responder, since he has the option. The comparison he makes is whether her continuation value exceeds the value of her option. Since the responder's continuation value is the proposer's ex-ante profit (discounted) we start by deriving an upper bound on v_P. The first thing to note is that, given that $x \geq u$, the proposer would always prefer to delay in state (ø, x). To see this, recall that $x \geq u$ implies that $\delta(v_R+v_P) \geq 1$. Now, if $x > \delta v_R$ -- and therefore the responder forces agreement at (1-x,x) -- we have $1-x < 1-\delta v_R \leq \delta v_P$. Given this, we can write the upper bound on the proposer's equilibrium payoff as

$$\bar{v}_P = \alpha x + \beta(1-y) + (\beta+\mu)\delta\bar{v}_R.$$

The upper bound on the responder's payoff is determined only implicitly:

$$\bar{v}_R = \alpha x + \beta y + \mu\delta\bar{v}_P + \beta\max\{x, \delta\bar{v}_P\}.$$

Resolving this system case by case, we obtain that

$$\bar{v}_P = \begin{cases} \dfrac{[\alpha-\beta\gamma+\delta(\beta+\mu)(\alpha+\beta\gamma)]x+\beta}{1-\delta^2(\beta+\mu)^2} > \dfrac{x}{\delta} & \text{, if } x < w' \\ \dfrac{[\alpha-\beta\gamma+\delta(\beta+\mu)(\alpha+\beta(1+\gamma))]x+\beta}{1-\delta^2(\beta+\mu)\mu} \leq \dfrac{x}{\delta} & \text{, if } x \geq w'. \end{cases}$$

Since delay can only occur when the proposers value is at least $\dfrac{x}{\delta}$, we have proved that if $x \leq w'$ then there exists an equilibrium where delay occurs in both state (ø, ø) and (ø, x) every period, with the payoff stated in the proposition. In addition we have also established that if $x > w'$ the only candidate for equilibrium is agreement in state (ø, x) in every period.

Now, assume that the equilibrium prescribes delay in state (ø, x) in periods t and t+1. Note that by the previous paragraph this implies that $x \leq w'$. We claim that this implies that in equilibrium they wait in period t-1 also. To see this, observe that

$$v^t_P = \alpha x + \beta(1-y) + (\beta+\mu)\delta v^{t+1}_R = \alpha x + \beta(1-y) + (\beta+\mu)\delta(\alpha x + \beta y + (\beta+\mu)\delta v^{t+2}_P).$$

Given that the responder chooses to wait in period t+1 we know that $\delta v^{t+2}_P \geq x$. Therefore we have that

$$v^t_P \geq \alpha x + \beta(1-y) + (\beta+\mu)\delta(\alpha x + \beta y + (\beta+\mu)\delta x).$$

Since the RHS is greater than x/δ if and only if w' is greater than x (what we assumed), we have that in period t-1 the responder will prefer to wait, as claimed.

Next, assume that the equilibrium prescribes agreement in state (ø, x) in periods t and t+1. We claim that this implies that, if x > w, in equilibrium they agree in period t-1 too. To see this, observe that

$$v^t_P = \alpha x + \beta(2\text{-}x\text{-}y) + \mu\delta v^{t+1}_R = \alpha x + \beta(2\text{-}x\text{-}y) + \mu\delta(\alpha x + \beta(x+y) + \mu\delta v^{t+2}_P).$$

Given that the responder chooses to agree in period t+1 we know that $\delta v^{t+2}_P \leq x$. Therefore we have that

$$v^t_P \leq \alpha x + \beta(2\text{-}x\text{-}y) + \mu\delta(\alpha x + \beta(x+y) + \mu\delta x).$$

Since the RHS is less than x/δ if and only if w is less than x, we have that for x > w, in period t-1 the responder will prefer to agree, as claimed.

The last two results imply that when x is between w and w', the only possible equilibrium beside the two stationary ones is one where after some T, depending on the identity of the proposer they agree or delay in state (ø, x). Observe that in this candidate for equilibrium we have that (past T) $v_p^t = v_p^{t+2}$. Let t be a period where they agree. Then we have that

$$v_p^t = \alpha x + \beta(2 - x - y) + \mu\delta\left(\alpha x + \beta y + (\beta + \mu)\delta v_p^t\right),$$

and therefore,

$$v_p^t = \frac{\left[\alpha - \beta\left(1 + y\right) + \mu\delta\left(\alpha + \beta y\right)\right]x + 2\beta}{1 - \mu(\beta + \mu)\delta^2}.$$

Since in period t+1 they disagree by assumption, it must be the case that $\delta v_p^t \geq x$. Straightforward calculations show that this is equivalent to the condition $x \leq w$.

Carrying out the same exercise for period t+1, we obtain,

$$v_p^{t+1} = \alpha x + \beta(1 - y) + (\beta + \mu)\delta\left(\alpha x + \beta(y + x) + \mu\delta v_p^{t+1}\right),$$

and therefore,

$$v_p^{t+1} = \frac{\left[\alpha - \beta y + (\beta + \mu)\delta\left(\alpha + \beta(1 + y)\right)\right]x + \beta}{1 - \mu(\beta + \mu)\delta^2}.$$

Since in period t they agree by assumption, it must be the case that $\delta v_p^{t+1} \leq x$. Straightforward calculations show that this is equivalent to the condition $x \geq w'$.

Routine calculations show that u ≤ w implies w < w' and therefore we cannot have for x ≥ u that w' ≤ x ≤ w. Consequently the strategies under scrutiny cannot constitute an equilibrium.

All we have left to prove is that, for $u \leq x < w$, we have only the stationary equilibrium involving delay in state (ø, x) in every period. We will prove this by contradiction. If there existed a different equilibrium, it necessarily would involve agreement in state (ø, x) in some period. Note that this would imply that there is some continuation payoff which does not exceed the option's value: $x \geq \delta \underline{v}_P$. Given this inequality we can write the lower bound on the responder's value as

$$\underline{v}_R = (\alpha+\beta)x + \beta y + \mu\delta\underline{v}_P.$$

The lower bound for the proposer's value is not as straightforward, since it depends on the responder's decision. We have that,

$$v_P \leq \alpha\, x + \beta(1-y) + \mu\delta\underline{v}_R + \beta \min\{1-x, \delta\underline{v}_R\}.$$

Analyzing both cases we obtain that a necessary condition for the existence of the candidate equilibrium is that $x \geq \min\{w,w'\}$ which given $u \leq x$ is equivalent to $x \geq w$.

Q.E.D.

Note that, if the outside opportunities of players increase or remain constant when their opponent takes the option (that is, $p \geq \alpha+\beta$) then $q < u$ and therefore the scenario of Proposition 4.3b does not arise. A typical scenario for the case with low p, is one where there are few outside options in the economy and therefore if one is taken the probability that there will be another one available decreases sharply.

Propositions 4.1, 4.2, 4.3a and 4.3b give a characterization of the set of equilibria without unnecessary delay. In addition, we also have inefficient equilibria, where the delay is sustained by mutual threats of "regime switching", not by the expectation of a high valued option. These equilibria are a standard consequence in (stationary) strategic bargaining models, whenever there exist multiple efficient equilibria (see, for example, Avery and Zemsky [1994b]). Whenever we have multiple equilibria with immediate agreement, delayed agreements can be constructed in the usual way.

5. Examples

The following examples, focusing on some especially relevant particular cases, should facilitate the reading and interpretation of our results:

Perfectly patient players: Consider the asymptotic properties of our model as $\delta \to 1$.

A) $p = \alpha + \beta$

Taking the limit we obtain that $\underline{x}_{IIa} \to 0$, and $\bar{x}_{IIa}$, q, $z \to {}^1/_2$, so that we have

i) if $x \quad [0, {}^1/_2]$, then we are in case iv of Proposition 4.1 and thus $[\underline{v}(x), \bar{v}(x)] = [x, 1-x]$,

ii) if $x > {}^1/_2$, then we are in case ii of Proposition 4.3 and thus $v(x) = x$.

Since in the limit the value of commitment is zero, any agreement is possible which gives both players at least their option.

B) p = 0

Taking the limit we obtain that $\underline{x}_{IIa} \to 0$, $z \to 1/2$, $\bar{x}_{IIa} \to \dfrac{\alpha + 2\beta}{2\alpha + 3\beta}$ and, assuming $\beta > 0$,[10] we have that:

i) if $x \quad [0, {}^1/_2)$, then $[\underline{v}, \bar{v}] = [\underline{v}_{IIa}(x), \bar{v}_{IIa}(x)]$,

ii) If $x \quad [{}^1/_2, \dfrac{2}{3+\mu})$, then there is a unique equilibrium with delay in states (ø, ø) and (ø, x),

iii) if $x \quad [\dfrac{2}{3+\mu}, \dfrac{1}{1+\mu+\beta}]$, the equilibrium with delay in states (ø, ø) and (ø, x) coexists with the equilibrium with delay only in state (ø, ø),

iv) if $x \quad (\dfrac{1}{1+\mu+\beta}, 1]$, then the equilibrium with delay only in state (ø, ø) is unique.

<u>Perfectly correlated, equally likely options:</u>

Consider $\alpha = \mu = {}^1/_2$. We solve for $p = \alpha + \beta$.[11] Note that the following inequalities hold:

$$\underline{x}_{IIc} = \frac{\delta^2 - \delta^3}{2 - \delta - \delta^2} < \bar{x}_{IIc} = \bar{x}_{IIa} = \frac{\delta^2 - \delta^3}{2} < \underline{x}_{IIa} = \frac{\delta}{1+\delta} = \bar{x}_{IVa} < \frac{1}{2} \quad ;$$

$$q = \frac{2-\delta}{2}, \; s = \frac{\delta}{2} < \frac{1}{2}, \text{ and } z = \frac{2-\delta}{2\delta}.$$

Therefore we have the following:

i) if $x \quad [0, \dfrac{\delta^2 - \delta^3}{2 - \delta - \delta^2})$, then there is a unique equilibrium with $v = \dfrac{1}{1+\delta}$,

[10] If $\beta = 0$, then (II), (III) and (IV) collapse into (IV) for $x \quad [1/2, 1]$.

[11] The qualitative features of the set of SPE outcomes are robust to changes in p.

ii) if x $\left[\frac{\delta^2-\delta^3}{2-\delta-\delta^2}, \frac{\delta^2-\delta^3}{2}\right)$, then $[\underline{v}(x), \bar{v}(x)] = \left[\frac{2-\delta(2-x)}{2-\delta^2}, \frac{2-\delta-x}{2-\delta^2}\right]$,

iii) if x $\left[\frac{\delta^2-\delta^3}{2}, \frac{1}{2}\right)$, then there is a unique equilibrium with $v(x) = \frac{2-x}{2+\delta}$,

iv) if x $\left[\frac{1}{2}, \frac{2-\delta}{2\delta}\right)$, then there is a unique equilibrium with $v_P(x) = \frac{1+x}{2+\delta}$,

v) if x $\left[\frac{2-\delta}{2\delta}, 1\right]$, then there is the unique equilibrium with $v_P(x) = v_R(x) = \frac{x}{2-\delta}$.

The following figure illustrates the case $\delta = .9$:

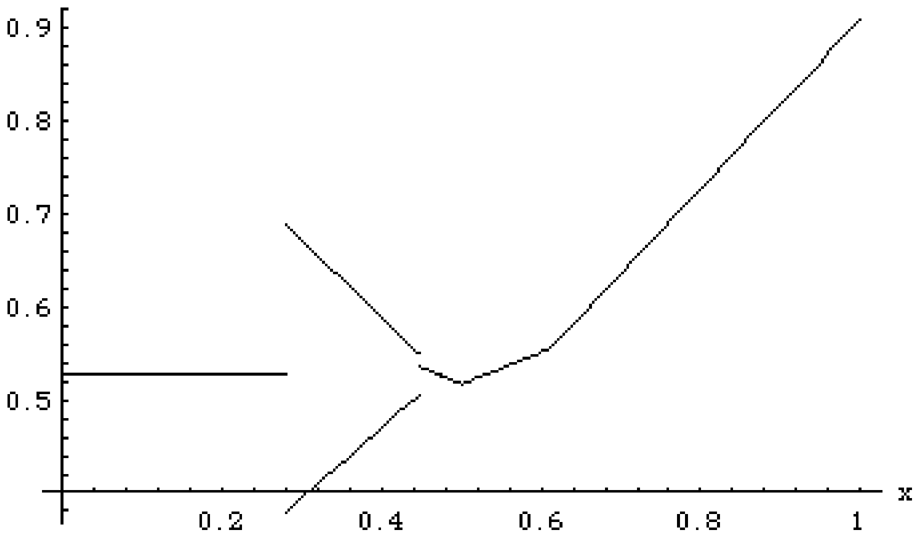

Figure 2
Player 1's payoffs with perfectly correlated and equally likely options ($\delta = .9$)

6. Final remarks

As we have seen, the set of equilibria is very sensitive to changes in the parameters. Nevertheless some general features can be identified. For example, it can be shown that for very low and very high values of the options the equilibrium is generically unique. As shown in Shaked [1994] and in Ponsatí and Sákovics [1998], multiplicity of equilibria arises if the proposer can be made indifferent between opting out and continuing to the next period upon rejection of her offer. In the present case, if x is small then opting out is (strictly) not credible for almost all the values of the parameters (unless $\delta = 1$) and we obtain the Rubinstein solution (c.f. the Outside Option

Principle). If x is large, the proposer's option is (strictly) credible for all parameter values, yielding uniqueness again. On the other hand, for any given parameter configuration, there always exists an interval of option values, for which there are multiple equilibria. This is so because the necessary conditions for the existence of Case IIc are always satisfied for an interval of x and the only possibility for these conditions not to be sufficient is that Case IIa invades this territory yielding an even larger set of equilibria. An additional curiosity -- that can be seen on Figure 2 -- is that the expected equilibrium share of the player to make the first offer is not necessarily increasing in the value of the options. This phenomenon actually has a simple explanation. When the equilibrium is such that only the proposer's threat of opting out is credible, the value of the options only enters her payoff directly in case (x, x), where it provides the value for the responder's payoff (c.f. Case IVa).

Proposition 4.3b is a result of some theoretical interest on its own right. The co-existence of exactly two equilibrium payoffs in an alternating offer bargaining game is absolutely novel. Usually, the in-between values can also be supported (by non-stationary strategies). In the current scenario, however, only stationary strategies exist, because in three of the states, the players have a strictly dominant strategy, while in the fourth state the necessary conditions for equilibrium impose stationarity.

Efficiency, or the lack of it, is an important characteristic of equilibria in bargaining games. In the context of outside opportunities it is not straightforward how to define it, though. Whether the outside opportunities are considered as potential gains to be realized (with some third economic agent) or they are already sunk and accounted for makes an important difference. For simplicity, let us evaluate our results using the latter criterion: therefore using a social welfare function which simply sums the payoffs of our two bargainers. The first thing to observe is that whenever they take their outside options in equilibrium (even if only one of them takes this decision) it is efficient to do so. Since every agreement is also efficient in the sense that they share the entire surplus, inefficiencies can only come about via the timing of agreement. As always, we have the inefficient equilibria generated by reversion to the extreme equilibria,[12] which entail delay (at least, with positive probability). However, not all equilibria which entail delay in some of the states are inefficient, since in the present context, the expected joint surplus may be maximized by waiting for the availability of the options. On the other hand, by the same token inefficiency may be caused by an agreement which is premature. Under the parameter configurations where in the periods where neither player has an option they delay agreement but in the state where only the responder has one they agree (see Proposition 4.3b), some surplus is wasted. The reason why this happens is clear: the responder can force a deal which for him is better than his continuation value, but his gains do not compensate for the proposer's losses.

[12] Whenever there are multiple equilibria with immediate agreement.

In spite of the uncertainty about future options, our bargaining environment is stationary, and this is crucial for our results. Yet, in many cases outside options arise and cease to be available over time in changing, non-stationary, environments. We address the role of options of uncertain value in non-stationary bargaining environments in Ponsatí and Sákovics [1999], where we have argued (among other things) that: i) the relevant attribute of an uncertain outside option is the time by which its value is revealed rather than the time up to which it is available; ii) the possibility of continuing bargaining after the option's value has become known has an option value[13] which biases the ex-ante valuation of the outside option upwards.

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Appendix

Proof of Proposition 4.1: We must explore cases I to IVb. Note that, in principle, for each parameter configuration, several cases (i.e. several types of equilibria) may coexist. Let us start by the analysis of the first case:

Case I: Note, that for both players, regardless of whether they are proposer or responder, taking the outside option is not only a credible threat but it is a strictly dominant action in case they do not reach agreement in a given period. Thus, in state (x, ø), the responder must accept all offers that give him more than y (the expected value that the players get when their opponent takes the option at a state in which they do not have one), while he is indifferent between accepting or rejecting y. Consequently, the unique possible equilibrium demand of the proposer is 1-y. Similarly, in states (ø, x) and (x, x), the responder will take the outside option if the proposer demands more than 1-x. At the same time, it is not credible that he would reject any other proposal ($x > \delta\bar{v}$), so agreement at 1-x is the unique possible equilibrium outcome. Finally, in state (ø, ø), since no player can take an outside option, payoffs for the proposer must lie in $[1-\delta\bar{v}, 1-\delta\underline{v}]$, just as in the original Rubinstein game. Therefore, conditional on being in case I, the following two equalities must hold:

$$\underline{v} = (\alpha+\beta)(1-x) + \beta(1-y) + \mu(1-\delta\bar{v}),$$

$$\bar{v} = (\alpha+\beta)(1-x) + \beta(1-y) + \mu(1-\delta\underline{v}).$$

Hence $\underline{v} = \bar{v} = v_I = \dfrac{1-(\alpha+\beta)x-\beta y}{1+\delta\mu} = \dfrac{1-(\alpha+\beta(1+\gamma))x}{1+\delta\mu}$. Substituting in the inequalities that define Case I, we obtain that the relevant one is the second, and thus this equilibrium exists if $x > x_I = \dfrac{\delta}{1+\delta(1-(1-\gamma)\beta)}$. Note however, that this does not imply that we have characterized a unique equilibrium for all x satisfying the inequality. It may well be the case that for the same parameter values we can satisfy the necessary conditions of a different case, resulting in additional equilibria.

Case IIa: Note that now all kinds of credibility combinations are possible. The worst case scenario for the proposer, which gives rise to the payoff $\underline{v}$, is $1-\delta\bar{v}$ in all states, since $x \leq \delta\bar{v}$, and thus when the proposer does not take her option the responder does not take it either, resulting in the Rubinstein continuation values.

For the upper bound, it is straightforward to see that the proposer can obtain up to 1-y and 1-x in states (x, ø) and (x, x), respectively. Since $x > \delta v$ for some v, by the Outside Option

Principle, the proposer can obtain up to 1-x in state (ø, x). Finally, when neither option is available, the maximum the proposer can get is $1-\delta\underline{v}$. Therefore, we have that

$$\underline{v} = 1 - \delta\bar{v},$$

$$\bar{v} = (\alpha+\beta)(1-x) + \beta(1-y) + \mu(1-\delta\underline{v}).$$

Hence $\underline{v} = \underline{v}_{IIa}(x) = \dfrac{1-\delta(1-(\alpha+\beta)x-\beta y)}{1-\delta^2\mu} = \dfrac{1-\delta(1-(\alpha+\beta(1+\gamma))x)}{1-\delta^2\mu}$ and $\bar{v} = \bar{v}_{IIa}(x) = \dfrac{1-(\alpha+\beta)x-\beta y-\delta\mu}{1-\delta^2\mu} = \dfrac{1-(\alpha+\beta(1+\gamma))x-\delta\mu}{1-\delta^2\mu}$.

Remark: Observe that in this case, since the relevant inequalities may go either way, by construction, we have the largest possible set of potential equilibrium values for every x.

To check the consistency of the equilibrium values with the inequalities defining this case, note that, as it can directly be seen from the equality defining $\underline{v}$, $\underline{v}_{IIa}(x) + \bar{v}_{IIa}(x) > 1$, and thus the relevant (that is, most stringent) inequalities to verify are $\delta\underline{v}_{IIa}(x) \le x$ and $x \le \delta(1-\underline{v}_{IIa}(x))$. Performing the calculations, we obtain the following conditions on the value of the option:

$$\underline{x}_{IIa} = \frac{\delta(1-\delta)}{1-\delta^2(1-\beta(1-\gamma))} \le x \le \frac{\delta^2(1-\delta\mu)}{1+\delta^2(\alpha+\beta(1+\gamma)-\mu)} = \bar{x}_{IIa}.$$

As a consequence of the Remark above, whenever x lies in the above interval, the set of equilibrium payoffs is $[\underline{v}_{IIa}(x), \bar{v}_{IIa}(x)]$.

To investigate what happens if x is outside that interval, assume that $x < \underline{x}_{IIa} = \delta\underline{v}_{IIa}$ first. If $x < \underline{x}_{IIa}$ then, by the maximality property of the bounds of case IIa (as mentioned in the Remark above, case IIa yields the minimal possible equilibrium values for every x), thus $x < \delta\underline{v}_{IIa}(x)$ implies $x < \delta\underline{v}_{I}(x)$, $x < \delta\underline{v}_{IIb}(x)$, $x < \delta\underline{v}_{IIIb}(x)$ and $x < \delta\underline{v}_{IVb}(x)$, contradicting the second inequality in cases I, IIb, IIIa, IIIb and IVb. Case IIc remains a candidate.

Case IIc: The corresponding equalities for the equilibrium bounds can easily be shown to be the following:

$$\underline{v}_{IIc} = 1 - \delta\bar{v}_{IIc},$$

$$\bar{v}_{IIc} = \alpha(1-x) + \beta(1-y) + (\beta+\mu)(1-\delta\underline{v}_{IIc}).$$

Solving the system, we obtain $\underline{v}_{IIc} = \frac{1-\delta(1-\alpha x-\beta y)}{1-\delta^2(\beta+\mu)} = \frac{1-\delta(1-(\alpha+\beta\gamma)x)}{1-\delta^2(\beta+\mu)}$ and $\bar{v}_{IIc} = \frac{1-\alpha x-\beta y-\delta(\beta+\mu)}{1-\delta^2(\beta+\mu)} = \frac{1-(\alpha+\beta\gamma)x-\delta(\beta+\mu)}{1-\delta^2(\beta+\mu)}$. Given that $x < \underline{x}_{IIa} = \delta\underline{v}_{IIa}$, $x < \delta\underline{v}_{IIc}$ is guaranteed, again by the extreme limits of case IIa. Therefore, the necessary conditions for the existence of this type of equilibrium follow from verifying $\delta(1-\bar{v}_{IIc}) \le x \le \delta(1-\underline{v}_{IIc})$, which yields

$$\underline{x}_{IIc} = \frac{\delta^2(1-\delta)(\beta+\mu)}{1-\delta^2(\beta+\mu)-\delta(\alpha+\beta\gamma)} \le x \le \frac{\delta^2(1-\delta(\beta+\mu))}{1-\delta^2(\beta(1-\gamma)+\mu-\alpha)} = \bar{x}_{IIc}.$$

Now, observe that these conditions are also sufficient, since the possible credibility configurations of the two remaining cases (IId and IVa) are a subset of those of case IIc.

If $x < \underline{x}_{IIc}$, we are left with case IId only, since case IVa would require $x > \delta(1-\underline{v}_{IVa}) \ge \delta(1-\bar{v}_{IIc})$.

<u>Case IId:</u> In this situation the outside options do not play any role, so we obtain the Rubinstein solution: $v_{IId} = \frac{1}{1+\delta}$. The necessary condition for existence is $x \le \frac{\delta^2}{1+\delta}$, which is guaranteed whenever $x \le \underline{x}_{IIc}$ $\left(<\frac{\delta^2}{1+\delta}\right)$, as assumed.

If $\underline{x}_{IIa} > x > \bar{x}_{IIc}$ $\left(>\frac{\delta^2}{1+\delta}\right)$ the necessary condition for case IId is violated, so our unique candidate case is IVa.

<u>Case IVa:</u> The corresponding equalities are

$$\underline{v} = \alpha(1-x) + \beta(1-y) + (\beta+\mu)(1-\delta\bar{v}),$$

$$\bar{v} = \alpha(1-x) + \beta(1-y) + (\beta+\mu)(1-\delta\underline{v}).$$

Hence, $\underline{v} = \bar{v} = v_{IVa} = \frac{1-\alpha x-\beta y}{1+\delta(\beta+\mu)} = \frac{1-(\alpha+\beta\gamma)x}{1+\delta(\beta+\mu)}$. The necessary conditions for existence are

$$\underline{x}_{IVa} = \frac{\delta^2(\beta+\mu)}{1+\delta(\beta(1-\gamma)+\mu-\alpha)} < x < \frac{\delta}{1+\delta(1-\beta(1-\gamma))} = \bar{x}_{IVa}.$$

It is straightforward to verify that $\underline{x}_{IVa} \le \bar{x}_{IIc}$ and $\underline{x}_{IIa} \le \bar{x}_{IVa}$ and therefore the necessary conditions are also sufficient.

It remains to be seen what happens if $x > \bar{x}_{IIa}$. By the extremal property of the limits of case IIa, we can directly discard all subcases of Cases II and III. A remaining candidate is case IVb.

Case IVb: The corresponding equalities are

$$\underline{v} = \alpha(1-x) + \beta(1-y) + (\beta+\mu)(1-\delta\bar{v}),$$

$$\bar{v} = (\alpha+\beta)(1-x) + \beta(1-y) + \mu(1-\delta\underline{v}).$$

Hence, $\underline{v}_{IVb} = \dfrac{1-\alpha x-\beta y-\delta(\beta+\mu)(1-(\alpha+\beta)x-\beta y)}{1-\delta^2(\beta+\mu)\mu}$ and $\bar{v}_{IVb} = \dfrac{1-(\alpha+\beta)x-\beta y-\delta\mu(1-\alpha x-\beta y)}{1-\delta^2(\beta+\mu)\mu}$. Checking the necessary conditions for existence, $\delta\underline{v}_{IVb} \leq x \leq \delta\bar{v}_{IVb}$ we obtain that this equilibrium may only exist at a single point:

$$\underline{x}_{IVb} = \bar{x}_{IVb} = \frac{\delta}{1+\delta(1-\beta(1-\gamma))}.$$

Note that at this point the equilibrium value is unique and equals $v_{IVb} = \dfrac{1}{1+\delta(1-\beta(1-\gamma))}$. The remaining necessary condition, $x > \delta(1-\underline{v}_{IVb})$, is easily seen to be satisfied at this point.

As we have seen before, x_{IVb} is the border between the region for the equilibria of cases IVa and I, whose value at x_{IVb} is also $\dfrac{1}{1+\delta(1-\beta(1-\gamma))}$.

Thus we have provided complete characterization of the subgame-perfect equilibria with immediate agreement for $x < 1/2$. Q.E.D.

Proof of Proposition 4.3a: When x lies in $[1/2, u]$ all efficient equilibrium payoffs can be implemented by immediate agreement in all states other that (x, x). Recall that in state (x, ø) the proposer obtains 1-y, in state (x, x) both players take their respective options and that the proposer's payoffs are in $[1-\delta\bar{v}_P, 1-\delta\underline{v}_P]$ in state (ø, ø). The exact values of $\underline{v}_P$, $\bar{v}_P$, $\underline{v}_R$ and $\bar{v}_R$ depend on the strategic capabilities of the proposer in state (ø, x). These are implicitly determined by whether $\delta\underline{v}_P \leq x < \delta\bar{v}_P$, $x < \delta\underline{v}_P$ or $x \geq \delta\bar{v}_P$. Thus, just as in our characterization for outside options $x < 1/2$, we need to check one by one each of these three cases.

Case I: If $\delta\underline{v}_P \leq x < \delta\bar{v}_P$ the proposer's payoffs in state (ø, x) are in $[1-\delta\bar{v}_P, 1-x]$. Therefore,

$$\underline{v}_P = \alpha\, x + \beta(1-y) + (\beta+\mu)\,(1-\delta\bar{v}_P),$$

$$\bar{v}_P = \alpha\, x + \beta(1-y) + \beta(1-x) + \mu(1-\delta\underline{v}_P).$$

Hence

$$\underline{v}_P = \frac{\left(\alpha x + \beta\left(2 - y\right) + \mu\right)\left(1 - \delta\left(\beta + \mu\right)\right) + \delta\beta\left(\beta + \mu\right)x}{1 - \delta^2\mu\left(\beta + \mu\right)},$$

$$\bar{v}_P = \frac{\left(\alpha x + \beta\left(2 - y\right) + \mu\right)\left(1 - \delta\mu\right) - \beta x}{1 - \delta^2\mu\left(\beta + \mu\right)}.$$

The necessary conditions for $\delta\underline{v}_P \le x < \delta\bar{v}_P$ are

$$x < \bar{x} \equiv \frac{\delta\left(1 - \alpha\right)}{1 + \delta\left(\mu + \alpha - \beta\left(1 + y\right)\right)},$$

$$x \ge \underline{x} \equiv \frac{\delta\left(1 - \alpha\right)}{1 + \delta\left(\mu - \alpha + \beta\left(1 + y\right)\right)},$$

Again, just as in the $x < {}^1/_2$ case, we can see that this is the case with the largest feasible set of equilibria and thus the conditions are also sufficient, as long as $x < u$.

<u>Case II:</u> If $x \ge \delta\bar{v}_P$ the proposer's payoffs in state (ø, x) are 1-x. Therefore,

$$\underline{v}_P = \alpha\, x + \beta(1-y) + \beta(1-x) + \mu\,(1-\delta\bar{v}_P),$$

$$\bar{v}_P = \alpha\, x + \beta(1-y) + \beta(1-x) + \mu\,(1-\delta\underline{v}_P).$$

This yields a unique solution

$$\underline{v}_P = \bar{v}_P = \frac{x\left(\alpha - \beta(1 + y)\right) + 1 - \alpha}{1 + \delta\mu}.$$

Verifying the necessary condition yields $x > \underline{x}$.

<u>Case III:</u> If $x < \delta\underline{v}_P$ the proposer's payoffs in state (ø, x) are in $[1-\delta\bar{v}_P,\ 1-\delta\underline{v}_P]$ .Therefore,

$$\underline{v}_P = \alpha\, x + \beta(1-y) + (\beta+\mu)\,(1-\delta\bar{v}_P),$$

$$\bar{v}_P = \alpha\, x + \beta(1-y) + (\beta+\mu)\,(1-\delta\underline{v}_P).$$

This yields a unique solution

$$\underline{v}_P = \bar{v}_P = \frac{x(\alpha - \beta\gamma) + 1 - \alpha}{1 - \delta(\beta + \mu)} \ .$$

Verifying the necessary condition yields $x > \underline{x}$.

Q.E.D.