Affilliation in Multi-Unit Discriminatory Auctions

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Summary. We extend Milgrom and Weber's affiliated valuations model to the multi-unit case with constant marginal valuations where 2 bidders compete for k identical objects. We show that the discriminatory auction has a unique equilibrium, that corresponds to Milgrom and Weber's first-price equilibrium. This unique equilibrium therefore leads to lower expected prices than the equilibrium of the English auction where the units are bundled together. Hence we show that in a common value auction of a single object where the object can be divided into k parts, it is not possible to increase revenue by using a multi-unit discriminatory auction. We discuss a possible application to Treasury auctions.

Keywords: Affiliated Valuations, Multi-Unit Auctions, Treasury Auctions.

JEL Classification Numbers: D44, D82, C72.

1 Introduction

In this paper, we extend the affiliated valuations model to the multi-unit demand case with 2 bidders and constant marginal valuations. We show that the discriminatory auction has a unique equilibrium, that corresponds to Milgrom and Weber's first-price equilibrium. This unique equilibrium therefore leads to lower expected prices than the equilibrium of an English auction where the units are bundled together.

There are some results on multi-unit discriminatory auctions in the literature. Back and Zender [1] extend Wilson's [15] share auction model to show that there is an equilibrium of the common values discriminatory auction that is equivalent to the first-price single unit auction. The share auction model is the continuous version of a multi-unit auction as the unit is continuously divisible and bidders can win any fraction of the unit. Viswanathan *et al* [14] use this framework to characterize the symmetric equilibria for decreasing marginal valuations with affiliation in the two bidder case. They also show that in the constant marginal valuations case, the first-price equilibrium is the unique symmetric equilibrium. However, they do not consider asymmetric equilibria.

We adopt a discrete model where a number of identical units are for sale and show that bidding for all units according to the single unit equilibrium strategy is the unique equilibrium. Engelbrecht-Wiggans and Kahn [2] use a discrete framework with 2 units and decreasing private values. They demonstrate that there must be regions where both units are bid at the same price in equilibrium even though the valuations are decreasing. Reny [9] proves existence in the k unit private values discriminatory auction. Lebrun and Tremblay [4] demonstrate uniqueness of the symmetric equilibrium in the two bidder private values case. Our model looks at the general case of affiliated valuations and therefore covers the private values model with constant marginal valuations.

In order to show that the multi-unit discriminatory auction has a unique equilibrium standard uniqueness results cannot be used because Lipschitz continuity does not hold at the lower boundary. In the single unit affiliated valuations case, Lizzeri and Persico [5] overcome this with a reserve price condition that ensures each bidder stays out of the auction with positive probability. Our uniqueness result only applies to the symmetric model but does not require the reserve price condition.

Although the results are of mainly theoretical interest, in section 3 we discuss a possible application to Treasury auctions.

2 Discriminatory Auction

We extend Milgrom and Weber's *affiliated valuations model* to look at the sale of k units of a good to 2 bidders. Let $X_i \epsilon [\underline{x}, \overline{x}]$, $i = 1, 2$ be the private information of bidder i of the value of each unit and S an additional variable (or vector) that affects the value of the good. The value of each unit to bidder i is $V^i = u(S, X_i, X_j)$ where u is nonnegative, continuous and strictly increasing in all its variables. Hence, we restrict attention to the case of constant marginal valuations. The variables S, X_1, X_2 are affiliated[1] with joint density $f(s, x_1, x_2)$ which is symmetric in x_1, x_2. Let $v(x, y) = E[V^1|X_1 = x, X_2 = y]$ and $f(y|x)$ and $F(y|x)$ be the conditional density and distribution of X_2 given $X_1 = x$. Affiliation implies that $F(y|x)/f(y|x)$ is decreasing in x[2].

Each bidder submits a bid detailing the price at which he is willing to purchase each of the k units. The total demand at price p is the number of units bid at or above p. The market clearing price, p^* is the highest price such that demand is at least k. All units bid above p^* are successful. If the demand at p^* is equal to k then all units bid at p^* are also successful. If however demand at p^* is greater than k then the units bid at p^* are rationed proportionally. The bidders pay what they bid on the units they win.

We look for Bayesian Nash equilibria where bidder i submits k bids according to $B^i(x) = \{b_1^i(x), \ldots.., b_k^i(x)\}$ where $b_1^i(x) \geqslant b_2^i(x)\ldots.., b_{k-1}^i(x) \geqslant b_k^i(x)$ and $b_m^i(x)$ is a strictly increasing differentiable function[3]. Let $\phi(b)$ be the inverse bid function, $\phi_j^i(.) = (b_j^i)^{-1}(.)$.

Now assume bidder 2 bids according to $B^2(y)$ and bidder 1 bids $\{b_1^1(w_1), \ldots.., b_k^1(w_k)\}$ where $b_1^1(w_1) \geqslant b_2^1(w_2)\ldots.., b_{k-1}^1(w_{k-1}) \geqslant b_k^1(w_k)$. Let $\widehat{m} = k - m + 1$. Note that bidder 1's demand for unit m is competing with bidder 2's demand for unit $\widehat{m}$. His highest bid will be successful if it is greater than bidder 2's lowest bid, $b_1^1(w_1) > b_k^2(y)$ or if $y < G_1^1(w_1)$ where $G_1^1(w_1) = \phi_k^2(b_1^1(w_1))$. Similarly, the m^{th} highest bid will be successful if $y < G_m^1(w_m)$ where

[1] For a formal definition of affiliation see Milgrom and Weber [6].

[2] Milgrom and Weber [6], lemma 1.

[3] In the appendix (corollary 2) we show that these regularity conditions must hold in the class of non-decreasing strategies.

$G^1_m(w_m) = \phi^2_{\widehat{m}}(b^1_m(w_m))$. The payoff to bidder 1 with signal x is

$$\Pi(x, w_1, w_2, ...w_k) = \sum_{j=1}^{k} \left(\int_{\underline{x}}^{G^1_j(w_j)} (v(x,y) - b^1_j(w_j)) f(y|x) dy \right)$$

and the partial derivative with respect to w_m is

$$\Pi'_m = \begin{pmatrix} (G^1_m)'(w_m) f(G^1_m(w_m)|x)(v(x, G^1_m(w_m)) - b^1_m(w_m)) \\ -(b^1_m)'(w_m) F(G^1_m(w_m)|x) \end{pmatrix} \tag{1}$$

If unit m is not constrained by the unit above or below then $\Pi'_m = 0$ when $w_m = x$. This gives the first order condition

$$(G^1_m)'(x) f(G^1_m(x)|x)(v(x, G^1_m(x)) - b^1_m(x)) - (b^1_m)'(x) F(G^1_m(x)|x) = 0 \tag{2}$$

If this first order condition is satisfied then $b^1_m(x)$ is an optimal bid in the range $[b^1_m(\underline{x}), b^1_m(\overline{x})]$. To see this re-arrange (1) to give

$$\begin{pmatrix} (G^1_m)'(w_m) f(G^1_m(w_m)|x) \\ \times \left[(v(x, G^1_m(w_m)) - b^1_m(w_m)) - \frac{(b^1_m)'(w_m)}{(G^1_m)'(w_m)} \frac{F(G^1_m(w_m)|x)}{f(G^1_m(w_m)|x)} \right] \end{pmatrix} \tag{3}$$

Since $F(G^1_m(w_m)|x)/f(G^1_m(w_m)|x)$ is decreasing in x and $v(x, G^1_m(w_m))$ is increasing in x, the above expression will be equal to zero when $w_m = x$, positive when $w_m < x$ and negative when $w_m > x$. Likewise the m^{th} first order condition for bidder 2 if bidder 1 uses $B^1(y)$ is

$$(G^2_m)'(x) f(G^2_m(x)|x)(v(x, G^2_m(x)) - b^2_m(x)) - (b^2_m)'(x) F(G^2_m(x)|x) = 0$$

This gives k pairs of non-linear differential equations,

$$(G^1_e)'(x) f(G^1_e(x)|x)(v(x, G^1_e(x)) - b^1_e(x)) - (b^1_e)'(x) F(G^1_e(x)|x) = 0 \tag{4}$$

$$(G^2_{\widehat{e}})'(x) f(G^2_{\widehat{e}}(x)|x)(v(x, G^2_{\widehat{e}}(x)) - b^2_{\widehat{e}}(x)) - (b^2_{\widehat{e}})'(x) F(G^2_{\widehat{e}}(x)|x) = 0 \tag{5}$$

where $e \epsilon [1, 2, ..., k]$. This pair of equations can be expressed in terms of the inverse functions,

$$(\phi^2_{\widehat{e}})'(b) = \frac{F(\phi^2_{\widehat{e}}(b)|\phi^1_e(b))}{f(\phi^2_{\widehat{e}}(b)|\phi^1_e(b))(v(\phi^1_e(b), \phi^2_{\widehat{e}}(b)) - b)} \tag{6}$$

$$(\phi^1_e)'(b) = \frac{F(\phi^1_e(b)|\phi^2_{\widehat{e}}(b))}{f(\phi^1_e(b)|\phi^2_{\widehat{e}}(b))(v(\phi^2_{\widehat{e}}(b), \phi^1_e(b)) - b)} \tag{7}$$

It is clear from (6) and (7) that we have a standard system of non-linear differential equations. Let $\Pi^i_m(x, b)$ be the profit bidder i with signal x makes on unit m if she bids b.

Definition 1 *Unit m is locally constrained over an interval of bids if $\Pi'_m \neq 0$ over this interval.*

Proposition 1 *A unit cannot be locally constrained in equilibrium.*

The proof is given in the appendix. Here we give a sketch of the proof. If unit e is constrained above by unit $e-1$ in equilibrium over the interval $b \in [c, d]$ then $\phi^2_{e-1}(b) = \phi^2_e(b) = \phi(b)$ over this interval. We begin by showing that this implies $b^j_{\widehat{e}+1}(x) = b^j_{\widehat{e}}(x)$ for all $x \in [\phi^j_{\widehat{e}}(c), \phi^j_{\widehat{e}}(d)]$. If $b^1_{\widehat{e}+1}(x) < b^1_{\widehat{e}}(x)$ then for $b \in [c, d]$,

$$\Pi^1_{\widehat{e}+1}(x, b) = \Pi^1_{\widehat{e}}(x, b) = \int_{\underline{x}}^{\phi(b)} (v(x, y) - b) f(y|x) dy$$

and

$$\left(\Pi^1_{\widehat{e}+1}\right)'(x, b) = \left(\Pi^1_{\widehat{e}}\right)'(x, b) = f(\phi(b)|x)\left[\phi'(b)(v(x, \phi(b)) - b) - \frac{F(\phi(b)|x)}{f(\phi(b)|x)}\right]$$

Let ζ and η be the signals at which $b^1_{\widehat{e}}(\zeta) = b^1_{\widehat{e}+1}(\eta) = c$. If $b^1_{\widehat{e}+1}(x) < b^1_{\widehat{e}}(x)$ for all $x \in [\phi^j_{\widehat{e}}(c), \phi^j_{\widehat{e}}(d)]$ then $\eta > \zeta$. In equilibrium $(\Pi^1_{\widehat{e}})'(x, b) \geqslant 0$ for $x \in [\phi^j_{\widehat{e}}(c), \phi^j_{\widehat{e}}(d)]$, ($\widehat{e}$ is possibly constrained from above) and $\left(\Pi^1_{\widehat{e}+1}\right)'(\eta, c) \leqslant 0$ ($\widehat{e}+1$ is possibly constrained from below). However, $F(\phi(b)|x)/f(\phi(b)|x)$ is decreasing in x and $v(x, \phi(b))$ is increasing in x. Therefore if $(\Pi^1_{\widehat{e}})'(\zeta, c) \geqslant 0$ then $\left(\Pi^1_{\widehat{e}+1}\right)'(\eta, c) > 0$, a contradiction. The intuition is simple. For bids in the interval $[c, d]$ the profit functions on units $\widehat{e}+1$ and $\widehat{e}$ are identical. Hence, whatever is optimal for unit $\widehat{e}+1$ in this interval is also optimal for unit $\widehat{e}$ and $b^j_{\widehat{e}+1}(x) = b^j_{\widehat{e}}(x)$ for all $x \in [\phi^j_{\widehat{e}}(c), \phi^j_{\widehat{e}}(d)]$.

Since the profit functions for units $e-1$ and e must be the same over this interval and the optimal bids must also be the same, unit e cannot be constrained above by $e-1$.

Hence the equilibrium must be a solution to the set of differential equations given by (4) and (5).

Lemma 1 *In equilibrium $b^1_1(\overline{x}) = b^2_1(\overline{x}) = ... = b^1_k(\overline{x}) = b^2_k(\overline{x})$.*

Proof. Assume that we have an equilibrium with a decreasing bid schedule at $\overline{x}$. Let $\overline{b}$ be the bid on the marginal unsuccessful bid when both signals are $\overline{x}$. This must also be the highest bid. Any bid at or above this is going to be successful with probability 1. Bidding above $\overline{b}$ only increases the amount the bidder pays. Hence there must be at least $k+1$ bids at $\overline{b}$. Let unit q be bidder i's highest ranked unit below $\overline{b}$, $b_q^i(\overline{x}) < \overline{b}$. Since there must be at least $k+1$ bids at $\overline{b}$, $b_{\widehat{q}+1}^j(\overline{x}) = b_{\widehat{q}}^j(\overline{x}) = \overline{b}$. Let x^* be the signal at which $b_{\widehat{q}}^j(x^*) = b_q^i(\overline{x})$. Then for $x \in [x^*, \overline{x}]$, unit $\widehat{q}$ is constrained below by $\widehat{q}+1$ (as bidder $j's$ unit $\widehat{q}$ is successful with probability 1 for $x > x^*$ and bidding above $b_q^i(\overline{x})$ only increases the amount the bidder pays). This is a contradiction to proposition 1. ■

Lemma 2 *The equilibrium is symmetric.*

Proof. From lemma 1 $b_1^1(\overline{x}) = b_1^2(\overline{x}) = ... = b_k^1(\overline{x}) = b_k^2(\overline{x}) = \overline{b}$. At the point $(\overline{x}, \overline{b})$, the system is locally Lipschitz and there is a unique solution over any interval $[G_e^1(\underline{x}) + \varepsilon, \overline{x}]$. Setting $b_e^1 = b_f^2$ results in equations (4) and (5) collapsing to

$$f(x|x)(v_p(x,x) - b(x)) - b'(x)F(x|x) = 0 \tag{8}$$

Imposing the boundary condition $b_e^1(\overline{x}) = b_f^2(\overline{x}) = \overline{b}$ implies that the unique solution of (4) and (5) is symmetric and is given by the solution of (8) with boundary condition $b(\overline{x}) = \overline{b}$. ■

Lemma 3 *The boundary condition for the system of differential equations must satisfy*

$$b_1^1(\underline{x}) = b_1^2(\underline{x}) = ... = b_k^1(\underline{x}) = b_k^2(\underline{x}) = v(\underline{x}, \underline{x}) \tag{9}$$

Proof. Assume we have a solution where $b_1^i(\underline{x}) > v(\underline{x}, \underline{x})$. Bidder i's expected surplus on unit 1 is

$$\int_{\underline{x}}^{x} (v(x,y) - b_1^1(x))f(y|x)dy$$

If $v(x,x) < b_1^i(x)$ then the expected surplus is negative so for $x \geqslant \underline{x}$, $v(x,x) \geqslant b_1^i(x)$. If $v(\underline{x}, \underline{x}) > b_1^i(\underline{x})$ then bidder i with signal $\underline{x}$ can bid above $b_1^i(\underline{x})$ and make a surplus with a positive probability. Hence $b_1^i(\underline{x}) = v(\underline{x}, \underline{x})$. Symmetry implies that this must be the boundary condition on all bids. ■

Note that the system is not Lipschitz continuous at the boundary. Hence, standard results on non-linear differential equations cannot be applied.

Proposition 2 *The unique equilibrium of the 2 bidder multi-unit discriminatory auction is* $b_1^1(x) = b_1^2(x) = b_2^1(x) = ... = b_k^2(x) = b(x)$ *where*

$$b(x) = v(x,x) - \int_{\underline{x}}^{x} \exp\left(-\int_{s}^{x} \frac{f(t|t)}{F(t|t)}dt\right) dv(s,s). \tag{10}$$

Proof. *As the Lipschitz condition does not hold, there may be many solutions with boundary condition (9). Assume there are two such solutions to (8),* $b_n(x)$ *and* $b_m(x)$*. Consider the solutions at* $\widehat{x}$ *where* $b_n(\widehat{x}) > b_m(\widehat{x})$*. From (8) this implies* $b_n'(\widehat{x}) < b_m'(\widehat{x})$*. The solution* $b_n(x)$ *must therefore be bounded away from* $b_m(x)$ *as* x *decreases which contradicts* $b_n(\underline{x}) = b_m(\underline{x})$*. Hence, if an equilibrium exists, it must be unique. Existence is established by solving (8) with boundary condition* $b(\underline{x}) = v(\underline{x}, \underline{x})$ *to give (10). We know that* $b(x)$ *is optimal in the range* $[b(\underline{x}), b(\overline{x})]$*. Bidding above* $b(\overline{x})$ *only increases the price paid and any bid below* $b(\underline{x})$ *has a zero probability of being successful.* ■

The proposition applies to the case where $k = 1$. Hence, this analysis shows that Milgrom and Weber's first-price model does not have an asymmetric equilibrium and that there is a unique symmetric equilibrium. In the multi-unit case with constant marginal valuations, the unique equilibrium simply involves submitting all k bids at this price.

Corollary 1 *The single unit rankings apply to the multi-unit discriminatory auction.*

This follows from uniqueness since the unique equilibrium is the same as in the single-unit first price auction and we can choose to bundle the units together and use any single unit auction. Milgrom and Weber [6] show that the first-price auction is dominated by the English auction in the single-unit case[4].

3 Treasury Auctions

Treasury auctions are commonly used around the world to sell Treasury bonds and other securities. The Treasury announces the number of bonds

[4] This ranking is limited to the symmetric equilibrium of the English auction. In the 2 bidder case there is also a continuum of asymmetric equilibria where one bidder plans to quit early and the other plans to quit late (for a given signal). Some of these equilibria are revenue dominated by the unique first-price equilibrium.

for sale, k. The bidders then compete for the bonds by submitting multiple price bids which are aggregated to give the market demand. The highest k bids win[5]. In practice two mechanisms have been used to price the bonds, a discriminatory auction where the successful bidders pay what they bid and a uniform price auction where they pay the bid on the marginal successful unit. The bonds are subsequently traded on the secondary market. If the secondary market is competitive then Treasury auctions can be modelled as multi-unit common value auctions. At the time of the auction there is uncertainty about the price at which they will trade on the secondary market. However, everyone agrees that the *true value* of each bond is the secondary market price.

The empirical evidence comparing the auction price to the secondary market price is inconclusive. Simon [11] estimates that the experiment of switching from a discriminatory auction to a uniform-price auction in the 1970s cost the US Treasury \$7000 to \$8000 for every \$1 million of bonds sold. However, Umlauf [13] estimates that the Mexican Treasury gained by switching to a uniform-price auction for the sale of 30-day bills although the gains were relatively small. Tenorio [12] looks at the Zambian government's sale of US dollars to importers who switched from a uniform-price auction to a discriminatory one. The conclusion after controlling for factors such as an increase in the number of dollars auctioned was that the switch resulted in a loss to the government even though the average price received per dollar was substantially increased. Goldreich [3] finds that the recent switch by the U.S. Treasury to a uniform-price auction[6] has increased revenue although the auction price remains significantly below the secondary market price.

Our results demonstrate that an ascending bid auction may do better than a discriminatory auction which has been used widely to sell Treasury securities. This simple model does not capture some important features of actual Treasury auctions such as asymmetric bidders (some bidders are better informed than others) and trading in the bonds before and after the auction. However, the intuition for the main result is compelling. Since the bidders do not learn anything of the other bidders' private signals, the winners' curse plays an important role in multi-unit sealed bid auctions just as it does in single-unit sealed bid auctions. If the bidders bid optimally, this leads to lower bids than the case where information about other signals is available.

[5] Or the bidders submit interest rate bids and the lowest bids win.

[6] In 1992 for 2 and 5 year bonds and 1998 for other securities.

Empirical support for the importance of private information in Treasury auctions is provided by Nyborg *et al* [8]. They study bidder behavior in Swedish Treasury auctions where a discriminatory rule is used. They find that increased volatility reduces the bids. This is consistent with bidders reducing their bids in response to the winners' curse. However, they also find that the bidders disperse their bids rather than submit flat demands as the model predicts and that this dispersion increases with volatility. The uniqueness of the equilibrium in the model follows from uniqueness of the single-unit auction equilibrium and because the constant marginal valuations assumption rules out any equilibrium where the units are constrained. The equilibrium is in flat demands whatever the level of volatility. The data therefore indicates decreasing marginal valuations which could be explained by risk aversion. If this is a significant factor in practice then apart from dispersing bids, risk aversion may reduce bid shading on the higher priced units relative to the risk-neutral case. Whether an ascending bid auction would increase revenue may then depend on the relative levels of private information and risk-aversion. There is clearly need for further theoretical and empirical research to address this question.

This paper shows that allowing agents to make multiple price/quantity bids rather than a single price bid makes no difference to the equilibrium in the 2-bidder constant marginal valuations model if a discriminatory auction is used. We know that enlarging the strategy space in this way can make things worse if a uniform-price rule is used (Wilson [15] and Back and Zender [1]). Although there are no examples of equilibria in the literature that we know of where the uniform auction does better than the first-price auction, such equilibria cannot be discounted. Hence a ranking of the ascending bid auction and uniform-price auction is not possible. Again further theoretical work is necessary.

A Appendix

Standard results rule out holes and mass points in the interior of the support of bid functions. However, we must be careful here as units may be constrained. Lemma 4 rules out holes when units are unconstrained below and lemma 5 rules out interior mass points. In what follows we are restricting attention to the class of non-decreasing strategies.

Lemma 4 *If in equilibrium units m and $\widehat{m}$ are unconstrained below over some interval of bids then there can be no holes in their supports over this interval.*

Proof. If only one firm has a hole over some interval of bids, the other firm will also not bid over such an interval as bidding less will not affect the probability of winning but will decrease the price. Assume that in equilibrium units m and $\widehat{m}$ are unconstrained below and there is a hole over the interval $[c, d]$ in both supports. First consider the case where both supports have a mass point at d. Let x be a signal at which bidder 1 bids unit m at d and $[s_1, s_2]$ the set of signals at which bidder 2 bids unit $\widehat{m}$ at d. The profit on unit m from bidding d minus the profit from bidding $d - \varepsilon$ converges to α times

$$\int_{s_1}^{s_2} (v(x, y) - d) f(y|x) dy \tag{11}$$

as ε converges to zero where α is the probability the unit is successful in a tie-break. If unit m is unconstrained above then the profit from bidding $d+\varepsilon$ minus the profit from bidding d converges to $(1 - \alpha)$ times the above and this must be infinitesimal for d to be an optimal bid. Since $(1 - \alpha)$ is not infinitesimal, this implies that (11) must be infinitesimal. However bidding $(d + c)/2$ strictly increases profit as the probability of winning is the same as bidding $d - \varepsilon$ but the price paid is significantly less. If unit m is constrained above then unit $m - 1$ must also have a mass point at d. Let $[t_1, t_2]$ be the (possibly empty) set of signals at which bidder 2 bids unit $\widehat{m} + 1$ at d. If unit $m - 1$ is unconstrained above then the profit from bidding $d + \varepsilon$ minus the profit from bidding d converges to $(1 - \beta)$ times

$$\int_{t_1}^{t_2} (v(x, y) - d) f(y|x) dy \tag{12}$$

as ε converges to zero where β is the probability that unit $m - 1$ is successful in a tie-break. Now if (11) is negative then d is not an optimal bid as bidding below d increases profit. If it is infinitesimal the previous argument applies and bidding $(d+c)/2$ strictly increases profit as the probability of winning is the same as bidding $d-\varepsilon$ but the price paid is significantly less. Since unit m is constrained above we must rule out the possibility that (11) is positive. If it is positive then (12) cannot be negative as $b^2_{\widehat{m}} \geqslant b^2_{\widehat{m}+1}$ implies $t_2 \geqslant s_2$ and $t_1 \geqslant s_1$ and $(v(x,y)-d)$ is increasing in y. Bidder 1 can then do better by bidding both units m and $m-1$ at $d+\varepsilon$ and d is not an optimal bid. If unit $m-1$ is constrained above then repeat the argument for unit $m-2$ and so on.

If only bidder i has a mass point at d then reducing the bid from d to $(d+c)/2$ reduces the price without affecting the probability of winning. Finally consider the case where there are no mass points at d. Then bidding $(d+c)/2$ dominates bidding just above d as the price is significantly less and the change in the probability of winning can be made sufficiently small by looking at a bid sufficiently close to d. ■

Lemma 5 *There can be no interior mass points in the supports of* $\{b^1_1(x), ..., b^1_k(x)\}$ *and* $\{b^2_1(x), ..., b^2_k(x)\}$.

Proof. Assume that bidder 2 has an interior mass point at d in the support of $b^2_{\widehat{m}}$ and that $b^2_{\widehat{m}}$ is not constrained below by $b^2_{\widehat{m}+1}$.

We begin by considering the case where the support of bidder 1's unit m does not have a mass point at d. If this support has a hole over some interval $(c,d]$ then bidder 2 will strictly prefer bidding $(d+c)/2$ to d as the probability of winning is the same as bidding d but the price paid is significantly less.

Let x be a signal at which bidder 1 bids unit m just below d and $[s_1, s_2]$ the set of signals at which bidder 2 bids unit $\widehat{m}$ at d. The expression in (11) is the limiting difference in the profit on unit m from bidding just below d and just above d. If this is positive and unit m is not constrained above then it will not be optimal for bidder 1 to submit bids just below d as bidding at d increases profit. If unit m is constrained above by unit $m-1$ then the expression in (12) is the limiting difference in the profit on unit $m-1$ from bidding just below d and just above d. This can't be negative if (11) is positive. Hence if unit $m-1$ is constrained above then setting both units at d will increase profits. If unit $m-1$ is constrained above then repeat the argument for unit $m-2$ and so on. Hence if (11) is positive then bidder 1 will not submit bids just below d. This implies a hole in the support of unit m below d which contradicts lemma 4 (as both units would be unconstrained below).

If (11) is negative then the integrand must be negative at the lower bound of the integral since $v(x,y)$ is increasing in y. The limiting profit from bidding just below d is

$$\int_0^{s_1} (v(x,y) - d) f(y|x) dy.$$

Since the integrand is negative at the upper bound of the integral, profits must be decreasing in the bid. Unit m must then be constrained below by unit $m+1$ (as otherwise we would have a hole below d which would contradict lemma 4). The limiting profit on unit $m+1$ from bidding just below d is

$$\int_0^{r_1} (v(x,y) - d) f(y|x) dy.$$

Now since unit $m+1$ is competing against a higher ranked unit, $r_1 \leqslant s_1$. Hence the integrand must also be negative at the upper bound of this integral since $v(x,y)$ is increasing in y. This implies unit $m+1$ must be constrained below by unit $m+2$ and repeating the argument we end up at unit k which cannot be constrained below. Hence it is not possible for (11) to be negative in equilibrium.

If bidder 1 also has a mass point at d then we can repeat the above argument by considering a signal x at which bidder 1 bids unit m at d. The limiting difference between bidding just above d and bidding at d is $(1-\alpha)$ times (11) where α is the probability the unit is successful in a tie-break. If (11) is positive and unit m is not constrained above then bidding above d is better than bidding at d. If it is constrained above by unit $m-1$ and unit $m-1$ is not constrained above then bidding both units above d increases profits and so on. If (11) is negative then the limiting difference from bidding at d and just below d is negative (α times (11)). If unit m is unconstrained below then d is not an optimal bid. If it is constrained below by unit $m+1$ then the limiting difference from bidding unit $m+1$ at d and just below d is γ times

$$\int_{r_2}^{r_1} (v(x,y) - d) f(y|x) dy$$

where γ is the probability that unit $m+1$ is successful in a tie-break. Since unit $m+1$ is competing against a higher ranked unit $r_1 \leqslant s_1$ and $r_2 \leqslant s_2$. Hence if (11) is negative then this integral cannot be positive and if unit $m+1$ is unconstrained below then d is not an optimal bid as bidder 1 can increase profit by bidding both units m and $m+1$ just below d. If unit $m+1$ is constrained below then repeat the argument for unit $m+2$ and so on.

Finally, consider the case where bidder 2 has an interior mass point at d in the support of $b^2_{\widehat{m}}$ and $b^2_{\widehat{m}}$ is constrained below by $b^2_{\widehat{m}+1}$. This implies that there must also be an interior mass point at d in the support of $b^2_{\widehat{m}+1}$. If $b^2_{\widehat{m}+1}$ is not constrained below we can rule out this mass point as we did with $b^2_{\widehat{m}}$. If it is constrained below then we can repeat the argument for $b^2_{\widehat{m}+2}$ and so on. Since unit k is not constrained below it is not possible to have an interior mass point in equilibrium. ■

A.1 Proof of proposition 1

Consider the largest interval $[c, d]$ where bidder 2's units k and $k-1$ are locally constrained. Then $\phi^2_{k-1}(b) = \phi^2_k(b) = \phi(b)$ for all $b \in [c, d]$. We begin by showing that this implies $b^1_2(x) = b^1_1(x)$ for all $x \in [\phi^1_1(c), \phi^1_1(d)]$.

Consider the lowest open interval $(\psi, \omega) \in [c, d]$ where $b^1_2(x) < b^1_1(x)$ for all $x \in (\phi^1_1(\psi), \phi^1_1(\omega))$. The lower bound of the support of b^1_1 cannot be above ψ as bidder 2's units k and $k-1$ cannot be locally constrained below this lower bound (since $(\Pi^1_k)' = 0$). Now observe that since b^1_1 and b^2_k are unconstrained from below, there can be no holes in their supports over (ψ, ω) lemma 4. Since $\phi^2_{k-1}(b) = \phi^2_k(b)$ the same is true for b^2_{k-1}. However, we must be more careful with b^1_2 as it may be constrained below by b^1_3. The upper bound of b^1_2 cannot be below ψ as bidder 2 will want to bid unit $k-1$ at the lowest possible price above this upper bound implying it must be constrained by unit k and by assumption bidder 2's units k and $k-1$ are not locally constrained below c and $b^1_2(x) = b^1_1(x)$ for $x \in [c, \psi]$[7].

We now show that ψ must be in the support of b^1_2. If $\psi > c$ then this is true by assumption. If $\psi = c$ and there is a hole in the support of b^1_2 over $[\psi - \varepsilon, \psi]$ then bidder 2 will not bid unit $k-1$ over $[\psi - \varepsilon, \psi]$. If there is a hole in the support of both b^1_2 and b^2_{k-1} over $[\psi - \varepsilon, \psi]$ then there is also a hole in the support of b^2_k over $[\psi - \varepsilon, \psi]$. However, this is not possible in equilibrium as both b^2_k and b^1_1 are unconstrained over $[\psi - \varepsilon, \psi]$ (lemma 4). Hence ψ must be in the support of both b^1_2 and b^1_1.

Let ζ and η be the (highest) signals at which $b^1_1(\zeta) = b^1_2(\eta) = \psi$. We want to show that $\zeta = \eta$. Assume $\zeta < \eta$. Consider a decreasing sequence $\{\zeta^n\} \downarrow \zeta$.

[7]If bidder 2's unit $k-1$ is always constrained by unit k because bidder 1's upper bound on unit 2 is below the lower bound on units $k-1$ and k then bidding unit 2 at c is the same as bidding at this upper bound since in either case it is never successful.

From continuity $b_1^1(\zeta^n) \downarrow \psi$. Type ζ^n prefers bidding $b_1^1(\zeta^n)$ to ψ,

$$\int_0^{\phi(b_1^1(\zeta^n))} (v(\zeta^n, y) - b_1^1(\zeta^n)) f(y|\zeta^n) dy \geqslant \int_0^{\phi(\psi)} (v(\zeta^n, y) - \psi) f(y|\zeta^n) dy$$

Subtracting $\int_0^{\phi(b_1^1(\zeta^n))}(v(\zeta^n, y) - \psi)f(y|\zeta^n)dy$ from both sides, dividing by $b_1^1(\zeta^n) - \psi$ and rearranging gives

$$F(\phi(b_1^1(\zeta^n))|\zeta^n)) \leqslant \frac{1}{b_1^1(\zeta^n) - \psi} \int_{\phi(\psi)}^{\phi(b_1^1(\zeta^n))} (v(\zeta^n, y) - \psi) f(y|\zeta^n) dy$$

Taking the limit $\zeta^n \downarrow \zeta$ and rearranging gives

$$\liminf \frac{\phi(b_1^1(\zeta^n)) - \phi(\psi)}{b_1^1(\zeta^n) - \psi} \geqslant \frac{F(\phi(\psi)|\zeta))}{(v(\zeta, \phi(\psi)) - \psi) f(\phi(\psi)|\zeta)}$$

Now consider a sequence $\{b^n\} \downarrow \psi$. A type η prefers bidding ψ to b^n,

$$\int_0^{\phi(\psi)} (v(\eta, y) - \psi) f(y|\eta) dy \geqslant \int_0^{\phi(b^n)} (v(\eta, y) - b^n) f(y|\eta) dy$$

Subtracting $\int_0^{\phi(\psi)}(v(\eta, y) - b^n)f(y|\eta)dy$ from both sides, dividing by $b^n - \psi$ and rearranging gives

$$F(\phi(\psi)|\eta)) \geqslant \frac{1}{b^n - \psi} \int_{\phi(\psi)}^{\phi(b^n)} (v(\eta, y) - b^n)) f(y|\eta) dy$$

Taking the limit $b^n \downarrow \psi$

$$\limsup \frac{\phi(b^n) - \phi(\psi)}{b^n - \psi} \leqslant \frac{F(\phi(\psi)|\eta))}{(v(\eta, \phi(\psi)) - \psi) f(\phi(\psi)|\eta)}$$

However, $F(\phi(b)|x)/f(\phi(b)|x)$ is decreasing in x and $v(x, \phi(b))$ is increasing in x. Hence for $\eta > \zeta$

$$\frac{F(\phi(\psi)|\eta))}{(v(\eta, \phi(\psi)) - \psi) f(\phi(\psi)|\eta)} < \frac{F(\phi(\psi)|\zeta))}{(v(\zeta, \phi(\psi)) - \psi) f(\phi(\psi)|\zeta)}$$

which leads to a contradiction.

We have established that $b_2^1(\eta) = b_1^1(\eta) = \psi$ where η is the highest signal at which the units are submitted at ψ. Now consider a signal $\theta \in (\eta, \phi_1^1(\omega))$

and let $b_2^1(\theta) = \rho$ which by the previous argument must be greater than ψ and we are assuming it is less than $b_1^1(\theta)$. Since b_1^1 has full support on (ψ, ω) there must be a signal $\vartheta \in (\phi_1^1(\psi), \phi_1^1(\omega))$ such that $b_1^1(\vartheta) = \rho$. Now if we replace (ζ, η, ψ) with $(\vartheta, \theta, \rho)$ in the above proof (since ρ is in the support of both b_1^1 and b_2^1) we have ϑ=θ. Hence it follows that $b_2^1(x) = b_1^1(x)$ for all $x \in (\eta, \phi_1^1(\omega))$ and since the profit functions on bidder 2's units k and $k-1$ are equivalent for bids over such an interval they cannot be locally constrained.

This also implies that the support of b_2^1 has no holes over an interval where unit $k-1$ and k are bid at the same price over an atomless interval. We can therefore repeat the argument for $\phi_{k-2}^2(b)$, $\phi_{k-1}^2(b)$, $b_3^1(x)$ and $b_2^1(x)$ and so on and use a symmetric argument for $b_2^2(x)$, $b_1^2(x)$, $\phi_{k-1}^1(b)$ and $\phi_k^1(b)$ and so on.

Corollary 2 *In the class of non-decreasing strategies, the equilibrium strategies must be strictly increasing and differentiable.*

In the proof of proposition 1 we do not assume that $b(x)$ is differentiable or continuous. The results from the single unit literature can be applied since the competition between bidder 1's eth unit and bidder 2's $\widehat{e}$th unit is not constrained by other units. In particular, in the class of non-decreasing strategies, the equilibrium must be strictly increasing and differentiable, Lizzeri and Persico[5].

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