Noise Matters in Heterogeneous Populations

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Abstract

The concept of boundedly rational agents in evolutionary game theory has succeeded in producing clear results when traditional methodology was failing. However the majority of such papers have obtained their results when this bounded rationality itself vanishes. This paper considers whether such results are actually a good reflection of a population whose bounded rationality is *small*, but *non-vanishing*. We also look at a heterogenous population who play a co-ordination game but have conflicting interests, and investigate the stability of an equilibria where two strategies co-exist together. Firstly, I find that results using the standard vanishing noise approach can be *very different* from those obtained when noise is small but persistent. Secondly, when the results differ it is the non-vanishing noise approach which selects the co-existence equilibria. As recent economic and psychology studies highlight the irrationality of their human subjects, this paper seeks to further demonstrate that the literature needs to concentrate more on the analysis of truly noisy populations.

Keywords: Non-vanishing noise, equilibrium selection, strategy co-existence.

1 Introduction

Nash Equilibrium has been the corner stone of game theory, however the existence of multiple Nash equilibria in even the simplest of games has proved a stubborn obstacle for theorists. When a population of rational players are in one of the Nash equilibria, the population is stuck there.[1] This equilibria could be the least efficient. And the only factor determining the equilibria selected are the preliminary beliefs of the population.

Evolutionary game theory has led the quest to find more appealing solutions. The ground-breaking papers of Kandori, Mailath and Rob(1993) and Young(1993)[2] produced a significant insight. By introducing boundedly rational agents who occasionally make mistakes, a population now had the potential to move between multiple equilibria. This persistent noise gives the process life, allowing for the investigation of which equilibria the population is more likely to be near, independent of the initial conditions.[3]

However when analysing a population subject to persistent noise, which by its nature is continually moving between states,[4] it is difficult to obtain clear results of whereabouts it will be in the long run. KMRY overcame this issue by producing all their results from analysis as the noise level decreases to 0. Here, for a population of any size, a single state is solely selected in the long run as noise vanishes.[5]

[1] By definition, no-one has an incentive to deviate.

[2] KMRY from here on.

[3] The introduction of boundedly rational agents also served as a step to address the criticisms of analysis with hyperrational players, players of god like intelligence and endless time to use it.

[4] Each state specifies a different combination of who is playing each strategy.

[5] The trend of vanishing noise analysis has continued. Early vanishing noise papers including Ellsion(1993), Samuelson(1994), Begin and lipman(1996), Fernando and Vega-

In this paper we also consider an alternative method, in which we allow a constant level of noise, but let the population size increase without limit. With non-vanishing noise a single state can never be selected as the process is always in motion. Nevertheless, for any population size, it is likely that process spends more time in one neighborhood of the state space than another. Indeed, as we let the population size increase without limit, we find a single neighborhood where the process spends all of its time is selected.

I feel that as we are dealing with bounded rationality, the second method makes more intuitive sense. As vanishing noise results require the source of the dynamics to disappear, it seems that such results are only justified if they reflect those that would be obtained under small non-vanishing noise, with boundedly rational agents who actually do make mistakes occasionally.

And so the primary aim of the paper is to assess whether the two methods agree on the long run location of the process. Specifically, in a large population, is the state selected under vanishing noise always within the neighborhood selected under small non-vanishing noise?

Surprisingly, in our main result we find that in some circumstances a large population with a only a small amount of non-vanishing noise[6] will never be where vanishing noise analysis tells us, the two methods yielding completely different results. The single state chosen under vanishing noise can be very far from the neighborhood selected under non-vanishing noise and an increasingly large population.

And so here we see vanishing noise analysis can present a very mislead-

Redondo(96) and Ellsion(2000) and more recently Kolstad(2003), Myatt and Wallace(2003), Norman(2003a) and Hojman(2004).

[6] One mistake every one hundred periods in the example of section 3

ing portrayal of an actual boundedly rational society. Consequently I believe that there exists a dangerous trend in the literature to conduct vanishing noise analysis alone, without any consideration of how significantly their results may change under non-vanishing noise. Often there are no simulations or calculations in the papers.

To explain the second aim of the paper and our choice of model, let us consider the mobile phone market in the UK. The existence of high call charges to networks other than your own, entails that each individual prefers the whole population to be on their own network. Therefore the most efficient market set-up would be one where just one network exists. And indeed, in the homogenous population of KMRY we find that the population is only stable when the entire population plays one strategy. Yet interestingly, when we look at the actual mobile phone market we continually observe many networks co-existing together. And there are other important examples of such strategy co-existence. Most notably, we often see many different political and religious beliefs existing within a population,[7] and this lack of co-ordination can sometimes produce severe inefficiency. On a smaller scale, different members of a town will often choose to invest in different public goods. Even towns which follow two sports teams could well be better off with everyone supporting just one.[8]

Although there are probably several reasons for strategy co-existence, this paper wishes to explain such observations by allowing different people to like different things. And so we may see one section of society playing the strategy they prefer, while the rest of the population play a

[7] As an individual you often prefer more people in the population agreeing with your own beliefs, than less.

[8] Some UK populations are known as either a football town, or a rugby town.

different strategy which they favor. Therefore I choose a model which has 2 types of players within a population, differing in their preferences for two strategy choices. This increased heterogeneity creates a third equilibrium, a co-existence equilibrium, where both strategies are played in the population.

We find under both methods that the co-existence equilibria can easily be the long-run location of the process. Furthermore, we find that when a discrepancy between the two methods exists, it is the non-vanishing noise approach that selects the co-existence equilibrium, vanishing noise selecting one of the monomorphic equilibria instead. And so by considering a population that is both heterogenous and boundedly rational, we reveal that observing several strategies together is possible, and indeed likely. Therefore this type of equilibrium most likely plays a much larger role in more realistic populations, than homogenous populations under vanishing noise would suggest.

Analysis with non-vanishing noise is not unique to this paper. For example Benaim and Weibull [2003a,b] also keep noise constant while taking population size to infinity, as do Binmore and Samuelson(1997). Myatt and Wallace(1998), and Beggs(2002) also devote some attention to the concept. The most similar paper is probably that of Sandholm(2005), which also looks at constant noise while taking population size to infinity, showing that in a in homogenous population there can exist some type of game where it is possible for constant noise results to differ from those of vanishing noise. The quantal response literature (McKelvey and Palfrey, 1995) has also given us the best experimental evidence that positive noise does matter. Separately, the co-existence of strategies exist in Kolstad(2003) and Anwar(1999) to name two. In Norman(2003) the introduction of switching costs creates many stable points of co-existence, although the vanishing noise analysis employed showed that in the long run no time would be spent in these states.

Here we look to investigate a model with heterogeneity, co-existence equilibria and positive noise, and the paper reads as follows. In Section 2 we present the model of a boundedly rational population with players of two conflicting types who have a choice of two strategies, with similarities to Kolstad's(2003) fluid interaction. They play a game of co-ordination as in any period an agent's payoff is monotonically increasing in the number of other agents playing her chosen strategy.

In Section 3, we give a quick and clear illustration of our main results.

In Section 4, the framework of the analysis is set out. The selection results pertaining to when the population spends all its time near the co-existence equilibrium are obtained using the usual vanishing noise approach, then selection results are instead found with small non-vanishing noise, and the population size being allowed to increase without limit. The results are then compared, producing the main result.

Section 4 illustrates a sample of real calculations of boundedly rational populations, showing that the larger the rate of individual mistakes, the more likely the two strategies will co-exist. We also take a look at the survival of minorities groups. Section 5 concludes.

2 The Model

Let a single population of N players consist of two types of players, type 1 denoted T_1 and type 2, T_2. The game is essentially one of co-ordination as in any period the more players in the population playing an agent's current strategy, the higher is that agent's payoff. However, there are two different types of players who receive different payoffs each period. The payoff in any period t for a T_1 agent playing strategy s_i, $\pi_1^{s_i}$, is given by

$$\pi_1^{s_1} = \beta_1(z(t)-1)^\rho \quad (1)$$

$$\pi_1^{s_2} = \gamma_1(N-z(t)-1)^\rho \quad (2)$$

where z(t) represents the number of agents playing s_1 in period t and $\rho \; \epsilon \; \mathbb{R}^+$[9]. $\beta_1 > \gamma_1$ indicates that all T_1 agents have the same preference to co-ordinate on strategy 1 rather than 2.

The essential difference between T_2 and T_1 players is that T_2 agents prefer the population to co-ordinate by playing s_2 rather than s_1, while T_1 agents have the opposite preference.

And so we have it that the payoff for a T_2 agent playing strategy s_i, $\pi_2^{s_i}$, is given by

$$\pi_2^{s_1} = \beta_2(z(t)-1)^\rho \quad (3)$$

$$\pi_2^{s_2} = \gamma_2(N-z(t)-1)^\rho \quad (4)$$

where $\beta_2 < \gamma_2$.

[9]For most applications we would have $\rho \; \epsilon \; (0,1]$ but we leave $\rho > 1$ open for generality.

2.1 $\rho = 1$ and Pairwise Matching

Here we see that a special case of the model is the familiar idea of pairwise patching, that in each period an agent has an equal chance of playing a stage game with any other agent in the population.

While demonstrating this, let us consider an example. Competing cell phone companies often have far higher charges for calls to other networks than to calls to the same network, thus each call is a co-ordination game. Consider a heterogenous world where T_1 agents (person or firm) prefer the network orange over T-mobile (perhaps due to differing sms packages, etc), and T_2 agents favor T-mobile. Then stage game between the two is given by[10]

T_1, T_2	$Orange$	$T-mobile$
$Orange$	a, c	e, f
$T-mobile$	g, h	b, d

where $a > e,\ b > g,\ c > f,\ d > h$ indicates the co-ordination of the game and $a - g > b - e$ and $c - h < d - f$ reveals the differences preferences between the two types. Without loss of generality e, f, g and h can be set to 0, and therefore the three stage games are

T_1, T_2	Or	Tm
Or	a, c	$0, 0$
Tm	$0, 0$	b, d

T_1, T_1	Or	Tm
Or	a, a	$0, 0$
Tm	$0, 0$	b, b

T_2, T_2	Or	Tm
Or	c, c	$0, 0$
Tm	$0, 0$	d, d

The matching process is one phone call each period to any other member of the population (equally likely). The payoff represents the cheapness of the call rate to the individual. Every period each agent will decide whether to change his network or not depending on how many people

[10] There are two other equally important stage games, one between two T_1 agents and another between two T_2 agents.

are on each network and his preferences, experimenting on occasions. Now, by setting $\rho = 1$, $\beta_1 = \frac{a}{N-1}, \beta_2 = \frac{c}{N-1}, \gamma_1 = \frac{b}{N-1}$, and $\gamma_2 = \frac{d}{N-1}$ in equations 1 to 4, we have it that $\pi_j^{s_i}$ becomes the average expected payoff for a T_j agent playing strategy s_i for pairwise matching. And therefore pairwise matching is just a special case of the general model.

2.2 $\rho < 1$: A Public Smoking Example

For a further example let s_1 represent the choice of going to a smoking area and let s_2 represent going to a non-smoking area. Label T_1 agents as smokers, and T_2 agents as non-smokers. Set $\rho < 1$ and for interest consider smokers to be in the minority.

Consider in each period that two groups form within the population, one containing all the people who choose to congregate in the smoking area, the other containing those who choose not to. For instance we could be in a familiar office setting where during daily breaks most smoker types often congregate in a different area to non-smokers. Here the co-ordination payoffs in equations 1 to 4 could represent the value of forming and enjoying relationships with other members of the group. The more people in your group the better it is for you, but as you are unlikely to to talk to everyone $\rho < 1$ indicates that the value of having 3 members in your group rather than 2 exceeds that of acquiring an extra 20th member. You do not interact with people in the other group during breaks and so gain no payoff from them.

Each day an agent decides whether to convene in the smoking or non-smoking area, depending on how many people were in which group yesterday and her preferences. Occasionally experimenting with a different strategy.

One question is how will employees take their breaks in the long run, all in the non-smoking or smoking area, or will a co-existence of the two groups prevail? Another question is whether the population's level of bounded rationality will effect the answer.

Alternatively, one could imagine the population to be the regular members of a bar. And if a co-existence of smoking and non-smoking groups is prevailing even though it is socially inferior, then there may well be cause for a government body to step in and ban one of the strategies.

2.3 Players and the Stochastic Dynamics

The players chosen here are myopic in the sense that they believe the state of play will be the same as the previous period, $z(t-1)$, and so last periods play is the only factor effecting a player's decision this period. Therefore a T_1 agent's best response this period is

$$\begin{cases} s_1 & \text{if } z(t-1) > \frac{1}{1+(\frac{\beta_1}{\gamma_1})^{\frac{1}{\rho}}} N + \frac{(\frac{\beta_1}{\gamma_1})^{\frac{1}{\rho}}-1}{(\frac{\beta_1}{\gamma_1})^{\frac{1}{\rho}}+1} \equiv pN + \delta \\ s_2 & \text{Otherwise} \end{cases} \tag{5}$$

Note that $p < 0.5 \ \forall \ \beta_1 > \gamma_1$.

And similarly a T_2 agent's best response this period is[11].

$$\begin{cases} s_1 & \text{if } z(t-1) > \frac{1}{1+(\frac{\beta_2}{\gamma_2})^{\frac{1}{\rho}}} N + \frac{(\frac{\beta_2}{\gamma_2})^{\frac{1}{\rho}}-1}{(\frac{\beta_2}{\gamma_2})^{\frac{1}{\rho}}+1} \equiv qN + \zeta \\ s_2 & \text{Otherwise} \end{cases} \tag{6}$$

[11]Note from the pairwise matching of section 2.1, with $\beta_1 = \frac{a}{N-1}, \beta_2 = \frac{c}{N-1}, \gamma_1 = \frac{b}{N-1}$, $\gamma_2 = \frac{d}{N-1}$, that $p = \frac{1}{1+(\frac{\beta_1}{\gamma_1})^{\frac{1}{\rho}}} = \frac{b}{a+b}$ is the mixed equilibrium of the T_1, T_2 stage game where T_2 agents play s_1 with probability p.
And $q = \frac{1}{1+(\frac{\beta_2}{\gamma_2})^{\frac{1}{\rho}}} = \frac{d}{c+d}$ is the other mixed equilibrium where T_1 agents play s_1 with probability q.

where $q > 0.5 \; \forall \; \beta_2 < \gamma_2$

We now continue by denoting N_1 as the number of T_1 agents in a given population, and N_2 as the number of T_2 agents, such that $N = N_1 + N_2$. Define the proportion of T_1 agents in the population as $\alpha = N_1/N$. In any period let $z_1(t)$ be the number of T_1 agents playing s_1, and let $z_2(t)$ be the number of T_2 agents playing s_1 such that $z(t) = z_1(t) + z_2(t)$. As agents do not differentiate between other players types $z(t) = \{0, 1, ..., N\}$ can be seen to define the state of the process at any time t. In each period every player is able to choose a best response to last periods state of play.

There exists at least two stable points for the process, E_1 where all agents choose to play $s_1(z = 1)$[12] and E_2 where all choose $s_2(z = 0)$. Furthermore, for $pN + \delta < \alpha N < qN + \zeta$, there exists a third stable point of the process E_m at $z_1 = \alpha N$ and $z_2 = 0$. Here all T_1 agent's best response is to play s_1 as they believe enough agents will join them to make it worthwhile, while all T_2 agents choose their preferred strategy s_2. E_m is a steady state of co-existence of both strategies.[13]

We can now define the basins of attraction of the stable points of the process. Firstly let the basin of attraction of E_i be denoted by B_i. Then B_2 is defined by any state $z(t) \; \epsilon \; \{0, ..., [pN + \delta]_-\}$.[14] At any point in B_2 *all* agent's best response is to play s_2 next period. Similarly, B_m is given by $z(t) \; \epsilon \; \{[pN + \delta]_+ ..., [qN + \zeta]_-\}$ and B_1 by $z(t) \; \epsilon \; \{[qN + \zeta]_+, ..., N\}$. The state space $\emptyset = \{z_1(t) = 0, ..., \alpha N, z_2(t) = 0, ..., (1 - \alpha)N\}$ and the basins of attraction can therefore be illustrated by

[12] Let $z \equiv z(t)$.

[13] I will consider cases only where $pN + \delta < \alpha N < qN + \zeta$ holds, as other cases essentially reduce to a homogenous population as in KMRY.

[14] $[x]_-$ is the largest integer below x, and $[x]_+$ is the smallest integer above x, $[x]_- = [x]_+ - 1$.

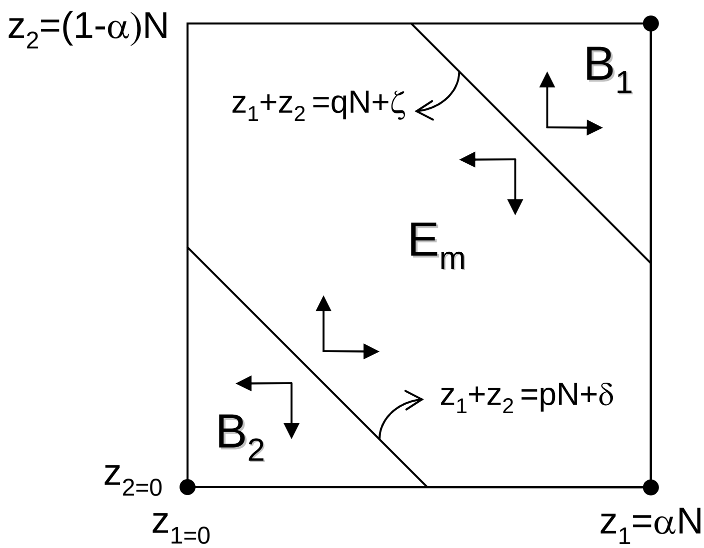


Figure 1: The State Space

Left alone the long run location of the process would depend only on which basin it was in initially. Instead, an element of bounded rationality is introduced. As in KMRY, an agent will select a strategy other than its best response with probability ε each period, I shall call such an event a mutation. Such mutations could be due to small temporary changes in circumstances for an individual. For instance, the smoking area is too cold for you one day so you go inside, your favorite football player is suspended and as a result you try a rugby game, or your mobile phone bill was unexpectedly expensive and so you change networks.[15]

Now via a certain number of mutations, it's possible for the process to leave its initial basin, and any other (often referred to as a basin jump). Indeed, the process is irreducible and aperiodic as it's possible to jump from any given state to any other state in one period, includ-

[15]This is my preferred interpretation. The more familiar story is that players experiment, just make mistakes or dye with probability 2ϵ and are replaced.

ing itself, and therefore the markov chain is ergodic. Perhaps the best way to visualize the markov chain is given by the simplified state space of $z(t) = \{0, 1, ..., N\}$ illustrated below, the larger arrows representing basin jumps, the smaller showing the flow of the basins.

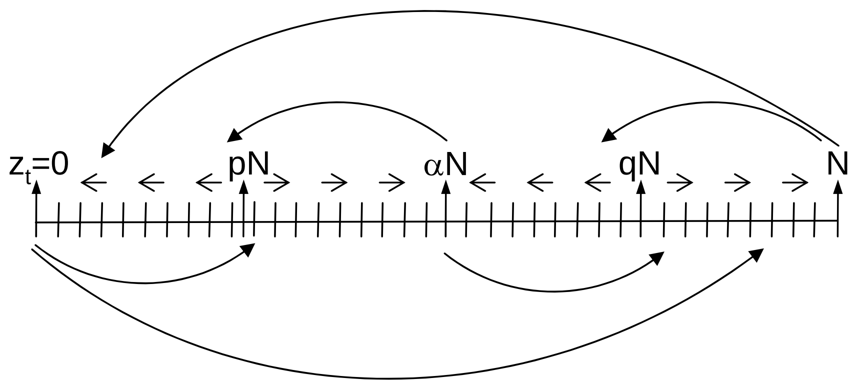


Figure 2: The Simplified State Space

And so we have a non-linear stochastic difference equation

$$z(t+1) = B(z(t)) + q(t) - r(t)$$

given $q(t) \sim Bin(N - B(z(t)), \varepsilon)$, $r(t) \sim Bin(B(z(t)), \varepsilon)$ and where $B(z(t))$ gives $z(t+1)$ when all agents (of both types) choose their best response to $z(t)$ last period without mutation. Thus we have a markov matrix Γ^ε with transition probabilities given by $\Gamma_{mn} = \mathbb{P}(z(t+1) = n|z(t) = m)$.

The long run behavior of the Markov chain is given by the stationary equations $\mu^\varepsilon\Gamma^\varepsilon = \mu^\varepsilon$, the solution μ^ϵ is stationary for fixed Γ^ε. Indeed, for an ergodic Markov chain μ^ε will be unique and therefore independent of the initial conditions. $\mu^\varepsilon = (\mu_1, \mu_2, ..., \mu_N)$ can be seen as the proportion of time society spends in each state $z = 1, 2, ..., N$.

Lemma 1 *The Markov chain on the finite state space* $Z = \{0,...N\}$ *defined by* Γ_{mn} *is ergodic . It therefore has a unique invariant distribution,* μ^{ε}.

Proof. This is a standard result. For example see Grimett and Stirzaker, 2001.

2.4 Welfare

Before continuing let us take the opportunity to discuss social welfare in different equilibria. Welfare in the co-existence equilibrium is often lower than the two pure equilibria as here the conflict between the two groups diminishes the network effect. Even though each agent type is playing their preferred strategy, they fail to co-ordinate with a whole section of society.

As the process is always dynamic when noise is allowed to stay constant, it is difficult to make precise statements on the welfare of the society in certain neighborhoods. However, for small values of noise we can say something of the total social welfare in each of the three stable states of the population.

Lemma 2 Let $\rho = 1$ and ε be small. Then,

a) for an equally distributed population such that $\alpha = 0.5$, if $\beta_1 \simeq \gamma_2$ then E_m is always the *worst* of the three equilibria in terms of total social welfare,

and,

b) for any value of α, β_1 and γ_2, E_m can never be the equilibria which maximises total social welfare.

Proof. Given in the Appendix. ■

3 An Illustration of the Results

Before we dive into the analysis, we present some calculations which demonstrate the stark difference in conclusions that vanishing and non-vanishing noise analysis can yield.

Consider a population with 50 T_1 and 50 T_2 players such that $\alpha = \frac{1}{2}$, who play the game below with the pairwise matching of section 2.1.[16]

T_1, T_2	s_1	s_2
s_1	$8,7$	$4,0$
s_2	$0,4$	$7,8$

The graph below shows that vanishing noise analysis concludes that the population will spend *none* of its time in the basin of attraction of E_m.

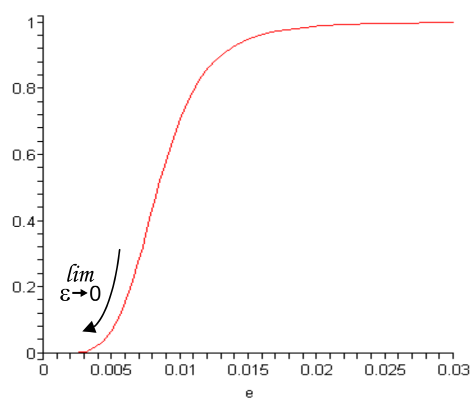


Figure 3: Time spent in the basin of E_m as noise vanishes

However, one can easily see from the graph that at an extremely small mutation rate of one in a hundred periods ($\varepsilon = 0.01$), the population in

[16]Note that the following results would be identical if instead the example was with $\rho = 0.5, \beta_1 = \gamma_2 = 1.63$ and $\beta_2 = \gamma_1 = 1$.

fact spends over 75% of its time in E_m's basin of attraction.

And so vanishing noise can portray a completely misleading picture of where a slightly noisy population will be in the long run.

In order to obtain clear results with positive levels of noise, we allow the population size to increase without limit. In fact, a large population with a mutation rate of one in a hundred periods will spend almost *all* of its time in the basin of E_m.[17]

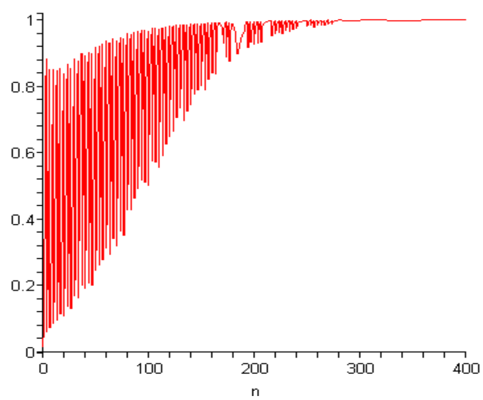


Figure 4: Time spent in the basin of E_m as the population increases, $\varepsilon = 0.01$.

And in terms of social welfare, this is the worst neighborhood for the population to be in.

[17]Note that in the graph below the large jumps in π_m as N increases are due to the discontinuous $[x]_+$ and $[x]_-$ functions which appear in all the markov probabilities. Also note the graph contains some non-integer values of N are not relevant, but do do little harm in illustrating the nature of the dynamics.

4 Analysis

As we will be dealing with positive levels of noise, the process will not converge to a single point. And so we divide the state space into three neighborhoods defined by the three basins of attraction, B_2, B_m and B_1. Indeed, under the best reply dynamics, at any state in a particular basin of attraction all agents of a particular type have the *same* best response next period. Therefore the number of mutations required to leave a basin, and the probability of this occurring, is the *same at any state* in that basin.

We can now consider just three states, $V = \{B_1, B_m, B_2\}$, and let $V(t)$ indicate which basin the population is in at time t. By defining the probability of leaving any state in basin i and entering any state in basin j by $p_{ij} = \mathbb{P}(V(t+1) = B_j | V(t) = B_i)$, we are able to simplify the whole state space into the three state markov chain below.

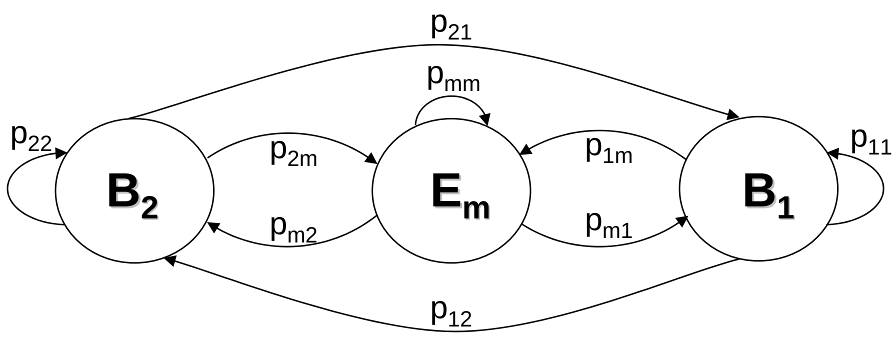


Figure 5: The Three State Markov Chain

Therefore we have a new ergodic markov chain whose long run behavior is given by the stationary equations $\pi^{\varepsilon} P^{\varepsilon} = \pi^{\varepsilon}$, where P^{ε} is the transition matrix containing the nine transition probabilities of p_{ij}, and π^{ε} is the unique solution for fixed P^{ε}. Here $\pi^{\varepsilon} = (\pi_1, \pi_m, \pi_2)$ can be seen as the proportion of time society spends in each neighborhood $V = B_1, B_m, B_2$.

The minimum number of mutations required to leave the basin of E_2 is $[pN]_+$, no matter which state of B_2 the process is in. Therefore the probability of escaping B_2 and entering B_m in any period is the simply the probability having between $[pN]_+$ and $[qN]_-$ mutations, and so

$$p_{2m} = \sum_{i=[pN+\delta]_+}^{[qN+\zeta]_-} (i\ N)\ \varepsilon^i\ (1-\varepsilon)^{N-i}$$

Similarly, the probability of escaping B_1 and entering B_m in any period requires between $N - [qN]_-$ and $N - [pN]_+$ mutations, and so

$$p_{1m} = \sum_{i=N-[qN+\zeta]_-}^{[N-[pN+\delta]_+} (i\ N)\ \varepsilon^i\ (1-\varepsilon)^{N-i}$$

Escaping from E_m is a more complicate affair. Two mutations from different types effectively cancel each other out as they will have no effect on the next period's state of play (there is no change in z). Therefore the process requires a net number of mutations in one direction to make such a jump.[18] And so the probability of escaping B_m and entering B_1 is given by

$$p_{m1} = \sum_{j=N-[qN+\zeta]_+}^{N} \sum_{k=max\{j-\alpha N,0\}}^{min\{j,N-i\}} (\alpha N, \alpha N + k - j)((1-\alpha)N - k\ N)$$
$$\varepsilon^{\alpha N+k-j}\ (1-\varepsilon)^{j-k}\ \varepsilon^k\ (1-\varepsilon)^{N-\alpha N-k}$$

The remaining transition probabilities, $p_{11}, p_{12}, p_{22}, p_{21}, p_{mm}$, and p_{m2} are given in the appendix.

[18] Consider that a minimum of 20 mutations are required to leave B_m and enter B_1. These 20 mutations must all be T_2 agents switching from their best response s_2 to s_1. Then, as each T_1 mutation 'cancels out' a T_2 mutation, the difference in the number T_2 and T_1 mutations must be at least 20 for the process to jump from B_m to B_1.

The time spent in the neighborhood B_m can be obtained from the transition probabilities alone.

Lemma 3

$$\pi_m = \frac{1}{1 + \frac{\frac{p_{m2}}{p_{2m}}(\frac{p_{12}}{p_{1m}}+1)+\frac{p_{21}}{p_{2m}}\frac{p_{m1}}{p_{1m}}}{\frac{p_{12}}{p_{1m}}+1+\frac{p_{21}}{p_{2m}}} + \frac{\frac{p_{m1}}{p_{1m}}(\frac{p_{21}}{p_{2m}}+1)+\frac{p_{12}}{p_{1m}}\frac{p_{m2}}{p_{2m}}}{\frac{p_{21}}{p_{2m}}+1+\frac{p_{12}}{p_{1m}}}}$$

Proof. See the appendix. ■

We divide the state space into neighborhoods as it allows the opportunity to obtain results with vanishing and non-vanishing non-vanishing noise. To achieve such results we now define the two most important terms of the paper.

For analysis the dynamics under vanishing noise we have the familiar notion of stochastic stability.

Definition 1 An equilibrium E_i is defined as being *stochastically stable* if [19]

$$\lim_{\varepsilon \to 0} \pi_i > 0$$

In order to investigate the dynamics under non-vanishing noise we introduce popular stability.

Definition 2 An equilibrium E_i is defined as being *popularly stable* at noise level ε if there exists ε such that

$$\lim_{N \to \infty} \pi_m > 0$$

As noise is not allowed to vanish for popular stability, the process is always dynamic and therefore no single state is selected as the population increases without limit. Instead an equilibrium's basin of attraction is

[19] In fact, for vanishing noise all the time is spent at the single state $z = \alpha N$. I have used π_i here for a clearer comparison of the two limiting techniques. Also, the usual definition stochastic stability states $\lim_{\varepsilon \to 0} \pi_z > 0$, for simplicity i wish to focus only on when *all* the time is spent in one area.

selected as we increase the population size.

The main aim of this paper is to investigate whether analysis of stochastic stability consistently yields the same conclusions as popular stability when ε is *small*.[20] In order to test this we focus on the conditions for which the long run location of the process is near E_m, for both limiting techniques.

We begin with the simpler case of stochastic stability.

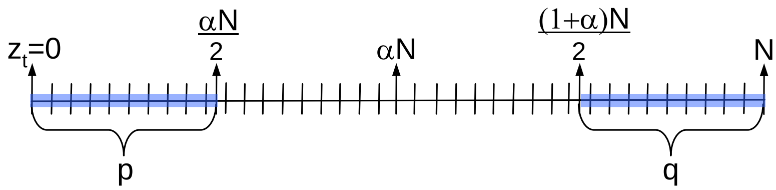
Proposition 1 *Under vanishing noise, E_m will be stochastically stable if and only if its basin B_m occupies at over half the markov space.*

$$\lim_{\varepsilon \to 0} \pi_m > 0 \text{ iff } p \leq \tfrac{\alpha}{2} \text{ and } q \geq \tfrac{1+\alpha}{2}.$$

Analysis under vanishing noise can be seen as simply counting and comparing the number of mutations needed to escape each basin. And so in order for E_m to be stochastically stable it must take more mutations to escape B_m(to B_1 or B_2) than any other adjacent basin escape. Let the transition (jump) from basin i into basin j be denoted by $B_{\overrightarrow{ij}}$. Then consider a scenario where the basin escape $B_{\overrightarrow{m1}}$ requires just one less mutation than $B_{\overrightarrow{1m}}$, for some N. Then in the limit of $\epsilon \to 0$, the 'cost' of this one extra mutation becomes infinitely large, overwhelming any other forces that could be in effect and ensuring that E_m is stochastically unstable. I shall refer to this force as the *basin size effect.* When obtaining results with vanishing noise the basin size effect is all that matters.

This can be seen by the illustration of Proposition 1, the shaded area representing the range of values that both p and q must take in order for E_m to be stochastically stable.

[20]The question of what constitutes small noise has no simple answer. With further experimental data we could perhaps replace the word small with realistic.



And the basin size effect remains a very powerful force when we examine popular stability. To see this consider that the basin escape $B_{\overrightarrow{m1}}$ requires 3 less mutations than $B_{\overrightarrow{1m}}$ for some $\tilde{N}$. Then at twice this population $B_{\overrightarrow{m1}}$ now requires 6 less mutations than $B_{\overrightarrow{1m}}$. 24 less at $4\tilde{N}$, and so on. Therefore in a boundedly rational population, the magnitude of the basin size effect rises linearly with N. And so it would seem that basin sizes again will be all that determines equilibrium selection.

However, there are other forces at work which are overwhelmed under vanishing noise, but have the ability under non-vanishing noise to alter selection against the basin size effect.

The first I shall call *the combination effect*. At any state in B_2 all N agents could experiment with s_1, while in B_m there are only αN agents able to experiment with s_2. This contributes towards there being many more combinations of mutations available for a $B_{\overrightarrow{2m}}$ jump than $B_{\overrightarrow{m2}}$. And as N rises this combinational difference also increases. In fact, this effect alone results in different popular and stochastic stability results. It is not hard to see that when a difference in equilibrium selection does occur it is popular stability that favors E_m.

The second effect I will call the *dis-coordination effect*. This is the effect, when in B_m only, of simultaneous mutations from both types canceling each other out, tending to make any B_m escape less probable at higher levels of noise. As N increases, the possible number of opposing mutations to any jump from B_m also increases. Again this effect strengthens E_m under small non-vanishing noise, and has the potential to change selection away from the stochastically stable equilibrium.

However, when trying to obtain popular stability results two main problems appear. Firstly, when analysing with vanishing noise, one may consider only the basin jumps requiring the minimum number of mutations, as all other possible jumps become negligible in the limit of $\varepsilon \to 0$. But under non-vanishing noise in the limit of $N \to \infty$, the probability of other possible jumps do not become negligible and so need to be included in the analysis. For instance, when analyzing the probability of jumping from $B_{\overrightarrow{1m}}$, one must consider the probability of jumping from B_1 to *any* state in B_m. As there exists many states in B_m, the number increasing in N, the calculation and analysis of basin escape probabilities can be complex.

The second main problem is that the positive probability of simultaneous mutations from both types complicates the basin escape probabilities from B_m further. These are not straight binomial probabilities, but the net of two binomials.

Such complications make the derivation of precise critical values (p, q, α and ε) for particular equilibrium selection under positive noise a complex task. However using the following lemma and proposition something can be said.

Lemma 4 *Let $Pr(S_n > r) = \sum_{v=0}^{\infty} b(r+v; v, l)$ where $b(r+v; v, l)$ is the binomial probability of exactly r+v successes from n trials with l being the probability of a success. Then*

$$P(S_n \geq r) \;\leq\; b(r; n, l)\, \tfrac{r(1-l)}{r-nl} \;\forall\; l < r.$$

Proof. This is a standard result. For example see Feller p.151. ■

Lemma 4 allows us to outline the general conditions for E_m to be popularly stable.

Proposition 2 $\lim_{N\to\infty} \pi_m \to 1$ *if and only if*

$$\begin{cases} \lim_{N\to\infty} \frac{p_{m2}}{p_{2m}} = 0 \ and \\ \lim_{N\to\infty} \frac{p_{m1}}{p_{1m}} = 0 \ \forall \ \epsilon < min(p, 1-q). \end{cases}$$

Proof. See the appendix. ■

Proposition 2 essentially explains that as p_{12} and p_{21} are relatively negligible, if the inflows into B_m progressively dominate outflows as the population increases, then the process will spend all its time in B_m.

We can now do more than just look at stochastic stability as we are able to determine a condition for the co-existence equilibrium to be popularly stable.

Proposition 3 If

$$\begin{aligned} x(\alpha, p, \varepsilon) &= \frac{\varepsilon^{2p-\alpha}(1-\varepsilon)^{1-2p}(\alpha-p)^{\alpha-p}}{\alpha^{\alpha}(1-p)^{(1-p)}} \geq 1 \ \ and \\ y(\alpha, q, \varepsilon) &= \frac{\varepsilon^{2q-(1+\alpha)}(1-\varepsilon)^{1-2q}(q-\alpha)^{q-\alpha}}{(1-\alpha)^{(1-\alpha)}q^q} \geq 1 \end{aligned}$$

then the time spent in B_m will be greater than that spent in B_1 or B_2 for $N > \tilde{N}$, p, q, α and $\varepsilon > 0$.

For increasingly large N, if the two above conditions are satisfied then

$$\lim_{N\to\infty} \pi_m = 1$$

and so E_m is popularly stable for p, q, α and noise level ε.

Proof. See Appendix ■

It should be understood that x and y are not the exact critical points determining equilibrium selection at noise level ε, for some $x < 0$ and $y < 0$ it is still very possible that $\lim_{N\to\infty} \pi_m \to 1$. Analysis in Proposition

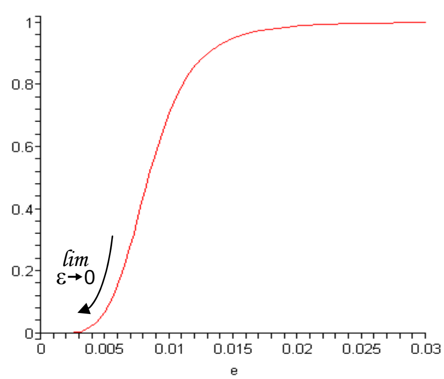
3 excludes the dis-coordination effect for tractability, and so does not capture the full strength of the co-existence equilibrium under small positive noise in an increasing population.

However Proposition 3 does allow us to establish our main result, that results of a process where noise vanishes completely can say very little of a population whose bounded rationality is innate.

Consider again the example in section 3 where $p = \frac{3}{11} = 0.27\dot{2}\dot{7}$ and $q = \frac{8}{11} = 0.72\dot{7}\dot{2}$.

T_1, T_2	s_1	s_2
s_1	$8, 7$	$4, 0$
s_2	$0, 4$	$7, 8$

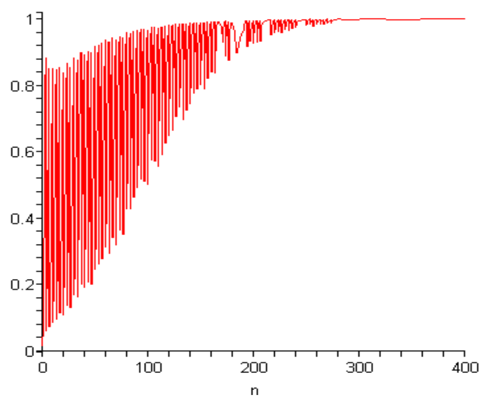
Here B_m contains less than half the markov space as both $p > \frac{\alpha}{2} = 0.25$ and $q < \frac{1+\alpha}{2} = 0.75$, from Proposition 2 we see that E_m is stochastically *un*stable. The calculations concur.



However, with a *small* non-vanishing experimentation rate of one in a 100 periods, $\varepsilon = 0.01$, we have it that

$$x(\alpha,p,\varepsilon) = \frac{0.01^{2.(0.273)-0.5}(1-0.01)^{1-2.(0.273)}(0.5-0.273)^{0.5-0.273}}{0.5^{0.5}(1-0.273)^{(1-0.273)}}$$
$$= y(\alpha,q,\varepsilon) = 0.0276 > 1$$

and from Proposition we see that E_m is popularly stable for $\varepsilon = 0.01$, and so the population will spend *all* its time in B_m as $N \to \infty$. The calculations agree.



Indeed, there exists a range of preferences for a large population where vanishing noise and small non-vanishing noise yield completely different results.

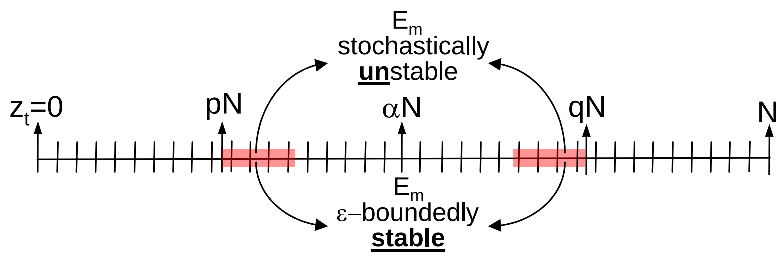
popularly

Theorem

For each $\varepsilon \,\epsilon\, (0, \min(\frac{\alpha}{2}, \frac{1+\alpha}{2}))$ *there exists a range of preferences corresponding to* $p \,\epsilon\, [\frac{\alpha}{2}, \frac{\alpha}{2} + \tau(\varepsilon))$ *and* $q \,\epsilon\, [\frac{1+\alpha}{2}, \frac{1+\alpha}{2} - \tau(\varepsilon))$, $\tau(\varepsilon) > 0$, *such that*

$$\begin{cases} \lim_{\epsilon \to 0} \pi_m = 0 & \forall\ \alpha, N,\ \textit{but} \\ \lim_{N \to \infty} \pi_m = 1 & \forall\ \alpha. \end{cases}$$

This theorem can be seen on the 1-dimensional diagram below, where the shaded region indicates for a given $\varepsilon > 0$ the range of p and q where the two limiting techniques give completely different results.



Corollary In the range $p \in [\frac{\alpha}{2}, \frac{\alpha}{2} + \tau(\varepsilon))$ and $q \in [\frac{1+\alpha}{2}, \frac{1+\alpha}{2} - \tau(\varepsilon))$, it is always popular stability that favors E_m as the long run equilibrium of the process, where as the stochastically stable state selects either E_1 or E_2.

Proof. As $\tau(\varepsilon) > 0$, this follows straight from the theorem. ■.

And so the technique of vanishing noise analysis does not show the true potential for boundedly rational populations to be caught in the neighborhood of the co-existence equilibria, often the worst place to be for the society to be. And so polymorphic states may play a much larger role than vanishing noise analysis would let us believe.

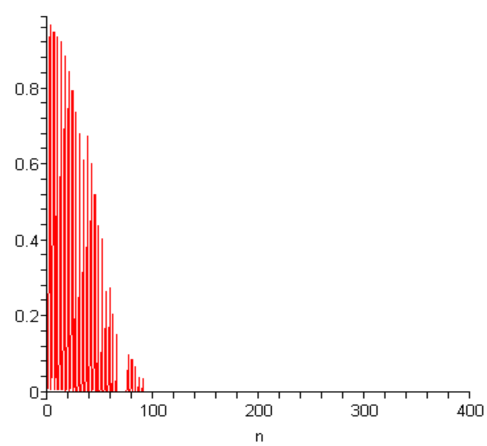
5 Calculations

In this section we examine a sample of calculations with $\rho = 0.5$. As will be seen, the more noisy a population becomes, the smaller B_m needs to be in order for the process to spend all its time there in a large population. Furthermore, we investigate a minority group who have strong preferences for their chosen strategy. In terms of social welfare, E_m is the worst equilibria for the society in every example.

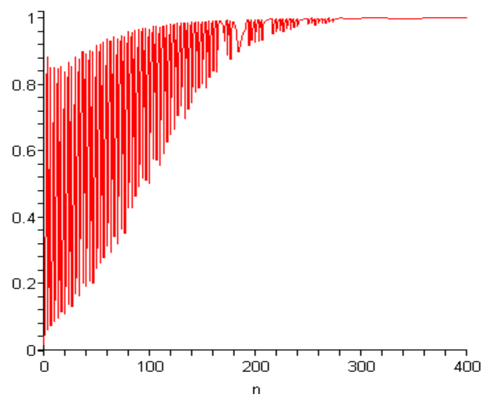
Consider a population where there exists a less intense difference in T_1 and T_2's preferences than in section 3 such that $\beta_1 = \gamma_2 = 1.58$ and $\beta_2 = \gamma_1 = 1$, giving $p = 0.286$ and $q = 0.714$.

Now at the experimentation rate of one per hundred periods, $\varepsilon = 0.01$, $\alpha = 0.5$ and N increasing we see that for large N the process will in fact spend none of its time in the basin of E_m.

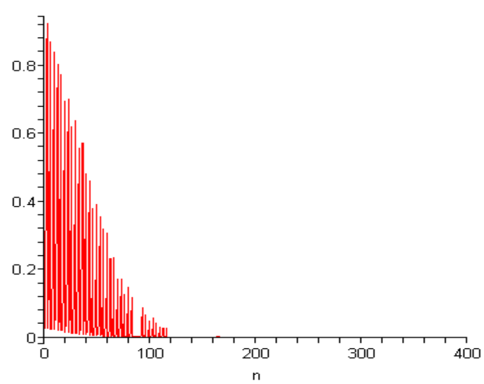
Note that $x(\frac{1}{2}, 0.286, 0.01) = y(\frac{1}{2}, 0.714, 0.01) = -0.0764 < 0$.



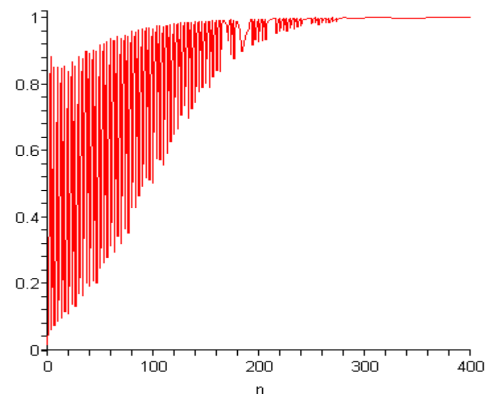
However, at a slightly larger mutation rate of 1 in 25, $\varepsilon = 0.04$, the population will again spend all of its time near E_m as the population grows large.
Here $x(\frac{1}{2}, 0.286, 0.01) = y(\frac{1}{2}, 0.714, 0.01) = 0.0094 > 0$.



But as the difference of T1 and T2's preferences become less intense still with $\beta_1 = \gamma_2 = 1.53$ and $\beta_2 = \gamma_1 = 1$, with non-vanishing noise $\varepsilon = 0.04$, again the process spends almost none of its time in the basin of E_m for large N.



Indeed the trend continues as when noise increases to $\varepsilon = 0.07$, E_m once more becomes the long run location of the population.

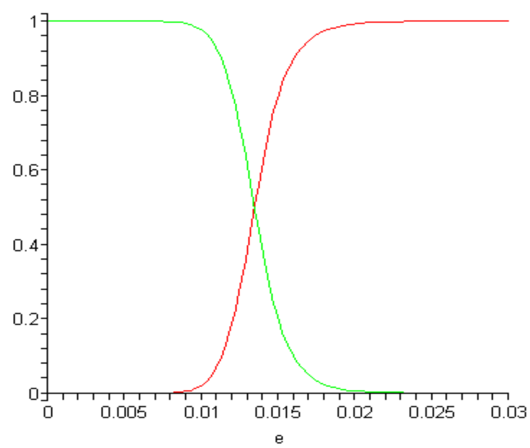


Consider that you are an observer, perhaps a member of a governing body who has no knowledge of the preferences of the population. Then one interpretation of the above trend is that the more noisy a population is, the more likely we are to find a large population spending nearly all of its time in a state of inefficient strategy co-existence.

5.1 Minority groups

So far our examples have focused upon an equal number of T_1 and T_2 agents with identically asymmetric preferences, but the model can easily lend itself to the analysis of minority groups. We again find that vanishing noise results can be a misleading portrayal of a boundedly rational population.

Consider twice as many T_2 agents than T_1 agents in a population of 90, but allow T_1 agents to have a stronger preference for their favoured strategy such that $\beta_1 = 1.83, \gamma_1 = 1$ and $\gamma_2 = 1.3, \beta_1 = 1$.



where π_m is given by the red line and π_2 by the green.

We see that results from vanishing noise analysis convey that the minority group's strong preferences have no influence over the state of the population, if they were indifferent between s_1 and s_2 they would be in the same position. However, in a population with some positive noise the minority group does have some sway in the population. At a small noise level of 0.02 we see that the population will spend almost all of its time near E_m, a significantly better location for the minority to be in.

6 Conclusions

This paper is motivated by the belief that conclusions drawn from vanishing noise results can often be surprisingly misleading when used to determine the nature of a truly boundedly rational population, even when this bounded rationality is small.

I investigate a typical KMRY type model with a population playing a 2x2 co-ordination game. The introduction of slight player heterogeneity creates a further steady state of co-existence of the two strategies.

By using a large population of agents who play a best response each period with probability 1-ε, I have been able to obtain results from both vanishing *and* non-vanishing noise techniques, and can therefore compare the two.

I find that the two do not yield the same results. Indeed, there exists a range of population preferences where the two methods produce *completely different conclusions.* Vanishing noise analysis telling us that the population will spend all of its time co-ordinating on one strategy, while under small non-vanishing noise the population will in fact always be close to the co-existence steady state.

The reason for the startling difference between the methods is that the limiting procedure of vanishing noise is somewhat overpowering. There are important forces at work in the population dynamics which are simply overwhelmed and ignored when noise completely vanishes. However these forces can be of influence when noise is small, even when the population is large, and this is why we observe the disparity in the results of the two techniques.

Given that there exists such a discrepancy, this paper seeks to highlight a dangerous trend in the past literature to conduct vanishing noise analysis alone, with little consideration of how significantly results would change with just a small amount of non-vanishing noise. As more and more studies emphasize the irrationality of their human subjects, perhaps the focus in the literature should be turning to truly noisy populations.

Appendix

Proof of Lemma 1:

At E_1 total social welfare(sw) is given by $\alpha N\beta_1(N-1)+(1-\alpha)N\gamma_1(N-1)$.

At $E_m,\ z=\alpha N$, $sw=\alpha N\beta_1(\alpha N-1)+(1-\alpha)N\gamma_1((1-\alpha)N-1)$.

Therefore by setting $\alpha_1=\gamma_2$ we have it that $E_m^{sw}>E_1^{sw}$ iff $2\gamma_2\alpha^2+\gamma_2-3\gamma_2\alpha-\gamma_1+\gamma_1\alpha>0$.

a) At $\alpha=0.5,\ E_m^{sw}-E_1^{sw}=-0.5\ \gamma_1<0$ such that $E_1^{sw}>E_m^{sw}$ for all γ_1,β_2 where $\beta_1\simeq\gamma_1$ and $\alpha=0.5$.

Similarly at $\alpha=0.5,\ E_m^{sw}-E_2^{sw}=-0.5\ \beta_2<0$ such that $E_2^{sw}>E_m^{sw}$ for all γ_1,β_2 where $\beta_1\simeq\gamma_1$ and $\alpha=0.5$.

b) For any α, β_1 and γ_2 let $\gamma_1=\beta_2=0$.

Then $E_m^{sw}>E_1^{sw}$ iff $\alpha<\frac{d}{a+d}$.

Similarly $E_m^{sw}>E_2^{sw}$ iff $\alpha>\frac{d}{a+d}$.

Therefore there will always exist another equilibrium which is superior to E_m in terms of total social welfare. ■

Proof of Lemma 3

The stationary equations of the 3 state markov process are given by

1. $\pi_1=\pi_1p_{11}+\pi_2p_{21}+\pi_mp_{m1}$; 2. $\pi_2=\pi_1p_{12}+\pi_2p_{22}+\pi_mp_{m2}$;

3. $\pi_m=\pi_1p_{1m}+\pi_2p_{2m}+\pi_mp_{mm}$; 4. $\pi_1+\pi_2+\pi_m=1$,

and also note,

5. $p_{11}+p_{12}+p_{1m}=1,$ 6. $p_{21}+p_{22}+p_{2m}=1,$ 7. $p_{m1}+p_{m2}+p_{mm}=1.$

$1 \Rightarrow \pi_1(1-p_{11})=\pi_2 p_{21}+\pi_m p_{m1}$, and

$2 \Rightarrow \pi_2=\frac{\pi_1 p_{12}+\pi_m p_{M2}}{1-p_{22}}$.

Therefore 1 and 2 $\Rightarrow \pi_1=[\frac{\frac{p_{m1}(1-p_{22})}{p_{21}}+p_{m2}}{\frac{(1-p_{11})(1-p_{22})}{p_{21}}-p_{12}}],\ \pi_2=[\frac{\frac{p_{m1}}{p_{1m}}(\frac{p_{21}}{p_{2m}}+1)+\frac{p_{12}}{p_{1m}}\frac{p_{m2}}{p_{2m}}}{\frac{p_{21}}{p_{2m}}+1+\frac{p_{12}}{p_{1m}}}]\,\pi_m,$

from substituting in 5 and 6 and rearranging.

Symmetrically we have $\pi_2=[\frac{\frac{p_{m2}}{p_{2m}}(\frac{p_{12}}{p_{1m}}+1)+\frac{p_{21}}{p_{2m}}\frac{p_{m1}}{p_{1m}}}{\frac{p_{12}}{p_{1m}}+1+\frac{p_{21}}{p_{2m}}}]\pi_M$

From substituting both expressions into 4 we obtain our result. ■

Proof of Proposition 1

First consider $\frac{p_{m2}}{p_{2m}}$.

$$\begin{aligned}
p_{m2} &= \sum_{j=0}^{[pN+\delta]_-}\ \sum_{k=max\{j-\alpha N,0\}}^{min\{j,N-i\}} (\alpha N, \alpha N-[pN+\delta]_-)((1-\alpha)N-k,\ N) \\
&\quad \varepsilon^{\alpha N+k-j}\ (1-\varepsilon)^{j-k}\ \epsilon^{k}\ (1-\varepsilon)^{N-\alpha N-k} \\
&= \varepsilon^{\alpha N-[pN+\delta]_-}[(\alpha N, \alpha N-[pN+\delta]_-(1-\varepsilon)^{[pN+\delta]_-}(1-\varepsilon)^{(1-\alpha)N} + \\
&\quad \sum_{j=1}^{[pN+\delta]_-}\sum_{k=0}^{j}(\alpha N, \alpha N-[pN+\delta]_-)((1-\alpha)N-k,\ N)\ \varepsilon^{\alpha N+k-j} \\
&\quad (1-\varepsilon)^{j-k}\varepsilon^{k}\ (1-\varepsilon)^{N-\alpha N-k} \\
&\quad +\sum_{k=1}^{j}(\alpha N, \alpha N-[pN+\delta]_-)((1-\alpha)N-k,\ N)\ \varepsilon^{\alpha N+k-j} \\
&\quad (1-\varepsilon)^{j-k}\ \varepsilon^{k}\ (1-\varepsilon)^{N-\alpha N-k} \\
&\quad \equiv \varepsilon^{\alpha N-[pN+\delta]_-}[\rho f(1-\varepsilon)+\rho f(\varepsilon)]
\end{aligned}$$

where ρ is some function independent of ϵ, and,

$$\begin{aligned} p_{2m} &= \sum_{i=[pN+\delta]_+}^{[qN+\zeta]_-} (i\ N)\ \varepsilon^i\ (1-\varepsilon)^{N-i} \\ &= \varepsilon^{[pN+\delta]_+}[(N,[pN+\delta]_+)(1-\varepsilon)^{N-[pN+\delta]_+} + \sum_{i=[pN+\delta]_++1}^{[qN+\zeta]_-} (i\ N)\ \varepsilon^i\ (1-\varepsilon)^{N-i} \\ &\equiv \varepsilon^{[pN+\delta]_+}[\rho f(1-\varepsilon)+\rho(\varepsilon)]. \end{aligned}$$

And so,

$$\frac{p_{m2}}{p_{2m}} = \frac{\varepsilon^{\alpha N-[pN+\delta]_-}[\rho f(1-\varepsilon)+\rho f(var\epsilon)]}{\varepsilon^{[pN+\delta]_+}[\rho f(1-\varepsilon)+\rho f(\varepsilon)]} = \varepsilon^{(\alpha-2p)N-2\delta+(\gamma_1-\gamma_2)}\frac{[\rho f(1-\varepsilon)+\rho f(\varepsilon)]}{[\rho f(1-\varepsilon)+\rho f(\varepsilon)]}$$

by letting $[x]_- = x-\gamma_1$ and $[x]_+ = x+\gamma_2$ where $\gamma_1, \gamma_2 < 1$ for any x.

Therefore

$$\lim_{\varepsilon\to 0}\frac{p_{m2}}{p_{2m}} = \begin{cases} 0 & \text{if } p < \frac{\alpha}{2} - \frac{\delta+(\gamma_1-\gamma_2)}{N} \\ \infty & \text{if } p > \frac{\alpha}{2} - \frac{\delta+(\gamma_1-\gamma_2)}{N} \end{cases}$$

which essentially corresponds to

$$\lim_{\varepsilon\to 0}\frac{p_{m2}}{p_{2m}} = \begin{cases} 0 & \text{if } p < \frac{\alpha}{2} \\ \infty & \text{if } p > \frac{\alpha}{2}. \end{cases}$$

Similarly we have

$$\lim_{\varepsilon\to 0}\frac{p_{m1}}{p_{1m}} = \begin{cases} 0 & \text{if } q > \frac{1+\alpha}{2} \\ \infty & \text{if } q < \frac{1+\alpha}{2}. \end{cases}$$

Therefore as $lim_{\varepsilon\to 0}\frac{p_{21}}{p_{2m}} = 0$ and $lim_{\varepsilon\to 0}\frac{p_{12}}{p_{1m}} = 0\ \forall\ N$, from lemma 3 we have it that $\lim_{\varepsilon\to 0}\pi_m = 1$ requires $p < \frac{\alpha}{2}$ and $q > \frac{1+\alpha}{2}$

Proof of Proposition 2

Recall from lemma 1 that

$$\pi_M = \frac{1}{1+\frac{\frac{p_{M1}}{p_{1M}}(\frac{p_{21}}{p_{2M}}+1)+\frac{p_{12}}{p_{1M}}\frac{p_{M2}}{p_{2M}}}{\frac{p_{21}}{p_{2M}}+1+\frac{p_{12}}{p_{1M}}}+\frac{\frac{p_{M2}}{p_{2M}}(\frac{p_{12}}{p_{1M}}+1)+\frac{p_{21}}{p_{2M}}\frac{p_{M1}}{p_{1M}}}{\frac{p_{12}}{p_{1M}}+1+\frac{p_{21}}{p_{2M}}}} \equiv \frac{1}{1+\pi_1+\pi_2}.$$

First note that $\frac{p_{21}}{p_{2M}} < c \; \forall \; \epsilon < p, \; \epsilon < 1-q, \; \alpha$ and N, where c is some constant.

To see this first define

$p_{2m}^{j} = ([pN+\delta]_{+}+j, \; N) \; \varepsilon^{[pN+\delta]_{+}+j} \; (1-\varepsilon)^{N-[pN+\delta]_{+}+j}$ and

$p_{21}^{j} = ([qN+\zeta]_{+}+j, \; N) \; \varepsilon^{[qN+\zeta]_{+}+j} \; (1-\varepsilon)^{N-[qN+\zeta]_{+}-j}$

and then consider

$$\frac{p_{21}}{p_{2m}} = \frac{p_{21}^{0}+p_{21}^{1}+...+p_{21}^{N-[qn+\zeta]_{+}}}{p_{2m}^{0}+p_{2m}^{1}+...+p_{2m}^{[qn+\zeta]_{-}-[pn+\delta]_{+}}} < \frac{p_{21}^{0}[qN+\zeta]_{+}(1-\varepsilon)}{p_{2m}^{0}[qN+\zeta]_{-}N\varepsilon}$$

using lemma 1.

As $\lim_{N\to\infty} \frac{[qN+\zeta]_{+}(1-\epsilon)}{[qN+\zeta]_{-}N\epsilon} = \frac{q(1-\epsilon)}{q-\epsilon} < c_1$ and $\frac{p_{21}^{0}}{p_{2m}^{0}} < 1 \;\; \forall \; \epsilon < p$,

we have it that $\frac{p_{21}}{p_{2M}} < c \; \forall \; \epsilon < p$. And symmetrically $\frac{p_{21}}{p_{2M}} < \acute{c} \; \forall \; \epsilon < p$.

Therefore if $\frac{p_{m2}}{p_{2m}} \to 0$ and $\frac{p_{m1}}{p_{1m}} \to 0$ as $N \to \infty$, then

$$\lim_{N\to\infty} \pi_1 = \lim_{N\to\infty} \frac{\frac{p_{m1}}{p_{1m}}(\frac{p_{21}}{p_{2m}}+1)+\frac{p_{12}}{p_{1m}}\frac{p_{m2}}{p_{2m}}}{\frac{p_{21}}{p_{2m}}+1+\frac{p_{12}}{p_{1m}}} = \frac{0+0c+0\acute{c}}{0+1+0} = 0.$$

Similarly $\lim_{N\to\infty} \pi_2 = 0$.

And therefore $\lim_{N\to\infty} \pi_m = \frac{1}{1+0+0} = 1$. ∎

Proof of Proposition 3

First consider when $\frac{p_{2m}}{p_{m2}}$ is rising with N.

By considering that the probability of a basin escape is at all times higher under a constraint that no opposing T_2 mutations can occur in a period, we can deduce an upper bound

$$\begin{aligned}
p_{m2} &= \sum_{j=0}^{[pN+\delta]_-} \sum_{k=max\{j-\alpha N,0\}}^{min\{j,N-i\}} (\alpha N, \alpha N - [pN+\delta]_-)((1-\alpha)N-k,\ N) \\
&\quad \varepsilon^{\alpha N+k-j}\ (1-\varepsilon)^{j-k}\ \varepsilon^k\ (1-\varepsilon)^{N-\alpha N-k} \\
&< \sum_{j=0}^{[pN+\delta]_-} (\alpha N, \alpha N - [pN+\delta]_- + j)\varepsilon^{\alpha N-[pN+\delta]_-+j}(1-\varepsilon)^{[pN+\delta]_--j} \\
&< (\alpha N, \alpha N - [pN+\delta]_-)\varepsilon^{\alpha N-[pN+\delta]_-}(1-\varepsilon)^{[pN+\delta]_-}\frac{[pN+\delta]_-(1-\varepsilon)}{[pN+\delta]_- - N\varepsilon} \\
&\quad \forall\ \varepsilon,\ p,\ \alpha\ and\ N,
\end{aligned}$$

the third part coming from lemma 4.

Also,

$$\begin{aligned}
p_{2m} &= \sum_{j=0}^{[pN+\delta]_-} (N, [pN+\delta]_+ + j)\epsilon^{[pN+\delta]_-+j}(1-\varepsilon)^{N-[pN+\delta]_--j} \\
&> (N, [pN+\delta]_+)\varepsilon^{\alpha N-[pN+\delta]_-}(1-\varepsilon)^{[pN+\delta]_-}\forall\ \varepsilon,\ p,\ \alpha\ and\ N.
\end{aligned}$$

And so,

$$\begin{aligned}
\frac{p_{2m}}{p_{m2}} &> \frac{(N, [pN+\delta]_+)\epsilon^{[pN+\delta]_-}(1-\varepsilon)^{N-[pN+\delta]_-}}{(\alpha N, \alpha N - [pN+\delta]_-)\varepsilon^{\alpha N-[pN+\delta]_-}(1-\varepsilon)^{[pN+\delta]_-}\frac{[pN+\delta]_-(1-\varepsilon)}{[pN+\delta]_- - N\varepsilon}} \\
&\quad \forall\ \varepsilon,\ p,\ \alpha\ and\ N.
\end{aligned}$$

And so when the right-hand side of this inequality, label it λ, increases without bound as N $\to \infty$, then so must $\frac{p_{2m}}{p_{m2}} \to \infty$ as N $\to \infty$. And so we now investigate under which parameter values the right hand side increases without bound as N rises.

Now,

$$\begin{aligned}
\lambda &= \frac{(N, [pN+\delta]_+)\varepsilon^{[pN+\delta]_+}(1-\varepsilon)^{N-[pN+\delta]_+}}{(\alpha N, \alpha N - [pN+\delta]_-)\varepsilon^{\alpha N - [pN+\delta]_-}(1-\varepsilon)^{[pN+\delta]_-}\frac{[pN+\delta]_-(1-\varepsilon)}{[pN+\delta]_- - N\varepsilon}} \\
&= \varepsilon^{[pN+\delta]_- + [pN+\delta]_+ - \alpha N}(1-\varepsilon)^{N-[pN+\delta]_+ - [pN+\delta]_-} \\
&\quad \frac{N!(\alpha N - [pN+\delta]_-)![pN+\delta]_-!}{(\alpha N)!(N-[pN+\delta]_+)![pN+\delta]_+!}\,\frac{[pN+\delta]_- - N\varepsilon}{[pN+\delta]_-(1-\varepsilon)} \\
&= \varepsilon^{[pN+\delta]_- + [pN+\delta]_+ - \alpha N}(1-\varepsilon)^{N-[pN+\delta]_+ - [pN+\delta]_-} \\
&\quad \frac{N!(\alpha N - [pN+\delta]_-)!}{(\alpha N)![N - pN + \delta]_+]!([pN+\delta]_+ + 1)}\,\frac{[pN+\delta]_- - N\varepsilon}{[pN+\delta]_-(1-\varepsilon)}
\end{aligned}$$

as $\frac{[x]_-!}{[x]_+!} = \frac{1}{[x]_+ + 1}$.
And

$$\begin{aligned}
\lambda &= \varepsilon^{[pN+\delta]_- + [pN+\delta]_+ - \alpha N}(1-\varepsilon)^{N-[pN+\delta]_+ - [pN+\delta]_-} \\
&\quad \frac{N!(\alpha N - [pN+\delta]_-)!(N - [pN+\delta]_-)}{(\alpha N)![N - pN + \delta]_-]!([pN+\delta]_+ + 1)}\,\frac{[pN+\delta]_- - N\varepsilon}{[pN+\delta]_-(1-\varepsilon)}
\end{aligned}$$

as $(x - [k]_+)! = (x - [k]_- - 1)! = \frac{(x-[k]_-)!}{x-[k]_-}$.
Using $[x]_- = x - \gamma_1$ and $[x]_+ = x + \gamma_2$ where $\gamma_1, \gamma_2 < 1$ for any x, and taking natural logarithms of both sides we obtain

$$\begin{aligned}
\ln\lambda &= ((2p-\alpha)N - 2\delta + \gamma_1 - \gamma_2)\ln\varepsilon + ((1-2p)N - 2\delta + \gamma_1 - \gamma_2)\ln(1-\varepsilon) \\
&\quad + \ln N! - \ln(\alpha N)! \; + \ln(N - pN + \delta - \gamma_1)! + \ln\frac{(1-p)N + \delta - \gamma_1}{pN + \delta + \gamma_2 + 1} \\
&\quad + \ln(\alpha N - pN + \delta - \gamma_1)! + \ln\frac{(p-\varepsilon)N + \delta - \gamma_1}{(pN + \delta - \gamma_1)(1-\varepsilon)}.
\end{aligned}$$

As we shall be taking the limit of $N \to \infty$, we are able to make use of stirling's formula which states

$$\lim_{x\to\infty}\frac{\ln x!}{x\ln x - x} = 1$$

Substituting this in gives

$$\begin{aligned}\ln\lambda &= ((2p-\alpha)N - 2\delta + \gamma_1 - \gamma_2)\ln\varepsilon + ((1-2p)N - 2\delta + \gamma_1 - \gamma_2)\ln(1-\varepsilon) \\ &\quad +N(lnN-1) \ + (\alpha N - pN + \delta - \gamma_1)(\ln(\alpha N - pN + \delta - \gamma_1) - 1) + \\ &\quad \ln\frac{N - pN + \delta - \gamma_1}{pN + \delta + \gamma_2 + 1} - \alpha N \ln(\alpha N - 1) - \\ &\quad (N - pN + \delta - \gamma_1)\ln((N - pN + \delta - \gamma_1) - 1) + \ln\frac{(p-\varepsilon)N + \delta - \gamma_1}{(pN + \delta - \gamma_1)(1-\varepsilon)}.\end{aligned}$$

Now implementing

$$\ln(x+\xi) = \ln x + \frac{\xi}{x} - \frac{\xi^2}{2x^2} + \frac{\xi^3}{3x^3} - ...$$

gives

$$\begin{aligned}\lambda &= (2p-\alpha)N\ln\varepsilon + (1-2p)N\ln(1-\varepsilon) - \alpha N\ln\alpha - (1-p)N\ln(1-p) \\ &\quad +(\alpha-p)N\ln(\alpha-p) + N\ln N + (\alpha-p)N\ln N - \alpha N\ln N - (1-p)N\ln N \\ &\quad -N - (1-p)N + (\alpha-p)N + (\gamma_1 - 2\delta - \gamma_2)\ln\varepsilon + (\gamma_1 - 2\delta - \gamma_2)\ln(1-\varepsilon) \\ &\quad -(1-p)N(\frac{\delta-\gamma_1}{N} + \frac{(\delta-\gamma_1)^2}{N^2} + ...) + (\alpha-p)N(\frac{\delta-\gamma_1}{N} - \frac{(\delta-\gamma_1)^2}{N^2} + ...) \\ &\quad +\ln\frac{(1-p)N + \delta - \gamma_1}{pN + \delta + \gamma_2 + 1} + \ln\frac{(p-\varepsilon)N + \delta - \gamma_1}{(pN + \delta - \gamma_1)(1-\varepsilon)} + \gamma_1\end{aligned}$$

or,

$$\begin{aligned}\lambda &= \ln[\frac{\varepsilon^{2p-\alpha}(1-\varepsilon)^{1-2p}(1-p)^{(1-p)}}{(\alpha-p)^{\alpha-p}\alpha^{\alpha}}]N + (\gamma_1 - 2\delta - \gamma_2)\ln\varepsilon + (\gamma_1 - 2\delta - \gamma_2)\ln(1-\varepsilon) \\ &\quad -(1-p)(\delta - \gamma_1 - \frac{(\delta-\gamma_1)^2}{N} + ...) + (\alpha-p)(\delta - \gamma_1 - \frac{(\delta-\gamma_1)^2}{N} + ...) \\ &\quad +\ln\frac{(1-p)N + \delta - \gamma_1}{pN + \delta + \gamma_2 + 1} + \ln\frac{(p-\varepsilon)N + \delta - \gamma_1}{(pN + \delta - \gamma_1)(1-\varepsilon)} + \gamma_1\end{aligned}$$

In the limit as $N \to \infty$ we have it that

$$\ln\frac{(1-p)N + \delta - \gamma_1}{pN + \delta + \gamma_2 + 1} + \ln\frac{(p-\varepsilon)N + \delta - \gamma_1}{(pN + \delta - \gamma_1)(1-\varepsilon)} \to \ln\frac{(1-p)}{p} + \ln\frac{(p-\varepsilon)}{p(1-\varepsilon)} < c_2.$$

And so $\lim_{N\to\infty}\lambda \to \infty$, and therefore $\lim_{N\to\infty}\frac{p_{2m}}{p_{m2}} \to \infty$, if and only if

$$\ln[\frac{\varepsilon^{2p-\alpha}(1-\varepsilon)^{1-2p}(\alpha-p)^{\alpha-p}}{\alpha^{\alpha}(1-p)^{(1-p)}}] > 0.$$

Symmetrical analysis yields that if

$$\ln[\frac{\varepsilon^{2q-(1+\alpha)}(1-\varepsilon)^{1-2q}(q-\alpha)^{q-\alpha}}{(1-\alpha)^{(1-\alpha)}q^{q}}] > 0 \text{ then,}$$
$$\lim_{N\to\infty} \frac{p_{1m}}{p_{m1}} \to \infty.$$

And so from Proposition 2, if both the above conditions are satisfied then $\lim_{N\to\infty} \pi_m \to 1$. ■

Proof of Theorem

To prove the theorem first consider the following lemma.

Lemma 5 *At* $p = \frac{\alpha}{2}$ *and* $q = \frac{1+\alpha}{2}$, $\lim_{N\to\infty} \pi_m = 1 \;\forall\; \alpha$ *and* $0 < \epsilon \leq \min(\frac{\alpha}{2}, \frac{1+\alpha}{2})$.

Proof.

First let us consider $x(\alpha, p, \epsilon)$ from Proposition 3.

At $p = \frac{\alpha}{2}$, we have it that $x(\alpha, \frac{\alpha}{2}, \varepsilon) = \frac{\alpha}{2} ln\frac{\alpha}{2} - \alpha ln\alpha - (1 - \frac{\alpha}{2})ln(1 - \frac{\alpha}{2}) + (1-\alpha)ln(1-\varepsilon)$

$$= -\frac{\alpha}{2} ln 2\alpha - (1 - \frac{\alpha}{2})ln(1 - \frac{\alpha}{2}) + (1-\alpha)ln(1-\varepsilon).$$

Let $\varepsilon = \frac{\alpha}{2}$ for each α, then we have

$$x(\alpha, \frac{\alpha}{2}, \frac{\alpha}{2}) = -\frac{\alpha}{2} ln 2\alpha - \frac{\alpha}{2} ln(1 - \frac{\alpha}{2}) = -\frac{\alpha}{2} ln(2\alpha - \alpha^2) > 0 \;\forall\; \alpha \;\varepsilon\; (0, 1).$$

And as $\frac{\partial x(\alpha, \frac{\alpha}{2}, \varepsilon)}{\partial \varepsilon} < 0$ we have it that $x(\alpha, \frac{\alpha}{2}, \varepsilon) > 0 \;\forall\; \varepsilon \leq \frac{\alpha}{2}$.

Symmetrically, at $q = \frac{1+\alpha}{2}$, $y(\alpha, \frac{1+\alpha}{2}, \varepsilon) > 0$ for any $\varepsilon < \frac{1+\alpha}{2}$.

Therefore $x(\alpha, \frac{\alpha}{2}, \varepsilon) > 0$ and $y(\alpha, \frac{1+\alpha}{2}, \varepsilon) > 0 \;\forall\; \varepsilon < min(\frac{\alpha}{2}, \frac{1+\alpha}{2})$, and by Proposition 3 we are done.

Now we can prove the theorem.

Proposition 2 shows that $\lim_{\epsilon\to 0} \pi_m = 0 \;\forall\; \alpha, N$ for any $p > \frac{\alpha}{2}$ and/or $q < \frac{1+\alpha}{2}$.

Now consider $\lim_{N\to\infty} \pi_m = 1$ and $x(\alpha, p, \epsilon)$.

Let $\alpha = \frac{2p}{k}$ and consider $k \;\epsilon\; [1, 2)$ such that $\alpha \;\epsilon\; (p, 2p\,]$.

Fix p and consider α varying.

For each p fix ε at some $\overline{\varepsilon} \;\epsilon\; (0, \min(p, \frac{1-2p}{2}))$. [21]

Then,

$x(\frac{2p}{k}, p, \overline{\varepsilon}) = 2p(1-k^{-1})ln\overline{\varepsilon} + (1-2p)ln(1-\overline{\varepsilon}) + (\frac{2p}{k} - p)ln(\frac{2p}{k} - p) - \frac{2p}{k}ln\frac{2p}{k} - (1-p)ln(1-p)$. and so,

$\frac{\partial x(\frac{2p}{k}, p, \overline{\varepsilon})}{\partial k} = \frac{2p}{k^2}ln\overline{\varepsilon} - \frac{2p}{k^2}ln(\frac{2p}{k} - p) - \frac{2p}{k^2} + \frac{2p}{k^2}ln\frac{2p}{k} - -\frac{2p}{k^2} = \frac{2p}{k^2}[ln\overline{\varepsilon} - ln(\frac{2p}{k} - p) + ln\frac{2p}{k}]$

which is continuous $\forall\; k \;\epsilon\; [1, 2)$ provided $\varepsilon > 0$.

At $k = 1$, $\frac{\partial x(\frac{2p}{k}, p, \overline{\varepsilon})}{\partial k} = 2p[ln2\overline{\varepsilon}] > -\infty \;\forall\;\; \overline{\varepsilon} > 0$.

From lemma 5 we have $x(\frac{2p}{k}, p, \overline{\varepsilon}) > 0$ at $k = 1$, therefore for each ε there must exist some range of $k > 1$, $\tau(\varepsilon)$, where x is positive. I.E, there exists some range of $p \;\epsilon\; [\frac{\alpha}{2}, \frac{\alpha}{2} + \tau(\varepsilon)]$ such that $x > 0$.

Symmetrical analysis gives $y > 0$ for some range of at least $q \;\epsilon\; [\frac{1+\alpha}{2} - \tau(\varepsilon), \frac{1+\alpha}{2}]$ and from Proposition 3 we are done.■

[21]This comes from $\varepsilon \leq min(min(2p, \frac{1-2p}{2}), min(p, \frac{1-p}{2}))$

The other markov probabilities are given by:

$$p_{11} = \sum_{i=0}^{N-[qN+\zeta]_+} (i\ N)\ \epsilon^i\ (1-\epsilon)^{N-i}.$$

$$p_{12} = \sum_{i=N-[pN+\delta]_-}^{N} (i\ N)\epsilon^i\ (1-\epsilon)^{N-i}$$

$$p_{22} = \sum_{i=0}^{[pN+\delta]_-} (i\ N)\ \epsilon^i\ (1-\epsilon)^{N-i}.$$

$$p_{2m} = \sum_{i=[pN+\delta]_+}^{[qN+\zeta]_-} (i\ N)\ \epsilon^i\ (1-\epsilon)^{N-i}.$$

$$p_{mm} = \sum_{j=[pN+\delta]_+}^{j=[qN+\zeta]_-} \sum_{k=max\{j-\alpha N,0\}}^{min\{j,N-i\}} (\alpha N, \alpha N+k-j)((1-\alpha)N-k\ N) \\ \varepsilon^{\alpha N+k-j}\ (1-\varepsilon)^{j-k}\ \varepsilon^k\ (1-\varepsilon)^{N-\alpha N-k}.$$

$$p_{m2} = \sum_{j=\alpha N-[pN+\delta]_-}^{\alpha N} \sum_{k=max\{j-\alpha N,0\}}^{min\{j,N-i\}} (\alpha N, \alpha N+k-j)((1-\alpha)N-k\ N) \\ \varepsilon^{\alpha N+k-j}\ (1-\varepsilon)^{j-k}\ \varepsilon^k\ (1-\varepsilon)^{N-\alpha N-k}.$$

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