The Important Thing Is not (Always) Winning but Taking Part: Funding Public Goods with Contests[*]

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Abstract

This paper considers a public good game with heterogeneous endowments and incomplete information affected by extreme free-riding. We overcome this problem through the implementation of a contest in which several prizes may be awarded. We identify a monotone equilibrium, in which the contribution is strictly increasing in the endowment. We prove that it is optimal for the social planner to set the last prize equal to zero, but otherwise total expected contribution is invariant to the prize structure. Finally, we show that private provision via a contest Pareto-dominates public provision and is higher than the total contribution raised through a lottery.

1 Introduction

This paper looks at multiple prize contests as a way to overcome the free-riding problem. It is well known that the public good provision resulting from individual voluntary contributions is generally sub-optimal, because of the incentive to free-ride associated with positive externalities (e.g. see Bergstrom et al., 1986; Andreoni, 1988). While fund-raising mechanisms based on tax rewards and penalties can be designed to solve this problem (e.g. Groves and Ledyard, 1977; Walker, 1981), they are not available to private organisations with no coercive power, such as charities or civic groups. Contests as incentive mechanisms are different from the above solutions because no power to enforce sanctions is required on the part of the institution conducting the tournament. Contests are competitions in which agents spend resources in order to win one or more prizes. The main characteristic is that, independently of success, all participants bear some costs.

A number of recent studies have explored different incentive based fund-raising mechanisms both theoretically and through laboratory experiments. These studies have focused on the use of lotteries (e.g. Morgan, 2000; Morgan and Sefton, 2000),

the comparison between winner-pay and all-pay auctions with one prize (e.g. Goeree et al., 2005) and between lotteries and one prize all-pay auctions (e.g. Orzen, 2005).

Morgan (2000) studies a lottery mechanism where one prize is awarded as a way to overcome the free-riding problem. Contributions to the public good entitle to lottery tickets, one ticket is randomly drawn and holder wins the prize. The public good consists of the revenue of the lottery net of the value of the prize. He considers a model with quasilinear preferences where all players contribute the same amount in equilibrium, independently of their income.

Goeree et al. (2005) compare the results of a winner-pay and an all-pay auction with one prize with a lottery. They study a public good game with a linear production function where agents are unconstrained and have heterogeneous preferences which are private information. The authors show that the all-pay auction dominates both the other mechanisms.

The puzzle with the equilibrium defined in Morgan (2000) is that it predicts that agents with different incomes contribute the same amount. This pattern does not seem to properly describe reality, while it seems more plausible that individuals with higher incomes contribute more than what poorer ones do[1]. On the other hand, Goeree et al. (2005) do not explore the behaviour of agents with heterogeneous endowments. Contrary to these analyses, in the present study we identify an equilibrium in which the contribution is strictly increasing in the endowment. Furthermore, although there exists a large literature which analyses the use of multiple prize contests and tournaments as incentive schemes[2], their use as fund-raising mechanisms has not been explored.

Moldovanu and Sela (2001) study a multiple prize contest where unconstrained agents differ in the ability to exert effort, and the ability is private information. They show that when costs are either linear or concave allocating only one prize maximises the total expected effort exerted by the bidders, while when costs are convex more prizes could be optimal. However, the paper on multiple prize contests most closely related to ours is the one by Barut and Kovenock (1998), who study symmetric multiple prize all-pay auctions with complete information. They extend the analysis of first price all-pay auctions with complete information[3] and show that, when players are not constrained, only mixed strategy equilibria exist. Further, expected expenditures are maximised by driving the value of the lowest prize to zero, but are invariant across all configurations leaving the lowest value fixed and the sum of the values constant.

In this paper we consider a public good game with a linear production func-

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[1]Interestingly, so far experiments on lotteries as a way to finance public goods have only focused on the case of subjects with homogeneous endowments (see, for instance, Morgan and Sefton, 2000; Orzen, 2005).

[2]Applications have been made to promotions in labour markets (Lazear and Rosen, 1981), technological and research races (Wright, 1983; Dasgupta, 1986; Taylor, 1995; Fullerton and McAfee, 1999; Windham, 1999), credit markets (Broecker, 1990), and rent seeking (Tullock, 1980) among others.

[3]The first price all-pay auction with complete information has been utilised extensively in the literature (Dasgupta, 1986; Hillman and Samet, 1987; Hillman and Riley, 1989; Ellingsen, 1991; Baye et al., 1993). There exists no pure strategy Nash equilibrium and a complete characterisation of its equilibria appears in Baye et al. (1996).

tion (as in Goeree et al., 2005) where agents have heterogenous endowments which are private information. Such a game is a modified version of the game with complete information which is typically employed in public good experiments. Each agent chooses how much of her wealth to allocate to the public good; this money is multiplied by a parameter, which takes a value between one and the number of players, and shared equally among all the agents. The unique Nash equilibrium is to contribute nothing, although it is socially optimal to contribute all the wealth. We overcome this extreme free-riding via a contest where several prizes may be awarded. We assume that the social planner has access to a small[4] budget, which can be allocated in form of prizes. The first prize is awarded to the player who contributes the most, the second prize to the player with the second highest contribution and so on until all prizes are awarded. The social planner determines the prize structure in order to maximise expected total welfare net of the value of the total prize sum.

Heterogeneity and incomplete information enable us to characterise a monotone equilibrium, in which the contribution is strictly increasing in the endowment. Such an equilibrium is a purification of the mixed strategy equilibrium identified by Barut and Kovenock (1998) in a symmetric setting with complete information and unconstrained agents. Our prediction seems more plausible than a completely symmetric equilibrium, either in mixed or in pure strategies. We prove that it is optimal for the social planner to set the last prize equal to zero, but otherwise total expected contribution is independent of the distribution of the total prize sum among the prizes. We show that the contest is a budget balanced incentive mechanism: expected total contribution is higher than the value of the total prize sum. There exists a critical level of the budget under which the monotone equilibrium exists independently of the prize structure. For any possible distribution of the endowments we identify necessary and sufficient conditions for the total prize sum to be below this critical level. Finally we prove that private provision via a contest Pareto-dominates public provision and is higher than total contribution in a lottery.

The paper is structured as follows. In Section 2 we introduce a linear public good game with complete information, as it is usually employed in public good experiments. In Section 3 we present the model and identify the Nash equilibrium. In Section 4 we solve the designer's problem and discuss the existence of the equilibrium. Section 5 compares private provision via a contest with both public provision and private provision in a lottery. Section 6 concludes.

2   A Linear Public Good Game with Complete Information

In this section we present the game which is typically used in public good experiments. n subjects take part in the experiment. Each subject is endowed with the same amount of money z and simultaneously chooses how much of her wealth to

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[4]We focus on cases where fund-raisers auction prizes of relatively low value. This seems to be the main source of revenue for most charities. See for example Goeree et al (2005), who provide data showing that the vast majority of fund-raisers seek small donations from a large number of donors.

allocate to the public good; this money is multiplied by a parameter α and shared equally among all the subjects. Agent i's payoff can be described by

$$U_i = z - g_i + \alpha \frac{G}{n}$$

where g_i is i's contribution to the public good and $G = \sum_{i=1}^{n} g_i$ is the total level of public good. If $\alpha \in (1, n)$ an individual's opportunity cost of contributing to the public good exceeds the marginal return of investing in the public good. Thus, the unique Nash equilibrium of the game is to contribute nothing, while it is efficient to contribute all the wealth.

3   The Model

Let us consider n players. Each player i is assumed to have endowment z_i, which is private information. Endowments are drawn independently of each other from the interval $[0, 1]$ according to the distribution function $F(z)$, which is common knowledge, with mean $E[z]$. We assume that $F(z)$ has a continuous and bounded density $F'(z) > 0$. Players play a public good game in which each individual has to choose how much to contribute to the public good. At the same time they take part in a contest in which n prizes are awarded such that $\pi_1 \geq \cdots \geq \pi_{m-1} > \pi_m = \cdots = \pi_n \geq 0$, $1 < m \leq n$ and $\sum_{j=1}^{n} \pi_j = \Pi$. This assumption rules out the possibility of awarding n equal prizes and will enable us find an equilibrium. We will call $\pi = (\pi_1, \cdots, \pi_n) \in \mathbb{R}^n$ the vector of prizes. The player with the highest contribution wins π_1, the player with the second highest contribution wins π_2, and so on until all the prizes are allocated. For each player, a strategy $g(z)$ will be the contribution to the public good as a function of the player's endowment and the action space for player i will be the interval $[0, z_i]$. If player i, who has endowment z_i and contributes g_i, wins prize j her payoff is

$$U_i = z_i - g_i + \alpha \frac{G}{n} + \pi_j$$

where $\alpha \in (1, n)$. Given the value assumed by α, notice that the equilibrium in the absence of a contest is the same as in the game described in Section 2.

Each player i chooses her contribution in order to maximise expected utility (given the other players' contributions and given the values of the different prizes). We will assume that Π is exogenously determined. For a given value of Π, the social planner determines the number of prizes having positive value and the distribution of the total prize sum among the different prizes in order to maximise the expected value of total welfare net of the value of Π (given the players' equilibrium strategy functions).

We will focus on the case in which the equilibrium strategy $g(z)$ is less than z for any type z on the interval $[0, 1]$. In order to find the equilibrium of the game it

is useful to introduce the function

$$K(F(z)) = \sum_{i=1}^{n} \pi_i \binom{n-1}{i-1} (F(z))^{n-i}(1-F(z))^{i-1} \qquad (1)$$

Given a vector of prizes π, $K(F(z))$ is a linear combination of n order statistics with weights equal to the prizes. If all agents adopt the same strictly increasing strategy $g(z)$, $K(F(z))$ represents the expected prize of the player with endowment z. The following result will help us identify the equilibrium of the game.

Lemma 1 *The function $K(F(z))$ is strictly monotonic increasing in z.*

Proof. Let's consider z_i and z_j such that $0 \leq z_i < z_j \leq 1$. Given that $F(z_i) < F(z_j)$, and given the assumption that $\pi_1 \geq \cdots \geq \pi_{m-1} > \pi_m = \cdots = \pi_n \geq 0$, $1 < m \leq n$, $K(F(z_j))$ assigns higher weights than $K(F(z_i))$ to higher prizes and lower weights than $K(F(z_i))$ to lower prizes. Therefore $K(F(z_i)) < K(F(z_j))$. ∎

Given Lemma (1), at interior solutions for all players we are able to characterise a monotone equilibrium, in which the contribution is strictly increasing in the endowment. Later on we will identify necessary and sufficient conditions for its existence independently of the prize structure.

Proposition 1 *Given a vector π of prizes, at an interior solution for all players the game has a symmetric pure strategy equilibrium given by*

$$g(z) = \frac{n}{n-\alpha}(K(F(z)) - \pi_n)$$

Proof. The expected utility of a player from a choice g can be calculated as

$$E[U(z - g, \pi) \mid g, g_{-i}] =$$
$$z - g + \alpha\frac{G}{n} + (\Pr[1 \mid g, g_{-i}]\pi_1 + \Pr[2 \mid g, g_{-i}]\pi_2 + \cdots + \Pr[n \mid g, g_{-i}]\pi_n)$$

where $\Pr[j \mid g, g_{-i}]$ is the probability of a choice g being j-th highest conditional on the other strategies g_{-i}. If all agents adopt the same strictly increasing strategy $g(z)$, then the probability that a candidate with endowment z_i is higher ranked than another randomly chosen candidate is $\Pr[g(z_i) > g(z)] = \Pr[z_i > z] = F(z_i)$. Therefore

$$(\Pr[1 \mid g, g_{-i}]\pi_1 + \Pr[2 \mid g, g_{-i}]\pi_2 + ... + \Pr[n \mid g, g_{-i}]\pi_n) =$$
$$K(F(z)) = \sum_{i=1}^{n} \pi_i \binom{n-1}{i-1} (F(z))^{n-i}(1-F(z))^{i-1}$$

Now, given the common strategy $g(z)$, we suppose that an individual with endowment z chooses $g(\hat{z})$ for some $\hat{z}$, then her expected utility will be

$$z - g(\hat{z}) + \alpha\frac{G_{-i} + g(\hat{z})}{n} + K(F(\hat{z}))$$

where G_{-i} is the sum of the contributions of all the other players. Differentiating with respect to $\hat{z}$ we obtain

$$\frac{\alpha - n}{n}g'(\hat{z}) + K'(F(\hat{z}))F'(\hat{z})$$

In equilibrium the individual with endowment z should choose $g(z)$ so that the above will be equal to zero when $\hat{z} = z$, and we have

$$g'(z) = \frac{n}{n - \alpha}K'(F(z))F'(z)$$

A player with the lowest possible endowment $z = 0$ does not contribute to the public good and wins the last prize. This yields the boundary condition $g(0) = 0$. Hence, the solution is

$$g(z) = \frac{n}{n - \alpha}(K(F(z)) - \pi_n)$$

From Lemma (1) we know that the candidate equilibrium function g is strictly monotonic increasing.

Assuming that all players rather than i play according to g, we finally need to show that, for any type z of player i, the contribution $g(z)$ maximises the expected utility of that type. Let us consider an individual with endowment z. If she plays $g(z) = \frac{n}{n-\alpha}(K(F(z)) - \pi_n)$ her expected utility is given by

$$E[U(z, g(z)) \mid g_{-i}] = z - \frac{\alpha}{n - \alpha}K(F(z)) + \frac{n}{n - \alpha}\pi_n + \frac{\alpha}{n}G$$

If she deviates and plays $\frac{n}{n-\alpha}(K(F(\hat{z})) - \pi_n)$ for some $\hat{z} \neq z$ her expected utility will be

$$
\begin{aligned}
E[U(z, g(\hat{z})) \quad \mid \quad g_{-i}] = {} & \\
& z - \frac{n}{n - \alpha}K(F(\hat{z})) + \frac{n}{n - \alpha}\pi_n + \frac{\alpha}{n}(G - \frac{n}{n - \alpha}K(F(z)) \\
& + \frac{n}{n - \alpha}\pi_n + \frac{n}{n - \alpha}K(F(\hat{z})) - \frac{n}{n - \alpha}\pi_n) + K(F(\hat{z})) \\
= {} & z - \frac{\alpha}{n - \alpha}K(F(z)) + \frac{n}{n - \alpha}\pi_n + \frac{\alpha}{n}G
\end{aligned}
$$

Therefore she is indifferent to play any other strategy $\frac{n}{n-\alpha}(K(F(\hat{z})) - \pi_n)$. If her action space $[0, z]$ is a subset of the set g_{-i} this rules out the possibility that she might be better off deviating from $g(z)$. If $z > g(1)$ it is easy to show that she would be worse off playing any strategy greater than $g(1)$. In fact, playing $g(1)$ would already guarantee π_1 and any higher contribution would result in a lower expected utility. ∎

Notice that the equilibrium strategy function defined in Proposition (1) can be rearranged as $g(z) = \frac{1}{1-\frac{\alpha}{n}}(K(F(z)) - \pi_n)$. The latter is the sum of a convergent series with reason $\frac{\alpha}{n}$ and can be expressed as

$$g(z) = (K(F(z)) - \pi_n)\sum_{m=1}^{\infty}(\frac{\alpha}{n})^{m-1}$$

The first part of the above equation represents the expected prize, in equilibrium, of a player with endowment z, net of the value of the last prize. This is because a contribution equal to zero would guarantee the agent to win the lowest prize. The multiplier $\sum_{m=1}^{\infty} (\frac{\alpha}{n})^{m-1}$ reflects the return to investment in the public good. In standard all-pay auctions an agent bids her expected prize in equilibrium. In our model, if an agent contributes up to her expected prize (net of the last prize) she receives back $\frac{\alpha}{n}$ times the value of her bid because of the public good. This implies that she will add to her contribution this remaining value, from which she will get back an equal proportion, and so on.

4 Designer's Problem and Revenue Equivalence

In this section we consider the maximisation problem faced by the designer, assuming that wealth constraints are non-binding for all players. We will then discuss the conditions which guarantee the existence of the equilibrium independently of the allocation of Π across the prizes.

Recall that the social planner determines the number of prizes having positive value and the distribution of the total prize sum among the different prizes in order to maximise expected total welfare net of the value of Π (given the players' equilibrium strategy functions). In order to analyse the maximisation problem we let the vector of prizes π be variable, maintaining the assumptions that $\pi_1 \geq \cdots \geq \pi_{m-1} > \pi_m = \cdots = \pi_n \geq 0$, $1 < m \leq n$ and $\sum_{j=1}^{n} \pi_j = \Pi$, and we study the family of functions

$$
\begin{aligned}
\phi(F(z), \pi \quad | \quad & \sum_{j=1}^{n} \pi_j = \Pi, \pi_1 \geq \cdots \geq \pi_{m-1} > \pi_m = \\
\cdots \quad = \quad & \pi_n \geq 0, 1 < m \leq n) = \\
& \frac{n}{n-\alpha} \sum_{i=1}^{n} \pi_i \binom{n-1}{i-1} (F(z))^{n-i}(1-F(z))^{i-1}
\end{aligned} \tag{2}
$$

Notice that, if π were fixed expression (2) would reduce to $K(F(z))$, as presented in equation (1). Letting the vector of prizes π be variable, at an interior solution for all players, the equilibrium strategy is represented by the following[5]

$$
\begin{aligned}
g(z, \pi) \quad = \quad & \frac{n}{n-\alpha}(\phi(F(z), \pi) - \pi_n) = \\
& \frac{n}{n-\alpha} \sum_{i=1}^{n} \pi_i \binom{n-1}{i-1}(F(z))^{n-i}(1-F(z))^{i-1} - \frac{n}{n-\alpha}\pi_n
\end{aligned}
$$

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[5] For simplicity of notation, unless differently specified, from now on we will refer to $\phi(F(z), \pi \mid \sum_{j=1}^{n} \pi_j = \Pi, \pi_1 \geq \cdots \geq \pi_{m-1} > \pi_m = \cdots = \pi_n \geq 0, 1 < m \leq n)$ as $\phi(F(z), \pi)$.

The social planner faces the following maximisation problem

$$\max_{\pi} W = n \int_0^1 (z - g(z, \pi) + \frac{\alpha}{n} G + \phi(F(z), \pi)) F'(z) dz - \Pi \qquad (3)$$

Recall that in equilibrium $\phi(F(z), \pi)$ represents the expected prize of a player with endowment z. This means that independently of the distribution of the total prize sum across the prizes the following holds

$$n \int_0^1 \phi(F(z), \pi) F'(z) dz = \Pi \qquad (4)$$

Therefore the expected total contribution in equilibrium is equal to

$$G = n \int_0^1 g(z, \pi) F'(z) dz = \frac{n}{n - \alpha} \Pi - \frac{n^2}{n - \alpha} \pi_n \qquad (5)$$

Note that the above expression depends only on the total prize sum and on the value of the last prize. Finally we can rearrange expression (3) as

$$\max_{\pi} W = n \int_0^1 (z - g(z, \pi) + \frac{\alpha}{n - \alpha} \Pi - \frac{\alpha n}{n - \alpha} \pi_n) F'(z) dz \qquad (6)$$

and we can state the following result.

Proposition 2 *At an interior solution for all players the social planner optimally sets the last prize equal to zero, but otherwise expected total contribution is $G = \frac{n}{n-\alpha} \Pi$ independently of the distribution of the total prize sum among the prizes.*

Proof. Expression (6) can be rewritten as

$$\begin{aligned}
\max_{\pi} W &= n \int_0^1 (z - \frac{n}{n - \alpha} \phi(F(z), \pi) + \qquad (7) \\
&\quad \frac{n}{n - \alpha} \pi_n + \frac{\alpha}{n - \alpha} \Pi - \\
\frac{\alpha n}{n - \alpha} \pi_n) F'(z) dz &= nE[z] + \frac{n(\alpha - 1)}{n - \alpha} (\Pi - n\pi_n)
\end{aligned}$$

It is obvious that $\pi_n = 0$ maximises the above expression. Further, from equation (5) we know that total expected contribution equals $\frac{n}{n-\alpha} \Pi$ when $\pi_n = 0$. ∎

Total expected contribution is higher than the total prize sum. While the standard result in all-pay auctions is the total dissipation of rent, in our model over-dissipation occurs because of the marginal return of investing in the public good. Furthermore, from expression (7) we can see that in equilibrium total expected welfare net of the value of Π equals $nE[z] + \frac{n(\alpha-1)}{n-\alpha} \Pi$, where $nE[z]$ represents initial welfare. This implies that the contest is a budget balanced incentive mechanism: the social planner does not need to possess Π in the first place, but can simply detract it from the total contribution.

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Corollary 1 *The contest is a budget balanced incentive mechanism. At an interior solution for all players, provided that the social planner optimally sets the last prize equal to zero, total expected welfare net of the value of Π is higher than initial welfare and equals $nE[z] + \frac{n(\alpha-1)}{n-\alpha}\Pi$.*

So far we have assumed that wealth constraints are non-binding for all agents. In order for the revenue equivalence to hold the solution must be interior for all players for any possible distribution of the total prize sum among the prizes. Continuity together with the assumption that $F(z)$ has a bounded density guarantee the existence of the equilibrium, independently of the allocation of prizes, if Π is small enough.

Proposition 3 *There exists a critical level $\bar{\Pi}$ such that the equilibrium strategy function is interior for all players independently of the distribution of the total prize sum across the first $n-1$ prizes if and only if $\Pi \leq \bar{\Pi}$.*

Proposition (8) in Appendix provides necessary and sufficient conditions for Π to be below such a critical value independently of the distribution of Π among the prizes, provided that the social planner optimally sets the last prize equal to zero.

5 Contest versus Public Provision and Lottery

We are going to compare the result obtained through a contest with both the welfare generated by public provision and the total contribution resulting from the use of a lottery.

When socially desirable public goods are not privately provided the obvious alternative is to publicly provide them. Suppose that the social planner has access to a budget equal to $\Pi \leq \bar{\Pi}$. Instead of allocating this sum in form of prizes he provides an amount of public good equal to Π. We want to analyse how the total expected welfare generated by public provision compares with that resulting form the use of a contest as an incentive scheme.

Proposition 4 *Private provision of public good via a contest, in which the total sum prize $\Pi \leq \bar{\Pi}$ is distributed among the $n-1$ players who contribute the most, Pareto-dominates public provision. If the social planner uses $\Pi \leq \bar{\Pi}$ to publicly provide the public good the expected total welfare net of the value of Π is $W^P = nE[z] + (\alpha-1)\Pi$.*

Proof. If the social planner uses $\Pi \leq \bar{\Pi}$ to provide the public good the expected total welfare net of the value of Π is given by

$$
\begin{aligned}
W^P &= n \int_0^1 (z + \frac{\alpha}{n}\Pi)F'(z)dz - \Pi = \\
& \quad nE[z] + (\alpha - 1)\Pi
\end{aligned}
\tag{8}
$$

From Corollary (1) we know that, if the last prize is equal to zero, the expected total welfare generated by a contest is equal to

$$
W = nE[z] + \frac{n(\alpha - 1)}{n - \alpha}\Pi
$$

which is strictly greater than expression (8). ∎

We now consider the case where the social planner resorts to a lottery to encourage contribution to the public good. In order to be able to compare the lottery mechanism with a contest we will restrict the analysis to interior solutions. To do this let us assume n players whose endowments are drawn independently of each other from the interval $[\underline{z}, \bar{z}]$, with $\underline{z}$ strictly positive, according to the distribution function $F(z)$, which is common knowledge. Assume that the social planner decides to award the sum Π in a lottery with the following properties. If player i with endowment z_i contributes $g_i \in [0, z_i]$ she wins Π with probability $\frac{g_i}{g_i + G_{-i}}$, where G_{-i} is the sum of the contributions of all the other agents. Player i's expected utility is given by

$$E[U(z_i - g_i, \Pi) \mid g_i, G_{-i}] = z_i - g_i + \alpha \frac{G_{-i} + g_i}{n} + \frac{g_i}{g_i + G_{-i}} \Pi$$

Differentiating with respect to g_i and setting this equal to zero we obtain the following

$$\frac{\alpha - n}{n} + \frac{G_{-i}}{(g_i + G_{-i})^2} \Pi = 0$$

Assuming that total contribution is different from zero[6] and rearranging we obtain player i's best response function, given by the following expression

$$g_i^* = -G_{-i} + \sqrt[2]{\frac{n}{n - \alpha} \Pi G_{-i}} \tag{9}$$

Based on equation (9) we can write an expression for the total contribution when player i plays according to her best response function

$$G(g_i^* \mid G_{-i}) = \sqrt[2]{\frac{n}{n - \alpha} \Pi G_{-i}}$$

Although endowments are private information, notice that z does not enter the first order condition. Each player will have the same best response function and the contribution in equilibrium will be the same for any z. Therefore we know that g_i^* will be $\frac{G(g_i^* \mid G_{-i})}{n}$ and we can express it as follows

$$g_i^* = \frac{\sqrt[2]{\frac{n}{n-\alpha} \Pi G_{-i}}}{n} \tag{10}$$

Setting equations (9) and (10) equal we obtain an expression for G_{-i} when all players play according to the best response function. This is represented by

$$G_{-i}^* = \frac{(n-1)^2}{n(n - \alpha)} \Pi$$

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[6]Notice that in equilibrium the total contribution will not be zero. In fact, if any other player different from i contributes zero, player i will contribute ε arbitrarily close to zero and win the prize.

Therefore, assuming that wealth constraints are non-binding, in equilibrium all agents will play according to the following

$$g^L = \frac{n-1}{n(n-\alpha)}\Pi$$

It is easy to see that $\Pi \leq \frac{n(n-\alpha)}{n-1}\underline{z}$ guarantees that the solution will be interior for all players. Contrary to the equilibrium we identified for the case of a contest, in a lottery all players contribute the same amount (as in Morgan, 2000). Total contribution in equilibrium is given by

$$G^L = \frac{n-1}{n-\alpha}\Pi$$

These results are summarised in the following proposition.

Proposition 5 *Assume n players whose endowments are drawn independently of each other from the interval $[\underline{z}, \bar{z}]$, with $\underline{z}$ strictly positive, according to the distribution function $F(z)$, which is common knowledge. Assume that z is private information. If $\Pi \leq \frac{n(n-\alpha)}{n-1}\underline{z}$ the lottery has a symmetric pure strategy equilibrium in which all players contribute $g^L = \frac{n-1}{n(n-\alpha)}\Pi$ and total contribution is $G^L = \frac{n-1}{n-\alpha}\Pi$.*

Note that in order to prove Proposition (7) in the Appendix we have not resorted to the support of z. The same conditions guarantee the existence of the equilibrium described in Proposition (1) also when endowments are drawn from the interval $[\underline{z}, \bar{z}]$, with $\underline{z}$ strictly positive, according to the distribution function $F(z)$, which is common knowledge, with a continuous and bounded density $F'(z) > 0$. In this case, provided that the social planner optimally sets the last prize equal to zero, the expected total contribution generated by a contest is given by

$$
\begin{aligned}
G &= n \int_{\underline{z}}^{\bar{z}} \frac{n}{n-\alpha}\phi(F(z), \pi \mid \sum_{j=1}^{n} \pi_j = \Pi, \pi_1 \geq \\
\cdots &\geq \pi_n, \pi_n = 0)F'(z)dz = \frac{n}{n-\alpha}\Pi
\end{aligned}
$$

We can conclude that for any finite n, when Π guarantees interior solutions for all players in both mechanisms, the expected total contribution raised with a contest is greater than that obtained through a lottery. The intuition behind this result is that a lottery can be thought of as a stochastic contest (see Tullock, 1980): the higher level of noise results in lower total revenue.

Proposition 6 *Assume n players whose endowments are drawn independently of each other from the interval $[\underline{z}, \bar{z}]$, with $\underline{z}$ strictly positive, according to the distribution function $F(z)$, which is common knowledge. Assume that z is private information and that $F(z)$ has a continuous and bounded density $F'(z) > 0$. If $\Pi \leq \min[\frac{n(n-\alpha)}{n-1}\underline{z}, \bar{\Pi}]$, the expected total contribution in a contest, where Π is allocated among the $n-1$ players who contribute the most, is greater than the total provision generated by a lottery.*

6 Conclusions

Exploring effective ways to fund public goods is a policy question of great importance, given the fundamental the role they play in society. There exists an extensive literature on fund-raising mechanisms based on taxes and penalties. However, solutions to the free-riding problem which do not require coercive power have only recently started to be studied. In the case of institutions which are unable to enforce sanctions, such as charities, this difference may be extremely important. To our knowledge, this is the first attempt to analyse multiple prize contests as incentive schemes to finance public goods. Further, this recent literature has either focused on cases where agents are unconstrained or have homogeneous endowments (e.g. Morgan and Sefton, 2000; Goeree et al. 2005; Orzen, 2005) or predicts that players with different incomes would contribute the same amount (Morgan, 2000). Contrary to these studies we identified an equilibrium in which the contribution is strictly increasing in the endowment.

We considered a linear public good game as it is often employed in laboratory experiments. The main characteristics of the model are the possibility of awarding multiple prizes on the one side, and heterogeneity of the endowments and incomplete information on the other. We assumed that the social planner has access to a small budget and uses it to implement a contest. The first prize is awarded to the player who contributes the most, the second prize to the player with the second highest contribution and so on until all prizes are awarded. The social planner's objective function is represented by the expected total welfare net of the total prize sum.

We concentrated our analysis on interior solutions. We found that there exists a critical level of budget under which wealth constraints are non-binding for all agents. For any possible distribution of wealth we identified necessary and sufficient conditions for the budget to be below this critical level. We found that it is optimal for the social planner to set the last prize equal to zero, but otherwise total expected contribution is invariant to all configurations leaving the lowest value fixed. Further, a contest is a budget balanced mechanism: the revenue generated is higher than the total prize sum. Provided interior solutions, we proved that a contest Pareto-dominates public provision of the public good and performs better than a lottery.

Heterogeneity of the endowments and incomplete information about income levels allowed us to characterise a monotone equilibrium, in which the higher the endowment of a player the higher her contribution. On the contrary, in the case of a lottery, a symmetric equilibrium arises (as in Morgan, 2000). This is an interesting difference which makes the equilibrium of a contest look more realistic than the latter. Indeed it does seem generally more plausible that richer people contribute more than individuals with lower incomes.

An interesting extension to the present work would be to test experimentally the main results of the model. An important question would be to check whether a contest actually generates a higher contribution than a lottery, and whether the revenue of a contest is independent of the prize structure. Further, it would be interesting to test whether a monotone equilibrium would arise, both in a contest and in a lottery.

Appendix

We want to find necessary and sufficient conditions for the value of Π such that $g(z)$ is interior for any z on the interval $[0, 1]$ for any possible allocation of Π among the first $n - 1$ prizes. In fact, assuming interior solutions, Proposition (2) assures us that the social planner will set $\pi_n = 0$.

If we let the vector of prizes π be variable, provided that the last prize is equal to zero and that the sum of the first $n - 1$ prizes is equal to Π, $g(z)$ is represented by the following[7]

$$\frac{n}{n - \alpha} \phi(F(z), \pi \mid \sum_{j=1}^{n} \pi_j = \Pi, \pi_1 \geq \cdots \geq \pi_n, \pi_n = 0) =$$
$$\frac{n}{n - \alpha} \sum_{i=1}^{n} \pi_i \binom{n - 1}{i - 1} (F(z))^{n-i} (1 - F(z))^{i-1}$$

Let us define the following object.

Definition 1 *Define the envelope function*

$$V(z) = \max_{\pi} \{ \phi(F(z), \pi) \mid \sum_{j=1}^{n} \pi_j = \Pi, \pi_1 \geq \cdots \geq \pi_n, \pi_n = 0 \}$$

for any z on the interval $[0, 1]$.

If we are able to provide necessary and sufficient conditions for $V(z)$ to be weakly less than z for any z on the interval $[0, 1]$, it will be easy to extend the result to $g(z)$. In order to do this we will define some useful concepts that will help us in the course of our analysis.

Definition 2 *For any i such that $1 \leq i \leq n - 1$:*
  1) define the set $Q^i \subset \mathbb{R}^n$ such that for every $\pi \in Q^i$ it holds that $\pi_1 \geq \cdots \geq \pi_i > \pi_{i+1} = \cdots = \pi_n = 0$ and $\sum_{l=1}^{i} \pi_l = \Pi$.
  2) call $\bar{\pi}^i$ the vector $\pi \in Q^i$ such that $\pi_1 = \cdots = \pi_i = \frac{\Pi}{i}$.

Definition 3 *For any i such that $2 \leq i \leq n - 1$ define the set $\tilde{Q}^i \subset Q^i$ such that for every $\pi \in \tilde{Q}^i$ it holds that $\pi_1 > \pi_i$.*

Obviously $\bar{\pi}^1 \in Q^1$, characterised by $\pi_1^1 = \Pi, \pi_l^1 = 0$ for $2 \leq l \leq n$, is the only element of the set Q^1 and $\phi(F(z), \bar{\pi}^1) = \Pi(F(z))^{n-1}$.

The next Proposition presents necessary and sufficient conditions for $V(z)$ to be weakly less than z on the interval $[0, 1]$.

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[7]Hereafter, unlike the rest of the paper, when writing $\phi(F(z), \pi)$ we will refer to $\phi(F(z), \pi \mid \sum_{j=1}^{n} \pi_j = \Pi, \pi_1 \geq \cdots \geq \pi_n, \pi_n = 0)$.

Proposition 7 $\phi(F(z), \bar{\pi}^i) \leq z$ on the interval $[0, 1]$ for $1 \leq i \leq n-1$ are necessary and sufficient conditions for $V(z) \leq z$.

Proof. The necessity of these conditions is obvious. In order to prove sufficiency we will have to present some technical results.

Lemma 2 Given a vector $\pi^R \in \mathbb{R}^n$ such that $\sum_{j=1}^{n} \pi_j^R = \Pi$ and $\pi_1^R \geq \cdots \geq \pi_n^R$, $\pi_n^R = 0$, consider a redistribution of the type $-\Delta\pi_i^R = \Delta\pi_{i+1}^R$, with $1 \leq i \leq n-1$ and $\Delta\pi_i^R > 0$, and call the resulting vector π^S. Then, $\phi(F(z), \pi^S) > \phi(F(z), \pi^R)$ for any z such that $F(z) < \frac{n-i}{n}$ and $\phi(F(z), \pi^S) < \phi(F(z), \pi^R)$ for any z such that $F(z) > \frac{n-i}{n}$.

Proof. Notice that $\frac{\partial\phi(F(z),\pi)}{\partial\pi_i} = \binom{n-1}{i-1}(F(z))^{n-i}(1 - F(z))^{i-1}$. To see how a redistribution of the type $-\Delta\pi_i = \Delta\pi_{i+1}$ affects $\phi(F(z), \pi)$ we have to study the sign of

$$-\frac{\partial\phi(F(z), \pi)}{\partial\pi_i} + \frac{\partial\phi(F(z), \pi)}{\partial\pi_{i+1}} \tag{11}$$

$$= (F(z))^{n-i}(1 - F(z))^1 \left(-\binom{n-1}{i-1}(F(z))\right.$$

$$\left. + \binom{n-1}{i}(1 - F(z))\right)$$

It is the case that expression $(11) > 0$ for any z such that $F(z) < \frac{\binom{n-1}{i}}{\binom{n-1}{i-1} + \binom{n-1}{i}}$ and $(11) < 0$ for any z such that $F(z) > \frac{\binom{n-1}{i}}{\binom{n-1}{i-1} + \binom{n-1}{i}}$. Further, it is easy to show that

$$\frac{\binom{n-1}{i}}{\binom{n-1}{i-1} + \binom{n-1}{i}} = \frac{\frac{(n-1)!}{i!(n-1-i)!}}{\frac{(n-1)!}{(i-1)!(n-i)!} + \frac{(n-1)!}{i!(n-1-i)!}} = \frac{n-i}{n}$$

∎

Lemma 3 Assume $1 \leq i \leq n-2$. Consider a vector $\pi^B \in \tilde{Q}^{i+1}$. If $2 \leq i \leq n-2$ then $\phi(F(z), \bar{\pi}^{i+1}) > \phi(F(z), \pi^B)$ for any z such that $F(z) \leq \frac{n-i}{n}$. If $i = 1$ then $\phi(F(z), \bar{\pi}^2) > \phi(F(z), \pi^B)$ for any z such that $F(z) < \frac{n-1}{n}$ and $\phi(\frac{n-1}{n}, \pi \mid \pi \in Q^2) = \phi(\frac{n-1}{n}, \bar{\pi}^1) = \Pi(\frac{n-1}{n})^{n-1}$.

Proof. Let us first consider the case in which $2 \leq i \leq n-2$. The vector $\bar{\pi}^{i+1}$ can be obtained from vector π^B applying the following algorithm in i steps.

Algorithm 1 Step 1. From vector π^B construct vector π^{B1} such that $\pi_1^{B1} = \frac{\Pi}{i+1}, \pi_2^{B1} = \pi_2^B + \pi_1^B - \frac{\Pi}{i+1}, \pi_j^{B1} = \pi_j^B, 3 \leq j \leq i+1$. Given that $\pi_2^B \geq \pi_3^B$ it will now be the case that $\pi_2^{B1} > \pi_3^{B1} \geq \cdots \geq \pi_{i+1}^{B1}$. Therefore $\frac{\Pi}{i+1} + i\pi_2^{B1} > \Pi$. The last inequality can be rewritten as $\pi_2^{B1} > \frac{\Pi}{i+1}$, therefore we can move to the next step and repeat the process.

Step j, with $2 \leq j \leq i-1$. From vector π^{Bj-1} construct vector π^{Bj} such that $\pi_j^{Bj} = \frac{\Pi}{i+1}, \pi_{j+1}^{Bj} = \pi_{j+1}^{Bj-1} + \pi_j^{Bj-1} - \frac{\Pi}{i+1}, \pi_l^{Bj} = \pi_l^{Bj-1}$ for $1 \leq l \leq j-1$

14

and $j + 1 \leq l \leq i + 1$. Given that $\pi_{j+1}^{Bj-1} \geq \pi_{j+2}^{Bj-1}$ it will now be the case that $\pi_{j+1}^{Bj} > \pi_{j+2}^{Bj} \geq \cdots \geq \pi_{i+1}^{Bj}$. Therefore it is the case that $2\frac{\Pi}{i+1} + (i + 1 - j)\ \pi_{j+1}^{Bj}$. Rearranging the last inequality we obtain $\pi_{j+1}^{Bj} > \frac{\Pi}{i+1}$. This means that we can move to the next step and repeat the process.

Step i. From vector π^{Bi-1} construct vector π^{Bi} such that $\pi_i^{Bi} = \frac{\Pi}{i+1}, \pi_{i+1}^{Bi} = \pi_{i+1}^{Bi-1} + \pi_i^{Bi-1} - \frac{\Pi}{i+1}, \pi_l^{Bi} = \pi_l^{Bi-1}$ for $1 \leq l \leq i - 1$. Notice that $\pi_l^{Bi-1} = \frac{\Pi}{i+1}$ for $1 \leq l \leq i - 1$. Therefore $\pi^{Bi} = \bar{\pi}^{i+1}$.

Notice that from Lemma 2 we know that $\phi(F(z), \pi^{Bj}) > \phi(F(z), \pi^{Bj-1})$ for any z such that $F(z) < \frac{n-j}{n}$ for $1 \leq j \leq i$. Therefore $\phi(F(z), \bar{\pi}^{i+1}) > \phi(F(z), \pi^B)$ for any z such that $F(z) \leq \frac{n-i}{n}$, that contradicts our assumption.

Consider now the case in which $i = 1$. Notice that $\bar{\pi}_1^2 < \pi_1^B$ and $\bar{\pi}_2^2 > \pi_2^B$. Applying the same algorithm as above from π^B we will obtain $\bar{\pi}_2^2$ after the first step. Applying Lemma 2 we know that $\phi(F(z), \bar{\pi}^2) > \phi(F(z), \pi^B)$ for any z such that $F(z) < \frac{n-1}{n}$. Further, from Lemma 2 we also know that $\phi(F(z), \pi \mid \pi \in Q^2) > \phi(F(z), \bar{\pi}^1)$ for any z such that $F(z) < \frac{n-1}{n}$ and $\phi(F(z), \pi \mid \pi \in Q^2) < \phi(F(z), \bar{\pi}^1)$ for any z such that $F(z) > \frac{n-1}{n}$. Therefore, by continuity, we can conclude that $\phi(\frac{n-1}{n}, \pi \mid \pi \in Q^2) = \phi(\frac{n-1}{n}, \bar{\pi}^1) = \Pi(\frac{n-1}{n})^{n-1}$. ∎

Lemma 4 Assume $2 \leq i \leq n - 2$. $\phi(F(z), \bar{\pi}^{i+1}) > \phi(F(z), \pi \mid \pi \in Q^j)$ for any z such that $F(z) \leq \frac{n-i}{n}$ and for $1 \leq j \leq i$.
Proof. The structure of this proof is in three parts.

First of all, from Lemma 3 we know that $\phi(F(z), \bar{\pi}^j) > \phi(F(z), \pi \mid \pi \in \tilde{Q}^j)$ for any z such that $F(z) \leq \frac{n-j-1}{n}$ and, given that $1 \leq j \leq i$, for any z such that $F(z) \leq \frac{n-i}{n}$.

For the second part of the proof, let us firs assume $j = 1$. Consider a vector $\pi^B \in \tilde{Q}^{i+1}$. We want to show that $\phi(F(z), \pi^B) > \phi(F(z), \bar{\pi}^1)$ for any z such that $F(z) \leq \frac{n-i}{n}$.

If $2 \leq j \leq i$, consider a vector $\pi^B \in \tilde{Q}^{i+1}$ such that $\pi_l^B = \bar{\pi}_l^j$ for $1 \leq l \leq j - 1$. Notice that, obviously, $\pi_j^B < \bar{\pi}_l^j$. We want to show that

$\phi(F(z), \pi^B) > \phi(F(z), \bar{\pi}^j)$ for any z such that $F(z) \leq \frac{n-i}{n}$ if $1 \leq j \leq i - 1$ and for any z such that $F(z) < \frac{n-i}{n}$ if $j = i$.

Vector π^B can be obtained from $\bar{\pi}^j$ through the following algorithm in $i + 1 - j$ steps.

Algorithm 2 Step 1. If $j = 1$, from vector $\bar{\pi}^1$ construct vector $\pi^{A1} \in \tilde{Q}^2$ such that $\pi_1^{A1} = \pi_1^B$ and $\pi_2^{A1} = \Pi - \pi_1^B$. If $2 \leq j \leq i$, from vector $\bar{\pi}^j$ construct vector $\pi^{A1} \in \tilde{Q}^{j+1}$ such that $\pi_l^{A1} = \bar{\pi}_l^j = \frac{\Pi}{j}$ for $1 \leq l \leq j - 1, \pi_j^{A1} = \pi_j^B$ and $\pi_{j+1}^{A1} = \bar{\pi}_j^j - \pi_j^B = \frac{\Pi}{j} - \pi_j^B$.

Step k, with $2 \leq k \leq i - j$. From vector π^{Ak-1} construct vector $\pi^{Ak} \in \tilde{Q}^{j+k}$ such that $\pi_l^{Ak} = \pi_l^{Ak-1}$ for $1 \leq l \leq j = k - 2, \pi_{j+k-1}^{Ak} = \pi_{j+k-1}^B$ and $\pi_{j+k}^{Ak} = \pi_{j+k-1}^{Ak-1} - \pi_{j+k-1}^B$.

Step $i + 1 - j$. From vector π^{Ai-j} construct vector $\pi^{Ai+1-j} \in \tilde{Q}^{i+1}$ such that $\pi_l^{Ai+1-j} = \pi_l^{Ai-j}$ for $1 \leq l \leq i - 2, \pi_i^{Ai+1-j} = \pi_i^B$ and $\pi_{i+1}^{Ai+1-j} = \pi_i^{Ai-j} - \pi_i^B$. Notice that $\pi_{i+1}^{Ai+1-j} = \pi_{i+1}^B$ and $\pi^{Ai+1-j} = \pi^B$ by construction.

From Lemma 2 we know that $\phi(F(z), \pi^{Ak}) > \phi(F(z), \pi^{Ak-1})$ for any z such that $F(z) < \frac{n+1-k}{n}$. Therefore if $1 \leq j \leq i-1$ then $\phi(F(z), \pi^B) > \phi(F(z), \bar{\pi}^j)$ for any z such that $F(z) \leq \frac{n-i}{n}$. If $j = i$ then $\phi(F(z), \pi^B) > \phi(F(z), \bar{\pi}^j)$ for any z such that $F(z) < \frac{n-i}{n}$ and $\phi(\frac{n-i}{n}, \pi^B) = \phi(\frac{n-i}{n}, \bar{\pi}^j)$.

Finally, from Lemma 3 we know that $\phi(F(z), \bar{\pi}^{i+1}) > \phi(F(z), \pi^B)$ for any z such that $F(z) \leq \frac{n-i}{n}$. Therefore $\phi(F(z), \bar{\pi}^{i+1}) > \phi(F(z), \pi \mid \pi \in \tilde{Q}^j)$ for any z such that $F(z) \leq \frac{n-1}{n}$. ■

Lemma 5 *Assume $2 \leq i \leq n-2$. Consider a vector $\pi^B \in \tilde{Q}^{i+1}$ such that $\pi_1^B > \pi_j^B$, with $2 \leq j \leq i$. Assume a vector $\pi^C \in \tilde{Q}^{i+1}$ such that $\pi_l^C = \pi_l^B$ for $j+1 \leq l \leq i+1$*

$$\text{and } \pi_1^C = \cdots = \pi_j^C = \frac{\Pi - \sum_{l=j+1}^{i+1} \pi_l^B}{j}.$$ *If $3 \leq j \leq n-2$ then $\phi(F(z), \pi^C) > \phi(F(z), \pi^B)$ for any z such that $F(z) \leq \frac{n-j+1}{n}$. If $j = 2$ then $\phi(F(z), \pi^C) > \phi(F(z), \pi^B)$ for any z such that $F(z) < \frac{n-1}{n}$ and $\phi(\frac{n-1}{n}, \pi^C) > \phi(\frac{n-1}{n}, \pi^B)$.*
Proof. Notice that $\pi_1^B > \pi_1^C$ and $\pi_j^B < \pi_j^C$. Vector π^C can be obtained from vector π^B applying the following algorithm in $j-1$ steps.

Algorithm 3 Step 1. *From vector π^B construct vector π^{B1} such that $\pi_1^{B1} = \pi_1^C, \pi_2^{B1} = \pi_2^B + \pi_1^B - \pi_1^C, \pi_l^{B1} = \pi_l^B$ for $3 \leq l \leq i+1$. Given that $\pi_2^B \geq \pi_3^B$ it will now be the*

case that $\pi_2^{B1} > \pi_3^{B1} \geq \cdots \geq \pi_{i+1}^{B1}$. Therefore $\pi_1^{B1} + (j-1)\pi_2^{B1} > \Pi - \sum_{l=j+1}^{i+1} \pi_l^B$. Since

$$\pi_1^{B1} = \pi_1^C = \frac{\Pi - \sum_{l=j+1}^{i+1} \pi_l^B}{j},$$ *the last inequality can be rearranged as $\pi_2^{B1} > \frac{\Pi - \sum_{l=j+1}^{i+1} \pi_l^B}{j}$. Therefore we can move to the next step and repeat the process.*

Step k, $2 \leq k \leq j-2$. *From vector π^{Bk-1} construct vector π^{Bk} such that $\pi_k^{Bk} = \pi_k^C$, $\pi_{k+1}^{Bk} = \pi_{k+1}^{Bk-1} + \pi_k^{Bk-1} - \pi_k^C, \pi_l^{Bk} = \pi_l^{Bk-1}$ for $1 \leq l \leq k-1$ and*

$k+2 \leq l \leq i+1$. *Notice that, by construction $\pi_l^{Bk} = \frac{\Pi - \sum_{l=j+1}^{i+1} \pi_l^B}{j}$ for $1 \leq l \leq k$ and $\pi_l^{Bk} = \pi_l^B$ for $k+2 \leq l \leq i+1$. Given that $\pi_{k+1}^{Bk-1} \geq \pi_{k+2}^{Bk-1}$ it will now be the case that*

$\pi_{k+1}^{Bk} > \pi_{k+1}^{Bk} \geq \cdots \geq \pi_{i+1}^{Bk}$. *Therefore $\frac{k}{j}(\Pi - \sum_{l=j+1}^{i+1} \pi_l^B) + (j-k)\pi_{k+1}^{Bk} > \Pi - \sum_{l=j+1}^{i+1} \pi_l^B$.*

The last inequality can be rearranged as $\pi_{k+1}^{Bk} > \frac{\Pi - \sum_{l=j+1}^{i+1} \pi_l^B}{j}$. Therefore we can move to the next step and repeat the process.

Step $j-1$. *From vector π^{Bj-2} construct vector π^{Bj-1} such that $\pi_{j-1}^{Bj-1} = \pi_{j-1}^C, \pi_j^{Bj-1} = \pi_j^{Bj-2} + \pi_{j-1}^{Bj-2} - \pi_{j-1}^C, \pi_l^{Bj-1} = \pi_l^{Bj-2}$ for $1 \leq l \leq j-2$ and $j+1 \leq l \leq i+1$. Notice that $\pi^{Bj-1} = \pi^C$ by construction.*

From Lemma 2 we know that $\phi(F(z), \pi^{Bk}) > \phi(F(z), \pi^{Bk-1})$ for any z such that $F(z) < \frac{n-k}{n}$ for $1 \leq k \leq j-1$. This means that if $3 \leq i \leq n-3$ then, by construction, we will have $\phi(F(z), \pi^C) > \phi(F(z), \pi^B)$ for any z such that $F(z) \leq$

16

$\frac{n-j+1}{n}$. If $i = 2$ then j will necessarily be equal to 3 and, by construction, we will have $\phi(F(z), \pi^C) > \phi(F(z), \pi^B)$ for any z such that $F(z) < \frac{n-1}{n}$. Further it will be the case that $\phi(\frac{n-1}{n}, \pi^C) > \phi(\frac{n-1}{n}, \pi^B)$. ∎

Lemma 6 *Consider a vector* $\pi^C \in \tilde{Q}^{i+1}$ *such that* $\pi_{i+1}^C = x, \pi_j^C = \frac{\Pi - x}{i}$ *with* $0 < x < \frac{\Pi}{i+1}$ *for* $1 \leq j \leq i$ *and* $2 \leq i \leq n - 2$. *If* $\phi(F(z), \pi^C) > \phi(F(z), \bar{\pi}^{i+1})$ *then* $\phi(F(z), \bar{\pi}^i) > \phi(F(z), \pi^C)$.
Proof. *The inequality* $\phi(F(z), \pi^C) > \phi(F(z), \bar{\pi}^{i+1})$ *can be rewritten as*

$$\frac{\Pi - x}{i}((F(z))^{n-1} + \cdots +$$

$$\binom{n-1}{i-1}(F(z))^{n-i}(1 - F(z))^{i-1}) +$$

$$x\binom{n-i}{i}(F(z))^{n-i-1}(1 - F(z))^i -$$

$$\frac{\Pi}{i+1}((F(z))^{n-1} + \cdots + (F(z))^{n-i-1}(1 - F(z))^i) > 0$$

The above expression can be rearranged as

$$(\frac{\Pi - x}{i} - \frac{\Pi}{i+1})((F(z))^{n-1} + \cdots + \qquad (12)$$

$$\binom{n-1}{i-1}(F(z))^{n-i}(1 - F(z))^{i-1}) >$$

$$(\frac{\Pi}{i+1} - x)(F(z))^{n-i-1}(1 - F(z))^i$$

Call A the expression $((F(z))^{n-1} + \cdots + \binom{n-1}{i-1}(F(z))^{n-i}(1 - F(z))^{i-1})$ *and call B the expression* $(F(z))^{n-i-1}(1 - F(z))^i$. *Inequality (12) is satisfied for* $\frac{A}{B} > i$.
The inequality $\phi(F(z), \bar{\pi}^i) > \phi(F(z), \pi^C)$ *can be rewritten as*

$$\frac{\Pi}{i}A - \frac{\Pi - x}{i}A - xB > 0 \qquad (13)$$

Inequality (13) is satisfied for $\frac{A}{B} > i$. ∎

From Lemma (4) we know that $\phi(F(z), \bar{\pi}^{i+1}) > \phi(F(z), \pi \mid \pi \in Q^j)$ for any z such that $F(z) \leq \frac{n-i}{n}$ and for $2 \leq i \leq n - 2$ and $1 \leq j \leq i$. In particular, this means that $V(z)$ will be equal to $\phi(F(z), \bar{\pi}^{n-1})$ for any z such that $0 \leq F(z) \leq \frac{2}{n}$. For those z such that $\frac{2}{n} \leq F(z) \leq \frac{3}{n}$ we will have to check the family of functions $\phi(F(z), \pi \mid \pi \in Q^{n-1})$ and $\phi(F(z), \bar{\pi}^{n-2})$. In general, assuming $0 \leq i \leq n - 3$, in order to find $V(z)$ for those z such that $\frac{n-i-1}{n} \leq F(z) \leq \frac{n-i}{n}$ we will have to check the families of functions $\phi(F(z), \pi \mid \pi \in Q^j)$ for $i + 2 \leq j \leq n - 1$ and the function $\phi(F(z), \bar{\pi}^{i+1})$.

Consider now a vector $\pi^C \in Q^{i+1}$ such that $\pi_1^C = \cdots = \pi_i^C > \pi_{i+1}^C$, for $2 \leq i \leq n - 2$. From Lemma (5) we know that, for those z such that $\frac{n-i}{n} < F(z) \leq \frac{n-i+1}{n}$, the function $\phi(F(z), \pi^C)$ is greater than any other function of the family $\phi(F(z), \pi \mid \pi \in Q^{i+1})$ with exclusion of $\phi(F(z), \bar{\pi}^{i+1})$.

17

From Lemma (6) though, we know that if $\phi(F(z), \pi^C) > \phi(F(z), \bar{\pi}^{i+1})$ then it is the case that $\phi(F(z), \bar{\pi}^i) > \phi(F(z), \pi^C)$.

Therefore, in order to find the envelope function $V(z)$ for those z such that $\frac{2}{n} \leq F(z) \leq \frac{3}{n}$, it will be sufficient to check the two functions $\phi(F(z), \bar{\pi}^{n-1})$ and $\phi(F(z), \bar{\pi}^{n-2})$. In general, assuming $0 \leq i \leq n-3$, in order to find $V(z)$ for those z such that $\frac{n-i-1}{n} \leq F(z) \leq \frac{n-i}{n}$ we will have to check the functions $\phi(F(z), \bar{\pi}^j)$ for $i+1 \leq j \leq n-1$.

From this follows that $\phi(F(z), \bar{\pi}^i) \leq z$ for $1 \leq i \leq n-1$ are sufficient conditions for $V(z) \leq z$ on the interval $[0,1]$. ∎

Finally, given Proposition (7), by continuity we can establish the following result.

Proposition 8 *Provided that the last prize is equal to zero, $g(z)$ is interior for any z on the interval $[0,1]$ independently of the distribution of Π among the first $n-1$ prizes if and only if $\frac{n}{n-\alpha}\phi(F(z), \bar{\pi}^i \mid \sum_{j=1}^{n} \bar{\pi}_j^i = \Pi) \leq z$ on the interval $[0,1]$ for $1 \leq i \leq n-1$.*

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