A Model of Inter-Regional Trade in Grains with Storage

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When there is spatial arbitrage between food markets, the price differential between the markets should equal the cost of transfer. This simple arbitrage rule has formed the basis of many empirical tests of the performance of developing countries' food markets. Here, a more complex structure to regional food markets is hypothesised. By including commodity storage into a model of inter-regional trade, it is shown that, in an optimal dynamic program, trade is intermittent. The possibility of periods without trade casts doubt on the validity of using tests for correlation in market prices as indicators of market performance.

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JEL Classification: Q13, C61
Keywords: food markets, optimal control, storage, spatial equilibrium

[*] The author was funded by an UK Economics and Social Research Council Studentship and would like to thank Brian Main for helpful comments.

1. Introduction

The Takayama-Judge model of spatial equilibrium underpins many of the tests of food market integration that have been used in developing countries by economists (Takayama and Judge (1971)). In the model, spatial arbitrage across regional food markets restricts regional price differentials to at most the cost of transfer between the regions. At times where transfer takes place, the price differential equals the cost of transfer. Analysts have then proceeded by testing this spatial arbitrage rule. When market prices are integrated (in the econometric co-integration sense), this is taken as a sign that the markets are trading and therefore, rather confusingly, integrated in the food marketing sense. This has generally been judged as 'good'. When market prices diverge, there may be short or long-term impediments to trade. This would be detrimental to welfare and the removal of the obstacles to trade would be suggested.

In this paper, a more complex structure to regional food markets is hypothesised. While the Takayama-Judge model allows the spatial aspects of food markets to be explored, the temporal features of food markets are not discussed. This paper analyses this aspect through more explicit reference to the dynamics of market trade. In particular, the departure from conventional modelling is the inclusion of commodity storage into a model of inter-regional trade. The addition of storage of grain yields interesting results. Most notably, trade is intermittent with periods where a region will consume its stored grains or purchase from another market. In such periods, tests for correlation in market prices would be inappropriate as inter-regional trading would not take place.

The paper will begin by discussing the motivation of the present research. Two streams in the literature are drawn together. Firstly, the widespread adoption by developing countries of food market liberalising policies has meant an explosion of interest in tests of market performance. A common test has been the checking of price time-series data for co-integration. Where co-integration is observed, the presence of arbitrage between spatially dispersed markets has been surmised. A second stream in the literature has identified empirical examples where arbitrage rules appear to breakdown. In their attempt to explain this, Wright and Williams (1989) model the spatial-temporal interaction which occurs when storage of a commodity as well as inter-market trade is allowed. They show that the empirical evidence can be explained without assuming some market problem causing arbitrage conditions to be broken.

The third and fourth sections of the paper present an application of optimal control where two regions are assumed to have an initial endowment of grain and access to some external market for both purchase and sale. The extent of inter-regional trade and trade with the external market is then modelled. Storage of grain is allowed but a positive discount rate implies that such storage incurs an opportunity cost. In the optimal program, there will be periods when inter-regional trade will not take place as regions either consume own stocks or source grain from the external market. At such times, prices will differ by amounts less than the cost of transfer between the region. In section four, how the various parameters affect the likelihood and the duration of inter-regional trade are explored. The implications of these results will be discussed in the final section.

2. Motivation for research

2.1 Tests of market integration

The logic behind testing for market efficiency in food markets of developing countries has been uncontested. Various agencies have pressed developing countries to liberalise food markets so that productive and allocative efficiency can be attained. There has been a need for tests on whether the policy has delivered its goals. Takayama and Judge (1971) provide a model of spatially dispersed markets which has provided the main underpinning in investigating the conditions for trade. In this model, equilibrium conditions are derived and a spatial arbitrage rule results. When trade occurs between a pair of markets, the price differential between the markets would be equal to the cost of transfer; when there is no trade, the price differential can at most equal the cost of transfer. The presence of trade would therefore result in co-movements of market prices.

Early empirical work involved the use of simple correlation techniques. However, it was apparent that market price data was not stationary and Ravallion (1986) uses an error correction specification to take account of this. Co-integration analysis has suggested methodological refinements. Palaskas and Harriss-White (1993) first note that two price series have to be co-integrated before an error correction model can be used citing the literature in econometrics on the Granger Representation Theory. When co-integration is observed, this is taken as evidence of long-run market integration. However, this is considered quite a weak test of market performance and analysts proceed by adopting an error correction model and testing for short-run market integration. This, however, relies on two key assumptions about market structure. Firstly, Alexander and Wyeth (1994) note that one market should be exogenous. They provide empirical tests of this assertion suggesting it should be a pre-cursor to Ravallion's tests.

The second assumption - that there is continuous, unidirectional trading - is more problematic. In Ravallion's case, both assumptions are satisfied by assuming that the Bangladeshi marketing system is radial. In such a system, a central market - perhaps, a large city - does not produce the foodcrop but its populace depends on the commodity as a staple. Therefore, movements of grain are continuous throughout the year and directed towards the central market. This trade means that the spatial arbitrage condition binds and a hypothesis that the price difference between a rural market and Dacca should equal the cost of transfer for the entire period studied can be tested using an error correction specification.

However, the assumptions about market structure needed for such analysis make the methodology difficult to generalise across a wide range of countries. The condition of continuous trading resulting in the arbitrage condition being a strict equality are relaxed in Baulch's parity bounds model Baulch (1995). He hypothesises that periodic market segmentation can occur. These would result due to impediments to trade raising transfer costs or due to a modest price spread between markets. Both would make inter-market arbitrage unprofitable. However, it is only in the first scenario that a market analyst should be concerned. Baulch presents a test which differentiates between the two different types of breaks of trade (Baulch (1997)). In such circumstances, the error correction models reject short-run integration unable to identify the intermittent breaks in trade; tests for co-integration, meanwhile, do not reject long-run market integration.

The Takayama-Judge model, however, remains behind all these empirical tests. In the error correction models, additional information regarding market structure is used to justify assuming that the arbitrage condition binds. Where breaks in trade are accepted, such as in Baulch's parity bounds model, these are not given a dynamic structure. Baulch (1997) models breaks in market trade as stochastic. This is satisfactory when the cause is a random event, such as damage to transport systems due to weather. However, as this paper shows, when the dynamics of grain trading are more explicitly modelled, breaks of trade result as an optimal solution without recourse to some exogenous shock.

2.2 Backwardation and the breaking of arbitrage rules

Backwardation, in the presence of significant stocks of a commodity, has been observed in the commodity markets by economists for many years - Kaldor (1939) being an early reference. Backwardation is when a commodity's price for future delivery is below the price for immediate delivery. If stocks are high, it appears that commodity storers would be losing money on stocks as the costs of storage would not be met through revenue from future sales. Backwardation in the presence of significant stores of grain is an empirical example of the breakdown of a temporal arbitrage rule. Early explanations of the phenomenon have identified some benefit of storage to the storer. For example, the storing of a commodity allows supply variations to be smoothed. This would justify the observed backwardation and the implicit negative price for storage as storers will effectively be paying for the additional benefit they gain through storage.

Wright and Williams (1989) propose an alternative explanation which relies on a model of storage across space. They consider storage under backwardation as an aggregation phenomenon. In their model, there are two commodities with the transformation of a quantity of the first into the second incurring a cost. Wright and Williams (1989) note that the most obvious example of a transformation is the transfer of the grain from one region to another so that the transformation cost becomes the transport cost. In the model, two periods are considered, the present and the future. While there is no physical costs to storing grain to be traded in the future market, a positive constant rate of interest insures that there is an opportunity cost to tying up capital in storing the commodity.

The deriving of an optimal solution is an application of the partial equilibrium theory of investment under uncertainty (Schienkman and Schechtman (1983)) and is solved by treating the two regions as one firm seeking to maximise the value of the future profit stream. The maximisation yields first-order conditions similar to Takayama-Judge conditions: transfer occurs until the marginal cost of transporting an additional unit equals the price differential across the regions. However, the additional feature of storage adds temporal conditions: where the expected price in the next period exceeds the present price by the rate of interest, storage takes place. When the temporal price difference is less than the costs of storage, no storage takes place.

These conditions set up a richer pricing and storage model. There are two sets of spot and futures prices associated with the storing behaviour of two different regions. This proves very important as comments about storage under backwardation never considered disaggregated price and stocks data. Wright and Williams (1989) then consider aggregation of commodity price and the storage levels across the regions. If spot and futures prices are quoted with delivery in some central market, as with Telser's (1958) use of grain prices, it is clear that only when both regions are storing would there be no backwardation. If there is no storage in the first regions, that region's future price will be below its spot price. If the stock levels are aggregated, however, the other region's stores will be included, backwardation would be observed in the presence of non-zero storage. In aggregate, the fact that there is no storage in one of the regions is hidden.

Empirical evidence supporting the model has been provided by Thompson (1986) who found that as the definition of coffee stocks and prices became more precise, observation of significant stockholding in times of backwardation diminish. This behaviour would be consistent with the model indicating the importance of the aggregation assumption (see also Benirschka and Binkley (1995) and Brennan et al. (1997)).

For the present research, the model offers an important insight into inter-regional trading. When one of the region's grain stocks are exhausted, the transport of grain from the other can be viewed as a period of inter-regional trading. Wright and Williams (1989) do not explore this aspect in any detail preferring to note the conditions under which exhaustion of a commodity will cause backwardation. Here, the issue of timing of trade and its determinants are discussed more fully. To do this, some important changes to the Wright-Williams model are made. In their model, the cost of transformation function is of a general form, but the treatment of time is greatly simplified by only considering a two-period model. In the following sections, a model is presented which pays more attention to the time when transformation takes place.

3. A model of trading when regions store grain

3.1 Overview

To analyse the grain marketing behaviour of two regions with stores of the commodity, this section presents an optimal control analysis. The section gives the necessary and sufficient conditions for an optimal program as detailed in Seierstad and Sydsaeter (1977). In the model, there are two regions which each have an initial harvest of grain. The regions must maintain non-negative grain stocks while using grain to meet their regional consumption needs. The regions can trade with each other and there is also an external market which buys and sells grain at a constant price throughout the program. However, this external market is some distance from the regions and purchases from and sales to it incur transfer costs.

In the optimisation, a revenue function is maximised over time with the revenues of the two regions added together. Combining revenue across regions in this manner means that the value of grains transferred between the regions is netted out. Only the costs associated with transfer are deducted from the total revenue. Storage of grains also incurs an opportunity cost. A positive discount rate, constant over the program, implies that storage of grain is at the expense of the earnings of some interest-bearing asset. Variables are as follows:

t	time;
h_i	harvested production of commodity in region i;
r	rate of interest;
m	cost of transport between markets;
$p_i(t)$	price of commodity at t in market i;
$y(t)$	quantity of commodity transferred from market 2 to market 1;
$x_i(t)$	grain sales by region i to the external market;
$z_i(t)$	grain purchases by region i from the external market;
p	the price of grain in the external market;
d_i	the cost of transfer from the external market;
*	initial value of a variable, i.e. at time, t=0.

3.2 Two-region model as a dynamic optimisation

Assume there are two regions, i=1,2, which each produce an initial harvest at time t=0 of grain, h_i. The regions must each then maintain non-negative stocks of grain, s_i, until the following harvest at the end of the program at t=T. For simplicity, grain consumption is modelled as a constant out-flow of one unit from the grain stock.

One source for grain is an external market which buys and sells grain at a price p. This price is constant throughout the program. The external market may be an international market for grain or may be some super-regional marketing institution, such as a grain marketing board. The regions procure grain, z_i, from this external market and such purchases will reduce total revenue. Apart from the price p charged, there is a cost of transfer, d_i. Thus, a unit of grain purchased from the external market will reduce revenue by $p+d_i$. When region i is selling x_i to the external market, the cost of transfer means that the revenue raised would be $p-d_i$ per unit sold.

Inter-regional trade can also occur. Without loss of generality, it is assumed that transfer only takes place from region 2 to region 1[1]. Transfer of an amount of grain will reduce revenue by the transfer cost, m, per unit multiplied by the amount which is transferred, y. Assumptions regarding the parameters and the initial level of regional harvests are as follows:

[1] In situations where the direction of the inequality is reversed, the region 1 would become region 2 and vice versa and the same analysis can be applied.

(A1) Transfers are costly, i.e. $d_1 > 0$, $d_2 > 0$, $m > 0$.

(A2) Transfer costs are less than the external market price, $d_1, d_2, m < p$.

(A3) The opportunity cost of storage, *r*, is a positive constant.

(A4) Transfer costs are such that $d_2 + m > d_1$.

(A5) Transfer costs are such that $d_1 + m > d_2$.

(A6) Regional harvests are such that $h_2 > h_1 - \frac{1}{r}\ln\frac{p+d_1}{p+d_2-m}$.

Assumptions A1 to A5 are used later to show that various onward sales by markets are not cost-effective. In appendix 1, it is shown that assumption A6 guarantees the direction of trade is from region 2 to region 1 and will later be relaxed to indicate how the model can be generalised for trade in both directions. Essentially, it indicates that while region 1's harvest level may be slightly greater than region 2, trade will occur from region 1 to region 2 if this disparity is too large. This would mean that the problem as set up would be inappropriate as the variable *y* only considers flows from 2 to region 1.

The optimal control problem is one of maximising the revenue from selling grain to the external market, less the costs of any purchases from it, for each of the regions and then deducting the costs associated with inter-regional trade:

$$\max \int_{t=0}^{T} e^{-rt}\left[(p-d_1)\cdot x_1 + (p-d_2)\cdot x_2 - (p+d_1)\cdot z_1 - (p+d_2)\cdot z_2 - my\right]dt \qquad (1)$$

The maximisation will be subject to the change in grain stock being equal to the various in-flows and out-flows:

$$\dot{s}_1 = z_1 - x_1 + y - 1 \qquad (2)$$

$$\dot{s}_2 = z_2 - x_2 - y - 1. \qquad (3)$$

At the end of the program, there is a new harvest and it is assumed that households use all grain by this time. Further, non-negativity is imposed on the level of stocks and flows:

$$s_i \geq 0; \qquad i=1,2 \qquad (4)$$

$$s_i(T) = 0; \qquad i=1,2 \qquad (5)$$

$$z_i, x_i, y \geq 0; \qquad i=1,2. \qquad (6)$$

Combining (1) with (2) and (3) gives an autonomous dynamic optimisation problem (Leonard and Van Long (1992)) so that a current value Hamiltonian can be used. This has five control variables - x_1, x_2, z_1, z_2 and *y* - and two current value costate variables associated with equation (2) and (3), ψ_1 and ψ_2 respectively:

$$\begin{aligned} L = &(p-d_1)x_1 + (p-d_2)x_2 - (p+d_1)z_1 - (p+d_2)z_2 - my + \\ &\psi_1(z_1 - x_1 + y - 1) + \psi_2(z_2 - x_2 - y - 1) + \mu_1 s_1 + \mu_2 s_2 \end{aligned}. \qquad (7)$$

The variable $\mu_i(.)$ is the Langrange multiplier associated with the non-negativity constraint on the stock of grain, equation (4). The necessary and sufficient conditions for the optimal dynamic program are detailed in Seierstad and Sydsaeter (1977).

Conditions (6) regarding the non-negativity of grain flows are integrated into the first order conditions for this problem[2]:

$$\frac{\partial L}{\partial x_i}.x_i = ((p-d_i)-\psi_i).x_i = 0\,;\quad x_i \ge 0\,;\quad \frac{\partial L}{\partial x_i} \le 0 \qquad i=1,2 \tag{8}$$

$$\frac{\partial L}{\partial z_i}.z_i = (-(p+d_i)+\psi_i).z_i = 0\,;\quad z_i \ge 0\,;\quad \frac{\partial L}{\partial z_i} \le 0 \qquad i=1,2 \tag{9}$$

$$\frac{\partial L}{\partial y}.y = (-m+\psi_1-\psi_2).y = 0\,;\quad y \ge 0\,;\quad \frac{\partial L}{\partial y} \le 0 \tag{10}$$

The non-negativity of the grain stocks imply that:

$$s_i.\mu_i = 0 \qquad i=1,2 \tag{11}$$

The dynamics of the two regional stocks of grain, given in equations (2) and (3), combines with the constraint that the grain stocks cannot be negative to complicate the dynamics of the costate variables. At most times, the following condition holds:

$$\dot{\psi}_i = -\frac{\partial L}{\partial s_i} = -\mu_i + r\psi_i \qquad i=1,2 \tag{12}$$

However, as observed by Jacobsen et al. (1971) (theorem 6), there can be jumps in the costate variable if the level of the state variable - grain stocks in this case - reaches the lower bound (zero). Such times in the optimal programme are called junction points. Defining t_j (j=1,2,...,k) as the values taken by t at the junction points in the particular program, additional conditions for the optimal solution are derived. The condition is, where $^-$ and $^+$ imply the instants before and after the jump:

$$\psi_i(t_j^-)-\psi_i(t_j^+) = \beta_i(t_j) \text{ with } \beta_i(t_j) \ge 0 \text{ and } \beta_i(t_j).s_i = 0 \tag{13}$$

Finally, the terminal level of grain stocks must be zero (5), so there are no terminal conditions on the value taken by the costate variables at the end of the program. The transversality conditions are hence ‘free’.

3.3 Preliminary model results

As with many optimal control programs, the essence of the model rests in the dynamics of the costate variables. The behaviour of the costates in this case are determined jointly by the level of the state variable (the stocks of grain in each of the regions) and which of the control variables takes a non-zero value. Consider the behaviour of the costate variable ψ_i when the region has grain stocks, i.e. $s_i > 0$. Condition (11) implies that the Lagrangean multiplier μ_i(.) takes the value zero so that (12) can be solved giving lemma 1. (Note that a superscript asterix refers to the value of a variable at the start of the program, i.e. $\psi_i^* = \psi_i(0)$).

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[2] The first order conditions include the non-negativity constraint on the flow variables. An alternative means to include this would be to add five additional terms to the Lagrangean - of the form $\lambda_y y$ for the variable y - for each of the control variables. For example, condition (3.6c) would then become:

$$\frac{\partial L}{\partial y} = -m+\psi_1-\psi_2+\lambda_y = 0\ ;\ \lambda_y \ge 0\,;\ \lambda_y y=0$$

The non-negativity of the Langrangean multiplier, λ_y implies the third part of condition (3.6c). The first part rewrites the standard Lagrangean condition $\lambda_y y = 0$.

Lemma 1. *When there are stocks of grain in region i* (=1,2), $s_i > 0$, *the costate variable is given by the function:*

$$\psi_i = \psi_i^* e^{rt} \qquad i=1,2. \qquad (14)$$

It is common to interpret the costate variable as the shadow price of the state variable, i.e. the price of a unit of grain in store. Therefore, this result can be viewed as a reiteration of the Hotelling *r*-percent rule (see Neher (1990)) which states that the value of stock of a resource must appreciate at the rate of discount. Any lower rate of appreciation would prompt storers to sell grain in favour of the interest-bearing asset while a higher rate would cause movement of funds in the opposite direction.

Some conditions on the control variables can also be derived easily using (7) to (13) and the assumptions of the model. Clearly, as $d_i > 0$ (A1), $p + d_i > p - d_i$. Therefore, using (8) and (9), the costate variable for a region cannot satisfy both $L_{x_i} = 0$ and $L_{z_i} = 0$. This implies that $z_i > 0$ and $x_i > 0$ cannot both be true simultaneously and leads to a second result.

Lemma 2. *A region will not buy from the external market at the same time as it is selling grain to the external market.*

The assumptions regarding the cost of transfers relative to the cost of accessing the external market - A4 and A5 - preclude other control variables from being positive contemporaneously. If inter-regional trade is occurring, $y > 0$, condition (10) implies that $\psi_1 - \psi_2 = m$. Firstly, consider whether region 2 could be buying from the external market, i.e. $z_2 > 0$, when this takes place. In such a case both conditions (9) and (10) would be equalities, that is both $\psi_1 - \psi_2 = m$ and $\psi_2 = p + d_2$ hold. The two are not consistent as, from A4, $\psi_1 = p + m + d_2 > p + d_1$. This contradicts the condition that $L_{z_1} \le 0$, condition (9). Secondly, given $y > 0$, consider whether region 1 would be simultaneously selling to the external market, $x_1 > 0$. For this to occur, (8) indicates that $\psi_1 - \psi_2 = m$ and $\psi_1 = p - d_1$ must hold. This cannot hold at the same time as $\psi_2 = p - m - d_1 < p - d_2$. This contradicts the condition that $L_{x_2} \le 0$, condition (8). These two results are combined to give lemma 3.

Lemma 3. *If inter-regional trade is occurring - region 2 selling grain to region 1 - then:*

i) *Region 2 is not buying from the external market;*
ii) *Region 1 is not selling to the external market.*

These general results would be present in models which do not jointly consider spatial and temporal aspects of grain markets. The first result, that the price in a region with grain stocks would obey the Hotelling *r*-percent rule, would be true for models where inter-regional trade did not take place, but storage was allowed. The second two results follow directly from the assumptions of the model and would be the case in a static, spatial model. Onward selling of grain is not cost-effective as assumptions A4 and A5 make the total cost of two transfers greater than that of a single transfer. The following propositions will indicate how the interaction of time and space in grain markets adds to these results. A first proposition of the dynamic model concerns the timing of any grain sales made by the regions. The external market purchases grain from the regions at a price fixed for the entire program. The positive rate of discount implies that if sales are to occur at all, it would be optimal to make the sales as early in the program as possible and then invest the revenue earned. This is proven as proposition 1.

Proposition 1. *In the model of inter-regional grain marketing described by A1 to A6 and (1) to (6), if grain sales to the external market occur, they will take place at time t=0.*

Proof. Conditions (2) to (5) indicate that for sales to the external market to take place, i.e. $x_i > 0$, either region *i* possesses grain stocks, $s_i > 0$, or it purchases grain from the external source, $z_i > 0$. Lemma 2 (above) rules this latter possibility out. In region 1, in addition to these possibilities, region 2 could supply grain, i.e. $y > 0$, to be sold to the external market, $x_2 > 0$. This is ruled out by lemma 3.

Consider then the case where $s_i > 0$. Lemma 1 indicates that, denoting the initial values of ψ_i with an asterix, $\psi_i = \psi_i^* e^{rt}$. The exponential growth in the costate means that its initial value is the lowest value taken during the entire period while there are positive grain stocks. For positive grain sales, (8) indicates that $\psi_i = p - d_i$. Also, from (8), $\frac{\partial L}{\partial x_i} \leq 0$ implies that $p - d_i - \psi_i \leq 0$. The value of the costate at times of grain sales must be the lowest value ψ_i takes. It therefore follows that this can only occur at *t*=0. ❑

Corollary 1. *If sales occur in region i,* $x_i > 0$*, then* $\psi_i^* = p - d_i$.

Proposition 1 greatly simplifies the discussion of the marketing system. Essentially, the selling of grain allows the region to earn revenue from any harvested grain which will be excess to the optimal program. Due to the positive rate of discount, storage of grains is costly and so any excess grain should be sold as early as possible.

Proposition 2. *In the model of inter-regional grain marketing described by A1 to A6 and (1) to (6), inter-regional trade occurs after region 1 runs out of grain.*

Proof. By contradiction. Taking equation (4) and condition (5) (i.e. $s_2 \geq 0$), grain sold to region 1 by region 2 must come from either: a) region 2's grain stock ($s_2 > 0$); or b) purchases from the external market ($z_2 > 0$). Lemma 2 precludes (b).

In the former case, $s_2 > 0 \Rightarrow \mu_2 = 0 \Rightarrow \psi_2 = \psi_2^* \cdot e^{rt}$. For inter-regional trade, $y > 0$ so that for condition (10) to be true, $\psi_1 - \psi_2 = m$. Combining,

$\psi_1 = \psi_2^* \cdot e^{rt} + m$ when region 2 has grain stocks and inter-regional trade is occurring. This behaviour of the costate variable, ψ_1, is not consistent with $s_1 > 0$. Lemma 1 indicates that the presence of stocks in region 1 would mean that the costate rises exponentially, i.e. $\psi_1 = \psi_1^* \cdot e^{rt}$. Thus, $s_2 > 0$ and $z_2 > 0$ is not consistent with $s_1 > 0$.❑

Corollary 2. *Region 2 must possess own stocks of grain for the entire period of inter-regional trade.*

Proposition 3. *In the model of inter-regional grain marketing described by A1 to A6 and (1) to (6), purchases from the external market occur from some time (here, t=T_2) until the end of the program, if at all.*

Proof. While region 2 has grain, lemma 1 gives $s_2 > 0 \Rightarrow \mu_2 = 0 \Rightarrow \psi_2 = \psi_2^* \cdot e^{rt}$. After this, with no stocks in region 2, condition (4) indicates that the region can only purchase from the external market. (9) gives the condition $\psi_2 = p + d_2$ on the costate when these purchases take place and the condition $\psi_2 < p + d_2$ when no purchases are taking place. The exponential growth of the costate when stocks exist in the region is consistent with the latter strict inequality and so must precede the equality on the costate. Consumption of own stocks must precede purchases from the external market.

For region 1, conditions (2) and (3) indicates that once own stocks are exhausted, consumption could be met through purchases from the region 2 or from the external market. When inter-regional trade is taking place, proposition 2 demonstrates that $\psi_1 = \psi_2^* \cdot e^{rt} + m$. This is a strictly increasing function and so the limit imposed on the costate variable of region 1 by (9, for region 1) would be a strict inequality, i.e. $\psi_1 < p + d_1$. Only after the exhaustion of stocks and any inter-regional trade can the equality hold, $\psi_1 = p + d_1$ and so $z_1 > 0$. ❑

Propositions 1 to 3 order the regions' actions. Grain sales to the external market occur at the start of the program, at t=0, and there follows a period of time when the regions consume their own stocks of grain. When region 1 runs out of grain, inter-regional trade may occur. The corollary to proposition 2 states that region 2's grain stocks must outlast this period of inter-regional trade. At the end of the program purchases will occur. The entire period of time can therefore be split into four periods. From t=0 to T_1, both regions are consuming own stocks. After this to T_m, inter-regional trade occurs. After this, until T_2, region 1 is purchasing from the external market while region 2 still has own stocks remaining. After T_2, both regions have exhausted their stocks of grain so that the external market is the only source of grain.

Proposition 4. *In the model of inter-regional grain marketing described by A1 to A6 and (1) to (6), the costate variable is continuous.*

Proof. The number of junction points where the Jacobson-Lele-Speyer conditions (13) may be satisfied is two in the present model, j=1 when $t = T_1$ (region 1 exhausts its stock of grain) and j=2 when $t = T_2$ (when region 2 exhausts its grain). Consider region 2. At time $t = T_2$, a discontinuity would imply that the costate will decrease at that instant. However, as observed in proposition 3, during the period prior to the junction point, $\psi_2 \leq p + d_2$. After the junction point, the region purchases from the external market so that $\psi_2 = p + d_2$ which is greater than any value the ψ_1 could take prior to T_1. The costate variable cannot instantaneously fall as (13) so $\beta_2(T_2) = 0$.

For region 1, if no inter-regional trade occurs, $\beta_1(T_1) = 0$ by an analogous argument to region 2. Otherwise, at time $t = T_1$, inter-regional trade will begin so that $\psi_1 - \psi_2 \leq m$ (condition 10) will actually bind. Corollary 2 notes that during and before trade, region 2 has stocks of grain so that $\psi_2 = \psi_2^* \cdot e^{rt}$ for the period before and after trade begins. The condition $\psi_1 \leq m + \psi_2^* \cdot e^{rt}$ prior to trade means that ψ_1 cannot fall after trade begins and the condition binds, i.e. $\psi_1 = m + \psi_2^* \cdot e^{rt}$. Again, the costate variable cannot instantaneously fall as (13) so $\beta_1(T_1) = 0$. ❑

3.4 Complete specification of the model of regional trade

The four propositions allows the optimal program to be completely solved. The first three propositions allow the ordering of the various grain flows to be characterised. The order after harvest is grain sales at t=0, followed by a period of consumption of harvested grain. When region 1 exhausts its grain stocks, there may follow a period of inter-regional trade with region 2 consuming its own stock of grain as well as selling grain to meet region 1's consumption needs. After grain trading, region 2 consumes any remaining own grain while region 1 sources all consumption from the external market. Both regions purchasing from the external market are the concluding flows. Some of the control variables may remain at zero throughout the program with the control variable associated quantifying inter-regional trade requiring particular attention. When a particular control variables takes a non-zero value, the associated function of the costate variable becomes an equality. Table 1 indicates the regional costate's functional form which results due to this for the entire program when there is trade between the two regions. Table 2 details the behaviour of the costate when there is no trade between the regions and the costates are independent.

Proposition 4 indicates what happens at the points in the program where there are switches between control variables. According to the maximum principle, the costate variable is continuous at all times except when the state variable reaches a bound, zero grain stocks being the case in this program. At such junction points, proposition 4 proves that the costate functions would be continuous. The continuity of the costate throughout the program allows the points in the program when costate switching occurs to be written in terms of costate equalities, i.e. an equality sign can be placed between the functional forms before and after the point in time when a switch occurs. This allows a set of conditions to be derived. When there is trade between the two regions, the conditions are:

At $t = T_1$, $\psi_1^* \cdot e^{rT_1} = \psi_2^* \cdot e^{rT_1} + m$; (15)

At $t = T_m$, $p + d_1 = \psi_2^* \cdot e^{rT_m} + m$; (16)

At $t = T_2$, $p + d_2 = \psi_2^* \cdot e^{rT_2}$. (17)

When there is no trade between the two regions, the continuity of the costate implies that:

For ψ_i at t=T_i , $\psi_i^* e^{rT_i} = p + d_i$ i=1,2. (18)

Further, the lack of trade implies that y =0 throughout the optimal program. From condition (10) - the spatial arbitrage condition - a further relationship can be noted. The lack of inter-regional trade implies that, at the time when inter-regional trade could begin (at the exhaustion of region 1's grain stocks (see proposition 2)) the difference between regional costates is less than the cost of transfer:

$\psi_1^* \cdot e^{rT_1} - \psi_2^* \cdot e^{rT_1} < m$. (19)

This states that when region 1 has exhausted its grain supply, inter-regional trade would not be initiated as the difference in the costate variables is less than the cost of transfer.

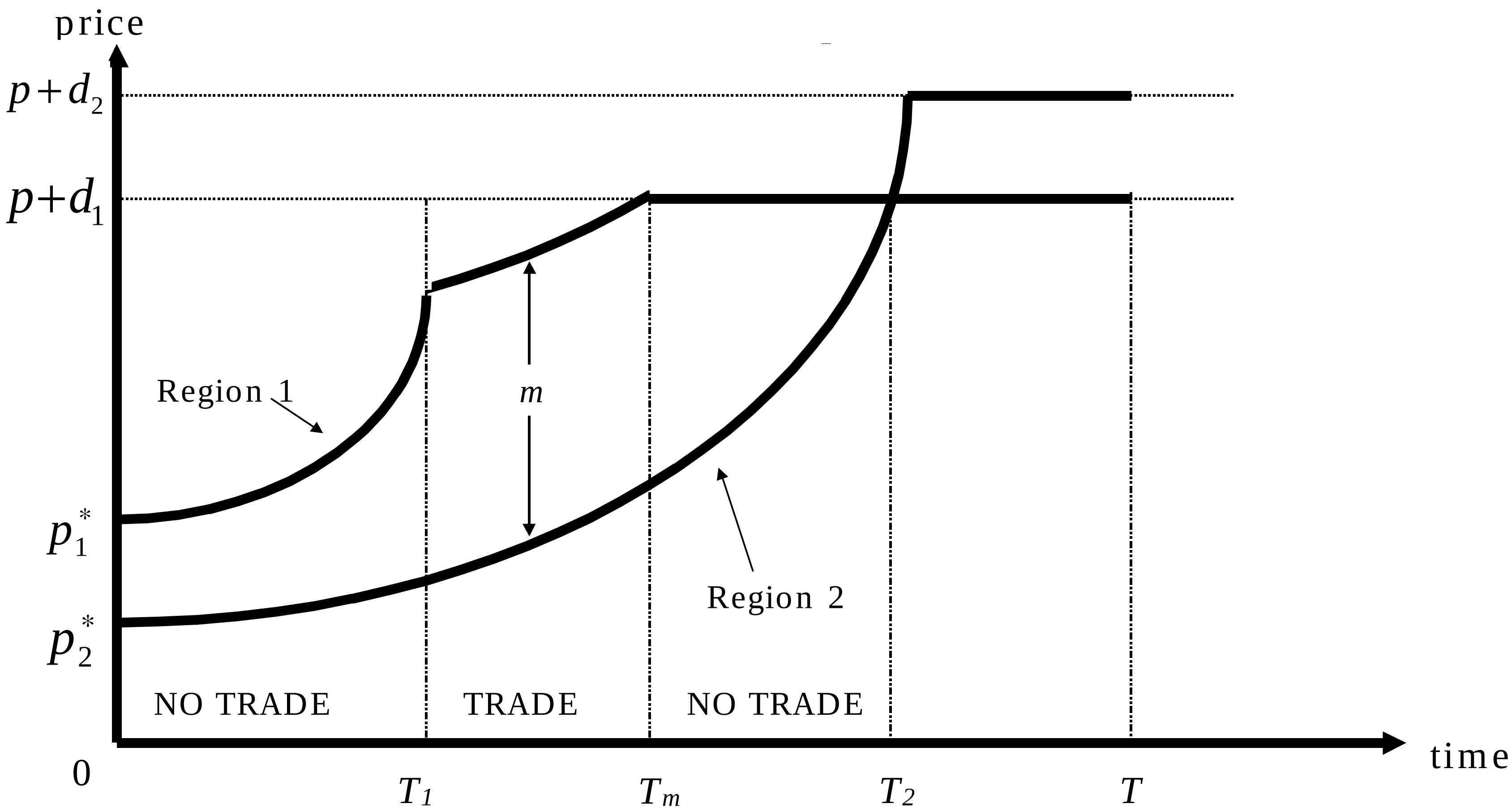
Table 1: Costate Variable Functional Forms during the Optimal Program when Trade Occurs

$t =$ 0	T_1	T_m	T_2	1
	NO TRADE	TRADE	NO TRADE	NO TRADE
ψ_1	$\psi_1^* \cdot e^{rt}$	$\psi_2^* \cdot e^{rt} + m$	$p + d_1$	$p + d_1$
ψ_2	$\psi_2^* \cdot e^{rt}$	$\psi_2^* \cdot e^{rt}$	$\psi_2^* \cdot e^{rt}$	$p + d_2$

Table 2: Costate Variable Functional Forms without Trade
i.e. $\psi_1^* \cdot e^{rT_1} - \psi_2^* \cdot e^{rT_1} < m$

$t =$ 0	T_1	T_2	1
ψ_1	$\psi_1^* \cdot e^{rt}$	$p + d_1$	$p + d_1$
ψ_2	$\psi_2^* \cdot e^{rt}$	$\psi_2^* \cdot e^{rt}$	$p + d_2$

Having derived the dynamics of the costates after harvest, the initial values taken by the costates can be considered. This depends on whether there are grain sales to the


price
p+d2
p+d1
p1*
p2*
Region 1
m
Region 2
NO TRADE
TRADE
NO TRADE
0
T1
Tm
T2
T
time

external market. Corollary 1 can be used to give the combination of values of the costate and the level of grain stocks at t=0. When one of the regions sells grain at the start of the program, the initial value the costate takes in that region is its minimum value, $p - d_i$. Also, the sales of grain by the region implies that the initial stocks of grain are some value less than the harvested level. The initial values taken be regional costate variables and stock levels differ when regions do not sell any grain to the external market. In such regions, the initial value the costate variable takes is greater that $p - d_i$. However, as no sales are made, initial grain stock levels must equal the harvested level of grains. Table 3 summarises the four possibilities.

Table 3: Initial Values of Costate and State Variables

Initial Scenario	y_1^*	y_2^*	s_1^*	s_2^*
Both regions sell grain	$p - d_1$	$p - d_2$	$h_1 - x_1$	$h_2 - x_2$
Only region 1 sells grain	$p - d_1$	$\geq p - d_2$	$h_1 - x_1$	h_2
Only region 2 sells grain	$\geq p - d_1$	$p - d_2$	h_1	$h_2 - x_2$
Neither region sell grain	$\geq p - d_1$	$\geq p - d_2$	h_1	h_2

Figure 1 indicates the behaviour of the costate variables in one of the four cases. In figure 1, initial harvest levels and parameter values are such that neither region sells grain but there is inter-regional trade. Region 1 is consuming it harvested grain until the period T_1. As there are stocks of grain in both region during this period, both regional costates rise exponentially. After this, region 1 begins purchasing grain from region 2. Until time T_m, region 1's costate differs from region 2's by the cost of transfer. During this period, price correlation may be expected as shocks in either region which affect the market clearing price would be transferred. Inter-regional trade ends when the external market becomes a cheaper source of grain for region 1 than region 2 so that region 1 purchases from the external market. From this point until T_2, region 2 is consuming its own grain and the costate continues its exponential rise. The external market becomes the sole supplier of grain after region 2 exhausts its stocks.

The total grain harvested at the start of the program provides regional constraints on the total level of flows out of, into and between the regions. As consumption of grain is identical in both regions at one per unit time, the times at which the various flows of grain switch off and on are equal to the quantity of grain transferred from a particular source. For example, in region 1, the quantity of grain available in the harvest must equal the sum of the amount region 1 sells to the external market and the length of time before the region exhausts its own supply (i.e. T_1):

$$h_1 = x_1^* + T_1. \tag{20}$$

For region 2, the picture is complicated by the region's supplying of region 1 if there is a period of inter-regional trade. In the present model, due to the simple way in which consumption is specified, the amount of grain transferred is equal to the length of time trade occurs, that is T_m - T_1:

$$h_2 = x_2^* + T_2 + T_m - T_1 . \quad (21)$$

If inter-regional trade does not take place, then the constraint on region 2 is similar to that of region 1:

$$h_2 = x_2^* + T_2 . \quad (22)$$

Equations (14)-(22) and the various combinations of costate and grain stock initial values provide systems of solvable equations for each of the four initial scenarios set out in table 2. However, the exact scenario which will occur depends on the levels of harvest in relation to the parameters of the model. These relationships are investigated further in the following section.

4. Behaviour of spatial commodity markets

4.1 Overview

This section considers the behaviour of regional markets when there is an external market and the possibility of inter-regional trade. The section focuses on the whether the two regions would trade in a particular season so that market integration would be observed. It will also consider how long such inter-regional trade will last. As might be expected, the parameters of the model and the level of harvest in the two regions jointly determine the extent of market integration.

An assumption of the previous section will be relaxed. Inter-regional trade was previously considered to be unidirectional - from region 2 to region 1. Appendix 1 indicates the conditions on the parameters and the level of the two regions' harvest when this assumption will be unsatisfactory. However, it also shows that in such circumstances, the model may be respecified with the regions exchanging places. This intuitively appealing property of the model allows the results of the following analysis to be generalised to inter-regional trade in either direction through a transformation of variables and parameters. In the two figures 2 and 3, figure 2 considers the combination of regional harvests which would yield trade from region 2 to 1. However, by reformulating the problem with the two regions switching places, figure 3 gives a more complete picture allowing trade to occur both from 2 to 1 and, in the upper left side of the diagram, from 1 to 2. It is worth noting that figure 3 is not symmetrical about the $h_1=h_2$ line unless the two regions are equidistant from the external market, i.e. $d_1=d_2$.

4.2 The occurrence of inter-regional trade

The conditions when trade do not take place provide an initial insight into the parameter relationships which determine the nature of regional grain stock management. If, after harvest, the regions do not trade with each other, the harvested grain is either consumed within the region or sold to the external market. Thus, constraints (20) and (22) are relevant to the program (i.e. harvested grain equals the sum of grain sales and the time before exhaustion of regional grain stocks). With regard to the conditions at the start of the program, the four potential scenarios detailed in table 2 need to be considered:

neither region selling grain to the external market, region 1 selling, region 2 selling and both regions making sales to the external market.

A start in analysing these various situations is to note the conditions on regional harvests for positive grain sales in both regions. Using (20) and (22), it is clear that when sales occur in regions i=1,2, and there is no inter-regional trade:

$$h_i > T_i \qquad i=1,2. \qquad (23)$$

Further, positive grain sales constrain the initial value of a region's costate variable as given by table 1b:

$$\psi_i^* = p - d_i \qquad i=1,2. \qquad (24)$$

Conditions (23) and (24) can be combined with equation (18):

For ψ_i at $t=T_i$, $\psi_i^* e^{rT_i} = p + d_i \qquad i=1,2. \qquad (18)$

For a region, this gives a minimum harvest level for grain sales to the external market from that region to occur when the two regions do not trade:

$$h_i > \frac{1}{r}\ln\frac{p+d_i}{p-d_i} \qquad i=1,2. \qquad (25)$$

These two inequalities partition the (h_1, h_2) space into distinct areas. However, as shown in figure 2, only in the situation where both regions are selling to the external markets are conditions (25) for regions i=1,2 sufficient to guarantee no inter-regional trade. This is area B in figure 2 and occurs when both areas have very high harvest levels. For this to be a sufficient condition, equation (19) needs to be considered:

$$\psi_1^* \cdot e^{rT_1} - \psi_2^* \cdot e^{rT_1} < m. \qquad (19)$$

(In appendix 2, this condition is explored further.)

Area C gives combinations of regional harvest levels where sales to the external market are made by region 2 and there is no inter-regional trade. Inter-regional trade does not occur only if region 2's sales to the external market could not be cheaply transferred to the other market. This is the case if condition (19) is met throughout the program. By noting that the maximum value the difference between regional costates can take will be at time, $t=T_1$, and, as there are no grain sales by region 1, this occurs at $t=h_1$, (19) can be re-written. Substituting into (18) using $t=h_1$, for region 1:

$$\psi_1^* \cdot e^{rh_1} = p + d_1 \qquad (18)$$

and the results from table 2:

$$\psi_2^* \cdot e^{rh_1} = (p - d_2) \cdot e^{rh_1} \qquad (26)$$

and solving gives the condition:

$$h_1 > \frac{1}{r}\ln\frac{(p+d_1-m)}{(p-d_2)}. \qquad (27)$$

Figure 2: Regional Harvest Levels and Inter-Regional Trade

A No Inter-regional Trade or Grain Sales to the External Market
B No Inter-regional Trade; Both Regions sell to the External Market
C No-Inter-regional Trade; Region 2 sells to the External Market
D Trade from Region 2 to 1 with Region 2 also selling to the External Market
E Trade from Region 2 to 1 with Neither Region selling to the External Market

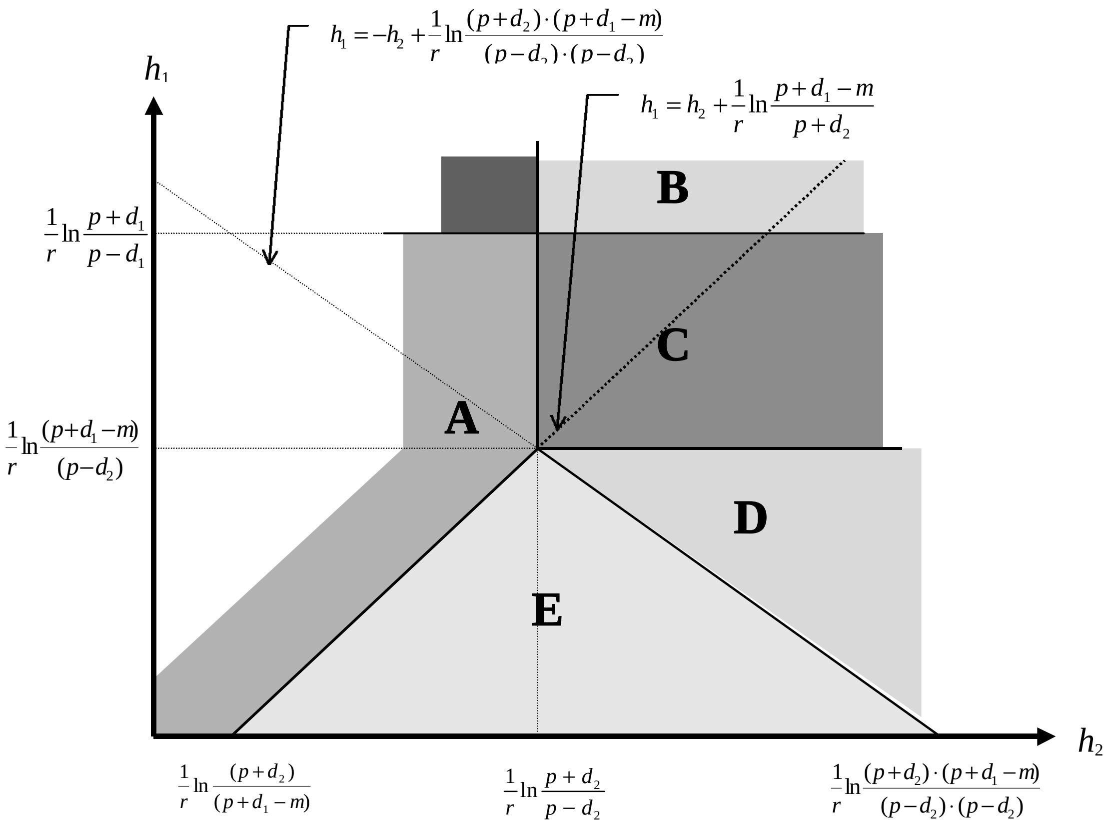
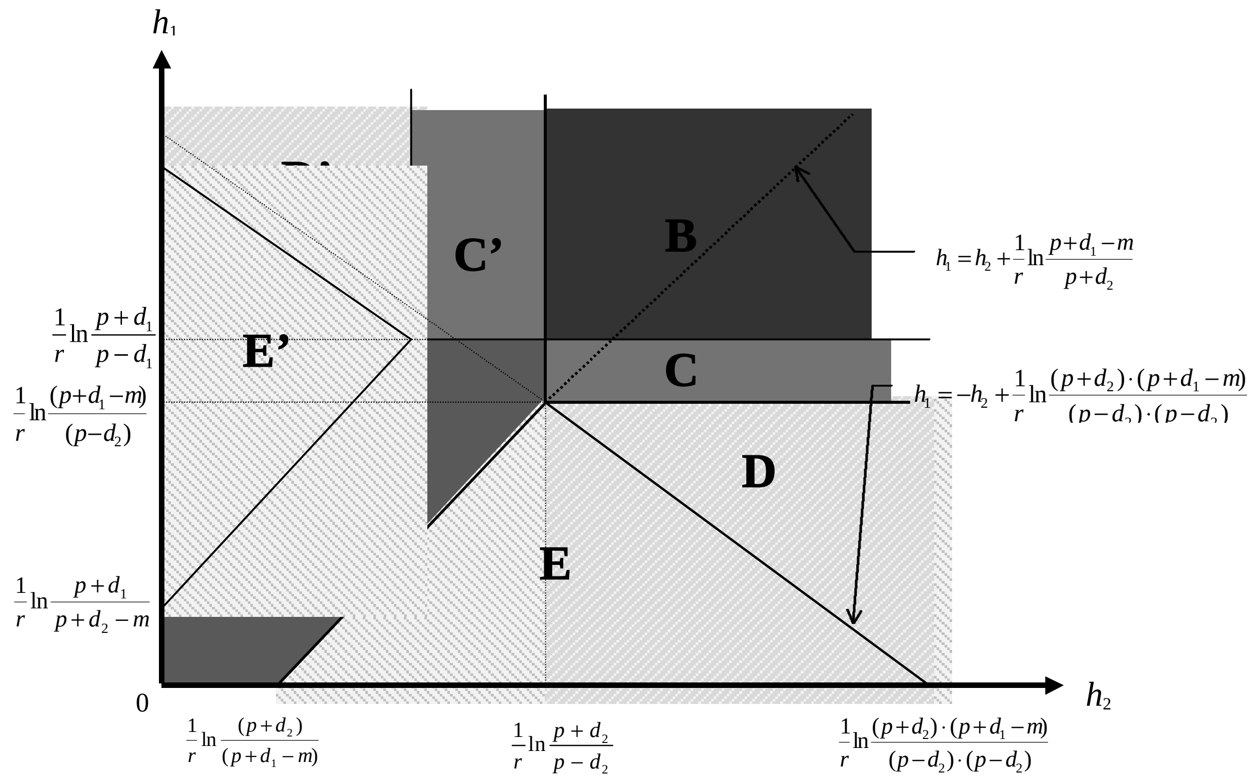


Figure 3: Regional Harvest Levels and Inter-Regional Trade

A	No Inter-regional Trade or Grain Sales to the External Market
B	No Inter-regional Trade; Both Regions sell to the External Market
C	No-Inter-regional Trade; Region 2 sells to the External Market
C’	No-Inter-regional Trade; Region 1 sells to the External Market
D	Trade from Region 2 to 1 with Region 2 also selling to the External Market
D’	Trade from Region 1 to 2 with Region 1 also selling to the External Market
E	Trade from Region 2 to 1 with Neither Region selling to the External Market
E’	Trade from Region 1 to 2 with Neither Region selling to the External Market



This is indicated as region C in figure 2. The reflection of this is indicated as region C' in figure 3.

Area A indicates combinations of harvest which result in no inter-regional trade and no sales to the external market. This area shows that equal or nearly equal harvests give rise to regional autarky. This is not surprising and can be explained by noting the identical consumption needs of the regions. When harvests are similar, both regions exhaust their grain stocks at a similar time and so it is unlikely that a region will sell to its neighbour. Appendix 1 indicates the derivation of the region. It is shown there that if inter-regional trade is only allowed in the direction of region 2 to 1, then the conditions on the harvest are:

$$h_1 \geq h_2 \tag{28}$$

$$h_1 \leq h_2 + \frac{1}{r}\ln\frac{p + d_2 - m}{p + d_1}. \tag{29}$$

The first condition notes that the distribution of harvests favours region 1. In such circumstances, there would be not trade in the direction of 2 to 1. The second condition is the bound put on the extent to which region 1's harvest can exceed 2's without causing price divergence to be great enough for trade *from* region 1 *to* region 2.

The dark area made up from A, B and C indicates combinations of regional harvest which would make regional trade unnecessary as own stocks and the external market would be adequate to meet the needs of the regions. Area A indicates that equal or near equal harvest levels would also make trade unlikely. The likelihood of falling into the regions C and D can be proxied by considering the two inequalities:

$$h_1 \geq \frac{1}{r}\ln\frac{(p + d_1 - m)}{(p - d_2)}; \tag{30}$$

$$h_2 \geq \frac{1}{r}\ln\frac{(p + d_2 - m)}{(p - d_1)}. \tag{31}$$

If both are true, then inter-regional trade does not occur as the harvest combination would be in the dark area of figure 2. From (30) and (31), it is apparent that the higher the rate of discount, the less likely trade is. Trade, however, is more likely to occur when regions are distant from the external market and the transfer costs between the regions is small.

Appendix 2 qualifies conditions (30) and (31) somewhat. Under certain parameter values, only one of the two conditions can be used to differentiate harvest levels where trading takes place from harvest levels without trading. When parameters take these values one of (30) and (31) will be replaced by one of the conditions (25).

4.3 The timing of trade

When there is inter-regional trade, figure 2 distinguishes between two possible scenarios. Inter-regional trade from region 2 to region 1 can take place in the presence of region 2 selling to the external market, indicated as area D. However, when region 2's harvest is low, such inter-regional trade can occur with no grain sales to the external market by either region. In figure 2, this is indicated as E. Trade from region 1 to region 2 is indicated by areas D’ (with region 1 selling to the external market) and E’ (no sales) in figure 3. The conditions on the harvest levels in the two regions to differentiate between the two can be derived by looking at the length of time that trade occurs and then comparing these results with the conditions for grain sales, that is, the pair of equations (25).

Consider firstly the situation where only region 2 sells grain both to the external market and the other region. Clearly, the lack of sales to the external market from region 1 means that its harvested grain is exhausted through consumption and the time taken by this is equal to the level of harvest, i.e.:

$$T_1 = h_1. \tag{32}$$

Trade begins at time T_1. Region 2’s sale of grain indicates that the initial value of the costate can be derived from equation (24) as $p - d_2$. This combines with (16) to give:

$$p - d_1 = (p - d_2) \cdot e^{rT_m} + m. \tag{33}$$

The time when trading ends can then be easily derived:

$$T_m = \frac{1}{r} \ln \frac{p + d_1 - m}{p - d_2}. \tag{34}$$

The time when region 2 exhausts its own supply of grain, T_2, is derived by combining (32) with (17) to give:

$$p + d_2 = (p - d_2) \cdot e^{rT_2}. \tag{35}$$

This solves to provide:

$$T_2 = \frac{1}{r} \ln \frac{p + d_1}{p - d_2}. \tag{36}$$

Now, consider the constraints (20) and (21) which, when combined, state that the total harvest must equal total grain sales, grain consumption and the amount of grain traded between the regions:

$$h_1 + h_2 = x_1^* + x_2^* + T_m + T_2. \tag{37}$$

Noting that for grain sales from region 2 to occur, $x_2^* > 0$, and that $x_1^* = 0$:

$$h_1 > -h_2 + \frac{1}{r} \ln \frac{(p + d_2) \cdot (p + d_1 - m)}{(p - d_2) \cdot (p - d_2)}. \tag{38}$$

The inequality (38) splits the region D from E - combinations of regional harvests which combine inter-regional trade with and without region 2 selling grain respectively.

The timing of trading derived can provide some indication of the length of time that markets trade. Trade occurs during the period T_m - T_1 and, when sales are made to the external market by region 2, this is equal to:

$$T_m - T_1 = \frac{1}{r} \ln \frac{p + d_1 - m}{p - d_2} - h_1. \tag{39}$$

When there is no grain sales to the external market, the length of time that trade takes place becomes:

$$T_m - T_1 = \frac{1}{2r} \ln \frac{p + d_1 - m}{p + d_2} + \frac{h_2 - h_1}{2}. \tag{40}$$

Equations (39) and (40) both indicate that the more expensive it is for region 1 to buy grain from the external market, the value $p+d_1$, the longer the trade. Cheapness in procuring grain from the neighbouring region, low m, and a low rate of discount, r, has a similar effect. The denominator in the logged fraction indicates the impact of grain sales on the level of trade. When region 2 is selling grain to the external market, the revenue gained through selling grain, $p - d_2$, is negatively related to the duration of inter-regional trade. When no grain sales occur, the opportunity cost of selling grain to region 1 becomes the cost of grain purchases from the external market, $p + d_2$.

5. Discussion and conclusions

In this paper, the Takayama-Judge spatial arbitrage model has been extended to consider the situation where there is storage of the commodity. A model presented by Williams and Wright (1989) highlights how commodity storage leads to breakdowns in temporal arbitrage rules. This paper extends this result into a dynamic seasonal model of storage. Previous models have assumed that inter-regional trade would occur throughout a year and then proceeded to test for price co-movements. Here, it is suggested that periods of time where there is no trade may be the result of storage substituting for trade as an efficient means to offset transportation costs. This substitutability depends crucially on the cost of transfer between the regions and the rate of discount.

The optimal control model and figures 2 and 3 highlight how likely it is that trade takes place in a particular year with a particular distribution of harvest. The lighter shaded parts of figure 3 give combinations of regional harvests where trade would take place and the Takayama-Judge condition on inter-regional could be expected to bind for some part of the year. Further as trade is taking place, market clearing prices in the two regions would be interdependent so that price correlation would be expected. However, the model presented here would only expect regional trade to occur for some portion of a year. Inter-regional trade throughout a year would not be optimal in the model so that periods where regional market prices are not correlated would be expected.

In years where harvest levels are in the dark area of figure 3, there is no inter-regional trade in the optimal program. In such years, there would be no reason to expect correlation between market prices in the two regions and market integration tests would be inappropriate. Because there would be no reason to suppose price changes would be transferred between markets, such an analysis of the price data may erroneously suggest market trade is in some way obstructed. The diagrams indicate that this is most likely to be the case when two regions have similar initial stocks levels (harvests). Trade would not be beneficial in such circumstances.

The results of this paper highlight the effect of spatial and temporal interaction on food market behaviour. In areas where storage of grain is significant, the market integration tests must be used with some caution. The primary result of the present analysis is that trade cannot be assumed to be a constant feature of food markets.

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Appendix 1

In this appendix, the conditions under which inter-regional flows of grain will be from region 2 to region 1 is discussed. A situation is considered where there is no trade and the inter-regional difference in costate variable is less than the cost of transfer. However, as $\psi_2 > \psi_1$, if trade were to occur, it would be *towards* region 2. Inter-regional trade does not occur as long as:

$$\psi_2^* \cdot e^{rt} - \psi_1^* \cdot e^{rt} < m \qquad (1.1)$$

is guaranteed throughout the program. If both regions do not trade, and assuming no grain sales to the external market, so that $h_i = T_i$, and using equation (18):

$$\psi_i^* = (p + d_i) \cdot e^{-rh_i} \qquad (1.2)$$

Combining (1.1) and (1.2) gives:

$$(p + d_2) \cdot e^{r(t-h_2)} - (p + d_1) \cdot e^{r(t-h_1)} < m \qquad (1.3)$$

Inter-regional trading from region 1 to 2 would occur when region 2 exhausts its grain stock if at all. Thus, if (1.3) holds at $t = h_2$ then it will have held during the entire program. This gives the condition (which is assumption 6):

$$h_2 > h_1 - \frac{1}{r} \ln \frac{p + d_1}{p + d_2 - m}. \qquad (A6)$$

Note that A6 can be rewritten so that region 1 and region 2 exchange places. This can then be rearranged so that h_2 appears on the left-hand side:

$$h_2 < h_1 + \frac{1}{r} \ln \frac{p + d_2}{p + d_1 - m} \qquad (1.4)$$

The areas given by (A6) and (1.4) overlap. This indicates that there is no pair of harvest levels which will not fall in one of the two areas. Some harvest pairs will lie in both. The direction of inter-regional trade - region 2 to 1 if A6 holds, region 1 to 2 if (1.4) holds, either direction is both are satisfied - can then be decided on this basis. This means that the present model can be applied to all harvest combinations.

Appendix 2

Figures 1 and 2 have been drawn assuming that:

$$\frac{p + d_i}{p - p_i} > \frac{p + d_i - m}{p - d_j} \qquad (i,j)=(1,2),(2,1) \qquad (2.1)$$

It can readily be derived that this will be the case, if:

$$d_1 \cdot (d_2 - d_1 + m) < p \cdot (d_1 - d_2 + m) \qquad (2.2)$$

in the case where $(i,j)=(1,2)$ and:

$$d_2 \cdot (d_1 - d_2 + m) < p \cdot (d_2 - d_1 + m) \qquad (2.3)$$

in the case where $(i,j)=(2,1)$.

It can readily be seen that if the two regions are the same distance from the external market, then conditions (2.2) and (2.3) will be satisfied because assumption 2 states that $d_1, d_2 < p$. The extent to which *p* exceeds the transfer costs for regions to the external market determines whether (2.2) and (2.3) are satisfied. For a wide range of parameter values, the two conditions are satisfied so that diagrams 2 and 3 can be viewed as representing the most likely scenario. However, when one of the two conditions is not satisfied, e.g. the sign of (2.2) is reversed:

$$d_1 \cdot (d_2 - d_1 + m) > p \cdot (d_1 - d_2 + m) \qquad (2.4)$$

The fact that $p > d_1$ and $p > d_2$ necessarily implies that (2.3) must hold. Thus, at least one of (2.2) and (2.3) is always true. Figure 4 indicates what the effect of (2.4) is on the extent of market trading. It indicates that region C in figure 3 no longer exists. That is, region 2 selling to the external market is no longer consistent with no inter-regional trade. The intuition behind this is that the parameters of the model are such that region 1 is more integrated with region 2 than with the external market. If region 2 has sufficient grain to sell to the external market, then some part of those sales would be firstly claimed by region 1. Note that the region C' still indicates: region 2 can sell grain to the external market and to the other region.

Figure 3: Regional Harvest Levels and Inter-Regional Trade

A	No Inter-regional Trade or Grain Sales to the External Market
B	No Inter-regional Trade; Both Regions sell to the External Market
C’	No-Inter-regional Trade; Region 1 sells to the External Market
D	Trade from Region 2 to 1 with Region 2 also selling to the External Market
D’	Trade from Region 1 to 2 with Region 1 also selling to the External Market
E	Trade from Region 2 to 1 with Neither Region selling to the External Market
E’	Trade from Region 1 to 2 with Neither Region selling to the External Market